Sturm-Liouville Differential Operators with an Interior Singularity

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Contents

Chapter 1

Introduction

In this paper the main goal is to examine the Sturm-Liouville eigenvalue problem

$$\big(- Dp(x)D + q(x)\big)u(x) = \lambda u(x) \tag{1.1}$$

on an interval $[a, b] \subset \mathbb{R}$ with interior singularity, as described in the article [3] by W.N. Everitt, J. Gunson and A. Zettl. In order to do so, we use two different approaches. The first approach uses Operator Theory, the second uses solutions in the sense of the so-called hyper-functions.

First of all, we consider the corresponding Sturm-Liouville differential operator(s) in the Hilbert-space $L^2(a, b)$ without any interior singularities.
In chapter 2 we derive certain (basic) properties of these operators, which enable us to describe the corresponding self-adjoint operator H, as well as in the case with regular endpoints (chapter 3), as in the case with singular endpoints (chapter 4).
In chapter 5 it is shown that the resolvent of the hence constructed self-adjoint differential operator H is an integral operator. Moreover, in special cases the resolvent is a Hilbert-Schmidt operator, which directly implies some beautiful properties of the eigenvalues and eigenfunctions of H. This is described in chapter 6.
In chapter 7 the (Sturm-Liouville) differential equations in the complex plane with interior singularity are briefly discussed, and the form of its possible solutions is derived.

After this, we are ready to return to our initial Sturm-Liouville eigenvalue-problem 1.1 with interior singularity. With the help of the following special case of such a problem

$$\big(- D^2 - \frac{1}{x}\big)u(x) = \lambda u(x), \quad \text{on the interval } [a, b]$$

with $-\infty < a < 0 < b < \infty$ (which is discussed in [3]), we describe the method which uses Operator Theory as discussed in the previous chapters in order to solve this problem. This is done in chapter 8.

In our last chapter (chapter 9) we use a totally different approach of problem 1.1 and we show that the use of solutions in the sense of hyper-functions can yield a similar result as described in chapter 8. Again, we are able to describe the desired properties of the eigenvalues and eigenfunctions.

Finally, two important theorems from the Operator Theory are presented in the appendices A and B, whereas the appendix C on page 81 deals with the concept of 'variation of parameters'. These complete the paper.

Chapter 2

The Sturm-Liouville Differential Operator

We consider the second-order differential expression

$$L = -Dp(x)D + q(x) \qquad \text{with } D = \frac{d}{dx} \quad \text{on an interval } (a,b) \subseteq \mathbb{R}\,.$$

with the following restrictions for p and q:

- $p(x)$, $q(x)$ are real-valued, measurable functions for $x \in \mathbb{R}$.
- $\frac{1}{p(x)}$, $q(x) \in L^1_{loc}(a,b)$.
- $p(x) > 0$ on (a,b).

Such a differential expression is called a Sturm-Liouville differential expression. Since p and q are real-valued, the differential expression L obviously is symmetric in the sense that

$$\int_a^b L\phi(x)\overline{\psi(x)}dx = \int_a^b \phi(x)\overline{L\psi(x)}dx$$

for $\phi, \psi \in C_0^\infty(a,b)$. The endpoint a is called regular if $a > -\infty$ and both $\frac{1}{p}$ and q belong to $L^1_{loc}[a,b)$. If not, the endpoint a is called singular. A similar definition holds for the regularity or singularity of the endpoint b. The differential expression L is regular if both a and b are regular, otherwise L is called singular.

The purpose of this section is to examine the above described differential expression as a Sturm-Liouville differential operator defined on the Hilbert-space $L^2(a,b)$ with inner product:

$$(f,h) = \int_a^b f(x)\overline{h}(x)dx, \quad f,h \in L^2(a,b).$$

In order to do so, we begin by defining the set $\mathcal{D}_p$ as follows:

$$\mathcal{D}_p = \{f : (a,b) \to \mathbb{C} \mid f, pf' \in AC_{loc}(a,b)\}.$$

We make the following definition for each function $f, h \in \mathcal{D}_p$:

$$\begin{aligned}
[f,h](x) :&= -(pf')(x)\overline{h}(x) + f(x)(\overline{ph}')(x) \\
&= \begin{pmatrix} h(x) \\ (ph')(x) \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(x) \\ (pf')(x) \end{pmatrix} \quad , \ x \in (a,b).
\end{aligned}$$

We are now ready to define a maximal domain for the differential operator L on $L^2(a,b)$. First of all, we demand the functions f in this domain to be in $\mathcal{D}_p$ so that the expression 'Lf' makes sense. Furthermore, we require L to map elements from $L^2(a,b)$ in $L^2(a,b)$, since we want L to be an operator on $L^2(a,b)$. To make this domain as large as possible, we require no boundary conditions whatsoever.We use the graph-notation which results in the following definition for the "maximal operator" L_{max}:

$$L_{max} := \Big\{ \{f, Lf\} \mid f \in \mathcal{D}_p,\ f, Lf \in L^2(a,b) \Big\}.$$

For each $f, h \in \operatorname{dom} L_{max}$ we can make the following definition:

$$\langle \{f, Lf\}, \{h, Lh\} \rangle := (Lf, h) - (f, Lh).$$

For all $f, h \in \operatorname{dom} L_{max}$ we can now deduce the following by partial integration for all sets $[c,d] \subset (a,b)$ (where we restrict the differential operator to the interval $[c,d]$):

$$\begin{aligned}
&\langle \{f, Lf\}, \{h, Lh\} \rangle|_c^d = \\
&= (Lf, h)|_c^d - (f, Lh)|_c^d \\
&= \int_c^d (Lf)(x)\overline{h}(x)dx - \int_c^d f(x)(\overline{Lh})(x)dx \\
&= \int_c^d \Big(-(p(x)f'(x))' + q(x)f(x) \Big) \overline{h}(x)dx \\
&\qquad - \int_c^d f(x) \Big(-(\overline{p(x)h'(x)})(x)' + (\overline{q(x)h(x)})(x) \Big) dx \\
&= [-(pf')(x)\overline{h}(x)]|_c^d + [f(x)(\overline{ph}')(x)]|_c^d \\
&= [f,h]|_c^d.
\end{aligned} \tag{2.1}$$

From equation 2.1 and the properties of the elements of L_{max} we can deduce Green's formula:

Lemma 2.1 (Green's formula.)
For $f, h \in \operatorname{dom} L_{max}$, the following limits exist:

$$\lim_{x \downarrow a}[f,h](x) =: [f,h](a) \qquad \lim_{x \uparrow b}[f,h](x) =: [f,h](b)$$

and

$$\langle \{f, Lf\}, \{h, Lh\} \rangle |_a^b = [f,h]|_a^b.$$

We consider the eigenvalue-problem $(L-\lambda)f = g$. When do we call a function f a solution for this problem?

Definition 2.2 *For a complex number λ, the equality*

$$(L-\lambda)f = g$$

means: $f \in \mathcal{D}_P$, g measurable and $(L-\lambda)f = g$ almost everywhere.

... and how do we know if there exists such a solution?

Lemma 2.3 *For all $g \in L^1_{loc}(a,b)$, for all $\lambda \in \mathbb{C}$ and for all $x_0 \in (a,b)$, $\xi_1, \xi_2 \in \mathbb{C}$ there exists a unique function $f : (a,b) \to \mathbb{C}$, which satisfies:*

$$(L-\lambda)f = g \qquad \text{and} \qquad \begin{aligned} f(x_0) &= \xi_1 \\ pf'(x_0) &= \xi_2. \end{aligned}$$

Outline of the proof.
Define $y_1 := f$ and $y_2 := pf' = py_1'$. We can now rewrite our differential equation $(L-\lambda)f = g$ as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{p} \\ (q-\lambda) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} \quad \text{with} \quad \begin{cases} y_1(x_0) = \xi_1 \\ y_2(x_0) = \xi_2 \end{cases}.$$

For this system of differential equations it can be shown that a solution as described in the lemma can be found by using Picard's successive approximation method. ■

Now we can construct a solution as described in lemma 2.3 by using solutions u_1, u_2 of the equation $Lu = 0$. First we derive a certain property of the so-called Wronskian determinant, defined as follows:

$$W(u_1, u_2, x) = \det \begin{pmatrix} u_1(x) & u_2(x) \\ pu_1'(x) & pu_2'(x) \end{pmatrix}.$$

Lemma 2.4 *Assume u_1 and u_2 are two solutions of the equation $Lu = 0$. Then $W(u_1, u_2, x)$ is constant (i.e. independent of x) for all $x \in (a, b)$ and the solutions are linearly independent if and only if this constant is not equal to 0.*

Proof.
The solutions u_1 and u_2 are linearly independent if and only if there exists an $x_1 \in (a, b)$ such that the Wronskian determinant in x_1 does not vanish:

$$W(u_1, u_2, x_1) = [u_1, \overline{u_2}](x_1) \neq 0.$$

Since u_1, u_2 are solutions of $Lu = 0$, we know from formula 2.1 on page 6 that $[u_1, u_2](c) = [u_1, u_2](d)$ for all $c, d \in (a, b)$ and hence $W(u_1, u_2, x)$ is a constant independent of x. Since this constant cannot be equal to 0, we have $W(u_1, u_2, x) \neq 0$ for all $x \in (a, b)$. ■

Assume $\{u_1, u_2\}$ is a fundamental system of the equation $Lu = 0$. We define the following matrix S:

$$\begin{aligned} S := &\begin{pmatrix} [u_1, u_1] & [u_2, u_1] \\ [u_1, u_2] & [u_2, u_2] \end{pmatrix} \\ = &\begin{pmatrix} \overline{u_1} & \overline{pu_1}' \\ \overline{u_2} & \overline{pu_2}' \end{pmatrix} \begin{pmatrix} -pu_1' & -pu_2' \\ u_1 & u_2 \end{pmatrix} \\ = &\begin{pmatrix} u_1 & u_2 \\ pu_1' & pu_2' \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ pu_1' & pu_2' \end{pmatrix}. \end{aligned}$$

From lemma 2.4 we know $\det S \neq 0$. Hence, the inverse of S can be computed. We define the kernel $K(x, y)$ as follows:

$$K(x, y) := \begin{pmatrix} u_1(x) & u_2(x) \end{pmatrix} S^{-1} \begin{pmatrix} u_1(y) & u_2(y) \end{pmatrix}^*.$$

Lemma 2.5 *For $g \in L^1_{loc}$ and $f(x) := \int_{x_0}^{x} K(x, y) g(y) dy$ we have: $f \in \mathcal{D}_p$ and $Lf = g$.*

Proof.
For an f as defined in the lemma we can write for its derivative:

$$f'(x) = K(x, x) g(x) + \int_{x_0}^{x} \frac{\partial K(x, y)}{\partial x} g(y) dy.$$

A simple calculation shows that

$$K(x, x) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}^* = 0$$

and

$$p(x)\frac{\partial K}{\partial x}(x,y)\Big|_{y=x} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}^* = -1.$$

This yields

$$\begin{aligned}(pf')'(x) &= \int_{x_o}^{x} \frac{\partial}{\partial x}\Big(p(x)\frac{\partial K}{\partial x}(x,y)\Big)dy + \Big(p(x)\frac{\partial K}{\partial x}(x,y)\Big)\Big|_{y=x} g(x)\\ &= \int_{x_0}^{x} q(x)K(x,y)g(y)dy - g(x)\\ &= q(x)f(x) - g(x)\end{aligned}$$

and thus $Lf(x) = (pf')'(x) - q(x)f(x) = g(x)$ and by the properties of u_1, u_2 (and therefore by the properties of $K(x,y)$) f belongs to $\mathcal{D}_p$. ■

Now we will take a closer look at our differential operator L_{max} and derive some useful properties of its elements. Note that for a $c \in (a,b)$, c is automatically a regular endpoint of our differential form restricted to (a,c) or (c,b). Moreover, we have the following lemma:

Lemma 2.6 *For any c_1, c_2, d_1, $d_2 \in \mathbb{C}$ there exists an $f \in \operatorname{dom} L_{max}$ such that for $[c,d] \subset (a,b)$ the following holds:*

$$\begin{aligned} f(c) &= c_1 & \qquad f(d) &= d_1\\ pf'(c) &= c_2 & \qquad pf'(d) &= d_2.\end{aligned}$$

Proof.
First we consider the case with $c_1 = c_2 = 0$. According to lemma 2.3 on page 7 there exist two linearly independent solutions u_1 and u_2 of the equation $Lu = 0$ with

$$\begin{aligned} u_1(d) &= 1 & \qquad u_2(d) &= 0\\ pu_1'(d) &= 0 & \qquad pu_2'(d) &= 1.\end{aligned}$$

Define $v := \gamma_1 u_1 + \gamma_2 u_2$ with $\gamma_1, \gamma_2 \in \mathbb{C}$ such that

$$\begin{aligned}(v,u_1) &= \gamma_1(u_1,u_1) + \gamma_2(u_2,u_1) = -d_2\\ (v,u_2) &= \gamma_1(u_1,u_2) + \gamma_2(u_2,u_2) = d_1.\end{aligned}$$

This can be done since:

$$\begin{pmatrix}(v,u_1)\\(v,u_2)\end{pmatrix} = \begin{pmatrix}(u_1,u_1) & (u_2,u_1)\\ (u_1,u_2) & (u_2,u_2)\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$$

where the invertibility of $\begin{pmatrix} (u_1, u_1) & (u_2, u_1) \\ (u_1, u_2) & (u_2, u_2) \end{pmatrix}$ follows from

$$\begin{pmatrix} \overline{x} & \overline{y} \end{pmatrix} \begin{pmatrix} (u_1, u_1) & (u_2, u_1) \\ (u_1, u_2) & (u_2, u_2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \|xu_1 + yu_2\|^2 \neq 0$$

for all $x, y \in \mathbb{C}$ since u_1 and u_2 are linearly independent.
According to lemma 2.3 on page 7 there exists a solution f_1 of the problem

$$Lf = v, \qquad f(c) = 0, \quad (pf')(c) = 0.$$

Then by using lemma 2.1 on page 7 restricted to the interval $[c, d]$ we have

$$\begin{aligned} d_1 &= \ \ (v, u_2) = \ \ (Lf_1, u_2) = \ \ [f_1, u_2]|_c^d + (f_1, Lu_2) = \ \ [f_1, u_2]|_c^d = f_1(d) \\ d_2 &= -(v, u_1) = -(Lf_1, u_1) = -[f_1, u_1]|_c^d - (f_1, Lu_1) = -[f_1, u_1]|_c^d = pf_1'(d). \end{aligned}$$

The same reasoning holds when we consider the case with $d_1 = d_2 = 0$ and $c_1, c_2 \in \mathbb{C}$. We then find a function f_2 which satisfies $f_2(c) = c_1$, $pf_2'(c) = c_2$ and $f_2(d) = pf_2'(d) = 0$. Combining these functions results in the function $f = f_1 + f_2$ we were looking for. ■

We introduce a minimal operator $L_{min} \subset L_{max}$ with the most 'strict' conditions possible:

$$L_{min} = \Big\{ \{f, Lf\} \in L_{max} \mid \text{supp}(f) \text{ is compact and contained in } (a, b) \Big\}^C.$$

A useful property of the elements f of dom L_{max} is described in the following lemma:

Lemma 2.7 *For an interval $[c, d] \subset (a, b)$ each $f \in \text{dom}\, L_{max}$ can be written as follows:*

$$f(x) = f_-(x) + f_+(x) - f_0(x) \quad \textit{with } f_-, f_+ \in \text{dom}\, L_{max}, \ f_0 \in \text{dom}\, L_{min}$$

and $f_0(x) = f(x)$ on $[c, d]$,

$$f_-(x) = \begin{cases} f(x) & \textit{on } (a, d) \\ f_0(x) & \textit{on } (d, b) \end{cases} \quad f_+(x) = \begin{cases} f(x) & \textit{on } (c, b) \\ f_0(x) & \textit{on } (a, c) \end{cases}.$$

Proof.
Our proof consists of defining $f_0(x)$ in such a way that it has the required properties. We begin by choosing an $\epsilon > 0$ with $[c - \epsilon, d + \epsilon] \subset (a, b)$ for an

interval $[c, d]$ as described in the lemma. By lemma 2.6 on page 9 we know that there exist functions u_1, u_2 such that the following holds:

$$\begin{array}{cccc} u_1(c-\epsilon) = 0 & u_1(c) = f(c) & u_2(d) = f(d) & u_2(d+\epsilon) = 0 \\ pu_1'(c-\epsilon) = 0 & pu_1'(c) = pf'(c) & pu_2'(d) = pf'(d) & pu_2'(d+\epsilon) = 0. \end{array}$$

We are now ready to define the function $f_0 \in \operatorname{dom} L_{min}$ with the required properties:

$$f_0(x) = \begin{cases} 0 & \text{on } (a, c-\epsilon) \\ u_1(x) & \text{on } [c-\epsilon, c) \\ f(x) & \text{on } [c, d] \\ u_2(x) & \text{on } (d, d+\epsilon] \\ 0 & \text{on } (d+\epsilon, b). \end{cases}$$

■

We recall the following definition of the adjoint of an operator S on a Hilbert-space $\mathcal{H}$:

$$S^* = \Big\{ \{u, v\} \mid \langle \{u, v\}, \{f, g\} \rangle = 0 \text{ for all } \{f, g\} \in S \Big\}.$$

It can easily be derived that $S^* = (S^c)^*$. We are ready to present the definitions of a symmetric operator S and a self-adjoint operator S on a Hilbert-space $\mathcal{H}$:

Definition 2.8 *An operator S defined on a Hilbert-space $\mathcal{H}$ is symmetric if the following holds:*

- $\operatorname{dom} S$ *is dense in* $\mathcal{H}$.
- S *is a closed operator.*
- $S \subset S^*$ *(i.e.:* $\operatorname{dom} S \subset \operatorname{dom} S^*$ *and* $Sf = S^* f$ *for all* $f \in \operatorname{dom} S$*).*

An operator S is self-adjoint if it is symmetric and the following holds:

$$S = S^*.$$

The following theorem states an important relation between the differential operators L_{min} and L_{max}:

Theorem 2.9 *L_{min} and L_{max} are densely defined, closed operators on $L^2(a, b)$ with $L_{min}^* = L_{max}$, and hence L_{min} is symmetric.*

Proof.
Choose an $f \in \text{dom } L_{min}$ with $\text{supp}(f) = [\alpha, \beta] \subset (a, b)$ and an $h \in \text{dom } L_{max}$. From lemma 2.1 on page 7 we know:

$$\langle \{f, Lf\}, \{h, Lh\} \rangle|_a^b = [f, h](b) - [f, h](a) = 0$$

and hence $L_{max} \subset L_{min}^*$.
Now assume $\{h, k\} \in L_{min}^*$. Then $(g, h) = (f, k)$ for all $\{f, g\} \in L_{min}$. We wish to prove $\{h, k\} \in L_{max}$. We know $h, k \in L^2(a, b)$. This leaves us with the proof of:

(a) $h \in \mathcal{D}_p$.

(b) $Lh = k$ almost everywhere on every set $[c, d] \subset (a, b)$.

We prove this with the use of the definition of the null space:

$$\widetilde{\ker} L = \{u \mid Lu = 0 \text{ on } [c, d]\}.$$

Choose

$$\tilde{g} \in L^2(a, b) := \begin{cases} g = 0 & \text{on } (a, b) \backslash [c, d], \\ g \perp \widetilde{\ker} L & \text{on } [c, d] \end{cases}$$

and

$$\tilde{k} := \begin{cases} k & \text{on } [c, d] \\ 0 & \text{on } (a, b) \backslash [c, d]. \end{cases}$$

We define

$$\tilde{f}(x) = \int_c^x K(x, y) \tilde{g}(y) dy$$
$$h_0(x) = \int_c^x K(x, y) \tilde{k}(y) dy.$$

Then $\tilde{f}$ has compact support which is contained in $[c, d]$ and we know by lemma 2.5 on page 8 that $L\tilde{f} = \tilde{g}$ and thus $\{\tilde{f}, \tilde{g}\} \in L_{min}$. Since $\{h, k\} \in L_{min}^*$ we have

$$(\tilde{g}, h)|_c^d = (\tilde{g}, h) = (\tilde{f}, k) = (\tilde{f}, \tilde{k}) = (\tilde{f}, \tilde{k})|_c^d.$$

By lemma 2.5 on page 8 $h_0 \in D_p$ and $Lh_0 = \tilde{k}$ and hence

$$(\tilde{g}, h_0)|_c^d = (L\tilde{f}, h_0)|_c^d = (\tilde{f}, Lh_0)|_c^d = (\tilde{f}, \tilde{k})|_c^d.$$

It follows that $(\tilde{g}, h - h_0)|_c^d = 0$ and thus $(h - h_0)|_{[c,d]} \in \widetilde{\ker} L$. For a certain $u \in \widetilde{\ker} L$ we can write $h = u + h_0$ on $[c, d]$. This holds for any arbitrary interval $[c, d] \in (a, b)$, hence (a) holds. Furthermore we have:

$$Lh = L(u + h_0) = Lh_0 = \tilde{k} = k \quad \text{on } [c, d].$$

This proves (b). This completes our proof of $L_{min}^* = L_{max}$, which also implies the closedness of L_{max}.
Obviously, the following holds:

$$\Big\{\{f, Lf\} \in L_{max} \mid \text{supp}(f) \text{ is compact and contained in } (a,b)\Big\} \subset L_{max},$$

which implies $L_{min} \subset L_{max} = L_{min}^*$. As L_{max} is an operator, so is L_{min}. Finally, from $L_{min}^* = L_{max}$ we know that L_{min} is densely defined and thus the same holds for L_{max}. Hence, L_{min} is a symmetric operator. ■

An other way to describe the minimal operator is the following:

Lemma 2.10

$$L_{min} = \Big\{\{f, Lf\} \in L_{max} \mid [f,h](a) = [f,h](b) = 0 \textit{ for all } h \in \text{dom}\, L_{max}\Big\}.$$

Proof.
Define

$$T := \Big\{\{f, Lf\} \in L_{max} \mid [f,h](a) = [f,h](b) = 0 \text{ for all } h \in \text{dom}\, L_{max}\Big\}.$$

We wish to prove $L_{min} = T$. With the help of lemma 2.1 on page 7 it can easily be seen that $T \subset L_{max}^* = L_{min}$, which leaves us with the proof of $L_{min} \subset T$. Choose $f \in \text{dom}\, L_{min}$. Since L_{min} is symmetric, we have $[f,h]|_a^b = 0$ for all $h \in \text{dom}\, L_{max}$. We can write h as in lemma 2.7 on page 10:

$$h(x) = h_-(x) + h_+(x) - h_0(x)$$

with $h_-, h_+ \in \text{dom}\, L_{max}$ and $h_0 \in \text{dom}\, L_{min}$ as described by the lemma. Then the following holds:

$$[f,h](a) = [f,h_-](a) = -([f,h_-](b) - [f,h_-](a)) = 0$$

since $h_- \in \text{dom}\, L_{max}$. In a similar way $[f,h](b) = 0$ can be derived and hence $L_{min} \subset T$. ■

We consider the following definition:

Definition 2.11 (a) *The defect space of an operator T is defined by the set* $\ker(T^* - \lambda)$, *for a* $\lambda \in \mathbb{C}$.

(b) *The defect index $\gamma(T,\lambda)$ of an operator T is the dimension of its defect space, i.e.* $\gamma(T,\lambda) = \dim\ker(T^* - \lambda)$.

We present the so-called first Von Neumann formula for a closed, symmetric operator which states that such an operator S can be represented as follows:

$$\operatorname{dom} S^* = \operatorname{dom} S \oplus \ker(S^* - \lambda) \oplus \ker(S^* - \overline{\lambda}).$$

It is known that for a symmetric operator S its defect indices are constant on $\mathbb{C}^+$ and $\mathbb{C}^-$ and therefore can be denoted by γ_+ and γ_- respectively.

Lemma 2.12 *The defect indices γ_+, γ_- of L_{min} are equal and $0, 1$ or 2.*

Proof.
Since L_{min} is symmetric, we know that the defect indices of L_{min} are constant in the upper and lower half-planes of $\mathbb{C}$. We denote these defect indices by $\gamma_+ = \dim\ker(L^*_{min} - i)$ and $\gamma_- = \dim\ker(L^*_{min} + i)$ respectively. In using $L^*_{min} = L_{max}$ we can rewrite the defect space as follows:

$$\ker(L^*_{min} - \lambda) = \ker(L_{max} - \lambda) = \{f \in \operatorname{dom} L_{max} | L_{max} f = \lambda f\},$$

which has either dimension 0,1 or 2. Assume $y \in \ker(L^*_{min} - \lambda$. Then y is a solution of the equation $L_{max} y - \lambda y = 0$ and (since L_{max} is real) $\overline{y}$ is a solution of the equation $L_{max}\overline{y} - \overline{\lambda y} = 0$. Hence, for every solution y with $\lambda \in \mathbb{C}^+$ there exists a solution $\overline{y}$ for the equation with $\overline{\lambda} \in \mathbb{C}^-$, i.e. $\gamma_+ = \gamma_-$. ∎

From now on, we use γ to denote the defect index $\gamma_+ = \gamma_-$. As self-adjoint differential operators have some very useful properties which are described in chapter 5, our goal is to construct a differential operator H as follows:

- $L_{min} \subset H \subset L_{max}$.
- H is self-adjoint.

Such an operator H is called a canonical, self-adjoint extension of L_{min} and H exists if and only if $\gamma_+ = \gamma_-$, thus lemma 2.12 shows that L_{min} has canonical self-adjoint extensions. The case where $\gamma = 0$ is a special case:

Corollary 2.13 *The differential operator L_{min} is self-adjoint if and only if its defect index γ is 0.*

Proof.
Since $\gamma_+ = \gamma_-$, the first Von Neumann formula gives us the following representation for L^*_{min}:

$$\text{dom } L^*_{min} = \text{dom } L_{min} \oplus \ker(L^*_{min} - \lambda) \oplus \ker(L^*_{min} - \overline{\lambda})$$

Since the dimensions of $\ker(L^*_{min} - \lambda)$ and $\ker(L^*_{min} - \overline{\lambda})$ are both zero, we can conclude $\text{dom } L^*_{min} = \text{dom } L_{min}$. Since for the operator and its adjoint the following holds: $L^*_{min} \subset L_{min}$, we can conclude $L_{min} = L^*_{min} = L_{max}$, hence, L_{min} is self-adjoint. ■

Finally we remark the fact that in both cases $\gamma = 1$ and $\gamma = 2$ the extension H is closed, since it is embedded in the two closed operators L_{min} and L_{max}.

Chapter 3

The Regular Case

In this chapter we will consider the case where a and b are regular endpoints. First, we will describe some properties of L_{min} and L_{max} in the case of a regular endpoint a (b respectively).

Lemma 3.1 *If a is a regular endpoint of the differential operator, then for every function $f \in \operatorname{dom} L_{max}$ the following limits exist:*

$$\lim_{x\to a} f(x) =: f(a) \qquad \lim_{x\to a}(pf')(x) =: (pf')(a)$$

and thus for all $f, h \in \operatorname{dom} L_{max}$

$$[f,h](a) = \lim_{x\to a}[f,h](x) = f(a)\overline{(ph')}(a) - (pf')(a)\overline{h}(a).$$

The same results holds with 'a' replaced by 'b' if b is a regular endpoint.

Proof.
We choose functions $\psi_1, \psi_2 \in \mathcal{D}_p$, such that the following holds:

$$\begin{array}{lll} L\psi_1 = 0 & \psi_1(a) = 1 & p\psi_1'(a) = 0 \\ L\psi_2 = 0 & \psi_2(a) = 0 & p\psi_2'(a) = 1 \end{array}$$

and we consider an interval $[c,d] \in (a,b)$. According to lemma 2.6 on page 9 there exist functions $u_1, u_2 \in \operatorname{dom} L_{max}$, such that

$$u_i(c) = \psi_i(c), \quad pu_i'(c) = p\psi_i'(c) \text{ and } u_i(d) = pu_i'(d) = 0, \quad i = 1,2.$$

We now define functions h_1, h_2 as follows:

$$h_i(x) = \begin{cases} \psi_i(x) & x \in (a,c) \\ u_i(x) & x \in [c,d] \\ 0 & x \in (d,b) \end{cases} \quad i = 1,2.$$

This implies $h_1, h_2 \in \operatorname{dom} L_{max}$. The limits $\lim_{x\to a}[f, h_i](x)$ exists for $i = 1, 2$ according to lemma 2.1 on page 7 and can be computed:

$$\begin{aligned}[f, h_i](a) &= \lim_{x\to a}[f, h_i](x) = \lim_{x\to a}[f, \psi_i](x) \\ &= \lim_{x\to a} -pf'(x)\overline{\psi}_i(x) + f(x)\overline{p\psi_i'}(x) \quad i = 1, 2.\end{aligned}$$

We can rewrite this as follows:

$$\begin{pmatrix}[f, h_1](x)\\ [f, h_2](x)\end{pmatrix} = \begin{pmatrix}\psi_1(x) & \psi_2(x)\\ p\psi_1'(x) & p\psi_2'(x)\end{pmatrix}^* \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}f(x)\\ pf'(x)\end{pmatrix}.$$

Since the limits for $x \to a$ on the lefthandside of this equation exist and the first matrix on the righthandside tends to the identity-matrix for $x \to a$ and is hence invertible on (a, c), we can rewrite this equation as follows:

$$\begin{pmatrix}f(x)\\ pf'(x)\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\psi_1(x) & \psi_2(x)\\ p\psi_1'(x) & p\psi_2'(x)\end{pmatrix}^{-*}\begin{pmatrix}[f, h_1](x)\\ [f, h_2](x)\end{pmatrix}.$$

This implies the existence of the limits as described in the lemma. ■

By extending lemma 2.6 on page 9 to the regular endpoint a the following can be shown:

Lemma 3.2 *If a is a regular endpoint then for all $\alpha, \beta \in \mathbb{C}$ there exists an $f \in \operatorname{dom} L_{max}$ such that $f(a) = \alpha$, $(pf')(a) = \beta$.*
A similar argument holds in the case of a regular endpoint b.

Thus for an $f \in \operatorname{dom} L_{max}$ both $f(x)$ and $(pf')(x)$ can be continuously extended in a (resp b). Moreover, lemma 3.1 on the page before implies the following:

$$[f, g](a) = 0 \text{ for all } g \in L_{max} \text{ if and only if } f(a) = (pf')(a) = 0$$

which implies the following definition for L_{min} in the case where both endpoints a, b are regular:

Lemma 3.3 *For a, b regular endpoints the following holds:*

$$L_{min} = \Big\{\{f, Lf\} \in L_{max} \mid f(a) = (pf')(a) = f(b) = (pf')(b) = 0\Big\}.$$

From now on we will consider the case where both endpoints a and b are regular and we will examine the form and existence of the self-adjoint extension H of L_{min} we are looking for. We start by remarking the fact that both defect indices γ of L_{min} are equal to 2, i.e.:

$$\gamma = \dim\ker(L_{max} - \lambda) = 2, \quad \lambda \in \mathbb{C},$$

since the equation $(L_{max} - \lambda)u = 0$ has two linearly independent (and obviously continuous) solutions $y_1, y_2 \in L^2(a,b)$. Hence, we know that a self-adjoint extension H of L_{min} exists. The following theorem describes its form:

Theorem 3.4 *An operator H is a self-adjoint canonical extension of L_{min} if and only if H can be written as follows:*

$$H = \left\{ \{f, Lf\} \in L_{max} \mid M \begin{pmatrix} f(a) \\ pf'(a) \end{pmatrix} + N \begin{pmatrix} f(b) \\ pf'(b) \end{pmatrix} = 0 \right\}$$

where $M, N \in \mathbb{C}^{2\times 2}$ satisfy the following equations:

$$MJM^* - NJN^* = 0, \; J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \operatorname{rank}(M:N) = 2.$$

Proof.

($\Leftarrow$) : Assume H has the form as described in the theorem. First of all, we wish to prove $H \subset H^*$. Since rank $(M : N) = 2$, there exist invertible matrices $M^+, N^+ \in \mathbb{C}^{2\times 2}$ such that the matrix $S = \begin{pmatrix} M & N \\ M^+ & N^+ \end{pmatrix} \in \mathbb{C}^{4\times 4}$ is invertible. We define the projection '$\hat{\ }$' as follows:

$$\hat{\ }: C^2[a,b] \to \mathbb{C}^4, \quad \hat{g} = \begin{pmatrix} g(a) \\ pg'(a) \\ g(b) \\ pg'(b) \end{pmatrix} \in \mathbb{C}^4.$$

According to lemma 3.2 on the preceding page there exist functions

$$g_1, \dots, g_4 \text{ in } \mathbb{C}^2[a,b]$$

such that

$$\hat{g}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \hat{g}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{g}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{g}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

which proves the surjectivity of the projection ' ^ '. But then for all $\alpha, \beta \in \mathbb{C}^2$ there exist functions $f, g \in C^2[a,b]$ such that

$$\hat{f} = S^{-1}\begin{pmatrix}0\\ \alpha\end{pmatrix}, \quad \hat{g} = S^{-1}\begin{pmatrix}0\\ \beta\end{pmatrix}. \tag{3.1}$$

For functions $f, g \in \operatorname{dom} H$ which satisfy 3.1 the following holds:

$$\begin{aligned}
\langle\{f, Lf\}, \{g, Lg\}\rangle|_a^b &= [f,g](b) - [f,g](a)\\
&= \hat{g}^* \begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix} \hat{f}\\
&= (S\hat{g})^* S^{*-1} \begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix} S^{-1}(S\hat{f})\\
&= \begin{pmatrix}0 & \beta^*\end{pmatrix} \begin{pmatrix}P & Q\\ R & T\end{pmatrix} \begin{pmatrix}0\\ \alpha\end{pmatrix}\\
&= 0.
\end{aligned}$$

The last equation holds since $T = 0$, which can be shown by writing out the expression $\left(S\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}S^*\right)^{-1}$. Hence, $H \subset H^*$.

Now assume $f \in \operatorname{dom} H^*$. Let $\beta \in \mathbb{C}^2$. Then by the surjectivity of the projection and the invertibility of S there is a $g \in \operatorname{dom} L_{max}$ such that $S\hat{g} = \begin{pmatrix}0\\ \beta\end{pmatrix}$, in particular $g \in \operatorname{dom} H$. We have:

$$\begin{aligned}
0 &= \langle\{f, Lf\}, \{g, Lg\}\rangle\\
&= \begin{pmatrix}0 & \beta^*\end{pmatrix} \begin{pmatrix}P & Q\\ R & 0\end{pmatrix} \begin{pmatrix}\gamma\\ \alpha\end{pmatrix}\\
&= \beta^* R\gamma.
\end{aligned}$$

Since β is arbitrary, $R\gamma = 0$ and since R is invertible, this implies $\gamma = 0$. Thus $f \in \operatorname{dom} H$ and $H^* \subset H$.

($\Rightarrow$) : Now we assume $H = H^*$ is a self-adjoint extension of L_{min}. First of all, we wish to prove that H is a two-dimensional extension of L_{min}, i.e.

$$H = L_{min} \oplus \mathcal{M}, \text{ with } \mathcal{M} = \text{L.S.}\big\{\{h_i, Lh_i\}, i = 1, 2\big\}, \tag{3.2}$$

where $\{h_i, Lh_i\}$ are linearly independent elements of L_{max}, such that

$$L_{min} \cap \mathcal{M} = \big\{\{0,0\}\big\}.$$

Note that this implies $H^* = L_{max} \cap \mathcal{M}^*$.

Assume $\mathcal{M}$ is k-dimensional, i.e. $\mathcal{M} = \text{L.S.}\{\{h_i.Lh_i\}, i = 1, \ldots, k\}$, where $H = L_{min} \oplus \mathcal{M}$ such that $\{h_i, Lh_i\}$ are linearly independent elements of L_{max}, with

$$L_{min} \cap \mathcal{M} = \{\{0,0\}\}. \tag{3.3}$$

Obviously, $\mathcal{M} \subset L_{max}$ and $k \leq \dim L_{max}/L_{min} = 4$ holds. The cases $k = 0$ and $k = 4$ cannot occur, since that would imply the self-adjointness of L_{min}. We consider the following space:

$$L_{max} \ominus (L_{max} \cap \mathcal{M}^*) = \text{L.S.}\{\{a_j, b_j\}\},\ j = 1, \ldots, l,$$

where the elements $\{a_j, b_j\}$ are linearly independent elements of L_{max}. The dimension l of this space is finite. We define a $k \times l$-matrix C with elements c_{ij} as follows:

$$c_{ij} = \langle \{a_j, b_j\}, \{h_i, Lh_i\} \rangle, \quad i = 1 \ldots, k,\ j = 1, \ldots, l.$$

Assume $l > k$. Then the l columns of the matrix are linearly dependent and hence there exists a nonzero vector $x \in \mathbb{C}^l$, such that $Cx = 0$. The following must hold for all $i = 1, \ldots, k$:

$$\langle \sum_{j=1}^{l} \{a_j, b_j\} x_j, \{h_i, Lh_i\} \rangle = 0,$$

which implies $\sum_{j=1}^{l} \{a_j, b_j\} x_j \in L_{max} \cap \mathcal{M}^*$. But we also have

$$\sum_{j=1}^{l} \{a_j, b_j\} x_j \in L_{max} \ominus (L_{max} \cap \mathcal{M}^*).$$

Hence, $\sum_{j=1}^{l} \{a_j, b_j\} x_j = 0$, which implies $x = 0$. But the vector x was nonzero, and we may conclude $l \leq k$.
Now we assume $k > l$. In that case, the rows of the matrix C must be linearly dependent and we can find a nonzero vector $y \in \mathbb{C}^k$ such that $y^* C = 0$. This implies

$$\langle \{a_j, b_j\}, \sum_{i=1}^{k} \{h_i, Lh_i\} \bar{y}_i \rangle = 0,$$

for all $j = 1, \dots, l$. Since for all $\{f, g\} \in L_{max} \cap \mathcal{M}^*$ we have

$$\langle \{f, g\}, \sum_{i=1}^{k} \{h_i, Lh_i\} \bar{y}_i \rangle = 0,$$

and moreover $L_{max} = (L_{max} \cap \mathcal{M}^*) \oplus \big(L_{max} \ominus (L_{max} \cap \mathcal{M}^*)\big)$, we can conclude that $\sum_{i=1}^{k} \bar{y}_i \{h_i, Lh_i\} \in L_{max}^* = L_{min}$. Hence by equation 3.3 we know

$$\sum_{i=1}^{k} \bar{y}_i \{h_i, Lh_i\} = 0,$$

which implies $y = 0$. Again, this contradicts the fact that y is a nonzero vector, and we may conclude that $k = l$. We have

$$L_{max} \ominus (L_{max} \cap \mathcal{M}^*) = L_{max} \ominus (L_{min} \oplus \mathcal{M}).$$

The space on the lefthandside has dimension k, the space on the righthand-side has dimension $4 - k$. Hence, $k = 2$, which proves 3.2.

Hence, there exist two linearly independent functions

$$h_1, h_2 \in \operatorname{dom} L_{max} / \operatorname{dom} L_{min}$$

such that $\operatorname{dom} H = \operatorname{dom} L_{min} \oplus \text{L.S.}\{h_1, h_2\}$. Then for $i = 1, 2$ we have for any $f \in \operatorname{dom} H = H^*$:

$$\begin{aligned}
0 &= \langle \{f, Lf\}, \{h_i, Lh_i\} \rangle \\
&= \hat{h}_i^* \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \hat{f} \\
&= \begin{pmatrix} \overline{h_i(a)} & \overline{ph_i'(a)} & \overline{h_i(b)} & \overline{ph_i'(b)} \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \hat{f} \\
&= \begin{pmatrix} \overline{-ph_i'(a)} & \overline{h_i(a)} & \overline{ph_i'(b)} & \overline{-h_i'(b)} \end{pmatrix} \hat{f},
\end{aligned}$$

which yields

$$0 = \begin{pmatrix} -ph_1'(a) & -ph_2'(a) \\ h_1(a) & h_2(a) \end{pmatrix}^* \begin{pmatrix} f(a) \\ pf'(a) \end{pmatrix} + \begin{pmatrix} -ph_1'(b) & -ph_2'(b) \\ h_1(b) & h_2(b) \end{pmatrix}^* \begin{pmatrix} f(b) \\ pf'(b) \end{pmatrix}.$$

We can make the following definitions:

$$M = \begin{pmatrix} -ph_1'(a) & -ph_2'(a) \\ h_1(a) & h_2(a) \end{pmatrix}^*, \quad N = \begin{pmatrix} -ph_1'(b) & -ph_2'(b) \\ h_1(b) & h_2(b) \end{pmatrix}^*.$$

The operator H can be written as follows:

$$H = \left\{ \{f, Lf\} \in L_{max} \mid M \begin{pmatrix} f(a) \\ pf'(a) \end{pmatrix} + N \begin{pmatrix} f(b) \\ pf'(b) \end{pmatrix} = 0 \right\}$$
$$= \left\{ \{f.Lf\} \in L_{max} \mid \langle \{f, Lf\}, \{h_i, Lh_i\} \rangle = 0, i = 1, 2 \right\}. \quad (3.4)$$

The matrices M and N can be replaced by PM and PN respectively for any invertible 2×2 matrix P. We take a closer look at the matrix

$$M : N = \begin{pmatrix} \overline{-ph_1'(a)} & \overline{h_1(a)} & \overline{-ph_1'(b)} & \overline{h_1(b)} \\ \overline{-ph_2'(a)} & \overline{h_2(a)} & \overline{-ph_2'(b)} & \overline{h_2(b)} \end{pmatrix}.$$

Its rows must be linearly independent, since if they were not, a non-trivial linear combination $\alpha h_1 + \beta h_2$ must belong to L_{min}. This is a contradiction since $L_{min} \cap \mathcal{M} = \{\{0,0\}\}$. Hence, rank $M : N = 2$. The last step in this proof is to show that the matrices M and N satisfy the following:

$$MJM^* - NJN^* = 0. \quad (3.5)$$

Since h_1 and h_2 belong to H, they satisfy $[h_i, h_j]|_a^b = 0$, for all $i, j = 1, 2$. Using this equality, we can proof the following:

$$MJM^* - NJN^* = \begin{pmatrix} [h_1, h_1]|_a^b & [h_2, h_1]|_a^b \\ [h_1, h_2]|_a^b & [h_2, h_2]|_a^b \end{pmatrix} = 0$$

which proves 3.5.

Hence all $f \in \operatorname{dom} H$ satisfy

$$M \begin{pmatrix} f(a) \\ pf'(a) \end{pmatrix} + N \begin{pmatrix} f(b) \\ pf'(b) \end{pmatrix} = 0$$

for certain $M, N \in \mathbb{C}^{2\times 2}$ with rank $M : N = 2$ and $MJM^* - NJN^* = 0$.

■

The corresponding boundary eigenvalue-problem, described as follows:

$$BEP \begin{cases} Lf = \lambda f \\ M \begin{pmatrix} f(a) \\ pf'(a) \end{pmatrix} + N \begin{pmatrix} f(b) \\ pf'(b) \end{pmatrix} = 0 \end{cases}$$

with M, N as described above, is also self-adjoint.

Chapter 4

The Singular Case

In this chapter we consider the case where either a or b or both endpoints of the differential operator are singular. As in the previous chapter, our goal is to examine the existence and form of the self-adjoint extension H of L_{min}. In order to say something about the existence of H, we need to compute the defect index γ of our operator L_{min}. The following theorem distinguishes two possibilities for a singular endpoint b:

Theorem 4.1 (Weyl's alternative) *Let L be a Sturm-Liouville differential expression defined on (a,b) as before and suppose b is a singular endpoint of L. Let $c \in (a,b)$. There are two options:*

Limit circle case: For every $\lambda \in \mathbb{C}$, all solutions u of $(L-\lambda)u = 0$ belong to $L^2(c,b)$.

Limit point case: For every $\lambda \in \mathbb{C}$, there exists at least one solution of $(L-\lambda)u = 0$ for which $u \notin L^2(c,b)$. In this case, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, there exists exactly one solution (up to a multiplicative constant) of the equation $(L-\lambda)u = 0$ which belongs to $L^2(c,b)$.

Proof.
First, we prove the following:

(*) *If for a given $\lambda_0 \in \mathbb{C}$, all solutions u of the equation $(L-\lambda_0)u = 0$ are of class $L^2(c,b)$, then for all $\lambda \in \mathbb{C}$ all solutions u of the equation $(L-\lambda)u = 0$ are in $L^2(c,b)$.*

Let v_1 and v_2 be two linearly independent solutions of $(L-\lambda_0)u = 0$. Then v_1, $v_2 \in L^2(c,b)$ as presumed. Without loss of generality we may assume

$W(v_1, v_2) = 1$ (for we know $W(v_1, v_2) = C$ for a $C \in \mathbb{C}\backslash\{0\}$). We now consider a $\lambda \in \mathbb{C}$ and a solution u of $(L-\lambda)u = 0$. Then u is also a solution of the following equation:

$$(L-\lambda_0)u = (\lambda - \lambda_0)u$$

and u can be written as follows, using the method of variation of parameters (see also appendix C):

$$\begin{aligned}
u(x) &= c_1 v_1(x) + c_2 v_2(x) + v_1(x)\int_c^x v_2(y) W(v_1, v_2)^{-1}\big((\lambda-\lambda_0)u(y)\big)dy \\
&\quad - v_2(x)\int_c^x v_1(y) W(v_1, v_2)^{-1}\big((\lambda-\lambda_0)u(y)\big)dy \\
&= c_1 v_1(x) + c_2 v_2(x) + (\lambda-\lambda_0)\int_c^x \big(v_1(x)v_2(y) - v_2(x)v_1(y)\big)u(y)dy.
\end{aligned}$$

Since $v_1, v_2 \in L^2(c,b)$, we only need to prove $w(x) \in L^2(c,b)$ with

$$w(x) := (\lambda-\lambda_0)\int_c^x \big(v_1(x)v_2(y) - v_2(x)v_1(y)\big)u(y)dy.$$

We make the following definitions:

$$\begin{aligned}
v(x) &:= |v_1(x)| + |v_2(x)| \\
M &:= |\lambda-\lambda_0|^2 \int_c^b v(y)^2 dy.
\end{aligned}$$

Using the Schwarz-inequality we get:

$$\begin{aligned}
|w(x)|^2 &\le |\lambda-\lambda_0|^2 \left| \int_c^x \big(v_1(x)v_2(y) - v_2(x)v_1(y)\big)u(y)dy \right|^2 \\
&\le |\lambda-\lambda_0|^2 \int_c^x |v_1(x)v_2(y) - v_2(x)v_1(y)|^2 dy \int_c^x |u(y)|^2 dy \\
&\le M v(x)^2 \int_c^x |u(y)|^2 dy.
\end{aligned}$$

Since $v(x) \in L^2(c,b)$, there exists a $d \in (c,b)$ such that $\int_d^b v(x)^2 dx \le \frac{1}{4M}$. Then for all $x_1 \in (d,b)$ we have:

$$\begin{aligned}
\int_d^{x_1} |w(x)|^2 dx &\le M \int_d^{x_1} v(x)^2 \left\{ \int_c^x |u(y)|^2 dy \right\} dx \\
&\le M \int_d^b v(x)^2 \left\{ \int_c^{x_1} |u(y)|^2 dy \right\} dx \\
&\le \frac{1}{4} \int_c^{x_1} |u(y)|^2 dy
\end{aligned}$$

Since $u(x) = c_1v_1(x) + c_2v_2(x) + w(x)$ and $v_1, v_2 \in L^2(c,b)$ there exists a constant $C > 0$ such that $\int_c^{x_1} |u(x)|^2dx \leq C + \frac{1}{2}\int_c^{x_1} |w(y)|^2dy$.
This yields the following equation:

$$\int_d^{x_1} |w(y)|^2dy \leq \frac{1}{2}C + \int_c^d |w(y)|^2dy.$$

Since $u(x)$ is a solution of $(L-\lambda)u = 0$, the function $w(x)$ is continuous on (a,b). Hence the integral on the righthandside of the equation is finite. Letting $x_1 \to b$ we see that $w(x) \in L^2(d,b)$, and thus $w(x) \in L^2(c,b)$. This proves (*).

In order to prove the last part of the theorem, we use a more geometric approach. This proof also gives a good explanation of the use of the names 'limit circle' and 'limit point'.

We assume $b = \infty$ is the singular endpoint (the case were b is a finite singular endpoint is similar to this case). We first choose a $b < \infty$, and then let $b \to \infty$ at the end of our proof.

Consider the equation $(L-\lambda)u = 0$ for an arbitrary $\lambda \in \mathbb{C}$. As L is a second-order differential form, it has at most two solutions, say, $v_1(x)$ and $v_2(x)$. Those solutions can be chosen in such a way that the following holds:

$$\begin{aligned} v_1(a) &= \sin(\alpha) & v_2(a) &= \cos(\alpha) & \\ p(a)v_1'(a) &= -\cos(\alpha) & p(a)v_2'(a) &= \sin(\alpha) & \text{with } 0 \leq \alpha < \pi. \end{aligned}$$

Then $v_1(x)$ and $v_2(x)$ are linearly independent and v_1, v_2, v_1' and v_2' are entire functions of λ. We can also compute $[v_1, v_2](a) = \cos^2\alpha + \sin^2\alpha = 1$. From this and equation 2.1 on page 6 we know that $[v_1, v_2](x) = 1$ for all $x \in (a,b)$. For every $\lambda \in \mathbb{R}$ the functions $v_1(x)$ and $v_2(x)$ are real and v_1 and v_2 satisfy the following equations:

$$\begin{aligned} \cos(\alpha)v_1(a) + \sin(\alpha)p(a)v_1'(a) &= 0 \\ \sin(\alpha)v_2(a) - \cos(\alpha)p(a)v_1'(a) &= 0. \end{aligned}$$

We now consider an arbitrary solution $u(x)$ ($\neq v_2(x)$) of $(L-\lambda)u = 0$. The solution $u(x)$ can be written as a linear combination of v_1 and v_2, e.g. $u(x) = v_1(x) + mv_2(x)$, for a certain $m \in \mathbb{C}$. In order to make $u(x)$ more specific, we want $u(x)$ to satisfy the following boundary condition in b:

$$\cos(\beta)u(b) + \sin(\beta)p(b)u'(b) = 0 \qquad \text{with } 0 \leq \beta < \pi. \tag{4.1}$$

Then m must satisfy

$$m = -\frac{\cot(\beta)v_1(b) + p(b)v_1'(b)}{\cot(\beta)v_2(b) + p(b)v_2'(b)} = m(\lambda, b, \beta).$$

As v_1, v_2, v_1', v_2' are entire functions of λ, m is a real, meromorphic function of λ for $\lambda \in \mathbb{R}$. For fixed λ and b, m can be written as follows:

$$m(z) = -\frac{Az + B}{Cz + D} \qquad \text{where } z := \cot\beta$$

$$A := v_1(b) \qquad C := v_2(b) \tag{4.2}$$

$$B := p(b)v_1'(b) \qquad D := p(b)v_2'(b). \tag{4.3}$$

For $0 \leq \beta < \pi$, z is real. The function $m(z)$ as described above maps the real axis ($z \in \mathbb{R}$) on the circle C_b. We can conclude that $u(x)$ satisfies the boundary condition 4.1 on the preceding page if and only if m is on the circle C_b. We rewrite the equation as follows:

$$z = -\frac{B + Dm}{A + Cm}$$

and consider the case where $\operatorname{Im} z = 0$. This is equal to the following equation, which is our new description of the circle C_b:

$$(B + Dm)(\overline{A + Cm}) - (\overline{B + Dm})(A + Cm) = 0. \tag{4.4}$$

From equation 4.4 we can derive the following equations for the center of C_b ($= \tilde{m}_b$) and its radius ($= r_b$):

$$\tilde{m}_b = \frac{A\overline{D} - B\overline{C}}{\overline{C}D - C\overline{D}} \tag{4.5}$$

$$r_b = \frac{|AD - BC|}{|\overline{C}D - C\overline{D}|}. \tag{4.6}$$

We can rewrite the equations 4.4, 4.5 and 4.6 with the help of equations 4.2 and 4.3 as follows:

$$\overline{u}(b)p(b)u'(b) - u(b)p(b)\overline{u}'(b) = 0 \Leftrightarrow [u, u](b) = 0 \tag{4.7}$$

$$\tilde{m}_b = -\frac{[v_1, v_2](b)}{[v_2, v_2](b)} \tag{4.8}$$

$$r_b = \frac{|[v_1, \overline{v}_2](b)|}{|[v_2, v_2](b)|} = \frac{1}{|[v_2, v_2](b)|} \quad (\text{since } [v_1, \overline{v}_2](x) = 1 \text{ for all } x). \tag{4.9}$$

The coefficient of $m\overline{m}$ in equation 4.4 is equal to

$$C\overline{D} - \overline{C}D = [v_2, v_2].$$

The interior of the circle C_b is hence given by:

$$\frac{[u,u](b)}{[v_2,v_2](b)} < 0. \tag{4.10}$$

From Green's formula 2.1 on page 7 it follows that:

$$[v_2, v_2]|_a^b = \int_a^b Lv_2\overline{v}_2 - v_2\overline{Lv_2}dx = (\lambda - \overline{\lambda}) \int_a^b \overline{v}_2 v_2 dt.$$

When we use $[v_2, v_2](a) = (v_2 p\overline{v}_2' - v_2' p\overline{v}_2)(a) = 0$ this equation results in:

$$[v_2, v_2](b) = 2i \operatorname{Im}\lambda \int_a^b |v_2|^2 dt. \tag{4.11}$$

In applying Green's formula to $[u,u]|_a^b$, and using the fact that

$$[u,u](a) = -2i \operatorname{Im} m$$

we get:

$$[u,u](b) = 2i \operatorname{Im}\lambda \int_a^b |u|^2 dt - 2i \operatorname{Im} m \tag{4.12}$$

With the help of equations 4.11 and 4.12 the interior of C_b 4.10 can be written as follows:

$$\frac{2i \operatorname{Im}\lambda \int_a^b |u|^2 dt - 2i \operatorname{Im} m}{2i \operatorname{Im}\lambda \int_a^b |v_2|^2 dt} < 0$$

if and only if

$$\int_a^b |u|^2 dt < \frac{\operatorname{Im} m}{\operatorname{Im}\lambda} \qquad \text{with } \operatorname{Im}\lambda \neq 0.$$

The point m is on the circle C_b if and only if

$$\int_a^b |u|^2 dt = \frac{\operatorname{Im} m}{\operatorname{Im}\lambda}. \tag{4.13}$$

We assume $\operatorname{Im}\lambda > 0$. We can rewrite equation 4.9 on page 28 as follows:

$$|[v_2, v_2](b)| = 2\operatorname{Im}\lambda \int_a^b |v_2|^2 dt = \frac{1}{r_b} \tag{4.14}$$

For an arbitrary d with $0 < d < b < \infty$ and a point m on or inside the circle C_b the following holds:

$$\int_a^d |u|^2 dt < \int_a^b |u|^2 dt \le \frac{\operatorname{Im} m}{\operatorname{Im}\lambda} \tag{4.15}$$

and hence m is also inside the circle C_d and $C_b \subset C_d$ for every d with $0 < d < b < \infty$.
Now a distinction between the two cases as described in the theorem can be made:

limit circle case: For $b \to \infty$: C_b converges to a circle C_∞.

limit point case: For $b \to \infty$: C_b converges to a point m_∞.

In the limit circle case the radius r_b of C_b converges to a positive constant $r_\infty > 0$ (radius of the circle C_∞) for $b \to \infty$. From equation 4.14 we can conclude that $v_2 \in L^2(a, \infty)$, since:

$$\int_a^\infty |v_2|^2 dt = \frac{1}{2r_\infty \operatorname{Im}\lambda} < \infty. \tag{4.16}$$

Let $\hat{m}_\infty$ be a point on the circle C_∞. Then $\hat{m}_\infty$ is inside or on the circle C_b for every $b > 0$. Thus $\hat{m}_\infty$ satisfies the following equation:

$$\int_a^b |u|^2 dt = \int_a^b |v_1 + \hat{m}_\infty v_2|^2 dt < \frac{\operatorname{Im}\hat{m}_\infty}{\operatorname{Im}\lambda} < \infty \text{ with } \operatorname{Im}\lambda \neq 0. \tag{4.17}$$

Letting $b \to \infty$ in equation 4.17, we can conclude that

$$u = v_1 + m v_2 \in L^2(a, \infty),$$

and therefore all solutions of $(L - \lambda)u = 0$ (with $\operatorname{Im}\lambda \neq 0$) are in $L^2(a, \infty)$, as proven before.
Moreover, we have the following result for the limit circle case:

$$\hat{m}_\infty \textit{ is on the limit circle } C_\infty \quad \Leftrightarrow \quad [u, u](\infty) = 0,$$

since we know $\operatorname{Im}\lambda \int_a^\infty |u|^2 dt = \operatorname{Im}\hat{m}_\infty = -\frac{1}{2i}[u,u](0)$ (from equation 4.17 on the page before) and $2i\operatorname{Im}\lambda \int_a^\infty |u|^2 dt - 2\operatorname{Im}\hat{m}_\infty = [u,u](\infty)$ (from equation 4.12 on page 29).

In the limit point case obviously $r_b \to 0$ for $b \to \infty$. Hence, we can conclude $v_2 \notin L^2(a,\infty)$, since $2\operatorname{Im}\lambda \int_a^\infty |v_2|^2 dt = \frac{1}{r_b}$. But, using the same argument as in the limit circle case, we can prove $u \in L^2(a,\infty)$, i.e. there exists exactly one solution u of the equation $(L-\lambda)u = 0$ with $u \in L^2(a,\infty)$ in the limit point case with $\operatorname{Im}\lambda \neq 0$. ∎

In the first case as described in the theorem 4.1 on page 25 the singular endpoint b is called limit circle and the defect index of L_{min} restricted to the interval (c,b), $c \in (a,b)$ is $\gamma = 2$, since there exist exactly two linearly independent solutions u_1, u_2 of the equation $(L-\lambda)u = 0$, with $u_1, u_2 \in L^2(c,b)$ for all $\lambda \in \mathbb{C}$.
In the second case as described in theorem 4.1 the singular endpoint b is called limit point and the defect index of L_{min} restricted to the interval (c,b), $c \in (a,b)$ is $\gamma = 1$, since obviously there exists only one solution u of the equation $(L-\lambda)u = 0$, with $u \in L^2(c,b)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.
Obviously, a similar argument holds for the case where a is a singular endpoint.

The following lemma will make an even more clear distinction between the limit circle, the limit point and the regular case:

Lemma 4.2 *For an endpoint b the following holds:*

(1) *The endpoint b is limit point if and only if for all $f, h \in \operatorname{dom} L_{max}$ we have $[f,h](b) = 0$.*

(2) *The endpoint b is limit circle or regular if and only if there exist functions $f, h \in \operatorname{dom} L_{max}$ such that $[f,h](b) \neq 0$.*

Proof.

(1) ($\Rightarrow$): Assume we have limit point in b and a is regular (if a is singular, simply choose a $c \in (a,b)$ and restrict the differential operator to the interval (c,b)). Consider two linearly independent solutions u_1, u_2 of $(L-\lambda)u = 0$ with $[u_1,u_2](a) \neq 0$ and choose $[c,d] \subset (a,b)$. With the help of lemma 2.6 on page 9 we can construct two functions $v_1, v_2 \in \operatorname{dom} L_{max}$ such that the following holds:

$$v_i(x) = 0 \quad \text{for } x \in (d,b), \qquad v_i(x) = u_i(x) \quad \text{for } x \in (a,c), \quad i = 1,2.$$

But then $[v_1, v_2](a) = [u_1, u_2](a) \neq 0$ and on account of lemma 2.10 on page 13 we have $v_1, v_2 \notin \text{dom}\, L_{min}$. Therefore obviously the functions v_1, v_2 belong to $\text{dom}\, L_{max}/\text{dom}\, L_{min}$ and they are linearly independent modulo $\text{dom}\, L_{min}$. Since the defect indices $\gamma_+ = \gamma_- = 1$, we know

$$\dim\left(\text{dom}\, L_{max}/\text{dom}\, L_{min}\right) = \gamma_+ + \gamma_- = 2$$

and L_{max} can hence be written as follows:

$$L_{max} = L_{min} \oplus \text{L.S.}\Big\{\{v_1, Lv_1\}, \{v_2, Lv_2\}\Big\}. \tag{4.18}$$

With the help of equation 4.18 we can rewrite all $f, g \in \text{dom}\, L_{max}$ as follows:

$$f = f_0 + c_1 v_1 + c_2 v_2, \quad g = g_0 + d_1 v_1 + d_2 v_2$$

with $f_0, g_0 \in \text{dom}\, L_{min}$ and $c_i, d_i \in \mathbb{C}$, $i = 1, 2$ and therefore on account of lemma 2.10 on page 13 we have

$$[f, g](b) = [f_0, g_0](b) = 0.$$

(2) ($\Rightarrow$): We now consider the endpoint a. We use the functions v_1, v_2 which belong to $\text{dom}\, L_{max}$ as constructed in the first part of our proof and we define $f = v_1$, $g = v_2$. These functions f, g have the desired property: $[f, g](a) \neq 0$.

Since an endpoint is either limit point, limit circle or regular, the above proven cases (1)($\Rightarrow$) and (2)($\Rightarrow$) also imply the remaining cases (1)($\Leftarrow$) and (2)($\Leftarrow$). ■

We are now ready to describe the different cases as they can occur when we consider the differential operator L_{min} on the entire interval (a, b):

Case 1. a regular/limit circle, b limit point

The equation $(L - \lambda)u = 0$ has exactly one solution $u \in L^2(a, b)$, namely the one which is L^2 in a neighbourhood of the limit point b and we have $\gamma = 1$. In this case we can write

$$L_{max} = L_{min} \oplus \text{L.S.}\Big\{\{v_1, Lv_1\}, \{v_2, Lv_2\}\Big\}$$

where v_1, v_2 are the functions as defined in the proof of lemma 4.2 on the preceding page.

In the case where a limit point and b regular/limit circle we can construct functions $w_1, w_2 \in \text{dom}\, L_{max}$ in a similar way such that the following holds:

$$w_i(x) = 0 \quad \text{for } x \in (a, c) \qquad w_i(x) = u_i(x) \quad \text{for } x \in (d, b), \quad i = 1, 2.$$

where u_1, u_2 are linearly independent solutions of the equation

$$(L - \lambda)u = 0$$

and $[w_1, w_2](b) = [u_1, u_2](b) \neq 0$. Again, $\gamma = 1$ and we can write:

$$L_{max} = L_{min} \oplus \text{L.S.}\Big\{\{w_1, Lw_1\}, \{w_2, Lw_2\}\Big\}.$$

Case 2. a, b **both limit circle/regular**

The case where both a and b are regular has already been described in chapter 3 and is a special case of the more general case considered here. For a, b not limit point, the defect index $\gamma = 2$ and therefore $\dim(\text{dom}\, L_{max}/\text{dom}\, L_{min}) = \gamma_+ + \gamma_- = 4$. We can rewrite L_{max} as follows:

$$L_{max} = L_{min} \oplus \text{L.S.}\Big\{\{v_1, Lv_1\}, \{v_2, Lv_2\}\Big\} \oplus \text{L.S.}\Big\{\{w_1, Lw_1\}, \{w_2, Lw_2\}\Big\}$$

with v_1, v_2, w_1, w_2 as described in the previous case.

Case 3. a, b **both limit point**

In this case, we have $[f, g](a) = [f, g](b) = 0$ for all $f, g \in \text{dom}\, L_{max}$ and hence $L_{min} = L_{max}$ as described in lemma 2.13 on page 14. The defect index γ is zero and the differential operator $L_{min} = L_{max}$ is self-adjoint.

Since self-adjoint extensions H as defined in chapter 2 only exist for $\gamma \in \{1, 2\}$, such extensions H only exist in the first and second case.

In the first case (with a regular/limit circle, b limit point), the self-adjoint differential operator H can be written as follows:

$$H = \Big\{\{f, Lf\} \in L_{max} \mid [f, v](a) = 0\Big\} \tag{4.19}$$

where $v \in \text{L.S.}\{v_1, v_2\} \subset \text{dom}\, L_{max}/\text{dom}\, L_{min}$, $v \neq 0$ satisfies $[v, v](a) = 0$, and the functions v_1, v_2 are as described in the proof of lemma 4.2 on page 31. When a is limit point and b is regular/limit circle we have

$$H = \Big\{\{f, Lf\} \in L_{max} \mid [f, w](b) = 0\Big\} \tag{4.20}$$

where $w \in \text{L.S.}\{w_1, w_2\} \subset \text{dom}\, L_{max}/\text{dom}\, L_{min}$, $w \neq 0$ satisfies $[w, w](b) = 0$, and the functions w_1, w_2 are as described in the proof of lemma 4.2.

In the second case, the self-adjoint differential operator H can be written as follows:

$$H = \left\{ \{f, Lf\} \in L_{max} \mid \langle \{f, Lf\}, \{h_i, Lh_i\} \rangle = 0, \ i = 1, 2 \right\} \quad (4.21)$$

where the functions h_1, h_2 are linearly independent linear combinations of v_1, v_2, w_1, w_2 as described in the first case, i.e.:

$$h_i \in \text{L.S.}\{v_1, v_2, w_1, w_2\}, \quad i = 1, 2.$$

Moreover, the functions h_1, h_2 satisfy the following equation:

$$\langle \{h_i, Lh_i\}, \{h_j, Lh_j\} \rangle = 0 \quad \text{for } i, j \in \{1, 2\}.$$

If for example we choose $h_1 = v + w$, with $v \in \text{L.S.}\{v_1, v_2\}$,$w \in \text{L.S.}\{w_1, w_2\}$ the following holds for all $f \in \text{dom}\, H$:

$$[f, h_1]|_a^b = [f, v]|_a^b + [f, w]|_a^b,$$

and we are back in a situation much like the description of H in the regular case as discussed in chapter 3. Finally, with the help of lemma 4.2 the description of H in the first case(s) (4.19 and 4.20) can be easily derived from the description of H in the second case (4.21).

Chapter 5

The Resolvent of the Self-Adjoint Extension H

As we are now able to describe the self-adjoint extension H of the differential operator L_{min} in all possible cases (regular and singular), we proceed by examining its resolvent $R(\lambda)$ in order to derive some useful properties of H and its eigenvalues.

We start by giving the definition of the resolvent operator $R(\lambda)$:

Definition 5.1 *The resolvent $R(\lambda)$ of an operator T is defined as follows:*

$$R(\lambda) = (T - \lambda)^{-1},$$

where λ is chosen in such a way that the expression in the definition makes sense. We will prove that the resolvent of the self-adjoint extension H is an integral operator. In order to do so, we need the help of two lemmas and the following definition:

Definition 5.2 *A mapping K of a Hilbert-space $\mathcal{H}$ onto itself is called a conjugation if:*

- $K(af + bg) = \overline{a}Kf + \overline{b}Kg, \quad$ *for all* $f, g \in \mathcal{H}, \; a, b \in \mathbb{C}$.
- $K^2 = I$.
- $(Kf, Kg) = (f, g)^*, \quad$ *for all* $f, g \in \mathcal{H}$.

An example of such a conjugation is the natural conjugation J on elements of L^2, defined by:

$$Jf = f^*.$$

An operator T on a Hilbert-space $\mathcal{H}$ is called K-real if the following holds:

$$TK(f) = KT(f) \quad \text{for all } f \in \operatorname{dom} T.$$

Now we will present and prove the following two lemmas:

Lemma 5.3 *If for all $g \in L^2(0,\infty)$ with compact support the resolvent $R(\lambda)$ of a J-real self-adjoint differential operator T with $\operatorname{dom} T \subset L^2(0,\infty)$ has the form:*

$$(R(\lambda))g(t) = \int_0^\infty k(t,s,\lambda)g(s)ds$$

with $k(t,s,\lambda)$ continuous for $t,s>0,\ t\neq s$, then we have

$$k(t,s,\lambda) = k(s,t,\lambda).$$

Proof.
Assume T is a J-real operator. This implies $(T-\lambda)R(\lambda) = I$ and hence $(T-\overline{\lambda})JR(\lambda) = J(T-\lambda)R(\lambda) = J$. This results in the following equation:

$$JR(\lambda) = (T-\overline{\lambda})^{-1}J = R(\overline{\lambda})J.$$

We use $J^2R = R$ and $R^*(\lambda) = R(\overline{\lambda})$ (this follows from the self-adjointness of the operator T) to obtain the following equation:

$$R(\lambda) = JR^*(\lambda)J, \tag{5.1}$$

For a function g as described in the lemma the following holds:

$$\begin{aligned}(R^*(\lambda)g)(s) :&= \int k^*(s,t,\lambda)g(t)dt\\ &= \int \overline{k(t,s,\lambda)}g(t)dt,\end{aligned}$$

since we have $k^*(s,t,\lambda) = \overline{k(t,s,\lambda)}$ as $R^*(\lambda) = R(\overline{\lambda})$. According to equation 5.1 we have:

$$\begin{aligned}(R^*(\lambda)g)(s) &= (JR(\lambda)Jg)(s)\\ &= J\int k(s,t,\lambda)Jg(t)dt\\ &= \int \overline{k(s,t,\lambda)}g(t)dt\end{aligned}$$

and hence $\overline{k(t,s,\lambda)} = \overline{k(s,t,\lambda)}$, i.e.:

$$k(t,s,\lambda) = k(s,t,\lambda).$$

■

Lemma 5.4 *Let T be a linear functional which is bounded on $L^2(0,a)$ for $a > 0$. Then for all $f \in (\operatorname{dom} T \cap L^2(0,a))$ with compact support there exists a function h such that:*

$$Tg = \int_0^\infty g(s)\overline{h(s)}ds$$

for all constants $a > 0$, $g \in \operatorname{dom} T$, where h is a uniquely determined by L and $h \in L^2(0,a)$.

Proof.
By the Riesz representation theorem (see appendix B for the proof of this theorem) applied to the linear functional T we know that there exists exactly one function $h_{a_i} \in L^2(0,a_i)$ such that for all $g \in L^2(0,a_i)$ we have

$$Tg = (g, h_{a_i}) = \int_0^{a_i} g(s)\overline{h_{a_i}(s)}ds \ \in \mathbb{C}$$

for a certain constant a_i, $i = 1, 2, \ldots$ Assume $a_1 < a_2 < \infty$. We can extend the function $g \in L^2(0,a_1)$ by defining

$$\tilde{g}(s) = g(s) \quad \text{for } s \in (0,a_1), \qquad \tilde{g}(s) = 0 \quad \text{for } s \in (a_1,a_2).$$

Obviously $\tilde{g} \in L^2(0,a_2)$ and we have

$$\int_0^{a1} g(s)\overline{h_{a_1}(s)}ds = Lg = L\tilde{g} = \int_0^{a2} g(s)\overline{h_{a_2}(s)}ds = \int_0^{a1} g(s)\overline{h_{a_2}(s)}ds$$

and thus $h_{a_1}(s) = h_{a_2}(s)$ on $(0,a_1)$. A similar argument holds for $a_2 < a_3 < \infty$ and hence the extension of h_{a_i} results in a function $h \in L^2(0,\infty)$ where

$$Tg = \int_0^\infty g(s)\overline{h(s)}ds$$

as desired. ∎

With the help of the above described lemmas, we are now able to prove the following theorem:

Theorem 5.5 *The resolvent $R(\lambda)$ of the self-adjoint extension H of a differential operator L with defect indices (γ,γ) is an integral operator.*

We will restrict the proof of this rather general theorem to the case where L is the Sturm-Liouville differential operator L_{min} as defined in chapter 2 on an interval $(a,b) = (0,\infty)$ with self-adjoint extension H. We assume the endpoint $a = 0$ to be regular. This immediately implies the singularity of the point $b = \infty$.

Proof.
First, we consider the case where H and λ are real, i.e. the case where the coefficients in the boundary conditions of H are real. Then we have the following for the resolvent $R(\lambda) = (L - \lambda)^{-1}$ and for a $g \in L^2(0, \infty)$:

$$(L - \lambda)R(\lambda)\, g = g \quad \text{with} \quad R(\lambda)\, g \in \operatorname{dom}(H - \lambda)$$

Consider two linearly independent solutions $u_1(t, \lambda), u_2(t, \lambda)$ of the equation $(L - \lambda)u = 0$ with $W(u_1, u_2, x) = 1$. We make a distinction according to the value of the defect indices as we did in the previous chapter:

Case 1: $\gamma = 1$. This is the limit point case as described before. We assume $u_1(t, \lambda) \in L^2(0, \infty)$, and hence we must have $u_2(t, \lambda) \notin L^2(0, \infty)$.

Case 2: $\gamma = 2$. This is the limit circle case, and both solutions $u_1(t, \lambda)$, $u_2(t, \lambda)$ belong to $L^2(0, \infty)$.

We wish to solve $(L - \lambda)u = g$ with $u = R(\lambda)\, g \in \operatorname{dom} H$. By variation of parameters (appendix C) the general solution u of the equation $(L - \lambda)u = g$ can be written as follows:

$$u(t) = u_2(t) \int_0^t u_1(s)g(s)ds - u_1(t) \int_0^t u_2(s)g(s)ds + k_1u_1(t) + k_2u_2(t)$$

where k_1, k_2 are uniquely determined constants if we demand $u \in \operatorname{dom} H$.

In the first case to be considered ($\gamma = 1$) we assume $g(t)$ has compact support. We choose $k_2 = -\int_0^\infty u_1(s)g(s)ds$ to get

$$u(t) = -u_1(t) \int_0^t u_2(s)g(s)ds - u_2(t) \int_t^\infty u_1(s)g(s)ds + k_1u_1(t),$$

which implies $u \in L^2(0, \infty)$. In the second case ($\gamma = 2$) the function u already belongs to $L^2(0, \infty)$ if we assume g to have compact support, since $u_1, u_2 \in L^2(0, \infty)$. We define $v_1(s) := -u_2(s)$ and $v_2(s) := u_1(s)$ so that our general solution $u = R(\lambda)\, g$ can be written as follows:

$$\begin{aligned} R(\lambda)\, g(t) = u(t) &= \sum_{i=1}^{\gamma} u_i(t) \int_0^t v_i(s)g(s)ds - \sum_{i=\gamma+1}^{2} u_i(t) \int_t^\infty v_i(s)g(s)ds \\ &\quad + \sum_{i=1}^{\gamma} k_iu_i(t) \\ &= \int_0^\infty K_\lambda(t, s)g(s)ds + \sum_{i=1}^{\gamma} k_iu_i(t) \end{aligned} \tag{5.2}$$

where $K_\lambda(t,s)$ is defined as follows:

$$K_\lambda(t,s) = \begin{cases} \sum_{i=1}^{\gamma} u_i(t)v_i(s) & \text{for } s \le t \\ -\sum_{i=\gamma+1}^{2} u_i(t)v_i(s) & \text{for } s > t. \end{cases} \tag{5.3}$$

This yields in our two cases ($\gamma = 1, \gamma = 2$) the following functions $K_\lambda(t,s)$:

$$\gamma = 1: \quad K_\lambda(t,s) = \begin{cases} -u_1(t)u_2(s) & s \le t \\ -u_2(t)u_1(s) & s > t \end{cases}$$

$$\gamma = 2: \quad K_\lambda(t,s) = \begin{cases} -u_1(t)u_2(s) + u_2(t)u_1(s) & s \le t \\ 0 & s > t \end{cases}$$

We are now ready to prove the fact that the k_i as described in equation 5.2 on the preceding page are linear functionals which satisfy the properties given in lemma 5.4 on page 37 for all $i = 1, 2, \ldots$. Obviously, k_i is a linear functional defined on all functions $g \in L^2(0,\infty)$ with compact support. Therefore, we only need to prove the boundedness of k_i on $L^2(0,a)$, for a constant $a > 0$. In order to do so, we consider the projection operator $P_a: \ L^2(0,\infty) \to L^2(0,a)$ defined as follows:

$$P_a g = g 1_{[0,a]} \qquad \text{for all } g \in L^2(0,\infty)$$

where $1_{[0,a]}$ denotes the function which is equal to 1 on $[0,a]$ and 0 outside the interval $[0,a]$. We apply P_a on the equation 5.2 on the preceding page and take the inner product with u_j, $j = 1, 2$:

$$\begin{aligned} (P_a R(\lambda)\, g, u_j) &= \left(P_a \int_0^a K_\lambda(t,s)g(s)ds, u_j\right) + \sum_{i=1}^{\gamma}\Big(P_a k_i u_i(t), u_j(t)\Big) \\ &= (K_a g, u_j) + \sum_{i=1}^{\gamma} k_i(P_a u_i(t), u_j(t)) \end{aligned} \tag{5.4}$$

with $K_a g = P_a \int_0^a K_\lambda(t,s)g(s)ds$.
In the first case ($\gamma = 1$) we have:

$$(P_a R(\lambda)\, g, u_j) = (K_a g, u_1) + k_1(P_a u_1, u_1).$$

We use the following (in)equalities in order to prove the boundedness of k_1 in this case:

$$\|K_a g\| = \|P_a \int_0^a K_\lambda(t,s)g(s)ds\| \le C\sqrt{\int_0^a |g(s)|^2 ds}, \quad C > 0$$

and

$$(P_a u_1, u_1) = \|P_a u_1\|^2 \text{ (since } P_a \text{ is a projection)}$$

which imply (assume $\|P_a u_1\|^2 \neq 0$):

$$\begin{aligned}
\|k_1\| &= \frac{1}{\|P_a u_1\|^2} \left\| \left((P_a R(\lambda)\, g, u_1) - (K_a g, u_1) \right) \right\| \\
&\leq \frac{1}{\|P_a u_1\|^2} \left(\|R(\lambda)\| \|g\|_{L^2(0,a)} \|u_1\| + (C\|g\|_{L^2(0,a)} \|u_1\| \right)
\end{aligned} \tag{5.5}$$

which proves the boundedness of k_1.
In the second case ($\gamma = 2$) the equation 5.4 on the page before results in the following two equations:

$$\begin{aligned}
(P_a R(\lambda)\, g, u_1) &= (K_a g, u_1) + k_1 (P_a u_1, u_1) + k_2 (P_a u_2, u_1) \\
(P_a R(\lambda)\, g, u_2) &= (K_a g, u_2) + k_1 (P_a u_1, u_2) + k_2 (P_a u_2, u_2)
\end{aligned}$$

which we can write as follows:

$$\begin{pmatrix} (P_a R(\lambda)\, g, u_1) - (K_a g, u_1) \\ (P_a R(\lambda)\, g, u_2) - (K_a g, u_2) \end{pmatrix} = \begin{pmatrix} (P_a u_1, u_1) & (P_a u_2, u_1) \\ (P_a u_1, u_2) & (P_a u_2, u_2) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} =: M \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Since u_1, u_2 are linearly independent, the matrix M is invertible and hence:

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = M^{-1} \begin{pmatrix} (P_a R(\lambda)\, g, u_1) - (K_a g, u_1) \\ (P_a R(\lambda)\, g, u_2) - (K_a g, u_2) \end{pmatrix}.$$

Proceeding in a similar way as in case 1, we can prove the boundedness of both k_1 and k_2.

Now we can apply lemma 5.4 on page 37 and find $\psi_i \in L^2(0, a)$ such that the following holds for all constants $a > 0$:

$$k_i(g) = \int_0^\infty g(s) \psi_i(s) ds \qquad i = 1, 2.$$

This means we can rewrite equation 5.2 on page 38 in the desired form:

$$\begin{aligned}
R(\lambda)\, g(t) = u(t) &= \int_0^\infty K_\lambda(t, s) g(s) ds + \sum_{i=1}^{\gamma} u_i(t) \int_0^\infty g(s) \psi_i(s) ds \\
&= \int_0^\infty K_0(t, s, \lambda) g(s) ds
\end{aligned}$$

$$\text{with } \; K_0(t, s, \lambda) = \begin{cases} \sum_{i=1}^{\gamma} u_i(t) (v_i(s) + \psi_i(s)) & s \leq t \\ \sum_{i=1}^{\gamma} u_i(t) \psi_i(s) - \sum_{i=\gamma+1}^{2} u_i(t) v_i(s) & s > t. \end{cases} \tag{5.6}$$

This yields in our first case ($\gamma = 1$) the following equation for the function u:

$$\begin{aligned} u(t) &= \int_0^\infty K_\lambda(t,s)g(s)ds + u_1(t)\int_0^\infty g(s)\psi_1(s)ds \\ &= \int_0^\infty K_0(t,s,\lambda)g(s)ds \end{aligned}$$

where $K_0(t,s,\lambda) = \begin{cases} u_1(t)\psi_1(s) - u_1(t)u_2(s) & \text{for } s \leq t \\ u_1(t)\psi_1(s) - u_2(t)u_1(s) & \text{for } s > t. \end{cases}$

In the second case ($\gamma = 2$) the equation 5.6 on the preceding page yields:

$$\begin{aligned} u(t) &= \int_0^\infty K_\lambda(t,s)g(s)ds + u_1(t)\int_0^\infty g(s)\psi_1(s)ds + u_2(t)\int_0^\infty g(s)\psi_2(s)ds \\ &= \int_0^\infty K_0(t,s,\lambda)g(s)ds \end{aligned}$$

where

$$K_0(t,s,\lambda) = \begin{cases} u_1(t)\psi_1(s) + u_2(t)\psi_2(s) - u_1(t)u_2(s) + u_2(t)u_1(s) & \text{for } s \leq t \\ u_1(t)\psi_1(s) + u_2(t)\psi_2(s) & \text{for } s > t. \end{cases}$$

We have now proven that for all functions $g \in L^2(0,\infty)$ with compact support, the resolvent $R(\lambda)$ is an integral operator with kernel $K_0(t,s,\lambda)$.

We will see that this result still holds for all functions $g \in L^2(0,\infty)$. Since H is a J-real self-adjoint differential operator with $\operatorname{dom} H \subset L^2(0,\infty)$ and a continuous kernel, we may apply lemma 5.3 on page 36. Hence, the kernel $K_0(t,s,\lambda)$ as a function of t belongs to $L^2(0,\infty)$ for all s, and the following holds:

$$K_0(t,s,\lambda) = K_0(s,t,\lambda),$$

which yields:

$$\int_0^\infty |K_0(s,t,\lambda)|^2 ds = \int_0^\infty |K_0(t,s,\lambda)|^2 ds < \infty.$$

Therefore we have the following for all $a > 0$::

$$\begin{aligned} \int_0^a |K_0(t,s,\lambda)g(s)|dt &\leq \sqrt{\int_0^a |K_0(t,s,\lambda)|^2 ds}\sqrt{\int_0^a |g(s)|^2 ds} \\ &\leq \|K_0(t,s,\lambda)\| \cdot \|g(s)\|_{L^2(0,\infty)} \end{aligned}$$

which implies the existence of the integral

$$\int_0^\infty K_0(t,s,\lambda)g(s)ds$$

for all $g \in L^2(0,\infty)$ and hence $R(\lambda)$ is an integral operator with kernel $K_0(t,s,\lambda)$.

This leaves us with the proof of the case where the self-adjoint extension H is not necessarily real, which proof is similar to the above described one, provided we use the equation $K(s,t,\lambda) = \overline{K(t,s,\lambda)}$ instead of the equation $K_0(s,t,\lambda) = K_0(t,s,\lambda)$ as used in the case of a real self-adjoint extension H. We conclude this proof by examining the case where $\lambda \notin \mathbb{R}$. In this case, we write $\widetilde{R(\lambda)}$ for the resolvent of H, and consider $\widetilde{R(\lambda)}\,g$, which satisfies the equation $(L-\lambda)u = g$ and therefore can be written as follows:

$$\widetilde{R(\lambda)}\,g = R(\lambda)\,g + \sum_{i=1}^{2} c_i u_i,$$

where $R(\lambda)$ is the resolvent of a real self-adjoint extension as described above. Since $u_i \notin L^2(0,\infty)$ for $i = \gamma+1,\ldots,2$, obviously the corresponding c_i must be equal to zero. The remaining c_i are linear functionals as described in lemma 5.4 on page 37 and hence:

$$c_i = \int_0^\infty g(s)\phi_i(s)ds \qquad \text{for } i = 1,\ldots,\gamma$$

with $\phi_i \in L^2(0,\infty)$ for all $i = 1,\ldots,\gamma$ since they can be expressed in terms of the functions $u_i \in L^2(0,\infty)$, with $i = 1,\ldots,\gamma$. In the case where $\gamma = 2$ this can be seen as follows:

$$\begin{pmatrix} ((\widetilde{R(\lambda)} - R(\lambda))g, u_1) \\ ((\widetilde{R(\lambda)} - R(\lambda))g, u_2) \end{pmatrix} = \begin{pmatrix} (u_1,u_1) & (u_2,u_1) \\ (u_1,u_2) & (u_2,u_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where the first matrix on the righthandside is invertible, since u_1, u_2 are linearly independent (see also the matrix S defined after lemma 2.4 in chapter 2). We define

$$\tilde{K}(t,s,\lambda) = K_0(t,s,\lambda) + \sum_{i=1}^{\gamma} u_i(t)\phi_i(s). \tag{5.7}$$

This results in

$$\widetilde{R(\lambda)}\,g = \int_0^\infty \tilde{K}(t,s,\lambda)g(s)ds, \qquad \text{for } g(s) \in L^2(0,\infty)$$

and again, the resolvent is proven to be an integral-operator. In the first case ($\gamma = 1$) we have $c_2 = 0$ and therefore

$$\widetilde{R(\lambda)}\, g = \int_0^\infty \big(K_0(t,s,\lambda) + u_1(t)\phi_1(s)\big) g(s) ds, \qquad \text{for } g(s) \in L^2(0,\infty).$$

In the case where $\gamma = 2$ the resolvent $\widetilde{R(\lambda)}\, g$ has the following form:

$$\widetilde{R(\lambda)}\, g = \int_0^\infty \big(K_0(t,s,\lambda) + u_1(t)\phi_1(s) + u_2(t)\phi_2(s)\big) g(s) ds$$

with $g(s) \in L^2(0,\infty)$. ■

Chapter 6

The Case $\gamma = 2$: Hilbert-Schmidt Operators

In this chapter we restrict ourselves to the case where $\gamma = 2$ and we take a closer look at the form and properties of the integral operator

$$R(\lambda) = (H - \lambda)^{-1}$$

with kernel $\tilde{K}(t, s)$ as computed in chapter 5, where H is the self-adjoint extension of the differential operator L on an interval $(a, b) \subset \mathbb{R}$.
The following properties of the kernel $\tilde{K}(t, s, \lambda)$ can be easily derived since both $K_0(t, s, \lambda)$ and ϕ_i, $i = 1, \ldots, \gamma$ as described in equation 5.7 on page 42 belong to $L^2(a, b)$:

$$\int_a^b |\tilde{K}(t,s,\lambda)|^2 dt = \int_a^b \left|K_0(t,s,\lambda) + \sum_{i=1}^{\gamma} u_i(t)\phi_i(s)\right|^2 dt < \infty \tag{6.1}$$

$$\int_a^b |\tilde{K}(t,s,\lambda)|^2 ds = \int_a^b \left|K_0(t,s,\lambda) + \sum_{i=1}^{\gamma} u_i(t)\phi_i(s)\right|^2 ds < \infty. \tag{6.2}$$

Moreover, in the case where $\gamma = 2$ the following holds:

Lemma 6.1 *In the case where $\gamma = 2$, the following holds for the kernel of the resolvent $R(\lambda)$ of the self-adjoint differential operator H:*

$$\int_a^b \int_a^b |\tilde{K}(t,s,\lambda)|^2 ds dt < \infty.$$

Proof.
Since the kernel $K_0(t, s, \lambda)$ satisfies the conditions as described in lemma 5.3,

we know that $K_0(t,s,\lambda)$ is symmetric, which means it satisfies the following equation:

$$K_0(t,s,\lambda) = K_0(s,t,\lambda).$$

With the help of equation 5.7 we can rewrite this equation, which yields the following for $s \leq t$:

$$\sum_{i=1}^{2} u_i(t)\psi_i(s) - u_1(t)u_2(s) + u_2(t)u_1(s) = \sum_{i=1}^{2} u_i(s)\psi_i(t). \qquad (6.3)$$

Since u_1, u_2 are linearly independent, there exist s_1, s_2 such that the following holds:

$$\det \begin{pmatrix} u_1(s_1) & u_1(s_2) \\ u_2(s_1) & u_2(s_2) \end{pmatrix} \neq 0$$

We consider the following system of equations:

$$\sum_{i=1}^{2} u_i(t)\psi_i(s_1) - u_1(t)u_2(s_1) + u_2(t)u_1(s_1) = \sum_{i=1}^{2} u_i(s_1)\psi_i(t)$$
$$\sum_{i=1}^{2} u_i(t)\psi_i(s_2) - u_1(t)u_2(s_2) + u_2(t)u_1(s_2) = \sum_{i=1}^{2} u_i(s_2)\psi_i(t)$$

With the help of these equations the functions ψ_1, ψ_2 can be expressed in terms of the functions $u_1, u_2 \in L^2(a,b)$, which implies $\psi_1, \psi_2 \in L^2(a,b)$. Since all functions on the righthandside of equation 5.7 on page 41 belong to $L^2(a,b)$ in both variables t, s, the following holds:

$$\int_a^b \int_a^b |K_0(t,s,\lambda)|^2 ds dt < \infty.$$

We combine this property of $K_0(t,s,\lambda)$ with equation 5.7 on page 42 to get:

$$\int_a^b \int_a^b |\tilde{K}(t,s,\lambda)|^2 ds dt < \infty. \qquad (6.4)$$

■

A kernel of an integral operator which satisfies equation 6.4 is called a Hilbert-Schmidt kernel. Hence, in the case of $\gamma = 2$ the kernel $\tilde{K}$ is such a Hilbert-Schmidt kernel and the operator $R(\lambda)$ is called a Hilbert-Schmidt operator. This results in some useful properties of the Hilbert-Schmidt operator:

Theorem 6.2 *Let K be a Hilbert-Schmidt operator with Hilbert-Schmidt kernel $k(s,t)$, i.e.:*

$$K : L^2(a,b) \to L^2(a,b),\ Kg(t) = \int_a^b k(s,t)g(s)ds.$$

The following holds:

(a) *The operator K is bounded: $\|K\| \leq \|k\|$.*

(b) *The operator K is a compact operator.*

(c) *The operator K is self-adjoint if and only if the following holds:*

$$k^*(t,s) = k(s,t).$$

(d) *If K is self-adjoint, there exist an orthonormal basis $\{\psi_i, i = 1, 2, \dots\}$ for $L^2(a,b) \ominus \ker K$ of eigenfunctions of K with corresponding eigenvalues $\gamma_i \neq 0$ such that the following holds:*

$$Kg(t) = \sum_{i=1}^{\infty} \gamma_i \left(\int_a^b g(s)\overline{\psi_i(s)}ds \right) \psi_i(t) = \sum_{i=1}^{\infty} \gamma_i (g, \psi_i) \psi_i(t).$$

Proof.

(a) The Schwarz-inequality yields

$$\int_a^b |K(s,t)f(s)|ds \leq \sqrt{\int_a^b |k(s,t)|^2 ds} \sqrt{\int_a^b |f(s)|^2 ds}$$

and hence

$$\begin{aligned} \|Kf\|^2 &\leq \int_a^b \left(\int_a^b |k(s,t)f(s)|ds \right)^2 dt \\ &\leq \|f\|^2 \int_a^b \int_a^b |k(s,t)|^2 ds dt. \end{aligned}$$

This yields:

$$\|K\|^2 \leq \int_a^b \int_a^b |k(s,t)|^2 dsdt = \|k(s,t)\|^2 < \infty, \qquad (6.5)$$

hence K is a bounded and (obviously) linear operator.

(b) Consider an orthonormal basis $\{\phi_1, i = 1, 2, \dots\}$ of $L^2(a,b)$. Then the set $\{\Phi_{i,j}(s,t) = \phi_i(s)\phi_j(t),\ i,j = 1,2,\dots\}$ represents an orthonormal basis for $L^2\big((a,b)\times(a,b)\big)$, which can be seen as follows:
we use Fubini's theorem to find:

$$\begin{aligned}(\Phi_{i,j}, \Phi_{k,l}) &= \int_a^b \int_a^b \phi_i(s)\phi_j(t)\overline{\phi_k}(s)\overline{\phi_l}(t)dsdt \\ &= \int_a^b \phi_i(s)\overline{\phi_k}(s)ds \int_a^b \phi_j(t)\overline{\phi_l}(t)dt = \delta_{ik}\delta_{jl} \qquad (6.6)\end{aligned}$$

where δ_{ij} denotes the Kronecker delta, satisfying

$$\delta_{ij} = 1 \quad \text{for } i = j, \qquad \delta_{ij} = 0 \quad \text{for } i \neq j.$$

Hence, the set $\Phi_{i,j}$ is orthonormal. Assume the following holds for a certain function $f(s,t) \in L^2\big((a,b),(a,b)\big)$:

$$(f, \Phi_{i,j}) = 0 \quad \text{for all } i,j = 1,2,\dots$$

Then we have for all $i, j = 1, 2\dots$:

$$\begin{aligned}0 = (f, \Phi_{i,j}) &= \int_a^b \int_a^b f(s,t)\overline{\phi_i}(s)\overline{\phi_j}(t)dsdt \\ &= \int_a^b \overline{\phi_j}(t)\left(\int_a^b f(s,t)\overline{\phi_i}(s)ds\right)dt.\end{aligned} \qquad (6.7)$$

Therefore the expression $\int_a^b f(s,t)\overline{\phi_i}(s)ds$ is equal to zero almost everywhere and there exists a set $\mathcal{N} \subset (a,b)$ of Lebesgue measure zero such that the following holds for all $t \in (a,b)\backslash\mathcal{N}$:

$$\int_a^b f(s,t)\overline{\phi_i}(s)ds = 0 \qquad (6.8)$$

hence $f(s,t) = 0$ for almost every s and each $t \in (a,b)\backslash\mathcal{N}$ and we have:

$$\int_a^b |f(s,t)|^2 ds = 0 \quad \text{for all } t \in (a,b)\backslash\mathcal{N}$$

and thus

$$\int_a^b \int_a^b |f(s,t)|^2 dsdt = 0 \quad \text{for all } t \in (a,b)\backslash\mathcal{N}$$

which implies $f(s,t) = 0$ almost everywhere on $(a,b) \times (a,b)$. Now we can conclude that $\{\Phi_{i,j}, i,j = 1,2,\dots\}$ as defined above is an orthonormal basis for $L^2\big((a,b),(a,b)\big)$.

The following holds for the kernel $k(s,t)$:

$$k(s,t) = \sum_{i,j} (k, \Phi_{i,j})\Phi_{i,j}.$$

We define $k_n = \sum_{i,j=1}^{n} (k, \Phi_{i,j})\Phi_{i,j}$ which implies $\|k - k_n\| \to 0$ and we define the corresponding integral operators

$$(K_n f)(t) = \int_a^b k_n(s,t) f(s) ds.$$

The operators K_n are bounded linear operators of finite rank (since $\operatorname{Im} K_n \subset \text{L.S.}\{\phi_1, \dots, \phi_n\}$) and hence compact for each n. The following holds:

$$\|K - K_n\| \leq \|k - k_n\| \to 0.$$

Let $\{f_n\}$ be a sequence in $L^2(a,b)$ with $\|f_n\| = 1$. Since the operator K_1 is compact there exists a subsequence $\{f_{1n}\}$ of the sequence $\{f_n\}$ such that the sequence $\{K_1 f_{1n}\}$ converges. Since the operator K_2 is also compact, a subsequence $\{f_{2n}\}$ of the sequence $\{f_{1n}\}$ can be found such that the sequence $\{K_2 f_{2n}\}$ converges. Proceeding in a similar way a convergent subsequence $\{f_{in}\} \subset \{f_{(i-1)n}\}$ can be found for each $i = 3,4,\dots$ such that the sequence $\{K_i f_{in}\}$ converges.

In order to prove the compactness of the operator K, we will show that the diagonal sequence $\{K f_{nn}\}$ is a Cauchy sequence and therefore converges in the complete space $\mathcal{H}$, which implies the compactness of K.

Let $\epsilon > 0$ be given. Then there exists an m such that

$$\|K - K_m\| < \frac{\epsilon}{2}. \tag{6.9}$$

For $n \leq m$ the sequence $\{K_m f_{nn}\}$ converges since it is a subsequence of the convergent sequence $\{K_m f_{mn}\}$. Equation 6.9 hence yields:

$$\begin{aligned}
\|K f_{nn} - K f_{kk}\| &\leq \|K f_{nn} - K_m f_{nn}\| + \|K_m f_{nn} - K_m f_{kk}\| \\
&\quad + \|K_m f_{kk} - K f_{kk}\| \\
&\leq 2\|K - K_m\| + \|K_m f_{nn} - K_m f_{kk}\| \\
&< \epsilon + \|K_m f_{nn} - K_m f_{kk}\| \to \epsilon
\end{aligned}$$

thus $\{K f_{nn}\}$ is a Cauchy sequence and the operator K is compact.

(c) Let f, g be two functions which belong to $L^2(a,b)$. We use Fubini's theorem which results in:

$$\begin{aligned}(Kf,g) &= \int_a^b \int_a^b k(s,t)f(s)\overline{g(t)}dsdt \\ &= \int_a^b \int_a^b \overline{\overline{k(s,t)}g(t)}dtf(s)ds = (f, K^*g(t)).\end{aligned}$$

Hence K is a self-adjoint operator if and only if

$$k(s,t) = \overline{k(t,s)} = k^*(t,s).$$

(d) Since the operator K is compact, this property follows immediately from the spectral theorem for compact, self-adjoint operators A.1 in appendix A. ■

We consider the case where $\gamma = 2$ and choose an $\lambda_0 \in \rho(H) \cap \mathbb{C}^+$ such that the definition of the resolvent $R(\lambda_0)$ makes sense. According to lemma 6.1 on page 45 we have:

$$||R(\lambda_0)|| \leq ||\tilde{K}(t,s,\lambda_0)|| < \infty, \tag{6.10}$$

and $R(\lambda_0)$ is a bounded, linear and compact integral operator on $L^2(a,b)$. Hence, the spectrum of $R(\lambda_0)$ consists of the point spectrum only, i.e.:

$$\sigma(R(\lambda_0)) \setminus \{0\} = \sigma_p(R(\lambda_0)) \setminus \{0\}.$$

This enables us to choose a $\gamma_i \in \sigma_p(R(\lambda_0)) \setminus \{0\}$ with corresponding eigenfunction ψ_i such that the following holds:

$$R(\lambda_0)\psi_i = (H - \lambda_0)^{-1}\psi_i = \gamma_j\psi_i.$$

This implies

$$\psi_i = \gamma_i(H - \lambda_0)\psi_i = \gamma_i H\psi_i - \gamma_i\lambda_0\psi_i.$$

The eigenvalues λ_i of H can thus be expressed in terms of the eigenvalues of $R(\lambda_0)$ as follows:

$$\lambda_i = \frac{\gamma_i\lambda_0 + 1}{\gamma_i} \tag{6.11}$$

and the eigenfunctions of $R(\lambda)$ are also the eigenfunctions of H. From equation 6.11 we can conclude that $\sigma(H) = \sigma_p(H)$. Since H is self-adjoint, its eigenvalues must be real, i.e. $\sigma(H) = \sigma_p(H) \subset \mathbb{R}$ and σ_p has a countable number of elements. Now we can pick a $\lambda \in \mathbb{R}$ such that $\lambda \notin \sigma(H)$. Then the corresponding resolvent $R(\lambda)$ is a real and self-adjoint operator. Now we can apply the previous theorem and conclude that there must exist an orthonormal basis of $L^2(a,b)$ (since $\ker R(\lambda) = \{0\}$) of eigenfunctions of $R(\lambda)$ (and hence of H) such that the property described in **(d)** holds. The eigenvalues γ_i must converge to zero for $i \to 0$ according to lemma A on page 73. We can conclude that the following holds for the eigenvalues λ_i of H:

$$\lim_{i \to \infty} \lambda_i = \pm\infty.$$

Chapter 7

The Sturm-Liouville Differential Equation in $\mathbb{C}$ with a Regular Singularity

In this chapter we will take a closer look at Sturm-Liouville differential equations with interior singularities in the complex plane and briefly discuss the corresponding theory and its solutions.

We consider the Sturm-Liouville differential equations with an isolated singularity only. Without loss of generality we may assume such a singularity to be in $x = 0$ and we consider differential equations of the form:

$$y''(x) + \frac{A(x)}{x} y'(x) + \frac{B(x)}{x^2} y(x) = 0 \tag{7.1}$$

with A, B analytic functions on a neighbourhood U of the point $x = 0$, $U \subset \mathbb{C}$. Then the point $x = 0$ is a regular singular point of the differential equation (7.1).
Our purpose is to find a solution $\tilde{y}(x)$ on $[a, b] \subset \mathbb{R}$, with $[a, b] \subset U$ and $a < 0 < b$, which satisfies the following boundary conditions in the (regular) endpoints a and b:

$$y(a) = 0, \; y(b) = 0.$$

Consider two linearly independent solutions $y_1(x), y_2(x)$ of (7.1) on the neighbourhood of a certain regular point $x \in U$. Such solutions can be analytically extended along any path in $U \backslash \{0\}$. In particular, a solution can be extended along the circle with center $x = 0$ in U (in counterclockwise direction) which results in the function $\tilde{y}_1$ ($\tilde{y}_2$ respectively). The map $\tau : y_i \to \tilde{y}_i, \; i = 1, 2$ is

both linear and invertible, and possesses with respect to the basis $\{y_1, y_2\}$ the matrix representation M.
Assume λ is the eigenvalue of τ with eigenvector $y(x)$, i.e. $\tau y(x) = \lambda y(x)$. Since $\lambda \neq 0$ (as τ is invertible) we can define $\alpha := \frac{1}{2\pi i}\ln(\lambda)$ and hence $\lambda = e^{2\pi i\alpha}$. We also know $\tau x^\alpha = e^{2\pi i\alpha}x^\alpha$. Therefore the function $x^{-\alpha}y(x)$ has Laurent-series $\sum_{n=-\infty}^{\infty} a_n x^n$ and thus $y(x) = x^\alpha \sum_{n=-\infty}^{\infty} a_n x^n$. In the case of a regular singular point ($x = 0$), it can be shown that the function $y(x)$ has finite principal part, and $y(x)$ can therefore be written as $y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n$.

It is easily seen that such an α must satisfy the following equation :

$$s(s-1) + A(0)s + B(0) = 0 \tag{7.2}$$

known as the indicial equation of the differential equation (7.1 on the preceding page). This indicial equation is obtained by substituting

$$y(x) = x^s = \exp s(\ln|x| + i\arg(x))$$

in the differential equation and considering the differential equation modulo x^{s-1}.

We can now make a distinction with respect to the eigenvalues of the matrix M:

M has two distinct eigenvalues λ_1, λ_2 :
In this case the matrix M can be diagonalized and there exists a fundamental system of solutions y_1, y_2 of the equation (7.1 on the page before) of the form:

$$y_1(x) = x^{\alpha_1}\sum_{n=0}^{\infty}\gamma_n x^n$$
$$y_2(x) = x^{\alpha_2}\sum_{n=0}^{\infty}\eta_n x^n$$

with $\alpha_1 := \frac{1}{2\pi i}\ln(\lambda_1)$, $\alpha_2 := \frac{1}{2\pi i}\ln(\lambda_2)$.

M has only one eigenvalue $\lambda = \lambda_1 = \lambda_2$:
In this case $\alpha_1 - \alpha_2 \in \mathbb{N}$ with $\alpha_1 \geq \alpha_2$. This leaves us with two possibilities: either M can be diagonalized in which case $\alpha_1 > \alpha_2$ and we can proceed as above, or M cannot be diagonalized, which is certainly the case for $\alpha_1 = \alpha_2$. The matrix M then has the form $\begin{pmatrix}\lambda & c\\ 0 & \lambda\end{pmatrix}$. The solution y_1 has the form $y_1(x) = x^{\alpha_1}\sum_{n=0}^{\infty}\gamma_n x^n$. The second solution y_2 of the fundamental-system satisfies $\tau y_2 = cy_1 + \lambda y_2$. Therefore the solution y_2 must have the following form: $y_2(x) = x^{\alpha_2}\sum_{n=0}^{\infty}\eta_n x^n + y_1(x)\frac{c}{2\pi i\lambda}\ln|x|$.

We are now ready to present the so called Frobenius theorem:

Theorem 7.1 *Consider the differential equation*

$$y''(x) + \frac{A(x)}{x}y'(x) + \frac{B(x)}{x^2}y(x) = 0$$

and the solutions α, β *of the indicial equation*

$$s(s-1) + A(0)s + B(0) = 0.$$

Then the differential equation (7.1 on page 53) has two linearly independent solutions of the form:

$$y_1(x) = x^\alpha \sum_{n=0}^{\infty} \gamma_n x^n, \; y_2(x) = x^\beta \sum_{n=0}^{\infty} \eta_n x^n + Cy_1(x) \ln|x|$$

with $\gamma_0 = \eta_0 = 1$ *and* $C \neq 0$ *if* $\alpha = \beta$ *and* $C = 0$ *if* $\alpha - \beta \neq 0, 1, 2, \ldots$ $(\mathrm{Re}\,\alpha \geq \mathrm{Re}\,\beta)$.

Chapter 8

The Case with an Interior Singularity: an Example

In this chapter we will show how to use the differential operator theory as described in the previous chapters in a concrete example. In doing so, we will also see how the results for singular endpoints can be extended and used in the case of interior singularities, by using the direct-sum method.

We consider the following eigenvalue problem as described in [3]:

$$\begin{cases} -y''(x) - \frac{1}{x}y(x) = \lambda y(x), & x \in [a,b] \\ y(a) = y(b) = 0 & -\infty < a < 0 < b < \infty \end{cases}$$

which is a Sturm-Liouville differential equation as described in chapter 2, with $p(x) = 1$ and $q(x) = -\frac{1}{x}$. This equation is a simplified version of the equation

$$u''(r) + \left(\frac{\zeta}{r} - \frac{l(l+1)}{r^2}\right)u(r) = \kappa^2 u(r),$$

which plays an important role in the study of energy levels of the hydrogen atom in quantum mechanics. According to the Frobenius theorem 7.1 on page 55 we find the following initial equation:

$$s(s-1) = 0$$

with solutions $\alpha = 1, \beta = 0$. This implies the existence of the solutions y_1 and y_2 with the following form:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^n + C y_1(x) \ln|x|.$$

In the case of the first solution y_1 we find the following recurrence relation for the coefficients a_n:

$$a_0 = 1, \qquad a_1 = -\frac{1}{2} \qquad \text{and} \qquad a_{n+2} = -\frac{\lambda a_n + a_{n+1}}{(n+3)(n+2)}$$

which yields a solution of the form:

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{(\frac{1}{2} - \lambda)}{6}x^3 + \dots \tag{8.1}$$

In the case of the second solution y_2 we must choose $C \neq 0$. Indeed, if we did choose $C = 0$, the coefficients b_n must satisfy

$$0 = \sum_{n=2}^{\infty} b_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} b_n x^{n-1} + \sum_{n=0}^{\infty} \lambda b_n x^n$$

which implies $b_0 = 0$ and hence it yields the same solution as in the first case. We choose $C = -1$ and $b_1 = 0$ (this can be done since the first solution has the form $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$), which yields the following recurrence relation for the coefficients b_n:

$$b_0 = -a_0, \qquad b_1 = 0 \qquad \text{and} \qquad b_{n+2} = -\frac{b_{n+1} + \lambda b_n + (2n+3)a_{n+1}}{(n+2)(n+1)}.$$

Hence, we have the following form for the solution y_2 :

$$y_2(x) = -1 + \frac{1}{2}(\lambda + \frac{3}{2})x^2 + \dots - (x - \frac{1}{2}x^2 + \dots)\ln x. \tag{8.2}$$

We take a closer look at the nature of differential equation 8.1. Obviously, the function $q(x)$ does not belong to $L^2[a,b]$ on account of the singularity in $x = 0$. However, the function $q(x)$ does belong to both $L^2[a,\epsilon)$ and $L^2(\epsilon,b]$ for all $\epsilon > 0$. This is why we choose to consider the differential equation on two separate intervals $[a,0)$ and $(0,b]$. In both cases, the endpoint $x = 0$ is a singular endpoint.

We concentrate on the interval $(0,b]$. As we did in chapter 2 on page 5, we examine the above described differential expression as a differential operator defined on the Hilbert-space $L^2(0,b]$. We recall the definition of the set $\mathcal{D}_p$:

$$\mathcal{D}_p = \{f : (0,b] \to \mathbb{C} \mid f, f' \in AC_{loc}(0,b]\}$$

which implies the following definition for the maximal operator L_{max}:

$$L_{max} := \Big\{\{f, Lf\} \mid f \in \mathcal{D}_p,\ f, Lf \in L^2(0,b]\Big\}.$$

We consider the two functions $u_1(x) = x$ and $u_2(x) = 1 - x \ln x$. It can easily be seen that both functions belong to dom $L_{max}(0, b]$. As we did in the proof of lemma 4.2 on page 31, we define the following functions $v_1, v_2 \in$ dom L_{max} for a certain interval $[c, d] \subsetneq (0, b]$:

$$v_i(x) = u_i \quad \text{for } x \in (0, c), \qquad v_i(x) = 0 \quad \text{for } x \in (d, b], \quad i = 1, 2.$$

These functions satisfy the following equations for $x \sim 0$:

$$v_1(x) \sim y_1(x), \qquad v_2(x) \sim y_2(x).$$

By lemma 2.1 on page 7 we know that the following limit exists:

$$\lim_{x \downarrow 0} [v_1, v_2](x) = \lim_{x \downarrow 0} -x - 1 = -1 \neq 0$$

and hence by lemma 4.2 on page 31 the endpoint $x = 0$ is limit circle. This implies $\gamma_+ = \gamma_- = 2$, and there exist a self-adjoint differential operator H_+ as defined in equation 4.21 on page 34. We choose $v(x) = v_1(x)$. Since the endpoint b is a regular endpoint, the function $w(x)$ in equation 4.21 can be chosen in such a way that

$$w(b) = 0, \; w'(b) = 1 \quad w(0) = 0, \; w'(0) = 0.$$

This yields the description of the self-adjoint differential operator H_+:

$$H_+ = \left\{ \{f, Lf\} \in L_{max} \mid \lim_{x \downarrow 0} f(x) - x f'(x) = 0, f(b) = 0 \right\}. \tag{8.3}$$

Since H_+ is a (J-)real self-adjoint extension, we can find in a way as described in chapter 6 a real λ such that the resolvent $R(\lambda) = (H_+ - \lambda)^{-1}$ is a compact, self-adjoint Hilbert-Schmidt operator according to theorem 6.2 on page 47. Hence, H_+ has a discrete spectrum. As we have seen at the end of chapter 6, the eigenfunctions ψ_i^+ of H_+ form an orthonormal basis of $L^2(0, b]$ with corresponding real eigenvalues λ_i^+. Since H^+ is a positive operator, its eigenvalues must converge to $+\infty$, i.e.

$$\lim_{i \to \infty} \lambda_i^+ = \infty.$$

We can prove the following for the eigenvalues of H_+:

Lemma 8.1 *The eigenvalues λ_i^+ of H_+ are simple.*

Proof.
Consider an eigenvalue λ of H_+ with

$$(H_+ - \lambda)\psi_1 = 0 \quad \text{and} \quad (H_+ - \lambda)\psi_2 = 0$$

where $\psi_1, \psi_2 \in \operatorname{dom} H_+$ are linearly independent eigenvalues. According to lemma 2.4 on page 8 the following holds:

$$\psi_1\psi_2' - \psi_1'\psi_2 \neq 0. \tag{8.4}$$

However, rewriting equation 8.4 also yields the following:

$$\psi_1\psi_2' - \psi_1'\psi_2 = \\ \frac{1}{1+x}\Big([\psi_1, v_2](x)(-[\psi_2, v_1](x)) - ([\psi_2, v_2](x)(-[\psi_1, v_1](x)))\Big) \tag{8.5}$$

which converges to zero for $x \to 0$, since $\psi_1, \psi_2 \in \operatorname{dom} H_+$. Thus, ψ_1, ψ_2 are not linearly independent and the eigenvalue λ is simple. ■

In a completely similar way, we are able to construct a self-adjoint differential operator H_- on the Hilbert-space $L^2[a, 0)$ as follows:

$$H_- = \left\{\{f, Lf\} \in L_{max} \mid \lim_{x\uparrow 0} f(x) - xf'(x) = 0, f(a) = 0\right\}. \tag{8.6}$$

where L_{max} is the maximal operator with $\operatorname{dom} L_{max} \subset L^2[a, 0)$. The operator H_- again has simple, real eigenvalues λ_i^- and corresponding eigenfunctions ψ_i^-, which form an orthonormal basis of $L^2[a, 0)$.

At this point, we wish to describe the self-adjoint extension H of the direct sum of the operators $L_{min,[a,0)} \oplus L_{min,(0,b]}$. In general, the boundary conditions in the singular point $x = 0$ for such an operator has the following form:

$$M\begin{pmatrix}\lim_{x\uparrow 0}[f, v_1](x)\\ \lim_{x\uparrow 0}[f, v_2](x)\end{pmatrix} + N\begin{pmatrix}\lim_{x\downarrow 0}[f, v_1](x)\\ \lim_{x\downarrow 0}[f, v_2](x)\end{pmatrix} = 0$$

where the matrices $M, N \in \mathbb{C}^{2\times 2}$, $\operatorname{rank}(M : N) = 2$ and M and N satisfy the following:

$$MJM^* - NJN^* = 0, \quad J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We can choose $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then we have $H = H_+ \oplus H_-$. We consider this operator $H_- \oplus H_+$ on the Hilbert-space $L^2[a,0) \oplus L^2(0,b]$ with inner product

$$(\{f_-, f_+\}, \{g_-, g_+\}) = (f_-, g_-) + (f_+, g_+).$$

This operator $H_- \oplus H_+$ on the Hilbert-space $L^2[a,0) \oplus L^2(0,b]$ is defined as follows:

- $\operatorname{dom}(H_- \oplus H_+) = \left\{ f \in L^2[a,b] \mid f = f_- \oplus f_+,\ f_\pm \in \operatorname{dom} H_\pm \right\}$.
- $(H_- \oplus H_+)f = H_- f_- \oplus H_+ f_+$.

Obviously, $H_- \oplus H_+$ is self-adjoint and its eigenvalues are

$$\{\lambda_i^-, i = 1, 2, \dots\} \cup \{\lambda_i^+, i = 1, 2, \dots\}$$

The corresponding eigenfunction for an eigenvalue λ_i^- is $\{\psi_i^-, 0\}$, the corresponding eigenfunction for a eigenvalue λ_i^+ is $\{0, \psi_i^+\}$. These eigenfunctions form an orthonormal basis for the Hilbert-space $L^2[a,0) \oplus L^2(0,b]$.

The last step in defining a self-adjoint operator H on $L^2[a,b]$ is to define an isometric, isomorphic mapping U as follows:

$$U : L^2[a,0) \oplus L^2(0,b] \to L^2[a,b]$$

$$U\{f_-, f_+\}(x) = f(x) = \begin{cases} f_-(x) & \text{for } x \in [a,0) \\ f_+(x) & \text{for } x \in (0,b], \end{cases}$$

and the operator H on the Hilbert-space $L^2[a,b]$:

- $f \in \operatorname{dom} H$ if and only if there exist an $f_- \in \operatorname{dom} H_-$, $f_+ \in \operatorname{dom} H_+$ such that the following holds:

  $$f(x) = f_-(x) \text{ for } x \in [a,0), \quad f(x) = f_+(x) \text{ for } x \in (0,b],$$

  and defining $f(0) = 0$,
- $Hf(x) = \begin{cases} H_- f_-(x) & \text{for } x \in [a,0) \\ H_+ f_+(x) & \text{for } x \in (0,b]. \end{cases}$

It can easily be seen that the operators $H_- \oplus H_+$ and H are unitarily equivalent under the mapping U. If we reorder the eigenvalues

$$\{\lambda_i^-, i = 1, 2, \dots\} \cup \{\lambda_i^+, i = 1, 2, \dots\}$$

on the real axis, we get the eigenvalues $\{\lambda_n, n = 1, 2, \dots\}$ of the operator H, which is a self-adjoint differential operator since $H_- \oplus H_+$ is self-adjoint. Since the operators H_+ and H_- are positive operators, the spectrum of H is real, discrete with its only possible cluster-point in ∞. The eigenvalues have multiplicity 2 at most, which occurs if for certain i, j the following holds:

$$\lambda_i^+ = \lambda_j^- \qquad \text{or} \qquad \lambda_j^+ = \lambda_i^- .$$

This implies the following for the eigenvalues λ_n of H:

$$-\infty < \lambda_1 \leq \lambda_1 \leq \dots \leq \lambda_n \leq \lambda_{n+1} \to \infty.$$

The corresponding eigenfunctions ψ_n form an orthonormal basis in $L^2[a, b]$ and have the following form:

$$\psi_n = \begin{cases} \psi_n^- & \text{for } x \in [a, 0) \\ 0 & \text{for } x \in (0, b] \end{cases} \quad \text{or} \quad \psi_n = \begin{cases} 0 & \text{for } x \in [a, 0) \\ \psi_n^+ & \text{for } x \in (0, b] \end{cases}$$

according to whether the corresponding λ_n is equal to either λ_i^+ or λ_i^- for some i.

Chapter 9

Solutions in the Sense of Hyper-Functions

In this chapter we will leave the theory of differential operators behind and use a totally different approach in order to find solutions, eigenfunctions and corresponding eigenvalues of Sturm-Liouville differential equations with interior singularity as discussed in chapter 8. We will show that similar solutions to the ones as found in the previous chapter can be found by using hyper-functions.

We return to our initial problem and consider the following eigenvalue-problem:

$$y''(x) + \frac{A(x)}{x} y'(x) + \frac{B(x)}{x^2} y(x) = \lambda y(x) \tag{9.1}$$

on $[a,b] \subset \mathbb{R}$, $a < 0 < b$ with boundary conditions:

$$y(a) = 0, \quad y(b) = 0$$

where $A(x)$, $B(x)$ are real-valued for $x \in [a,b] \subset \mathbb{R}$. Consider a neighbourhood M of part of the real axis which intersects the real axis in $[a,b]$. When we remove the interval $[a,b]$ from M, we get two separate areas $M_+ = \{z \in M \mid \operatorname{Im} z > 0\}$ and $M_- = \{z \in M \mid \operatorname{Im} z < 0\}$, and hence

$$M = M_+ \cup M_- \cup [a,b].$$

With $A(M_+)$ we denote the set of functions y_+ which are analytic on the intersection of a neighbourhood of the interval $[a,b]$ and the set M_+. The set $A(M_-)$ is defined in a similar way.
Two linearly independent solutions $y_1(z,\lambda), y_2(z,\lambda)$ of our differential equation 9.1 can be found on $M \backslash [a,0]$ with the help of the Frobenius theorem

7.1. The general solution of equation 9.1 on the page before on $M\backslash[a,0]$ can hence be written as follows:

$$y(z,\lambda) = c_1(\lambda)y_1(z,\lambda) + c_2(\lambda)y_2(z,\lambda). \tag{9.2}$$

This function $y(z,\lambda)$ is analytic except for the branch point in $z = 0$. The coefficients c_1, c_2 can now be determined up to a certain factor by solving the equation:

$$y(b,\lambda) = c_1(\lambda)y_1(b,\lambda) + c_2(\lambda)y_2(b,\lambda) = 0 \tag{9.3}$$

(take for instance $c_1 = y_2(b,\lambda), c_2 = -y_1(b,\lambda)$). The function $y(z,\lambda)$ thus obtained by solving equation 9.3 can be analytically extended towards a going either counterclockwise or clockwise around the singularity in $z = 0$. This results in the functions $y_+(z,\lambda) \in A(M_+)$ and $y_-(z,\lambda) \in A(M_-)$ respectively. These functions $y_+(z,\lambda)$ and $y_-(z,\lambda)$ coincide for $\operatorname{Re} z > 0$, but in general they need not coincide for $\operatorname{Re} z < 0$.

The following holds for every $\zeta \in [a,0) \cup (0,b]$:

$$\text{the limits} \quad \lim_{\xi\to 0} y_+(\zeta + i\xi) \quad \text{and} \quad \lim_{\xi\to 0} y_-(\zeta - i\xi) \quad \text{exist for } \xi > 0.$$

Moreover, if we choose $\zeta \in (0,b]$, we have

$$\lim_{\xi\to 0} y_+(\zeta + i\xi,\lambda) = \lim_{\xi\to 0} y_-(\zeta - i\xi,\lambda) = y_0(\zeta,\lambda) \text{ for } \xi > 0. \tag{9.4}$$

Now we consider the hyper-function Y, which is defined as follows:

$$Y = \{y_+, y_-\} \in A(M_+) \cup A(M_-),$$

and we consider Y modulo functions which are analytic on the whole of M (i.e. mod $A(M)$). We define the following function :

$$w(z,\lambda) = y_+(z,\lambda) - y_-(z,\lambda), \tag{9.5}$$

which is well-defined on the interval $[a,0)$ and zero on the interval $(0,b]$, which can be deduced from equation 9.4. We now define for an $\epsilon > 0$:

$$h_\epsilon(z,\lambda) = \frac{1}{2\pi i}\int_a^{-\epsilon} \frac{w(t,\lambda)}{z-t}dt, \quad z \notin [a,-\epsilon].$$

For all $z \notin [a,-\epsilon]$, $h_\epsilon(z,\lambda)$ is an analytic function for every $\epsilon > 0$. The following limits exist for $\zeta \notin \{a,-\epsilon\}$:

$$\lim_{\xi\downarrow 0} h_\epsilon(\zeta + i\xi,\lambda) = h_{+\epsilon}(\zeta,\lambda), \quad \lim_{\xi\uparrow 0} h_\epsilon(\zeta + i\xi,\lambda) = h_{-\epsilon}(\zeta,\lambda). \tag{9.6}$$

From equation 9.6 on the page before we can deduce the following equation with the help of the Plemelj-formula:

$$h_{+\epsilon}(z,\lambda) - h_{-\epsilon}(z,\lambda) = w(z,\lambda) = y_+(z,\lambda) - y_-(z,\lambda), \quad \text{for } z \in (a,-\epsilon),$$

which implies

$$h_{+\epsilon}(z,\lambda) - y_+(z,\lambda) = h_{-\epsilon}(z,\lambda) - y_-(z,\lambda) := v_\epsilon(z,\lambda) \quad \text{for } z \in (a,-\epsilon).$$

Hence, the expression $v_\epsilon(z,\lambda)$ is analytic on $M\backslash[-\epsilon,0]$ Since the above described arguments hold for all $\epsilon > 0$, we can using a Mittag-Leffler construction to obtain a function $h(z,\lambda)$, analytic for all $z \notin [a,0]$, such that

$$h(z,\lambda) \equiv y(z,\lambda) \mod A(M \setminus \{0\}).$$

Hence, Y can be represented by $h(z,\lambda)$.

Summarising: not only is our hyper-function Y defined with the help of the functions $y_+(z,\lambda), y_-(z,\lambda)$, Y is also represented by the function $w(z,\lambda)$ as defined in equation 9.5 on the preceding page. Moreover, Y satisfies the following:

$$Y|_{(0,b]} = 0 \quad \text{and} \quad Y|_{[a,0)} \neq 0$$

which can be deduced from equation 9.4 on the page before. We hence have constructed a solution Y of our differential equation 9.1 on page 63 with

$$\left(\operatorname{supp}(Y) \cap \mathbb{R}\right) \subset [a,0].$$

Since we have already satisfied the boundary condition in $z = b$, we now only need to solve

$$Y(a,\lambda) = y_+(a,\lambda) - y_-(a,\lambda) = 0.$$

Obviously, $Y(a,\lambda)$ is an analytic function of λ and therefore has distinct zeros which form the eigenvalues $\lambda_n, n = 1,2,...$ of our differential equation. There are either zero or a finite or countable number of eigenvalues λ_n, and therefore there are either zero or a finite or countable number of *real* eigenvalues λ_n. Since $y_\pm(a,\lambda)$ are analytic functions in $\mathbb{C}$, the only possible cluster point for the eigenvalues must be in ∞ (for if it were elsewhere, both $y_\pm(a,\lambda)$ would be equal to a constant!).

Finally, we wish to examine whether our function $w(z,\lambda)$ is real for $z,\lambda \in \mathbb{R}$. In order to do so, we use theorem 7.1 on page 55 in order to examine the functions y_1, y_2. We make a distinction with respect to the solutions α, β of the indicial equation 7.2 on page 54 and consider $\lambda \in \mathbb{R}$ only:

Case 1: α, β real, $\alpha - \beta \neq 0, 1, 2, \ldots$:
First of all we consider the case where α, β satisfy $\alpha - \beta \neq 0, 1, 2, \ldots$ and both $\alpha, \beta \in \mathbb{R}$. The two linearly independent solutions y_1 and y_2 of our differential equation 9.1 have the following form according to the Frobenius theorem 7.1 on page 55:

$$y_1 = x^\alpha \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n, \quad y_2 = x^\beta \sum_{n=0}^{\infty} \eta_n(\lambda) x^n,$$

which are both real-valued for $x \in \mathbb{R}$. The analytical extension of the general solution

$$y(z, \lambda) = c_1(\lambda) y_1(z, \lambda) + c_2(\lambda) y_2(z, \lambda)$$

as described in equation 9.2 on page 64 results in the two functions $y_+(z, \lambda), y_-(z, \lambda)$ which have the following form for real $x < 0$:

$$\begin{aligned} y_+(x, \lambda) = (-x)^\alpha e^{\pi i \alpha} c_1(\lambda) \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n \\ + (-x)^\beta e^{\pi i \beta} c_2(\lambda) \sum_{n=0}^{\infty} \eta_n(\lambda) x^n \quad \text{with } \arg(-x) = 0, \end{aligned}$$

$$\begin{aligned} y_-(x, \lambda) = (-x)^\alpha e^{-\pi i \alpha} c_1(\lambda) \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n \\ + (-x)^\beta e^{-\pi i \beta} c_2(\lambda) \sum_{n=0}^{\infty} \eta_n(\lambda) x^n \quad \text{with } \arg(-x) = 0. \end{aligned}$$

These functions coincide for real $x > 0$. We divide by a factor 2i, which yields the following form for our function $w(x, \lambda)$ for real x:

$$\begin{aligned} \frac{w(x, \lambda)}{2i} &= \sin(\pi\alpha) c_1(\lambda) (-x)^\alpha \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n & (9.7) \\ &+ \sin(\pi\beta) c_2(\lambda) (-x)^\beta \sum_{n=0}^{\infty} \eta_n(\lambda) x^n \quad x < 0 & (9.8) \\ &= 0 \qquad\qquad x > 0. & (9.9) \end{aligned}$$

Hence $\frac{w(x,\lambda)}{2i}$ is a real-valued function for real x (note that $\frac{w(x,\lambda)}{2i}$ still satisfies the differential equation).

Case 2: α, β real, $\alpha - \beta \in \{0, 1, 2, \dots\}$:

Now we consider the case where the solutions α, β of the indicial equation 7.2 on page 54 satisfy $\alpha - \beta = 0, 1, 2, \dots$ and both $\alpha, \beta \in \mathbb{R}$. In general, $C \neq 0$. We may assume $\alpha = k + \beta$ for a certain $k \in \mathbb{N}$. Again two linearly independent solutions of our differential equation 9.1 on page 63 can be found with the help of the Frobenius theorem 7.1 on page 55 (in general, $C \neq 0$):

$$y_1 = x^\alpha \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n, \quad y_2 = x^\beta \sum_{n=0}^{\infty} \eta_n(\lambda) x^n + C y_1(x) \ln x$$

where both y_1, y_2 are real-valued solutions for $x \in \mathbb{R}$.
The general solution of the differential equation 9.1 as described in the equation 9.2 on page 64 then has the form

$$y(x, \lambda) = c_1(\lambda) y_1(x, \lambda) + c_2(\lambda) y_2(x, \lambda).$$

Analytical extension as described above results in the functions y_+, y_- which have the following form for real $x < 0$:

$$\begin{aligned} y_+(x, \lambda) = {} & (-x)^\alpha e^{\pi i \alpha} c_1(\lambda) \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n \\ & + (-x)^\beta e^{\pi i \alpha} (-1)^k c_2(\lambda) \sum_{n=0}^{\infty} \eta_n(\lambda) x^n \\ & + (-x)^\alpha e^{\pi i \alpha} c_2(\lambda) \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n C (\ln |x| + i\pi), \ \arg(-x) = 0, \end{aligned}$$

$$\begin{aligned} y_-(x, \lambda) = {} & (-x)^\alpha e^{-\pi i \alpha} c_1(\lambda) \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n \\ & + (-x)^\beta e^{-\pi i \alpha} (-1)^k c_2(\lambda) \sum_{n=0}^{\infty} \eta_n(\lambda) x^n \\ & + (-x)^\alpha e^{-\pi i \alpha} c_2(\lambda) \sum_{n=0}^{\infty} \gamma_n(\lambda) x^n C (\ln |x| - i\pi), \ \arg(-x) = 0 \end{aligned}$$

which coincide for $x > 0$, but in general they need not coincide for $x < 0$. As in the previous case, we divide by $2i$. This yields the following form for our solution $\frac{w(x,\lambda)}{2i}$ for real x:

$$\begin{aligned}\frac{w(x,\lambda)}{2i} &= \sin(\pi\alpha)c_1(\lambda)(-x)^\alpha \sum_{n=0}^{\infty} \gamma_n(\lambda)x^n \\ &+ (-1)^k \sin(\pi\alpha)c_2(\lambda)(-x)^\beta \sum_{n=0}^{\infty} \eta_n(\lambda)x^n \\ &+ C\sin(\pi\alpha)c_2(\lambda)(-x)^\alpha \sum_{n=0}^{\infty} \gamma_n(\lambda)x^n \ln|x| \\ &+ C\cos(\pi\alpha)c_2(\lambda)(-x)^\alpha \sum_{n=0}^{\infty} \gamma_n(\lambda)x^n \qquad x < 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \geqslant 0\end{aligned}$$

which obviously is a real-valued function for real x.

Case 3: $\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$:

For α, β in $\mathbb{C}$, we have $\alpha = \overline{\beta} = \nu + i\omega$, with $\omega \neq 0$. Obviously, $C = 0$ in all cases (since $\alpha - \beta = 2i\omega \neq 1, 2, \ldots$). According to the Frobenius theorem 7.1 on page 55 we have the following solutions:

$$y_1 = x^\alpha \sum_{n=0}^{\infty} \gamma_n x^n, \quad y_2 = x^\beta \sum_{n=0}^{\infty} \eta_n x^n$$

However, both y_1, y_2 need not be real-valued for $x \in (0, b]$, since the coefficients γ_n, η_n are complex (and moreover: $\gamma_n = \overline{\eta}_n$, $n = 0, 1, 2, \ldots$). For $x \in \mathbb{R}$ we have $y_1(x,\lambda) = \overline{y_2(x,\lambda)}$.

Again, we consider the general solution

$$y(x,\lambda) = c_1(\lambda)y_1(x,\lambda) + c_2(\lambda)y_2(x,\lambda).$$

In order to satisfy $y(b,\lambda) = 0$, we need to choose $c_1 = -\overline{c}_2 = -\overline{y_1(b,\lambda)}$, since $y_2(b,\lambda) = \overline{y_1(b,\lambda)}$. This yields:

$$y(x,\lambda) = x^\alpha \sum_{n=0}^{\infty} c_1\gamma_n x^n - x^\beta \sum_{n=0}^{\infty} \overline{c_1\gamma_n} x^n.$$

Some calculation shows that no real-valued solutions of the differential equation 9.1 can be obtained unless $\omega = 0$, in which case $\alpha, \beta \in \mathbb{R}$ and solutions can be obtained as in the previous cases.

Conclusion: the function $\frac{w(x,\lambda)}{2i}$ is real-valued for real x if and only if the solutions α, β of the indicial equation are real.

Obviously, if we start from the endpoint a (instead of b) and analytically extend our solutions towards b, we can find in a similar way hyper-function-solutions V which satisfy:

$$V|_{(0,b]} \neq 0 \quad \text{and} \quad V|_{[a,0)} = 0$$

with corresponding eigenvalues.

The hyper-function-solutions Y, V of our differential equation 9.1 on page 63 as described above can hence be identified with the solutions as found in chapter 8 which are equal to 0 on one side of the singularity, and not necessarily zero on the other side. In the case of our special example

$$\begin{cases} -y''(x) - \frac{1}{x}y(x) = \lambda y(x), & x \in [a,b] \\ y(a) = y(b) = 0 & -\infty < a < 0 < b < \infty \end{cases}$$

we have real α, β. Hence the solution $w(x, \lambda)$ is real for real λ and the eigenfunctions ψ_n have the same form as described in chapter 8:

$$\psi_n = \begin{cases} \psi_n^- & \text{for } x \in [a,0) \\ 0 & \text{for } x \in (0,b] \end{cases} \quad \text{or} \quad \psi_n = \begin{cases} 0 & \text{for } x \in [a,0) \\ \psi_n^+ & \text{for } x \in (0,b]. \end{cases}$$

In the same way as in chapter 8, it can be seen that the corresponding eigenvalues λ_n form a real, discrete set and have multiplicity 2 at most, and converge to ∞ for $n \to \infty$:

$$-\infty < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \to \infty.$$

It is possible to give a different construction of hyper-function-solutions starting from an analytic solution y on $\operatorname{Im} x > 0$ and considering a pair $\{y, \tilde{y}\}$ where $\tilde{y} = \sigma y_+ + (1-\sigma)y_-$, where y_+ denotes the analytically continuation of y around 0 in the positive sense, and y_- denotes the analytically continuation of y in the negative sense. The method we have used above is a special case of this more general method.

Chapter 10

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august 1996, Groningen.

Appendix A

The Spectral Theorem for Compact Self-Adjoint Operators

The following theorem is known as the spectral theorem for compact self-adjoint operators:

Theorem A.1 *If T a compact, self-adjoint operator on Hilbert-space $\mathcal{H}$, there exists an orthonormal basis $\{\psi_i, i = 1, 2, \dots\}$ of eigenfunctions of T with corresponding eigenvalues $\{\gamma_i, i = 1, 2, \dots\}$ such that for each $f \in \mathcal{H}$ the following holds:*

$$Tf = \sum_{i=1}^{\infty} \gamma_i (f, \psi_i) \psi_i.$$

If there are an infinite number of eigenvalues γ_i, the eigenvalues converge to zero:

$$\lim_{i \to \infty} \gamma_i = 0.$$

In order to prove this theorem, we use the following lemmas:

Lemma A.2 *If T is a self-adjoint operator on Hilbert-space $\mathcal{H}$, we have*

$$||T|| = \sup_{||f||=1} |(Tf, f)|, \quad f \in \mathcal{H}.$$

Proof.
We define $c = \sup_{||f||=1} (Tf, f)$, which implies for all f with $||f|| = 1$:

$$|(Tf, f)| \leq ||Tf|| \leq ||T||,$$

hence $c \leq ||T||$. In order to prove $||T|| \leq c$, we consider

$$\big(T(f+h),(f+h)\big) - \big(T(f-h),(f-h)\big) = 4\mathrm{Re}\,(Tf,h).$$

We use the parallelogram law to get

$$\begin{aligned} 4|\mathrm{Re}\,(Tf,h)| &\leq |\big(T(f+h),(f+h)\big)| + |\big(T(f-h),(f-h)\big)| \\ &\leq c||f+h||^2 + c||f-h||^2 = 2m(||f||^2 + ||h||^2). \end{aligned} \tag{A.1}$$

There exists a $\theta \in \mathbb{R}$ such that $(Tf,h) = |(Tf,h)|e^{i\theta}$. We substitute $f = fe^{-i\theta}$ in inequality A.1 which yields:

$$\begin{aligned} 4|\mathrm{Re}\,(Tf,h)e^{-i\theta}| &= 4|\mathrm{Re}\,|(Tf,h)|| \\ &= 4|(Tf,h)| \\ &\leq 2m(||f||^2 + ||h||^2) \end{aligned}$$

and hence

$$|\mathrm{Re}\,(Tf,h)e^{-i\theta}| \leq \frac{m}{2}(||f||^2 + ||h||^2) \quad \text{for all } f,h \in \mathcal{H}. \tag{A.2}$$

We take $h = \frac{||f||}{||Tf||}Tf$ in inequality A.2 to get

$$||Tf||||f|| \leq c||f||^2$$

and hence $||Tf|| \leq c||f||$ for all $f \in \mathcal{H}$. This proves $||T|| \leq c$. ■

Lemma A.3 *For a compact self-adjoint operator T on a Hilbert-space $\mathcal{H}$ at least one of the numbers $||T||$ or $-||T||$ is an eigenvalue of T.*

Proof.
If $T = 0$, obviously $0 = ||0||$ is an eigenvalue. For $T \neq 0$, we know from lemma A.2 on the page before $0 \neq ||T|| = \sup_{||f||=1} |(Tf,f)|$. Hence there exists a sequence $\{f_n\}$ with $||f_n|| = 1$ such that

$$|(Tf_n,f_n)| \to ||T||.$$

There exists a subsequence, say $\{(Tf_{ni},f_{ni})\} \subset \{(Tf_n,f_n)\}$ such that the following holds:

$$\{(Tf_{ni},f_{ni})\} \to ||T|| \text{ or } \{(Tf_{ni},f_{ni})\} \to -||T||. \tag{A.3}$$

This can be seen as follows: assume such a subsequence does not exist. Then there exists an $\epsilon > 0$ such that for the neighbourhood $\mathcal{M}$ of $||T||$ defined as follows

$$\mathcal{M} = (-||T|| - \epsilon, -||T|| + \epsilon) \cup (||T|| - \epsilon, ||T|| + \epsilon) \in \mathbb{R}$$

the following holds for all $n = 1, 2, \ldots$:

$$\{(Tf_n, f_n)\} \notin \mathcal{M}$$

and hence $\{|(Tf_n, f_n)|\} \notin (||T|| - \epsilon, ||T|| + \epsilon)$, which contradicts the fact that $|(Tf_n, f_n)| \to ||T||$. This proves equation A.3 on the preceding page. There exists a real number λ such that $|\lambda| = ||T||$ and the following holds:

$$\{(Tf_{ni}, f_{ni})\} \to \lambda.$$

We use $T = T^*$ to obtain

$$\begin{aligned} 0 &\leq ||Tf_{ni} - \lambda f_{ni}||^2 \\ &= ||Tf_{ni}||^2 - 2\lambda(Tf_{ni}, f_{ni}) + |\lambda|^2 \\ &\leq 2\lambda^2 - 2\lambda(Tf_{ni}, f_{ni}) \to 0. \end{aligned}$$

Thus $Tf_{ni} - \lambda f_{ni} \to 0$. Since T is compact, there exists a converging subsequence $\{Tf_{ni'}\}$ of $\{Tf_{ni}\}$ with $\{Tf_{ni'}\} \to g$ for some $g \in \mathcal{H}$. Hence $Tf_{ni'} - \lambda f_{ni'} \to 0$ and therefore

$$||\lambda f_{ni'} - g|| \leq ||\lambda f_{ni'} - Tf_{ni'}|| + ||Tf_{ni'} - g|| \to 0$$

which implies $f_{ni'} \to \frac{g}{\lambda}$. Since T is a bounded operator we have

$$Tf_{ni'} \to \frac{Tg}{\lambda}$$

and hence $Tg = \lambda g$. Moreover, $g \neq 0$ since we have $||f_{ni'}|| \to ||\frac{g}{\lambda}||$ which implies $||g|| = |\lambda| \neq 0$. Thus λ is an eigenvalue of T. ■

Proof of theorem A.
We define $\mathcal{H}_1 = \mathcal{H}$ and $T_1 = T$. We know by lemma A.3 that there exists an eigenvalue γ_1 of T_1 with corresponding eigenvector $\psi_1, ||\psi_1|| = 1$ such that $T_1\psi_1 = \gamma_1\psi_1$ and $\gamma_1 = ||T_1||$ or $\gamma_1 = -||T_1||$.
Now we define $\mathcal{H}_2 = \mathcal{H}_1 \ominus \text{L.S.}\{\psi_1\}$ and $T_2 = T_1|_{\mathcal{H}_1}$. Again, the operator T_2 is self-adjoint and compact, and by lemma A.3 we can find an eigenvalue γ_2 of T_2 with corresponding eigenvector $\psi_2, ||\psi_2|| = 1$ such that $T_2\psi_2 = \gamma_2\psi_2$

and $\gamma_2 = \|T_2\|$ or $\gamma_2 = -\|T_2\|$. Since $\mathcal{H}_2 \subset \mathcal{H}_1$, we have $\|T_2\| \le \|T_1\|$. The set $\{\psi_1, \psi_2\}$ obviously is an orthonormal set.

Continuing this process, we define $\mathcal{H}_3 = \mathcal{H}_1 \ominus \text{L.S.}\{\psi_1, \psi_2\} = \mathcal{H}_2 \ominus \text{L.S.}\{\psi_2\}$ and $T_3 = T_2|_{\mathcal{H}_3} = T|_{\mathcal{H}_3}$. The operator T_3 is still self-adjoint and compact, and again we can find an eigenvalue γ_3 of T_3 with corresponding eigenvector $\psi_3, \|\psi_3\| = 1$ such that $T_3\psi_3 = \gamma_3\psi_3$ and $\gamma_3 = \|T_3\|$ or $\gamma_3 = -\|T_3\|$. Again we have $\|T_3\| \le \|T_2\|$.

Proceeding this routine for as long as possible, we either find an n for which $T_n = 0$, or an infinite number of eigenvalues $\{\gamma_n\}$ and an orthonormal set of corresponding eigenvectors $\{\psi_n\}$ such that the following holds:

$$|\gamma_{n-1}| = \|T_{n-1}\| \ge \|T_n\| = |\gamma_n|, \qquad n = 2, 3, \ldots$$

In this case, the property $\gamma_n \to 0$ can be seen as follows: assume $|\gamma_n| \not\to 0$. Then there must exist an $\epsilon > 0$ such that $|\gamma_n| \le \epsilon$ for all $n = 1, 2, \ldots$. We use Pythagoras to get for all m, n:

$$\begin{aligned} \|T\psi_n - T\psi_m\|^2 &= \|\gamma_n\psi_n - \gamma_m\psi_m\|^2 \\ &= \|\gamma_n\psi_n\|^2 + \|\gamma_m\psi_m\|^2 \\ &= |\gamma_n|^2 + |\gamma_m|^2 \ge 2\epsilon^2. \end{aligned}$$

Since T is compact, there must exist a convergent subsequence of $T\psi_n$, and hence for a certain n, m we must have $\|T\psi_n - T\psi_m\| < \epsilon$. This contradicts our assumption and thus proves the fact that for an infinite number of γ_n we have $\lim_{n\to\infty} \gamma_n = 0$.

We make a distinction in two cases in order to prove the representation of T as described in the theorem:

Case 1: $T_{n+1} = 0$ for certain n. In this case, we have

$$f_{n+1} = f - \sum_{j=1}^{n} (f, \psi_j)\psi_j \in \mathcal{H}_{n+1}$$

for an $f \in \mathcal{H}$. This results in

$$0 = T_{n+1} f_{n+1} = Tf - \sum_{j=1}^{n} (f, \psi_j) T\psi_j$$

and hence

$$Tf = \sum_{j=1}^{n} \gamma_j (f, \psi_j)\psi_j.$$

Case 2: $T_{n+1} \neq 0$ for all $n = 0, 1, \ldots$. Similarly to case one we can derive $f_{n+1} = f - \sum_{j=1}^{n}(f, \psi_j)\psi_j$ and hence:

$$\begin{aligned}\|Tf - \sum_{j=1}^{n} \gamma_j(f, \psi_j)\psi_j\| = \|Tf_{n+1}\| &= \|T_{n+1}f_{n+1}\| \\ &\leq \|T_{n+1}\|\|f_{n+1}\| = |\gamma_{n+1}|\|f_{n+1}\| \\ &\leq |\gamma_{n+1}|\|f\| \to 0\end{aligned}$$

which implies $Tf = \sum_{j=1}^{n} \gamma_j(f, \psi_j)\psi_j$, as stated in the theorem. ■

Appendix B

The Riesz Representation Theorem

Theorem B.1 *Let f be a bounded linear functional on the Hilbert-space $\mathcal{H}$,i.e. $f \in \mathrm{L}(\mathcal{H}, \mathbb{C})$. Then there exists a unique $y \in \mathcal{H}$ such that for all $x \in \mathcal{H}$ the following holds:*

$$f(x) = (x, y) \qquad \textit{and} \qquad \|f\| = \|y\|.$$

Proof.
Define $\mathcal{N}$ as follows:

$$\mathcal{N} = \{x \in H \mid f(x) = 0\}.$$

If $\mathcal{N} = \mathcal{H}$, we have $f = 0$ for all x and $y = 0$. Suppose $f \neq 0$. Since f is continuous, $\mathcal{N}$ is a closed subspace of $\mathcal{H}$, and we can find a $x_0 \in \mathcal{N}^\perp$ with $x_0 \neq 0$. We make the following definition:

$$y = \frac{\overline{f(x_0)}x_0}{\|x_0\|^2}.$$

Then for all $x \in \mathcal{N}$ we have $f(x) = 0 = (x, y)$. For an $x = cx_0$ the following holds:

$$f(x) = f(cx_0) = cf(x_0) = \left(cx_0, \frac{\overline{f(x_0)}x_0}{\|x_0\|^2}\right) = (cx_0, y).$$

Thus, $f(x) = (x, y)$ for $x \in \mathcal{N}$ or $x = x_0$. Moreover, $f(x) = (x, y)$ for all $x \in \mathrm{L.S.}\{\mathcal{N}, x_0\}$. However, for all $w \in \mathcal{H}$ the following holds:

$$w = \left(w - \frac{f(w)}{f(x_0)}x_0\right) + \frac{f(w)}{f(x_0)}x_0 \ \in \mathrm{L.S.}\{\mathcal{N}, x_0\},$$

hence $\mathcal{H} = \text{L.S.}\{\mathcal{N}, x_0\}$ and $f(x) = (x, y)$ for all $x \in \mathcal{H}$. To show the uniqueness of y, we assume there exists a $w \in \mathcal{H}$ with the same property as y has, i.e.

$$(x, y) = f(x) = (x, w).$$

But this implies

$$0 = (x, y) - (x, w) = (x, y - w) \quad \text{for all } x \in \mathcal{H}$$

and hence $(y - w, y - w) = 0$ and $w = y$.
Finally, we have

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| \leq 1} |(x, y)| \leq \sup_{\|x\| \leq 1} \|x\| \|y\| = \|y\|$$

and also

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| \geq \left| f\left(\frac{y}{\|y\|}\right) \right| = \left(y, \frac{y}{\|y\|}\right) = \|y\|$$

which proves the property $\|f\| = \|y\|$. ∎

Appendix C

The Method of Variation of Parameters

In this appendix we will show the concept of variation of parameters when applied to a certain Sturm-Liouville differential expression L.

We wish to solve

$$(L-\lambda)u = g$$

by variation of parameters and therefore consider the general solution $u(t)$ of $(L-\lambda)u = 0$. Let u_1, u_2 be two linearly independent solutions of this equation. The general solution $u(t,\lambda)$ then has the following form:

$$u(t,\lambda) = c_1(t)u_1(t,\lambda) + c_2(t)u_2(t,\lambda).$$

We set $c_1'u_1 + c_2'u_2 = 0$, where the prime and D denote differentiation with respect to the variable t. From now on we will write $u(t)$ instead of $u(t,\lambda)$ and just keep the dependency of all solutions and parameters on λ in mind. Hence:

$$\begin{aligned}(L-\lambda)u &= c_1(-DpDu_1 + (q-\lambda)u_1) + c_2(-DpDu_2 + (q-\lambda)u_2) \\ &\quad - c_1'pu_1' - c_2'pu_2' \\ &= -c_1'pu_1' - c_2'pu_2' = g.\end{aligned}$$

We can rewrite this equation and use $c_1'u_1 + c_2'u_2 = 0$ to get:

$$\begin{pmatrix} u_1 & u_2 \\ -pu_1' & -pu_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

By lemma 2.4 on page 8 the first matrix in this equation is invertible. This yields the following:

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = W(u_1,u_2)^{-1} \begin{pmatrix} -pu_2' & -u_2 \\ pu_1' & u_1 \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} -u_2 g \\ u_1 g \end{pmatrix}$$

and hence

$$c_1 = -\int_0^t W(u_1, u_2)^{-1} u_2(t) g(t) dt + k_1$$
$$c_2 = \int_0^t W(u_1, u_2)^{-1} u_1(t) g(t) dt + k_2.$$

The general solution u of the equation $(L - \lambda)u = g$ can be written as follows:

$$u(t) = u_2(t) \int_0^t W(u_1, u_2)^{-1} u_1(s) g(s) ds - u_1(t) \int_0^t W(u_1, u_2)^{-1} u_2(s) g(s) ds$$
$$+ k_1 u_1(t) + k_2 u_2(t)$$

with $k_1, k_2 \in \mathbb{C}$.

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