Fractional linear transformations of Nevanlinna functions

Abe Douwe Ids Willemsma

Vakgroep
Wiskunde

RuG

Afstudeerverslag

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Abe Douwe Ids Willemsma

Rijksuniversiteit Groningen
Vakgroep Wiskunde
Postbus 800
9700 AV Groningen

Augustus 1996

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CONTENTS

0. Introduction

In this report we concider subclasses of Nevanlinna functions and their behavior under the fractional linear transformation.

$$(0.1) \quad Q_\tau(\ell) = \frac{Q(\ell) - \tau(\operatorname{Im} Q(\mu))^2}{1 + \tau Q(\ell)}, \quad \ell \in \mathbf{C} \setminus \mathbf{R},$$

where $\tau \in \mathbf{R} \cup \{\infty\}$. Each Nevanlinna function is the Q-function of a selfadjoint relation A and a one-dimensional symmetric restriction S of A. The fractional linear transformation (0.1) is equivalent to the Kreĭn formula parametrizing all selfadjoint extensions of S.

In particular, if A is a selfadjoint operator and S is not densely defined, then all but one of the selfadjoint extensions $A(\tau)$, $\tau \in \mathbf{R} \cup \{\infty\}$, of S are rank one perturbations of S, whose spectral behavior is described by the above mentioned subclasses of Nevanlinna functions.

The work in this report is a continuation of [HS1] and [HS2]. As in [HS1], the main difficulty is in the treatment of certain quadratic perturbation terms. Here we use the Hamburger-Nevanlinna asymptotic expansions (see [A]) and corresponding divided differences to estimate them. A simplification of the main arguments avoiding divided differences is based of a converse of the Hamburger-Nevanlinna expansions for both even and odd order expansions (see [HSW]).

The report is more or less self-contained. The main facts about Nevanlinna functions have been included in the appendix, see also [KK].

1. Nevanlinna functions & fractional linear transformations

We will consider functions $Q(\ell)$ in the class N of Nevanlinna functions. A function $Q(\ell)$ belongs to N if $Q(\ell)$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}$ and satisfies:

$$Q(\ell)^* = Q(\bar{\ell}), \quad \operatorname{Im} Q(\ell)/\operatorname{Im} \ell \geq 0, \qquad \ell \in \mathbf{C} \setminus \mathbf{R}.$$

Each function $Q(\ell) \in \mathrm{N}$ has an integral representation:

$$(1.1) \quad Q(\ell) = \alpha + \beta\ell + \int_{\mathbf{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) d\sigma(t),$$

where $\alpha \in \mathrm{R}$, $\beta \geq 0$ and where the function $\sigma(t)$ is nondecreasing on R and satisfies

$$(1.2) \quad \int_{\mathbf{R}} \frac{d\sigma(t)}{t^2+1} < \infty.$$

Conversely, if (1.1) and (1.2) are satisfied, then $Q(\ell) \in \mathrm{N}$. For more details about the integral representation, see the appendix. For a Nevanlinna function with integral representation (1.1), it follows that

$$(1.3) \quad \operatorname{Re} Q(iy) = \alpha + \int_{\mathbf{R}} \left(\frac{t}{t^2+y^2} - \frac{t}{t^2+1} \right) d\sigma(t),$$

and

$$(1.4) \quad \operatorname{Im} Q(iy) = \beta y + \int_{\mathbf{R}} \frac{y}{t^2+y^2} d\sigma(t).$$

This gives

$$(1.5) \quad \alpha = \operatorname{Re} Q(i), \quad \lim_{y \to \infty} \frac{\operatorname{Re} Q(iy)}{y} = 0,$$

while

$$(1.6) \quad \frac{\operatorname{Im} Q(iy)}{y} = \beta + \int_{\mathbf{R}} \frac{1}{t^2+y^2} d\sigma(t),$$

is nonincreasing for $y > 0$ and

$$(1.7) \quad \beta = \lim_{y \to \infty} \frac{\operatorname{Im} Q(iy)}{y} = \lim_{y \to \infty} \frac{Q(iy)}{y}.$$

The function $\sigma(t)$ may be recovered from $Q(\ell)$ by the Stieltjes inversion formula:

$$(1.8) \quad \sigma(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_0^t \operatorname{Im} Q(s+i\varepsilon) ds.$$

For more details about the Stieltjes inversion formula, see the appendix.

We will now define two subclasses $\mathbf{N}_1$ and $\mathbf{N}_0$ of $\mathbf{N} \supset \mathbf{N}_1 \supset \mathbf{N}_0$. Let $Q(\ell) \in \mathbf{N}$ and define

$$(1.9) \quad Q^{[0]}(y) = \frac{\operatorname{Im} Q(iy)}{y}.$$

Then $\mathbf{N}_1$ is the set of functions $Q(\ell)$ which belong to $\mathbf{N}$ and satisfy

$$\int_1^\infty Q^{[0]}(y)dy < \infty,$$

and $\mathbf{N}_0$ is the set of functions $Q(\ell)$ which belong to $\mathbf{N}$ and satisfy

$$\sup_{y>0} y^2 Q^{[0]}(y) < \infty.$$

For a function $Q(\ell) \in \mathbf{N}_1$ we have the following integral representation:

$$(1.10) \quad Q(\ell) = \gamma + \int_{\mathbf{R}} \frac{d\sigma(t)}{t-\ell},$$

where

$$(1.11) \quad \int_{\mathbf{R}} \frac{d\sigma(t)}{|t|+1} < \infty$$

and $\gamma \in \mathbf{R}$. Note that

$$(1.12) \quad \gamma = \lim_{y\to\infty} Q(iy) = \lim_{y\to\infty} \operatorname{Re} Q(iy).$$

The class of functions $\mathbf{N}_0$ coincides with the functions with integral representation (1.10) where

$$(1.13) \quad \int_{\mathbf{R}} d\sigma(t) < \infty.$$

For a proof of the integral representation (1.10) with (1.11) or (1.13), see the appendix.

Let $Q(\ell)$ belong to $\mathbf{N}$, and assume that $Q(\ell)$ does not reduce to a real constant. The fractional linear transformation $Q_\tau(\ell)$ is defined by

$$(1.14) \quad Q_\tau(\ell) = \frac{Q(\ell) - \tau\,(\operatorname{Im} Q(\mu))^2}{\tau Q(\ell) + 1},$$

see [HLS]. Here $\tau \in \mathbf{R} \cup \{\infty\}$, $\mu \in \mathbf{C} \setminus \mathbf{R}$ fixed. When $\tau = \infty$ we mean that

$$(1.15) \quad Q_\infty(\ell) = -\frac{(\operatorname{Im} Q(\mu))^2}{Q(\ell)}.$$

Thus $Q_\infty(\ell)$ can be seen as the limiting case of (1.14). Observe that

$$(1.16) \quad \operatorname{Re} Q_\tau(\ell) = \frac{\tau\left(|Q(\ell)|^2 - (\operatorname{Im} Q(\mu))^2\right) + \operatorname{Re} Q(\ell)\,(1 - \tau^2(\operatorname{Im} Q(\mu))^2)}{|1 + \tau Q(\ell)|^2}$$

and

$$(1.17)\quad \operatorname{Im} Q_\tau(\ell) = \operatorname{Im} Q(\ell)\frac{1+\tau^2(\operatorname{Im} Q(\mu))^2}{|1+\tau Q(\ell)|^2}.$$

For each $\tau \in \mathrm{R} \cup \{\infty\}$ the function $Q_\tau(\ell)$ belongs to N. Moreover, if $Q(\ell)$ belongs to N_1 or N_0, then for all but one $\tau \in \mathrm{R} \cup \{\infty\}$ the corresponding function $Q_\tau(\ell)$ belongs to N_1 or N_0, respectively. The exceptional value of τ is given by $1/\tau + \gamma = 0$, where γ is given by (1.12). If $Q(\ell) \in \mathrm{N}_0$ it follows that

$$(1.18)\quad \lim_{y\to\infty} \frac{\operatorname{Im} Q_\tau(y)}{y} = 0, \quad \frac{1}{\tau} + \gamma \neq 0$$

and

$$(1.19)\quad \beta = \lim_{y\to\infty} \frac{\operatorname{Im} Q_\tau(y)}{y} > 0, \quad \frac{1}{\tau} + \gamma = 0.$$

2. The subclasses $\mathbf{N}_{-k}$ for $k \in \mathbf{N} \cup \{0\}$

Let $Q(\ell)$ be a Nevanlinna function. We assume that $Q(\ell) \in \mathbf{N}_0$ and define the subclasses $\mathbf{N}_{-k}$, $k \in \mathbf{N}_0$ $(= \mathbf{N} \cup \{0\})$ inductively. In the previous section we started in $\mathbf{N}$ and defined the subclasses $\mathbf{N}_1$ and $\mathbf{N}_0$ by means of (1.9). If we proceed in $\mathbf{N}_0$ and define:

$$Q^{[2]}(y) = \sup_{y>0} y^2 Q^{[0]}(y) - y^2 Q^{[0]}(y), \tag{2.1}$$

then we may define the classes $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$ by

$$Q(\ell) \in \mathbf{N}_{-1} \text{ if and only if } Q(\ell) \in \mathbf{N}_0 \text{ and } \int_1^\infty Q^{[2]}(y)dy < \infty,$$

and

$$Q(\ell) \in \mathbf{N}_{-2} \text{ if and only if } Q(\ell) \in \mathbf{N}_0 \text{ and } \sup_{y>0} y^2 Q^{[2]}(y) < \infty.$$

Assume that $Q(\ell)$ belongs to $\mathbf{N}_{-2k}$ for some $k \in \mathbf{N}_0$ and that the function $Q^{[2k]}(y)$ is given such that

$$\sup_{y>0} y^2 Q^{[2k]}(y) < \infty.$$

We define

$$Q^{[2(k+1)]}(y) = \sup_{y>0} y^2 Q^{[2k]}(y) - y^2 Q^{[2k]}(y), \tag{2.2}$$

and we proceed in the following way:

$$Q(\ell) \in \mathbf{N}_{-2k-1} \text{ if and only if } Q(\ell) \in \mathbf{N}_{-2k} \text{ and } \int_1^\infty Q^{[2k+2]}(y)dy < \infty,$$

and

$$Q(\ell) \in \mathbf{N}_{-2k-2} \text{ if and only if } Q(\ell) \in \mathbf{N}_{-2k} \text{ and } \sup_{y>0} y^2 Q^{[2k+2]}(y) < \infty.$$

THEOREM 2.1. *Let $Q(\ell) \in \mathbf{N}_0$ with integral representation (1.10) and let $k \in \mathbf{N}$. Then $Q(\ell) \in \mathbf{N}_{-k}$ if and only if*

$$\int_{\mathbf{R}} \left(|t|^k + 1\right) d\sigma(t) < \infty. \tag{2.3}$$

Proof. We wil first prove the following claim:

$$Q(\ell) \in \mathbf{N}_{-2k} \Longleftrightarrow \int_{\mathbf{R}} \left(t^{2k} + 1\right) d\sigma(t) < \infty, \tag{2.4}$$

in which case

$$Q^{[2k+2]}(y) = \int_{\mathbf{R}} \frac{t^{2k+2}}{t^2 + y^2} d\sigma(t). \tag{2.5}$$

We will consider $k = 0$. Since for any $Q(\ell) \in \mathbf{N}$

$$Q^{[0]}(y) = \frac{\operatorname{Im} Q(iy)}{y} = \int_{\mathbf{R}} \frac{1}{t^2+y^2} d\sigma(t),$$

it follows that

$$\sup_{y>0} y^2 Q^{[0]} = \sup_{y>0} y \operatorname{Im} Q(iy) = \int_{\mathbf{R}} d\sigma(t) \ (\leq \infty),$$

so that

$$Q(\ell) \in \mathbf{N}_0 \Longleftrightarrow \int_{\mathbf{R}} d\sigma(t) \quad < \infty.$$

Moreover, in this case

$$Q^{[2]}(y) = \int_{\mathbf{R}} d\sigma(t) - \int_{\mathbf{R}} \frac{y^2}{t^2+y^2} d\sigma(t) = \int_{\mathbf{R}} \frac{t^2}{t^2+y^2} d\sigma(t).$$

Hence (2.4) and (2.5) are valid for $k = 0$.

Now assume that (2.4) holds for some k, then

$$\sup_{y>0} y^2 Q^{[2k+2]}(y) = \int_{\mathbf{R}} t^{2k+2} d\sigma(t) \ \ (\leq \infty).$$

Hence $Q(\ell) \in \mathbf{N}_{-2k-2} \Longleftrightarrow \int_{\mathbf{R}} \left(t^{2k+2}+1\right) \sigma(t) < \infty$, in which case

$$Q^{[2k+4]}(y) = \int_{\mathbf{R}} t^{2k+2} d\sigma(t) - \int_{\mathbf{R}} \frac{t^{2k+2} y^2}{t^2+y^2} d\sigma(t) = \int_{\mathbf{R}} \frac{t^{2k+4}}{t^2+y^2} d\sigma(t).$$

Finally observe that for $Q(\ell) \in \mathbf{N}_{-2k}$, it follows from (2.5) that

$$\begin{aligned}
\int_1^{\infty} Q^{[2k+2]}(y) &= \int_1^{\infty} \left(\int_{\mathbf{R}} \frac{t^{2k+2}}{t^2+y^2} d\sigma(t) \right) dy \\
&= \int_{\mathbf{R}} t^{2k+2} \left(\int_1^{\infty} \frac{1}{t^2+y^2} dy \right) d\sigma(t) \\
&= \int_{\mathbf{R}} t^{2k+2} \frac{1}{|t|} \left(\frac{\pi}{2} - \arctan \frac{1}{|t|} \right) d\sigma(t) \ (\leq \infty).
\end{aligned}$$

We conclude from this:

$$Q(\ell) \in \mathbf{N}_{-2k-1} \Longleftrightarrow \int_{\mathbf{R}} \left(|t|^{2k+1} + 1 \right) d\sigma(t) < \infty.$$

This proves the theorem.

Hence, if $Q(\ell) \in \mathbf{N}_{-k}$, for some $k \in \mathbb{N}_0$, then the moments $m_0, \ldots, m_k$

$$m_i = \int_{\mathbf{R}} t^i d\sigma(t), \qquad 0 \leq i \leq k,$$

are defined as absolutely convergent integrals. The following theorem with k even can be found in [A], where also a converse statement is given.

THEOREM 2.2. *Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbf{N}$. Then $Q(\ell)$ has the asymptotic expansion:*

$$(2.6)\quad Q(\ell) = \gamma - \sum_{i=0}^{k} \frac{m_i}{\ell^{\,i+1}} + o\left(\frac{1}{\ell^{\,k+1}}\right),\ \ell \to \infty,$$

uniformly in ℓ for $\delta \le \arg \ell \le \pi - \delta$, for any $0 < \delta < \pi$.

Proof. Let $Q(\ell) \in \mathbf{N}_{-k}$, then

$$(2.7)\quad \ell^{\,k+1}\left(Q(\ell) - \gamma + \sum_{i=0}^{k} \frac{m_i}{\ell^{\,i+1}}\right) = \int_{\mathbf{R}} \frac{t^{k+1}}{t-\ell} d\sigma(t)$$

holds, since

$$\begin{aligned} Q(\ell) - \gamma + \sum_{i=0}^{k} \frac{m_i}{\ell^{\,i+1}} &= \int_{\mathbf{R}} \frac{d\sigma(t)}{t-\ell} - \sum_{i=0}^{k} \frac{1}{\ell^{\,i+1}} \int_{\mathbf{R}} t^i d\sigma(t) \\ &= \int_{\mathbf{R}} \left(\frac{1}{t-\ell} - \frac{1}{\ell}\left(\frac{1-\frac{t^{k+1}}{\ell^{k+1}}}{1-\frac{t}{\ell}}\right)\right) d\sigma(t) \\ &= \frac{1}{\ell^{\,k+1}} \int_{\mathbf{R}} \frac{t^{k+1}}{t-\ell} d\sigma(t). \end{aligned}$$

Now we show

$$\int_{\mathbf{R}} \frac{t^{k+1}}{t-\ell} d\sigma(t) \to 0,\ \ell \to \infty.$$

We assume $\ell \in \mathbf{C}^+$ and $0 < \delta < \frac{\pi}{2}$. For $\delta \le \arg \ell \le \pi - \delta$, we obtain the inequalities [1]

$$(2.8)\quad |\ell - t| \ge |\ell| \sin \delta, \quad |\ell - t| \ge |t| \sin \delta.$$

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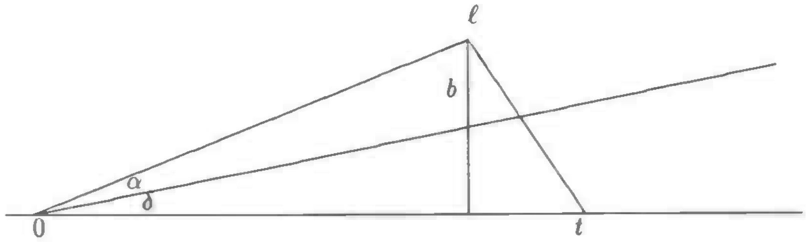
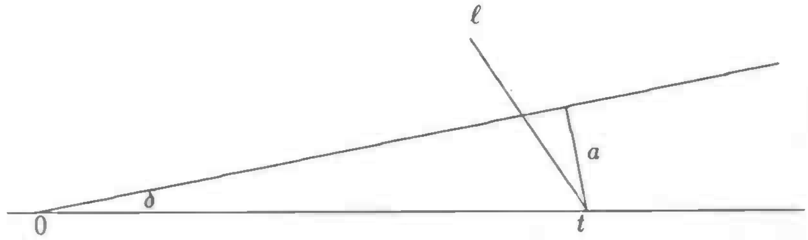
[1] For ℓ fixed in the sector, it follows that $|t - \ell| \ge b$. Moreover $\sin \delta \le \sin \alpha = \frac{b}{|\ell|}$.



For $t \in \mathbf{R}$, $t \neq 0$, fixed, it follows that $|t - \ell| \ge a$. Moreover, $a = |t| \sin \delta$.



From (2.8) we obtain for any $A > 0$

$$\left|\int_{\mathbb{R}} \frac{t^{k+1}}{t-\ell} d\sigma(t)\right| \leq \frac{1}{|\ell|\sin\delta}\int_{-A}^{A} |t|^{k+1} d\sigma(t) + \frac{1}{\sin\delta}\int_{|t|>A} |t|^{k} d\sigma(t).$$

For $\varepsilon > 0$, we choose $A > 0$, such that

$$\frac{1}{\sin\delta}\int_{|t|>A} |t|^{k} d\sigma(t) < \frac{\varepsilon}{2}.$$

Now choose ℓ in the sector so large that

$$\frac{1}{|\ell|\sin\delta}\int_{-A}^{A} |t|^{k+1} d\sigma(t) < \frac{\varepsilon}{2}.$$

Hence we have shown that (2.6) holds uniformly in ℓ in the indicated sector.

3. THE NONEXCEPTIONAL CASE

For $Q(\ell) \in \mathrm{N}$ and $1/\tau + \gamma \neq 0$ we define the function

$$(3.1) \quad \mathcal{B}_\tau(y) = \frac{1}{|1 + \tau Q(iy)|^2}, \quad y \in \mathbb{R}.$$

LEMMA 3.1. *Let $Q(\ell) \in \mathrm{N}_{-2k-1}$ for some $k \in \mathbb{N}_0$, and $1/\tau + \gamma \neq 0$. Then $\mathcal{B}_\tau(y)$ has an asymptotic expansion:*

$$(3.2) \quad \mathcal{B}_\tau(y) = \sum_{i=0}^{k+1} \frac{\beta_{2i}}{y^{2i}} + o\left(\frac{1}{y^{2k+2}}\right).$$

The asymptotic expansion (3.2) is equivalent to the limiting behavior of the successive divided differences:

$$\mathcal{B}_\tau^{\{0\}}(y) = \frac{1}{|1 + \tau Q(iy)|^2} \longrightarrow \beta_0,$$

$$\mathcal{B}_\tau^{\{2\}}(y) = y^2\left(\mathcal{B}_\tau^{\{0\}}(y) - \beta_0\right) \longrightarrow \beta_2,$$

$$\ldots$$

$$\mathcal{B}_\tau^{\{2k+2\}}(y) = y^2\left(\mathcal{B}_\tau^{\{2k\}}(y) - \beta_{2k}\right) \longrightarrow \beta_{2k+2}.$$

Proof. For $Q(\ell) \in \mathrm{N}_{-2k-1}$ and for $1 + \tau\gamma \neq 0$, we define

$$\mathcal{A}(y) = |1 + \tau Q(iy)|^2.$$

Then $\mathcal{A}(y)$ has an asymptotic expansion:

$$(3.3) \quad \mathcal{A}(y) = \sum_{j=0}^{k+1} \frac{a_j}{(iy)^j} + o\left(\frac{1}{(iy)^{k+1}}\right),$$

where the coefficients a_n are given by:

$$(3.4) \quad a_n = \tau^2 \sum_{j=0}^{n} (-1)^j m_{j-1} m_{n-j-1},$$

with $m_{-1} = -1/\tau - \gamma$. To see this we define $F(\ell) = 1 + \tau Q(\ell)$, so that for $Q(\ell) \in \mathrm{N}_{-2k-1}$ we have:

$$F(\ell) = \sum_{j=0}^{2k+2} \frac{f_j}{\ell^j} + o\left(\frac{1}{\ell^{2k+2}}\right),$$

where the coefficients are given by:

$$f_0 = 1 + \tau\gamma, \quad f_j = -\tau m_{j-1}, \quad j > 0.$$

Furthermore, we know that for $\ell = iy$, $\tilde{F}(\ell) := \overline{F(\ell)}$:

$$\tilde{f}_0 = 1 + \tau\gamma, \quad \tilde{f}_j = (-1)^{j+1}\tau m_{j-1}, \quad j > 0.$$

As $\mathcal{A}(y) = F(iy)\tilde{F}(iy)$, we only need to define the f_0 and the $\tilde{f}_0$ such that they satisfy the general formula for their coefficients. Notice that $f_0 = \tilde{f}_0 = -\tau m_{-1}$. Notice that $a_0 \neq 0$. Furthermore notice that the a_j in (3.4) have the following property: $a_{2j+1} = 0$. Also $\alpha_{2j} = (-1)^j a_{2j}$ and thus $\alpha_0 \neq 0$.

We know that $\mathcal{A}(y)$ has a finite asymptotic expansion, and as $\alpha_0 \neq 0$ we also know that $\mathcal{B}_\tau(y)$ has an asymptotic expansion of the same order. Hence we obtain (3.2). This proves the lemma.

THEOREM 3.2. *Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbf{N}_0$. If $1/\tau + \gamma \neq 0$, then $Q_\tau(\ell) \in \mathbf{N}_{-k}$.*

Proof. We define $c(\tau) = 1 + \tau^2 (\operatorname{Im} Q(\mu))^2$. We have the following claim.

For $k \geq 0$, $Q(\ell) \in \mathbf{N}_{-2k} \Longrightarrow Q_\tau \in \mathbf{N}_{-2k}$, and:

$$\frac{1}{c(\tau)} m_{2k}(\tau) = \sum_{i=0}^{k} (-1)^i \beta_{2i} m_{2k-2i}. \tag{3.5}$$

and

$$\frac{1}{c(\tau)} Q_\tau^{[2k+2]}(y) = \sum_{i=0}^{k} (-1)^i \beta_{2i} Q^{[2k+2-2i]}(y) + (-1)^{k+1} Q^{[0]}(y) \underbrace{y^2 \left(\mathcal{B}_\tau^{\{2k\}}(y) - \beta_{2k}\right)}_{\mathcal{B}_\tau^{\{2k+2\}}(y)}. \tag{3.6}$$

We will prove this claim by induction. Let $Q(\ell) \in \mathbf{N}_0$, then we know that:

$$Q_\tau^{[0]}(y) = \frac{Q^{[0]}(y)}{\mathcal{A}(y)} c(\tau).$$

Thus we have:

$$\sup_{y>0} y^2 Q_\tau^{[0]}(y) = c(\tau)\beta_0 \sup_{y>0} y^2 Q^{[0]}(y),$$

which implies that $Q_\tau \in \mathbf{N}_0$ and also that

$$\frac{1}{c(\tau)} m_0(\tau) = m_0 \beta_0.$$

By definition the following holds:

$$Q_\tau^{[2]}(y) = m_0(\tau) - y^2 Q_\tau^{[0]}(y),$$

which implies that:

$$\frac{1}{c(\tau)} Q_\tau^{[2]}(y) = m_0\beta_0 - y^2 \mathcal{B}_\tau^{\{0\}}(y) Q^{[0]}(y)$$

$$= \left(m_0 - y^2 Q^{[0]}(y)\right)\beta_0 + y^2\left(\beta_0 - \mathcal{B}_\tau^{\{0\}}(y)\right) Q^{[0]}(y) = \beta_0 Q^{[2]}(y) - Q^{[0]}(y)\mathcal{B}_\tau^{\{2\}}(y).$$

Now we assume $Q(\ell) \in \mathbf{N}_{-2k} \Longrightarrow Q_\tau(\ell) \in \mathbf{N}_{-2k}$ and that (3.5) and (3.6) hold. Let $Q(\ell) \in \mathbf{N}_{-2k-2}$. Then it follows from (3.6) that

$$\frac{1}{c(\tau)} \sup_{y>0} y^2 Q_\tau^{[2k+2]}(y) = \sum_{i=0}^{k} (-1)^i \beta_{2i} \sup_{y>0} y^2 Q^{[2k+2-2i]}(y).$$

The righthand side is finite and therefore $Q_\tau(\ell) \in \mathbf{N}_{-2k-2}$. Moreover,

$$\frac{1}{c(\tau)} m_{2k+2}(\tau) = \sum_{i=0}^{k+1} (-1)^i \beta_{2i} m_{2k+2-2i},$$

and by definition:

$$Q_\tau^{[2k+4]}(y) = m_{2k+2}(\tau) - y^2 Q_\tau^{[2k+2]}(y),$$

so that

$$\begin{aligned}
\frac{1}{c(\tau)} Q_\tau^{[2k+4]}(y) &= \frac{1}{c(\tau)} m_{2k+2}(\tau) - \frac{1}{c(\tau)} y^2 Q_\tau^{[2k+2]}(y) \\
&= \sum_{i=0}^{k} (-1)^i \beta_{2i} m_{2k+2-2i} - \sum_{i=0}^{k} (-1)^i \beta_{2i} y^2 Q^{[2k+2-2i]}(y) + \\
&\quad (-1)^{k+1} \left(\beta_{2k+2} m_0 - \mathcal{B}_\tau^{\{2k+2\}}(y) y^2 Q^{[0]}(y) \right) \\
&= \sum_{i=0}^{k} (-1)^i \beta_{2i} Q^{[2k+4-2i]}(y) + \\
&\quad (-1)^{k+1} \left(\underbrace{y^2 \left(\beta_{2k+2} - \mathcal{B}_\tau^{\{2k+2\}}(y) \right)}_{-\mathcal{B}_\tau^{\{2k+4\}}(y)} Q^{[0]}(y) \right. \\
&\qquad\qquad \left. + \underbrace{\left(m_0 - y^2 Q^{[0]}(y) \right)}_{Q^{[2]}(y)} \beta_{2k+2} \right) \\
&= \sum_{i=0}^{k+1} (-1)^i \beta_{2i} y^2 Q^{[2k+4-2i]}(y) + (-1)^{k+2} \mathcal{B}_\tau^{\{2k+4\}}(y) Q^{[0]}(y).
\end{aligned}$$

This proves the claim.

Let $Q(\ell) \in \mathbf{N}_{-2k-1}$. Then

$$\frac{1}{c(\tau)} \int_1^\infty Q_\tau^{[2k+2]}(y)dy = \sum_{i=0}^{k} (-1)^i \beta_{2i} \int_1^\infty Q^{[2k+2-2i]}(y)dy$$
$$+(-1)^{k+1} \mathcal{B}_\tau^{2k+2}(y) \int_1^\infty Q^{[0]}(y)dy.$$

As $Q(\ell) \in \mathbf{N}_{-2k-1}$, the righthand side is finite and therefore the lefthand side is finite. We conclude that $Q_\tau(\ell) \in \mathbf{N}_{-2k-1}$. This proves the theorem.

Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbb{N}_0$, so that also $Q_\tau(\ell) \in \mathbf{N}_{-k}$ when $1/\tau + \gamma \neq 0$. Then it follows from Theorem 3.2 that

$$(3.7) \quad Q_\tau(\ell) = \gamma(\tau) - \sum_{i=0}^{k} \frac{m_i(\tau)}{\ell^{i+1}} + o\left(\frac{1}{\ell^{k+1}}\right).$$

THEOREM 3.3. *Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbb{N}_0$ and assume that $1/\tau + \gamma \neq 0$. Then $\gamma(\tau)$, $m_0(\tau), \ldots, m_k(\tau)$, in (3.7) are given by*

$$(3.8) \quad \gamma(\tau) = \frac{\gamma - \tau(\operatorname{Im} Q(\mu))^2}{1 + \tau\gamma},$$

and

$$(3.9) \quad \begin{pmatrix} m_0(\tau) \\ m_1(\tau) \\ \vdots \\ m_{k-1}(\tau) \\ m_k(\tau) \end{pmatrix} = \frac{1 + \tau^2 (\operatorname{Im} Q(\mu))^2}{(1+\tau\gamma)^2}.$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\frac{\tau m_0}{1+\tau\gamma} & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\frac{\tau m_{k-2}}{1+\tau\gamma} & & \ddots & 1 & 0 \\ -\frac{\tau m_{k-1}}{1+\tau\gamma} & -\frac{\tau m_{k-2}}{1+\tau\gamma} & \cdots & -\frac{\tau m_0}{1+\tau\gamma} & 1 \end{pmatrix}^{-1} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{k-1} \\ m_k \end{pmatrix}.$$

Proof. From (1.14) we have the following equation:

$$(3.10) \quad Q_\tau(iy) - Q(iy) + \tau Q(iy) Q_\tau(iy) + \tau (\operatorname{Im} Q(\mu))^2 = 0.$$

We substitute (2.6) and (3.7) in (3.10). This leads to

$$\gamma(\tau) - \gamma + \tau (\operatorname{Im} Q(\mu))^2 + \sum_{i=0}^{k} (m_i - m_i(\tau)) \frac{1}{\ell^{i+1}}$$

$$+\tau \left(\sum_{i=-1}^{k} \left(\sum_{n=-1}^{i} m_n m_{i-n-1}(\tau) \right) \frac{1}{\ell^{i+1}} \right) = o\left(\frac{1}{\ell^{k+1}}\right).$$

Here $m_{-1} = -\gamma$, $\quad m_{-1}(\tau) = -\gamma(\tau)$. From this we have:

$$(3.11) \quad (1+\tau\gamma)\gamma(\tau) = \gamma - \tau (\operatorname{Im} Q(\mu))^2,$$

and

$$\sum_{i=0}^{k}(m_i - m_i(\tau))\frac{1}{\ell^{\,i+1}} + \tau\left(\sum_{i=0}^{k}\left(\sum_{n=-1}^{i} m_n m_{i-n-1}(\tau)\right)\frac{1}{\ell^{\,i+1}}\right) = o\left(\frac{1}{\ell^{\,k+1}}\right),$$

or equivalently,

$$(1-\tau\gamma(\tau))\sum_{i=0}^{k} m_i\frac{1}{\ell^{\,i+1}} + (-1-\tau\gamma)\sum_{i=0}^{k} m_i(\tau)\frac{1}{\ell^{\,i+1}}$$

$$+\tau\left(\sum_{i=0}^{k}\left(\sum_{n=0}^{i-1} m_n m_{i-n-1}(\tau)\right)\frac{1}{\ell^{\,i+1}}\right) = o\left(\frac{1}{\ell^{\,k+1}}\right).$$

This implies for fixed powers of ℓ for $n \in \mathbb{N}_0$

$$(1+\tau\gamma)m_n(\tau) = (1-\tau\gamma(\tau))m_n + \tau\sum_{i=1}^{n} m_{i-1}m_{n-i}(\tau).$$

Notice that

$$1-\tau\gamma(\tau) = \frac{1+\tau^2\left(\operatorname{Im} Q(\mu)\right)^2}{1+\tau\gamma},$$

and thus

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\frac{\tau m_0}{1+\tau\gamma} & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\frac{\tau m_{k-2}}{1+\tau\gamma} & & \ddots & 1 & 0 \\ -\frac{\tau m_{k-1}}{1+\tau\gamma} & -\frac{\tau m_{k-2}}{1+\tau\gamma} & \cdots & -\frac{\tau m_0}{1+\tau\gamma} & 1 \end{pmatrix}\begin{pmatrix} m_0(\tau) \\ m_1(\tau) \\ \vdots \\ m_{k-1}(\tau) \\ m_k(\tau)\end{pmatrix}$$

$$= \frac{1+\tau^2\left(\operatorname{Im} Q(\mu)\right)^2}{(1+\tau\gamma)^2}\begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{k-1} \\ m_k\end{pmatrix}.$$

Thus we can successively solve the τ-moments.

4. The exceptional case

For $Q(\ell) \in \mathbf{N}$ we define the function

(4.1) $\mathcal{C}(y) = y^2|Q(iy) - \gamma|^2$

LEMMA 4.1. *If $Q(\ell) \in \mathbf{N}_{-2k-1}$ then $|Q(iy) - \gamma|^2$ has an asymptotic expansion:*

(4.2) $$|Q(iy) - \gamma|^2 = \sum_{n=1}^{k+1} \frac{\gamma_{2n}}{y^{2n}},$$

where the coefficients are given by:

(4.3) $$\gamma_{2n} = \sum_{i=1}^{2n} (-1)^{n+i+1} m_{i-1} m_{2n-i}.$$

The asymptotic expansion (4.3) is equivalent to the limiting behaviour of the succesive divided differences:

$$\mathcal{C}_0(y) = |Q(iy) - \gamma|^2 \longrightarrow \gamma_0,$$

$$\mathcal{C}_2(y) = y^2\left(\mathcal{C}_0(y) - \gamma_0\right) \longrightarrow \gamma_2,$$

$$\dots$$

$$\mathcal{C}_{2k+2}(y) = y^2\left(\mathcal{C}_{2k}(y) - \gamma_{2k}\right) \longrightarrow \gamma_{2k+2}.$$

Proof. This is an easy consequence of the asymptotic expansion of $Q(iy)$.

For $1/\tau + \gamma = 0$ the following function:

(4.4) $$Q_\tau(\ell) - \beta\ell, \quad \beta = \lim_{y\to\infty} \frac{\operatorname{Im} Q_\tau(iy)}{y},$$

is a Nevanlinna function.

THEOREM 4.2. *Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbb{N}_0$. If $1/\tau + \gamma = 0$, then $Q_\tau(\ell) - \beta\ell \in \mathbf{N}_{-k+2}$.*

Proof. Define $d = \gamma^2 + (\operatorname{Im} Q(\mu))^2$, and denote the righthand side of (4.4) by $R(\ell)$. We have the following claim.

If $Q(\ell) \in \mathbf{N}_{-2k} \Longrightarrow R(\ell) \in \mathbf{N}_{-2k+2}$ and:

(4.5) $$\frac{1}{d} y^2 R^{[2k]}(y)\mathcal{C}(y) = -y^2 Q^{[2k+2]}(y) + \frac{1}{m_0}\mathcal{C}_{2k+2} + \frac{1}{d}\sum_{i=0}^{k-1}(-1)^{i+1} r_{2i}\mathcal{C}_{2k-2i}(y).$$

We will prove this claim by induction. For $Q(\ell) \in \mathbf{N}_0$ it follows from [HS1] that $R(\ell) \in \mathbf{N}$. Moreover

$$(4.6) \quad \operatorname{Im} R(iy) = \left(\frac{\operatorname{Im} Q(iy) m_0 - \frac{1}{y}\mathcal{C}(y)}{\frac{1}{y^2}\mathcal{C}(y) m_0} \right) d.$$

Rewriting this, gives

$$(4.7) \quad \frac{1}{d} y^2 R^{[0]}(y)\mathcal{C}(y) = -y^2 Q^{[2]}(y) + \frac{1}{m_0}\mathcal{C}_2(y).$$

Now assume the claim holds for k. Let $Q(\ell) \in \mathbf{N}_{-2k-2}$. Then $\sup_{y>0} y^2 Q^{[2k+2]}(y) < \infty$, thus the righthand side of (4.5) is finite, and thus also the lefthand side is finite. As $\sup_{y>0} y^2 R^{[2k]}(y) < \infty$, we conclude that $R(\ell) \in \mathbf{N}_{-2k}$ and

$$R^{[2k+2]}(y) = \sup_{y>0} y^2 R^{[2k]}(y) - y^2 R^{[2k]}(y).$$

Consider

$$\frac{1}{d} y^2 R^{[2k+2]}(y)\mathcal{C}(y).$$

From (4.5) we have

$$\frac{1}{d} r_{2k}\gamma_0 = -m_{2k+2} + \frac{1}{m_0}\gamma + \frac{1}{d}\sum_{i=0}^{k-1}(-1)^{i+1} r_{2i}\gamma_{2k-2i},$$

which implies

$$\frac{1}{d}\left[r_{2k} - y^2 R^{[2k]}(y)\right]\mathcal{C}(y) =$$

$$\frac{r_{2k}}{d}\mathcal{C}(y) + y^2 Q^{[2k+2]}(y) - \frac{1}{m_0}\mathcal{C}_{2k+2}(y) - \frac{1}{d}\sum_{i=0}^{k-1}(-1)^{i+1} r_{2i}\mathcal{C}_{2k-2i}(y)$$

$$-\frac{r_{2k}}{d}\gamma_0 - m_{2k+2} + \frac{1}{m_0}\gamma_{2k+2} + \frac{1}{d}\sum_{i=0}^{k-1}(-1)^{i+1} r_{2i}\gamma_{2k-2i} =$$

$$-\frac{r_{2k}}{d}\mathcal{C}_2(y) - Q^{[2k+4]}(y) + \frac{1}{m_0}\mathcal{C}_{2k+4}(y) + \frac{1}{d}\sum_{i=0}^{k-1}(-1)^{i+1} r_{2i}\mathcal{C}_{2k+2-2i}(y) =$$

$$\frac{1}{m_0}\mathcal{C}_{2k+4}(y) - Q^{[2k+4]}(y) + \frac{1}{d}\sum_{i=0}^{k}(-1)^{i+1} r_{2i}\mathcal{C}_{2k+2-2i}(y).$$

This concludes the proof of our claim.

Let $Q(\ell) \in \mathbf{N}_{-2k-1}$, so that

$$\int_1^\infty Q^{[2k+2]}(y)dy < \infty.$$

From this and (4.5) we conclude

$$\int_1^\infty R^{[2k]}(y)dy < \infty,$$

and thus that $R(\ell) \in \mathbf{N}_{-2k+1}$. This completes the proof of the theorem.

Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbf{N}_0$. Then $Q_\tau(\ell) - \beta\ell \in \mathbf{N}_{-k+2}$ when $1/\tau + \gamma = 0$, and it follows from Theorem 2.2 that

$$(4.8) \quad Q_\tau(\ell) - \beta\ell = \gamma(\tau) - \sum_{i=0}^{k-2} \frac{m_i(\tau)}{\ell^{i+1}} + o\left(\frac{1}{\ell^{k-1}}\right), \quad \ell \to \infty,$$

for real numbers $\gamma(\tau)$ and $m_0(\tau), \ldots, m_{k-2}(\tau)$.

THEOREM 4.3. *Let $Q(\ell) \in \mathbf{N}_{-k}$ for some $k \in \mathbf{N}_0$ and assume that $1/\tau + \gamma = 0$. Then β, $\gamma(\tau)$, and $m_0(\tau), \ldots, m_{k-2}(\tau)$ in (4.8) are given by*

$$(4.9) \quad \beta = \frac{1 + \tau^2(\operatorname{Im} Q(\mu))^2}{\tau^2 m_0^2},$$

$$(4.10) \quad \gamma(\tau) = \frac{1}{\tau} - \frac{m_1}{\tau^2 m_0^2}\left(1 + \tau^2 (\operatorname{Im} Q(\mu))^2\right),$$

and

$$(4.11) \quad \begin{pmatrix} m_0(\tau) \\ m_1(\tau) \\ \vdots \\ m_{k-1}(\tau) \\ m_k(\tau) \end{pmatrix} = \frac{1 + \tau^2 (\operatorname{Im} Q(\mu))^2}{\tau^2 m_0^2}.$$

$$\begin{pmatrix} m_0 & 0 & \cdots & 0 & 0 \\ m_1 & m_0 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ m_{k-3} & & \ddots & m_0 & 0 \\ m_{k-2} & m_{k-3} & \cdots & m_1 & m_0 \end{pmatrix}^{-1} \begin{pmatrix} m_0 m_2 - m_1^2 \\ m_0 m_3 - m_1 m_2 \\ \vdots \\ m_0 m_{k-1} - m_1 m_{k-2} \\ m_0 m_k - m_1 m_{k-1} \end{pmatrix}.$$

Proof. We substitute (2.6) and (4.8) in (3.10). By evaluating the coefficients of the decreasing powers of ℓ we find the formulas for β and $\gamma(\tau)$. Moreover, we obtain the recurrence relations

$$(4.12)\quad \sum_{j=0}^{n-1} m_j m_{n-j-1}(\tau) = \frac{1+\tau^2(\operatorname{Im} Q(\mu))^2}{\tau^2 m_0^2}\,(m_0 m_{n+1} - m_1 m_n)\,,$$

or, equivalently,

$$\begin{pmatrix} m_0 & 0 & \cdots & 0 & 0 \\ m_1 & m_0 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ m_{k-3} & & \ddots & m_0 & 0 \\ m_{k-2} & m_{k-3} & \cdots & m_1 & m_0 \end{pmatrix} \begin{pmatrix} m_0(\tau) \\ m_1(\tau) \\ \vdots \\ m_{k-3}(\tau) \\ m_{k-2}(\tau) \end{pmatrix}$$

$$= \frac{1+\tau^2\left(\operatorname{Im} Q(\mu)\right)^2}{\tau^2 m_0^2} \begin{pmatrix} m_0 m_2 - m_1^2 \\ m_0 m_3 - m_1 m_2 \\ \vdots \\ m_0 m_{k-1} - m_{k-2} \\ m_0 m_k - m_1 m_{k-1} \end{pmatrix}.$$

From this we can successively solve the τ-moments.

For a function $H(\ell)$ with $H(\ell) - \beta\ell \in \mathrm{N}_{-2k+1}$, we define the function

$$\mathcal{P}(y) = \frac{1}{|H(iy)|^2}, \quad y \in \mathbb{R}.$$

LEMMA 4.4. *If $H(\ell)-\beta\ell \in \mathrm{N}_{-2k+1}$, then the function $\mathcal{P}(y)$ has the asymptotic expansion:*

$$(4.13)\quad \mathcal{P}(y) = \sum_{i=1}^{2k} p_{-2i} y^{-2i} + o\left(\frac{1}{y^{2k}}\right).$$

The asymptotic expansion (4.13) is equivalent to the limiting behaviour of the successive divided differences:

$$\mathcal{P}_0(y) \longrightarrow p_0,$$

$$\mathcal{P}_2(y) = y^2(\mathcal{P}_0(y) - p_0) \longrightarrow p_2,$$

$$\dots$$

$$\mathcal{P}_{2k+2}(y) = y^2(\mathcal{P}_{2k}(y) - p_{2k}) \longrightarrow p_{2k+2}.$$

Proof. For $\ell = iy$ we have

$$H(\ell) = \beta\ell + \gamma - \sum_{i=0}^{2k+1} \frac{m_i}{\ell^{i+1}} + o\left(\frac{1}{y^{2k+2}}\right),$$

and also

$$|H(iy)|^2 = H(iy)H(-iy) = \sum_{j=-1}^{k} h_{2j} y^{-2j},$$

for some h_{2j}, $j = -1, \ldots, k$. If we choose $\beta = -m_{-2}$ and $\gamma = -m_{-1}$ we find

$$h_{2j} = \sum_{i=-2}^{j} (-1)^{i-j} m_i m_{j-i-2}.$$

The function $|H(iy)|^{-2}$ has a similar expansion. This proves the lemma.

Let $H(\ell)$ be a Nevanlinna function such that $\beta = \lim_{y\to\infty} \frac{\operatorname{Im} H(iy)}{y}$ is positive, so that $H(\ell) - \beta\ell$ is a Nevanlinna function. We define

$$(4.14)\quad Q(\ell) = -\frac{|H(\mu)|^4}{(\operatorname{Im} H(\mu))^2 H(\ell)},$$

so that

$$(4.15)\quad \operatorname{Im} Q(\ell) = \frac{|H(\mu)|^4}{(\operatorname{Im} H(\mu))^2} \frac{\operatorname{Im} H(\ell)}{H(\ell)^2}.$$

In [HS1] it is shown that $Q(\ell) \in \mathrm{N}_0$ and that $\lim_{y\to\infty} Q(iy) = 0$. Hence, $H(\ell) = Q_\infty(\ell)$, the exceptional function corresponding to the exceptional value $\tau = \infty$ of $Q(\ell)$. The following theorem refines this result.

THEOREM 4.5. *Let $H(\ell)$ be a Nevanlinna function for which $\lim_{y\to\infty} \frac{\operatorname{Im} H(iy)}{y}$ is positive. If the function*

$$H(\ell) - \beta\ell$$

belongs to N_{-k}, then the function $Q(\ell)$ in (4.14) belongs to N_{-k-2}.

Proof. Assume that the function $H(\ell)$ has the integral representation (1.1) with (1.2) and that $\beta = \lim_{y\to\infty} \frac{\operatorname{Im} H(iy)}{y}$. Let the function $Q(\ell)$ be defined as in (4.14), and introduce the function $T(\ell)$ by

$$(4.16)\quad T(\ell) = \frac{(\operatorname{Im} H(\mu))^2}{|H(\mu)|^4} Q(\ell).$$

and also

$$(4.17)\quad T^{[2k]}(y) = \frac{(\operatorname{Im} H(\mu))^2}{|H(\mu)|^4} Q^{[2k]}(y).$$

Define $R(\ell) = H(\ell) - \beta\ell$. We will prove the following claim:

$$R(\ell) \in \mathrm{N}_{-2k} \Longrightarrow Q(\ell) \in \mathrm{N}_{-2k-2},$$

and

$$\sup_{y>0} y^2 T^{[2k+2]}(y) = (-1)^{k+1} p_{2k}\beta + \mathcal{P}_0(y) m_{2k+2},$$

and also

$$T^{[2k+4]}(y) = (-1)^{k+2}\mathcal{P}_{2k+4}(y)\beta + \mathcal{P}_0(y)\int_{\mathbf{R}} \frac{t^{2k+4}}{t^2+y^2} d\sigma(t).$$

For $k = 0$

$$\sup_{y>0} y^2 T^{[2]}(y) = -p_2\beta + \sup_{y>0} y^2 \mathcal{P}_0(y)\int_{\mathbf{R}} \frac{t^2}{t^2+y^2} d\sigma(t).$$

As $R(\ell) \in \mathrm{N}_0$, we know that $\int_{\mathbf{R}} \frac{t^2}{t^2+y^2} d\sigma(t)$, p_2 and $\sup_{y>0} y^2\mathcal{P}_0(y)$ are finite. Thus $\frac{(\operatorname{Im} H(\mu))^2}{|H(\mu)|^4} Q(\ell) \in \mathrm{N}_{-2}$, which implies that $Q(\ell) \in \mathrm{N}_{-2}$,

$$\sup_{y>0} y^2 T^{[2]}(y) = -p_2\beta + \mathcal{P}_0(y) m_2 < \infty,$$

and

$$T^{[4]}(y) = \mathcal{P}_4(y)\beta + \mathcal{P}_0(y)\int_{\mathbf{R}} \frac{t^4}{t^2+y^2} d\sigma(t).$$

Now assume that the claim holds for k. Then

$$\sup_{y>0} y^2 T^{[2k+4]}(y) = (-1)^{k+2} p_{2k+4}\beta + \sup_{y>0} y^2 \mathcal{P}_0(y)\int_{\mathbf{R}} \frac{t^{2k+4}}{t^2+y^2} d\sigma(t).$$

As $R(\ell) \in \mathrm{N}_{-2k-2}$ we know that $\int_{\mathbf{R}} \frac{t^{2k+4}}{t^2+y^2} d\sigma(t)$, p_{2k+2} and $\sup_{y>0} y^2\mathcal{P}_0(y)$ are finite. Thus $\frac{(\operatorname{Im} H(\mu))^2}{|H(\mu)|^4} Q(\ell) \in \mathrm{N}_{-2k-4}$, which implies that $Q(\ell) \in \mathrm{N}_{-2k-4}$,

$$\sup_{y>0} y^2 T^{[2k+4]}(y) = (-1)^{k+2} p_{2k+4}\beta + \mathcal{P}_0(y) m_{2k+4} < \infty,$$

and

$$T^{[2k+6]}(y) = (-1)^{k+1}\mathcal{P}_{2k+6}(y)\beta + \mathcal{P}_0(y)\int_{\mathbf{R}} \frac{t^{2k+6}}{t^2+y^2} d\sigma(t).$$

This proves the claim.

If $H(\ell) \in \mathrm{N}_{-2k+1}$, then

$$\int_1^\infty T^{[2k+2]}(y)dy = \int_1^\infty \left((-1)^{k+1}\mathcal{P}_{2k+2}(y)\beta + \mathcal{P}_0(y)\int_{\mathbf{R}} \frac{t^{2k+2}}{t^2+y^2} d\sigma(t) \right) dy < \infty,$$

thus $Q(\ell) \in \mathrm{N}_{-2k-1}$. This proves the theorem.

5. Q-FUNCTIONS

In this section we give a short introduction and fix some notation and terminology, concerning relations in Hilbert spaces and Q-functions.

Consider a Hilbert space $\mathfrak{H}$ with inner product $[.,.]$. A linear relation S in $\mathfrak{H}$ is just a linear subspace of the orthogonal sum $\mathfrak{H} \oplus \mathfrak{H}$ of $\mathfrak{H}$ with itself. For instance, a linear operator is a linear relation when we identify the operator and its graph. The relation is said to be closed if it is closed as a subspace of $\mathfrak{H} \oplus \mathfrak{H}$. The nullspace $\ker S$ is given by $\ker S = \{f \in \mathfrak{H} \mid \{f, 0\} \in S\}$ and the multivalued part $S(0)$ is given by $S(0) = \{g \in \mathfrak{H} \mid \{0, g\} \in S\}$. A linear relation S is (the graph of) an operator if and only if $S(0) = \{0\}$. Its domain and range are given by $\operatorname{dom} S = \{f \mid \{f, g\} \in S\}$ and $\operatorname{ran} S = \{g \mid \{f, g\} \in S\}$, respectively. By S^{-1} we mean the linear relation $S^{-1} = \{\{g, f\} \mid \{f, g\} \in S\}$. Note that $\operatorname{dom} S^{-1} = \operatorname{ran} S$. Clearly, $\ker S^{-1} = S(0)$. We define the linear relation S_∞ by $S_\infty = \{0\} \oplus S(0)$. The operator part S_s of S is then defined by

$$S_s = S \ominus S_\infty = \{\{f, g\} \in S \mid g \perp S(0)\}.$$

In this sense a closed relation $S = S_s \oplus S_\infty$ is a multivalued linear operator.

For each $\ell \in \mathbf{C}$ we define the linear relation $S - \ell = \{\{f, g - \ell f\} \mid \{f, g\} \in S\}$. Clearly, $\ker (S - \ell)^{-1} = S(0)$. When S is closed, the resolvent set $\rho(S)$ is the set of all $\ell \in \mathbf{C}$ for which $(S - \ell)^{-1} = \{\{g - \ell f, f\} \mid \{f, g\} \in S\}$ is (the graph of) a bounded linear operator defined on all of $\mathfrak{H}$. The resolvent set $\rho(S)$ is open and the resolvent operator $(S - \ell)^{-1}$, $\ell \in \rho(S)$, satisfies

$$(S - \ell)^{-1} - (S - \lambda)^{-1} = (\ell - \lambda)(S - \ell)^{-1}(S - \lambda)^{-1}, \quad \ell, \lambda \in \rho(S),$$

which is the usual resolvent identity.

For any subset S in $\mathfrak{H}^2$ we define the adjoint S^* by

$$S^* = \{\{h, k\} \in \mathfrak{H}^2 \mid \langle \{h, k\}, \{f, g\} \rangle = 0, \quad \text{for all } \{f, g\} \in S\},$$

where the form $\langle \{f, g\}, \{h, k\} \rangle$ is defined by $[g, h] - [f, k]$. Then S^* is automatically a closed linear relation and $(S^*)^*$ is the closure of the linear subspace spanned by S. A linear relation S is called symmetric if $S \subset S^*$ and selfadjoint if $S = S^*$. For a closed symmetric relation S we have von Neumann's formula, which expresses S^* as a direct sum:

$$(5.1) \quad S^* = S \dot{+} M_{\bar{\mu}}(S^*) \dot{+} M_\mu(S^*), \quad \mu \in \mathbf{C} \setminus \mathbf{R},$$

where $M_\lambda(S^*) = \{\{f, g\} \in S^* \mid g = \lambda f\}$, $\lambda \in \mathbf{C}$; note that $\ker (S^* - \lambda) = \operatorname{dom} M_\lambda(S^*)$. The symbol $\dot{+}$ denotes the componentwise sum in $\mathfrak{H} \oplus \mathfrak{H}$. The defect numbers (m, n) of S are the dimensions of the defect subspaces $\ker (S^* - \bar{\mu})$ and of $\ker (S^* - \mu)$, respectively, if $\mu \in \mathbf{C}^+$. In general, the symmetric relation S has selfadjoint extensions if we extend the original Hilbert space. The operator S has canonical selfadjoint extensions, i.e., selfadjoint

extensions A in the original Hilbert space, i.e., $S \subset A$ if and only if the defect numbers of S are equal.
We recall that a symmetric relation S is called completely nonselfadjoint, if there is no nontrivial orthogonal decomposition of S in which one of the summands is selfadjoint. A completely nonselfadjoint symmetric relation is necessarily an operator. Let S be a closed symmetric relation with defect numbers $(1,1)$. Choose a fixed canonical selfadjoint extension $\mathring{A}$ of S, a point $\mu \in \mathbb{C} \setminus \mathbb{R}$ and a nontrivial element $\chi(\mu) \in \ker(S^* - \mu)$. We define the element

$$(5.2) \quad \chi(\ell) = \left(I + (\ell - \mu)(\mathring{A} - \ell)^{-1}\right)\chi(\mu),$$

or equivalently

$$\frac{\chi(\ell) - \chi(\mu)}{\ell - \mu} = (\mathring{A} - \ell)^{-1}\chi(\mu),$$

so that $\{\chi(\ell) - \chi(\mu), \ell\chi(\ell) - \mu\chi(\mu)\} \in \mathring{A} \subset S^*$. Therefore $\chi(\ell)$ belongs to $\ker(S^* - \ell)$. Let $\mathfrak{H}_\chi = \overline{\operatorname{span}}\{\chi(\ell) \mid \ell \in \mathbb{C} \setminus \mathbb{R}\}$. Recall that the extension $\mathring{A}$ is called minimal if $\mathfrak{H}_\chi = \mathfrak{H}$. The Hilbert space $\mathfrak{H}$ decomposes as $\mathfrak{H} = \mathfrak{H}_\chi \oplus \mathfrak{H}_0$, and $\mathring{A}$ is the orthogonal sum of selfadjoint relations $\mathring{A}_\chi$ in $\mathfrak{H}_\chi$ and $\mathring{A}_0$ in $\mathfrak{H}_0$, respectively. Moreover, S is the orthogonal sum of $S_\chi = \{\{f,g\} \in \mathring{A}_\chi \mid [g - \ell f, \chi(\bar{\ell})] = 0\}$, a completely nonselfadjoint symmetric operator, and $\mathring{A}_0$. Hence, the symmetric relation S is completely nonselfadjoint if and only if $\mathring{A}$ is minimal. Moreover, S and $\mathring{A}$ are connected by

$$S = \{\{f,g\} \in \mathring{A} \mid [g - \ell f, \chi(\bar{\ell})] = 0\}.$$

The Q-function of S and $\mathring{A}$ is defined as the solution $Q(\ell)$ of the equation

$$(5.3) \quad \frac{Q(\ell) - Q(\lambda)^*}{\ell - \bar{\lambda}} = [\chi(\ell), \chi(\lambda)], \quad \ell, \lambda \in \rho(\mathring{A}).$$

Clearly, this function is uniquely defined, up to real constants. In fact we have

$$(5.4) \quad Q(\ell) = Q(\mu)^* + (\ell - \bar{\mu})[(I + (\ell - \mu)(\mathring{A} - \ell)^{-1})\chi(\mu), \chi(\mu)], \quad \ell \in \rho(\mathring{A}).$$

We will call the operator representation (5.4) minimal if S is completely nonselfadjoint or, equivalently, if $\mathring{A}$ is minimal. If this is the case, then $\mathfrak{H}$, S and $\mathring{A}$ are uniquely determined by $Q(\ell)$, up to unitary equivalence. It follows from (5.3) that the function $Q(\ell)$ belongs to the class $\mathbf{N}$ of Nevanlinna functions. Hence, it also has an integral representation (1.1). In order to see the connection between the representations (5.4) and (1.1) we introduce the following terminology. Denote the multivalued part of the relation $\mathring{A}$ by $\mathring{A}(0)$ and the orthogonal projection on $\mathring{A}(0)^\perp$ by R. Let $E(t)$ denote the spectral family of the operator part $\mathring{A}_s$ of $\mathring{A}$ in $\mathfrak{H} \ominus \mathring{A}(0) = \overline{\operatorname{dom} \mathring{A}}$, the closure of $\operatorname{dom} \mathring{A}$.

PROPOSITION 5.1. *Let $Q(\ell)$ be the Q-function of S and $\mathring{A}$ with the operator representation (5.4). The parameters α, β and the function σ in the integral representation (1.1) are connected to the operator representation as follows:*

(i) $\alpha = Q(i)^* + i[\chi(i), \chi(i)] = \operatorname{Re} Q(i)$.

(ii) $\beta = [(I-R)\chi(i), (I-R)\chi(i)]$.

(iii) $\frac{d\sigma(t)}{t^2+1} = d([E(t)R\chi(i), R\chi(i)])$.

Proof. We rewrite the operator representation (5.4) using the spectral family of $\mathring{A}$. Note that $\mathring{A}(0) = \ker(\mathring{A} - \ell)^{-1}$ for all $\ell \in \rho(\mathring{A})$ and decompose $\chi(i)$: $\chi(i) = (I-R)\chi(i) + R\chi(i)$. Hence, from (5.4) it follows that

$$\begin{aligned} Q(\ell) &= Q(i)^* + (\ell + i)[(I + (\ell - i)(\mathring{A} - \ell)^{-1})\chi(i), \chi(i)] \\ &= Q(i)^* + (\ell + i)[(I - R)\chi(i), (I - R)\chi(i)] \\ &\quad + (\ell + i)[(\mathring{A}_s - i)(\mathring{A}_s - \ell)^{-1} R\chi(i), R\chi(i)] \\ &= Q(i)^* + (\ell + i)[(I - R)\chi(i), (I - R)\chi(i)] \\ &\quad + i[R\chi(i), R\chi(i)] + \int_{\mathbf{R}} \frac{\ell t + 1}{t - \ell} d([E(t)R\chi(i), R\chi(i)]) \\ &= Q(i)^* + i[\chi(i), \chi(i)] + \ell[(I - R)\chi(i), (I - R)\chi(i)] \\ &\quad + \int_{\mathbf{R}} \left(\frac{1}{t - \ell} - \frac{t}{t^2 + 1}\right)(t^2 + 1) d([E(t)R\chi(i), R\chi(i)]). \end{aligned}$$

By the essential uniqueness of the Nevanlinna integral representation, comparison of the various terms with the corresponding terms in (1.1) leads to the desired result. This completes the proof of the proposition.

The following result gives necessary and sufficient conditions for a Q-function to belong to the class $\mathbf{N}_1$ or to the class $\mathbf{N}_0$ in terms of the defect vectors. The characterization of the case that $\mathring{A}$ is an operator goes back to [LT].

PROPOSITION 5.2. *Let $Q(\ell)$ be the Q-function belonging to S and $\mathring{A}$. Then*

(i) $\lim_{y\to\infty} \operatorname{Im} Q(iy)/y = 0$ *if and only if* $\chi(\ell) \in \overline{\operatorname{dom} \mathring{A}}$ *for some (and, hence, for all)* $\ell \in \rho(\mathring{A})$.

Assume that the operator representation (5.4) is minimal. Then

(ii) $\lim_{y\to\infty} \operatorname{Im} Q(iy)/y = 0$ *if and only if* $\mathring{A}$ *is an operator.*

(iii) $Q(\ell) \in \mathbf{N}_1$ *if and only if* $\mathring{A}$ *is an operator and* $\chi(\ell) \in \operatorname{dom} |\mathring{A}|^{\frac{1}{2}}$ *for some (and, hence, for all)* $\ell \in \rho(\mathring{A})$.

(iv) $Q(\ell) \in \mathbf{N}_0$ *if and only if* $\mathring{A}$ *is an operator and* $\chi(\ell) \in \operatorname{dom} \mathring{A}$ *for some (and, hence, for all)* $\ell \in \rho(\mathring{A})$.

(v) $Q(\ell) \in \mathbf{N}_{-k}$ *if and only if* $\mathring{A}$ *is an operator and* $\chi(\ell) \in \operatorname{dom} |\mathring{A}|^{\frac{k+2}{2}}$ *for some (and, hence, for all)* $\ell \in \rho(\mathring{A})$.

Proof. According to (1.7) and (i) of Proposition 5.1 we see that $\lim_{y\to\infty} \operatorname{Im} Q(iy)/y = 0$ if and only if $\chi(i) \in \overline{\operatorname{dom} \mathring{A}}$. The definition (5.2) then implies that $\chi(\ell) \in \overline{\operatorname{dom} \mathring{A}}$ for all $\ell \in \rho(\mathring{A})$. Next we prove (ii). Assume that $\mathring{A}$ is an operator. Then $R = I$ and hence, according to Proposition 5.1, we obtain $\beta = 0$. Conversely, if $\beta = 0$, then again according to (i) $\chi(\ell) \in \mathring{A}(0)^{\perp}$. The minimality shows then that $\mathfrak{H} \subset \mathring{A}(0)^{\perp}$, and thus that $\mathring{A}(0) = \{0\}$. In order to show (iii) we also use the integral representation (1.1) of $Q(\ell)$ with the integrability condition (1.2). According to (ii) and Proposition 5.1, the following statements are equivalent:

(a) $Q(\ell) \in \mathbf{N}_1$.

(b) $\beta = 0$ and $\int_{\mathbb{R}} \frac{d\sigma(t)}{|t|+1} < \infty$.

(c) $\beta = 0$ and $\int_{\mathbb{R}} |t| d([E(t)R\chi(i), R\chi(i)]) < \infty$.

(d) $\mathring{A}$ is an operator and $\chi(i) \in \operatorname{dom} |\mathring{A}|^{\frac{1}{2}}$.

Note that if $\chi(i) \in \operatorname{dom} |\mathring{A}|^{\frac{1}{2}}$, then it follows from the definition of $\chi(\ell)$ that also $\chi(\ell) \in \operatorname{dom} |\mathring{A}|^{\frac{1}{2}}$ for all $\ell \in \rho(\mathring{A})$, since $\operatorname{dom} \mathring{A} \subset \operatorname{dom} |\mathring{A}|^{\frac{1}{2}}$. Finally, we show (iv). Again according to (ii) and Proposition 5.1 the following statements are equivalent:

(e) $Q(\ell) \in \mathbf{N}_0$.

(f) $\beta = 0$ and $\int_{\mathbb{R}} d\sigma(t) < \infty$.

(g) $\beta = 0$ and $\int_{\mathbb{R}} t^2 d([E(t)R\chi(i), R\chi(i)]) < \infty$.

(h) $\mathring{A}$ is an operator and $\chi(i) \in \operatorname{dom} \mathring{A}$.

Note that if $\chi(i) \in \operatorname{dom} \mathring{A}$, then it follows from the definition of $\chi(\ell)$ that also $\chi(\ell) \in \operatorname{dom} \mathring{A}$ for all $\ell \in \rho(\mathring{A})$. Finally, we show (v). Again according to (ii) and Proposition 5.1 the following statements are equivalent:

(i) $Q(\ell) \in \mathbf{N}_{-k}$.

(j) $\beta = 0$ and $\int_{\mathbb{R}} (|t^k| + 1) d\sigma(t) < \infty$.

(k) $\beta = 0$ and $\int_{\mathbb{R}} t^{k+2} d([E(t)R\chi(i), R\chi(i)]) < \infty$.

(l) $\mathring{A}$ is an operator and $\chi(i) \in \operatorname{dom} |\mathring{A}|^{\frac{k+2}{2}}$.

Note that if $\chi(i) \in \operatorname{dom} |\mathring{A}|^{\frac{k+2}{2}}$, then it follows from the definition of $\chi(\ell)$ that also $\chi(\ell) \in \operatorname{dom} \mathring{A}$ for all $\ell \in \rho(\mathring{A})$. This completes the proof of the proposition.

6. Kreĭn's formula

In this section we will give an operator-theoretic interpretation of the fractional linear transform (1.14), (1.15).

Let S be a symmetric relation with defect numbers $(1,1)$. Since the defect numbers are equal, S has canonical selfadjoint extensions, i.e., selfadjoint extensions within the Hilbert space $\mathfrak{H}$. Let $\mathring{A}$ be a fixed canonical selfadjoint extension with corresponding Q-function $Q(\ell)$. It is possible to parametrize all canonical selfadjoint extensions by means of $\mathring{A}$ using Kreĭn's formula, see [HLS] for a simple proof.

Theorem 6.1. *Let S be a closed symmetric relation, with defect numbers (1,1) and let $\mathring{A}$ be a fixed canonical selfadjoint extension with corresponding Q-function $Q(\ell)$. There is a one-to-one correspondence between (the resolvents $(A(\tau)-\ell)^{-1}$ of) all canonical selfadjoint extensions $A(\tau)$ of S and all parameters $\tau \in \mathbb{R}\cup\{\infty\}$, given by*

$$(6.1)\quad (A(\tau)-\ell)^{-1} = (\mathring{A}-\ell)^{-1} - (Q(\ell)+\frac{1}{\tau})^{-1}[.,\chi(\bar{\ell})]\chi(\ell), \quad \tau \neq 0,$$

and by $A(0)=\mathring{A}$.

The Kreĭn formula provides an enumeration of all canonical selfadjoint extensions $A(\tau)$ of S, in terms of one canonical selfadjoint extension $\mathring{A}$ and the corresponding Q-function $Q(\ell)$. The interpretation of $\tau=0$ in the above theorem is that it corresponds to the canonical extension $\mathring{A}$. To S and each extension $A(\tau)$ corresponds a Q-function $Q_\tau(\ell)$. All these Q-functions are uniquely defined up to real constants. When we fix all these Q-functions by the condition that their real parts vanish at a fixed point $\mu \in \mathbb{C}^+$ then the next proposition shows that they are all described by (6.1). For a simple proof, we refer to [HLS].

Proposition 6.2. *Let $Q(\ell)$ be the Q-function of S and $\mathring{A}$, and assume that $\operatorname{Re} Q(\mu)=0$. For each $\tau \in \mathbb{R}\cup\{\infty\}$, the function $Q_\tau(\ell)$ in (1.14) is the Q-function of the canonical selfadjoint extension $A(\tau)$ and S with the additional property that $\operatorname{Re} Q_\tau(\mu)=0$.*

We conclude that all canonical selfadjoint extensions of S can be parametrized by Kreĭn's formula (6.1) via the corresponding resolvent operators, or, equivalently, by the fractional linear transform (1.14), (1.15), via the corresponding Q-functions.

A Helly's selection theorems

Proposition A1. *Let $\sigma_n : [a,b] \to \mathbb{R}$ nondecreasing functions, such that $|\sigma_n(x)| \leq A$, $x \in [a,b]$. Then there exists a subsequence, σ_n and a nondecreasing function $\sigma : [a,b] \to \mathbb{R}$, such that*

$$\lim_{n\to\infty} \sigma_n(x) = \sigma(x), \quad x \in [a,b].$$

Proposition A2. *Let $\sigma_n, \sigma : [a,b] \to \mathbb{R}$ being nondecreasing functions, such that*

(i) $\qquad |\sigma_n(x)| \leq A, \quad x \in [a,b],$

(ii) $\qquad \lim_{n\to\infty} \sigma_n(x) = \sigma(x), \quad x \in [a,b].$

Then for every continuous function $h : [a,b] \to \mathbb{C}$ we have

$$\lim_{n\to\infty} \int_a^b h(x) d\sigma_n(x) = \int_a^b h(x) d\sigma(x).$$

For the proofs of these propositions the reader is suggested to consult [At] and [W].

B THE SCHWARZ FORMULA

First we derive the Schwarz formula, using the Cauchy integral formula.

PROPOSITION B1. *(The Schwarz Formula) Let f be holomorphic on $\mathbb{D}$, the unit disc, and let $0 < R < 1$. Then for $|z| < R$ the following holds*

$$f(z) = i\mathrm{Im}\, f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R\,e^{it} + z}{R\,e^{it} - z}(\mathrm{Re}\, f)(R\,e^{it})dt$$

Proof. We will proof this stepwise.
Step 1. As f is holomorphic on $\mathbb{D}$, f can be represented by the power series

$$f(\zeta) = \sum_{m=0}^{\infty} a_m\zeta^m, \quad |\zeta| < 1.$$

We can express the coefficients a_m in terms of $F = \mathrm{Re}\, f$, since the power series lead to:

$$2F(\zeta) = \sum_{m=0}^{\infty} a_m\zeta^m + \sum_{m=0}^{\infty} \bar{a}_m\bar{\zeta}^m, \quad |\zeta| < 1.$$

For $0 < R < 1$ we integrate $F(\zeta)/\zeta^{n+1}$, $n \geq 0$, in the positive direction over the circle C_R given by $|\zeta| = R$:

$$\frac{1}{\pi i}\int_{C_R} \frac{F(\zeta)}{\zeta^n}\frac{d\zeta}{\zeta} = \sum_{m=0}^{\infty} a_m \frac{1}{2\pi i}\int_{C_R} \frac{\zeta^m}{\zeta^n}\frac{d\zeta}{\zeta} + \sum_{m=0}^{\infty} \bar{a}_m \frac{1}{2\pi i}\int_{C_R} \frac{\bar{\zeta}^m}{\zeta^n}\frac{d\zeta}{\zeta}.$$

The first term on the righthand side equals a_n, the second term equals 0 if $n \geq 1$ and equals a_0 for $n = 0$. Thus we have integral expressions for the coefficients a_n in the expansion of f:

$$a_n = \frac{1}{\pi i}\int_{C_R} \frac{F(\zeta)}{\zeta^n}\frac{d\zeta}{\zeta}, \quad n \geq 1,$$

and

$$a_0 + \bar{a}_0 = \frac{1}{\pi i}\int_{C_R} F(\zeta)\frac{d\zeta}{\zeta}.$$

Step 2. Substitution in the power series of f gives

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{\pi i}\int_{C_R} F(\zeta)\left(\frac{z}{\zeta}\right)^n \frac{d\zeta}{\zeta} - \bar{a}_0, \quad |z| < R.$$

Since $\sum_{n=0}^{\infty}(z/\zeta)^n = \frac{\zeta}{\zeta - z}$, this gives

$$f(z) = \frac{1}{\pi i}\int_{C_R} \frac{F(\zeta)}{\zeta - z}d\zeta - \mathrm{Re}\, a_0 + i\mathrm{Im}\, a_0.$$

Especially in case of $z = 0$ we have

$$f(0) = \frac{1}{\pi i} \int_{C_R} \frac{F(\zeta)}{\zeta} d\zeta - \operatorname{Re} a_0 + i \operatorname{Im} a_0,$$

and since $a_0 = f(0)$, it follows that

$$\operatorname{Re} a_0 = \frac{1}{2\pi i} \int_{C_R} \frac{F(\zeta)}{\zeta} d\zeta.$$

Step 3. We conclude

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \left(\frac{2F(\zeta)}{\zeta - z} - \frac{F(\zeta)}{\zeta} \right) d\zeta + i \operatorname{Im} a_0,$$

and so

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{\zeta + z}{\zeta - z} F(\zeta) \frac{d\zeta}{\zeta} + i \operatorname{Im} a_0, \quad |z| < R.$$

By the parametrization $\zeta = R e^{it}$, $-\pi \le t \le \pi$, and with $d\zeta/\zeta = i dt$ we have the Schwarz formula.

C THE RIESZ-HERGLOTZ FORMULA

PROPOSITION C1. *(The Riesz-Herglotz Formula) The function f is holomorphic on $\mathbb{D}$ and $\operatorname{Re} f(z) \geq 0$, $z \in \mathbb{D}$, if and only if there exists a $\gamma \in \mathbb{R}$ and a non decreasing bounded function τ on $[-\pi, \pi]$ exists, such that*

$$f(z) = i\gamma + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\tau(t), \quad |z| < 1.$$

The function τ is (up to a constant) uniquely determined by f.

Proof. We will prove this stepwise.
Step 1. Assume f has the given representation. Then f is holomorphic on $\mathbb{D}$ and as $z = x + iy$ it follows:

$$\operatorname{Re} f(z) = \int_{-\pi}^{\pi} \frac{1 - x^2 - y^2}{(\cos t - x)^2 + (\sin t - y)^2} d\tau(t) \geq 0.$$

Step 2. Conversely, for f holomorphic on $\mathbb{D}$ and $\operatorname{Re} f(z) \geq 0$, $z \in \mathbb{D}$, we have the Schwarz formula:

$$f(z) = i\operatorname{Im} f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R\,e^{it}+z}{R\,e^{it}-z} (\operatorname{Re} f)(R\,e^{it})dt, \quad |z| < R < 1,$$

and in particular for $z = 0$

$$\operatorname{Re} f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\operatorname{Re} f)(R\,e^{it})dt.$$

When $\gamma = \operatorname{Im} f(0)$ and z is replaced by $R\,z$, this gives for $|z| < 1$,

$$\text{(C1)} \quad f(R\,z) = i\gamma + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} (\operatorname{Re} f)(R\,e^{it})dt = i\gamma + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\tau_R(t),$$

where the function

$$\tau_R(t) = \int_0^t (\operatorname{Re} f)(R\,e^{is})ds,$$

is monotonous non decreasing, since $(\operatorname{Re} f)(R\,e^{is}) \geq 0$. Furthermore

$$0 \leq \tau_R(t) \leq \tau_R(2\pi) = 2\pi \operatorname{Re} f(0).$$

We choose a sequence of numbers R monotonously increasing up to 1. By applying Helly's selection principle we find a subsequence such that τ_R converges. The validity of the conjecture follows from applying one of the previous mentioned theorems of Helly to (C1).
Step 3. To see that τ is uniquely determined in the integral representation we look at the integral series expansion. To see this notice that

$$\frac{e^{it}+z}{e^{it}-z} = \frac{1+ze^{-it}}{1-ze^{-it}} = (1+ze^{-it})\sum_{k=0}^{\infty} z^k e^{-itk}$$
$$= \sum_{k=0}^{\infty} z^k e^{-itk} + \sum_{k=1}^{\infty} z^k e^{-itk} = 1 + 2\sum_{k=1}^{\infty} z^k e^{-itk},$$

and thus

$$f(z) = i\gamma + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\tau(t) = i\gamma + \frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau(t) + \sum_{k=1}^{\infty} z^k \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ikt} d\tau(t).$$

Hence, if we set

$$f(z) = c + c_1 z + c_2 z^2 + \cdots$$

then the following equalities hold

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-itk} d\tau(t), \quad k = 1, 2, \cdots$$

$$c = i\gamma + \frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau(t), \quad c + \bar{c} = \frac{1}{\pi}\int_{-\pi}^{\pi} d\tau(t).$$

We assume there occur two functions τ in the representation and then we find for the difference $\tilde{\tau}$

$$\int_{-\pi}^{\pi} e^{-itk} d\tilde{\tau}(t) = 0, \quad k = 0, 1, 2, \cdots .$$

Taking the complex conjungates we see that the same holds for $k = 0, \pm 1, \pm 2, \cdots$. By integration by parts one infers that the following holds for $k = \pm 1, \pm 2, \cdots$

$$0 = \int_{-\pi}^{\pi} e^{-itk} d\tilde{\tau}(t) = e^{-itk}\tilde{\tau}(t)|_{t=\pi}^{\pi} + ik\int_{-\pi}^{\pi} e^{-itk}\tilde{\tau}(t)dt$$

$$= \int_{-\pi}^{\pi} d\tilde{\tau}(t) + ik\int_{-\pi}^{\pi} e^{-itk}\tilde{\tau}(t)dt.$$

Since $\int_{-\pi}^{\pi} d\tilde{\tau}(t) = 0$ we see

$$\int_{-\pi}^{\pi} e^{-itk}\tilde{\tau}(t)dt = 0, \quad k = \pm 1, \pm 2, \cdots$$

such that for $k = \pm 1, \pm 2, \cdots$, but also for $k = 0$ follows

$$\int_{-\pi}^{\pi} e^{-itk}(\tilde{\tau}(t) - C)dt = 0, \quad C = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\tau}(t)dt.$$

The completeness of the used orthonormal sequence in $L^2(-\pi, \pi)$ implies that $\tilde{\tau}(t) = C$.

D THE INTEGRAL REPRESENTATION FOR A NEVANLINNA FUNCTION

We denote the class of Nevanlinna functions by N.

PROPOSITION D1. *The function $\varphi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ belongs to N if and only if there exist $\alpha \in \mathbb{R}, \beta \geq 0$ and a monotonous nondecreasing function σ on $\mathbb{R}$ with $\int_{\mathbb{R}} d\sigma(t)/(t^2+1) < \infty$, such that*

$$\varphi(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t).$$

The function σ is uniquely determined modulo a constant by φ.

Proof. We divide the proof in four steps.

Step 1. If $\varphi(z)$ has the indicated representation, then it is rather obvious that φ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and that $\overline{\varphi(z)} = \varphi(\bar{z})$ is valid. Furthermore, from

$$\frac{\varphi(z) - \varphi(w)^*}{z - \bar{w}} = \int_{\mathbb{R}} \frac{1}{t-z} \frac{1}{t-\bar{w}} d\sigma(t) + \beta.$$

we conclude that φ belongs to N.

Step 2. Now suppose that φ belongs to N. We introduce de following transforms

$$z = i\frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D}, \quad \varphi(z) = if(\zeta),$$

note that the first formula is a Cayley transform, so $\zeta \in \mathbb{D}$ is mapped bijectively to $z \in \mathbb{C}^+$ and

$$\operatorname{Im} \varphi(z) \geq 0 \Rightarrow \operatorname{Re} f(\zeta) \geq 0.$$

By using the theorem of Riesz-Herglotz, which can be found in the appendix, we infer that the following equality holds:

$$\begin{aligned} f(\zeta) &= i\gamma + \int_0^{2\pi} \frac{e^{is}+\zeta}{e^{is}-\zeta} d\tau(s) \\ &= i\gamma + \int_{0+}^{2\pi-} \frac{e^{is}+\zeta}{e^{is}-\zeta} d\tau(s) + \frac{1+\zeta}{1-\zeta}(\tau(2\pi) - \tau(2\pi-) + \tau(0+) - \tau(0)) \\ &= i\gamma + \beta\frac{1+\zeta}{1-\zeta} + \int_{0+}^{2\pi-} \frac{e^{is}+\zeta}{e^{is}-\zeta} d\tau(s). \end{aligned}$$

One can rewrite this with aid of the definitions of φ and ζ:

$$\varphi(z) = -\gamma + i\beta\frac{1+\zeta}{1-\zeta} + i\int_{0+}^{2\pi-} \frac{e^{is}+\zeta}{e^{is}-\zeta} d\tau(s).$$

Step 3. We transform the integrand using

$$\begin{aligned} i\frac{e^{is}+\zeta}{e^{is}-\zeta} &= i\frac{(z+i)e^{is} + z - i}{(z+i)e^{is} - (z-i)} \\ &= i\frac{z(e^{is}+1) + i(e^{is}-1)}{z(e^{is}-1) + i(e^{is}+1)} = i\frac{z\frac{e^{is}+1}{e^{is}-1} + i}{z + i\frac{e^{is}+1}{e^{is}-1}} \\ &= \frac{iz\frac{e^{is}+1}{e^{is}-1} - 1}{z + i\frac{e^{is}+1}{e^{is}-1}} = \frac{z\cot s/2 - 1}{\cot s/2 + z}. \end{aligned}$$

This gives

$$\varphi(z) = -\gamma + \beta z + i \int_{0+}^{2\pi-} \frac{z \cot s/2 - 1}{\cot s/2 + z} d\tau(s).$$

Step 4. Substitute $\alpha = -\gamma$, $-\cot s/2 = t$ and $\tau(s) = \rho(t)$ in the previous equation to obtain:

$$\varphi(z) = \alpha + \beta z + \int_{\mathbf{R}} \frac{1 + tz}{t - z} d\rho(t),$$

and it is readily verified that this is equal to

$$\varphi(z) = \alpha + \beta z + \int_{\mathbf{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) (t^2 + 1) d\rho(t).$$

Introduce $d\sigma(t) = (t^2 + 1) d\rho(t)$, and the conjecture is proven.

E The Stieltjes inversion formula

The function σ in the integral representation of a function $\varphi \in \mathrm{N}$ is uniquely determined up to a constant. It is important to be able to deduce this function directly from φ. Since σ can possess jump discontinuities, we use the following normalization:

$$\sigma(\lambda) = \frac{\sigma(\lambda+0)+\sigma(\lambda-0)}{2}.$$

In addition we set σ by demanding $\sigma(0) = 0$. Under these circumstances we have the following result, cf. [D].

Proposition E1. *Let $\varphi \in \mathrm{N}$, then if $\lambda_1 \le \lambda_2$:*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} \varphi(x+i\varepsilon)dx = \sigma(\lambda_2) - \sigma(\lambda_1).$$

Proof. Again we divide the proof into steps.
Step 1. From the integral representation we can deduce the following identity

$$\operatorname{Im} \varphi(x+i\varepsilon) = \beta\varepsilon + \int_{\mathbf{R}} \frac{\varepsilon}{(t-x)^2+\varepsilon^2} d\sigma(t),$$

so

$$\frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} \varphi(x+i\varepsilon)dx = \frac{1}{\pi}\beta\varepsilon(\lambda_2-\lambda_1) + \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \left(\int_{\mathbf{R}} \frac{\varepsilon}{(t-x)^2+\varepsilon^2} d\sigma(t) \right) dx.$$

The first term on the righthand side has limit 0. Looking more carefully at the second term we see that by reversing the order of integration it equals

$$\frac{1}{\pi}\int_{\mathbf{R}} \left(\int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(t-x)^2+\varepsilon^2} dx \right) d\sigma(t) = \int_{\mathbf{R}} \frac{1}{\pi} \left(\arctan \frac{\lambda_2 - t}{\varepsilon} - \arctan \frac{\lambda_1 - t}{\varepsilon} \right) d\sigma(t).$$

The integrand on the righthand side is dominated by 1. If $\varepsilon \downarrow 0$ the integrand has a pointwise limit that is actually the characteristic function of the interval $[\lambda_1, \lambda_2]$, and attains averaged values in the endpoints. Since σ isn't necessarily bounded, we can not directly use the dominated convergence theorem.
Step 2. We choose an interval $[-R, R]$ containing $[\lambda_1, \lambda_2]$ and write φ as $\varphi = \varphi_1 + \varphi_2$ where

$$\varphi_1(z) = \alpha + \beta z + \int_{-R}^{R} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t)$$

and

$$\varphi_2(z) = \int_{|t|\ge R} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t).$$

Step 3. The function φ_1 has weight σ whose support is contained in $[-R, R]$. Now we can apply the dominated convergence theorem in step 1:

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} \varphi_1(x+i\varepsilon)dx = \sigma(\lambda_2) - \sigma(\lambda_1).$$

Step 4. Next we show that $\operatorname{Im}\varphi_2(x+i\varepsilon)$ has limit 0 uniformly on $\lambda_1 \le x \le \lambda_2$. Thus we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\operatorname{Im}\varphi_2(x+i\varepsilon)dx = 0,$$

this proves the proposition. Notice that

$$\begin{aligned}
\operatorname{Im}\varphi_2(x+i\varepsilon) &= \int_{|t|\ge R}\frac{\varepsilon}{(t-x)^2+\varepsilon^2}d\sigma(t)\\
&= \varepsilon\left(\int_{t>R}\frac{1}{(t-x)^2+\varepsilon^2}d\sigma(t)+\int_{t<-R}\frac{1}{(t-x)^2+\varepsilon^2}d\sigma(t)\right)\\
&\le \varepsilon\left(\int_{t>R}\frac{1}{(t-x)^2}d\sigma(t)+\int_{t<-R}\frac{1}{(t-x)^2}d\sigma(t)\right)\\
&\le \varepsilon\left(\int_{t>R}\frac{1}{(t-\lambda_2)^2}d\sigma(t)+\int_{t<-R}\frac{1}{(t-\lambda_1)^2}d\sigma(t)\right),
\end{aligned}$$

uniformly on $\lambda_1 \le x \le \lambda_2$. The last inequality uses

$$-\infty < -R \le \lambda_1 \le x \le \lambda_2 < R < \infty,$$

so for $t \ge R$ we have $t-x \ge t-\lambda_2$ and for $t \le -R$ we have $t-x \ge t-\lambda_1$.

F The integral representation for Nevanlinna functions belonging to the classes $\mathbf{N}_1$ and $\mathbf{N}_0$

Proposition F1. *A function $\varphi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ belongs to $\mathbf{N}_1$ if and only if there are $\gamma \in \mathbb{R}$ and a nondecreasing function σ on $\mathbb{R}$, satisfying $\int_{\mathbb{R}} d\sigma(t)/(|t|+1) < \infty$ such that*

$$\varphi(z) = \gamma + \int_{\mathbb{R}} \frac{1}{t-z} d\sigma(t).$$

Proof. We will prove this stepwise.
Step 1. If $\varphi(z)$ satisfies the indicated representation, then it is clear that φ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and $\overline{\varphi(z)} = \varphi(\bar{z})$ holds. Furthermore we observe

$$\frac{\operatorname{Im} \varphi(iy)}{y} = \int_{\mathbb{R}} \frac{1}{t^2+y^2} d\sigma(t).$$

This implies

$$\int_1^\infty \frac{\operatorname{Im} \varphi(iy)}{y} dy = \int_1^\infty \left(\int_{\mathbb{R}} \frac{1}{t^2+y^2} d\sigma(t) \right) dy = \int_{\mathbb{R}} \left(\int_1^\infty \frac{1}{t^2+y^2} dy \right) d\sigma(t),$$

whereas the function

$$\int_1^\infty \frac{1}{t^2+y^2} dy = \frac{1}{|t|} \left(\frac{\pi}{2} - \arctan \frac{1}{|t|} \right)$$

is bounded for $-1 \le t \le 1$ and attains values between $\pi/4|t|$ and $\pi/2|t|$ for $|t| \ge 1$. We conclude that φ belongs to $\mathbf{N}_1$.
Step 2. Assume that φ belongs to $\mathbf{N}_1$. The function φ has the integral representation

$$\varphi(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t).$$

where $\int_{\mathbb{R}} d\sigma(t)/(t^2+1) < \infty$. We so have

$$\frac{\operatorname{Im} \varphi(iy)}{y} = \beta + \int_{\mathbb{R}} \frac{1}{t^2+y^2} d\sigma(t).$$

The lefthand side is integrable, so $\beta = 0$ and also

$$\int_1^\infty \left(\int_{\mathbb{R}} \frac{1}{t^2+y^2} d\sigma(t) \right) dy < \infty = \int_{\mathbb{R}} \left(\int_1^\infty \frac{1}{t^2+y^2} dy \right) d\sigma(t).$$

From the behaviour of $\int_1^\infty \frac{1}{t^2+y^2} dy$ we deduce $\int_{\mathbb{R}} d\sigma(t)/(|t|+1) < \infty$. We conclude with $\gamma = \alpha - \int_{\mathbb{R}} \frac{t}{t^2+1} d\sigma(t)$, that the integral representation for φ reduces to

$$\varphi(z) = \gamma + \int_{\mathbb{R}} \frac{1}{t-z} d\sigma(t).$$

PROPOSITION F2. *A function $\varphi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ belongs to* $\mathbf{N}_0$ *if and only if there are* $\gamma \in \mathbb{R}$ *and a nondecreasing function* σ *on* $\mathbb{R}$ *satisfying* $\int_{\mathbb{R}} d\sigma(t) < \infty$ *such that*

$$\varphi(z) = \gamma + \int_{\mathbb{R}} \frac{1}{t-z} d\sigma(t).$$

Proof. We will prove this stepwise.
Step 1. If $\varphi(z)$ satisfies the indicated representation, then it is clear that φ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and $\overline{\varphi(z)} = \varphi(\bar{z})$ holds. Furthermore we observe

$$\frac{\operatorname{Im} \varphi(iy)}{y} = \int_{\mathbb{R}} \frac{1}{t^2 + y^2} d\sigma(t).$$

This implies

$$\sup_{y>0} y^2 \frac{\operatorname{Im} \varphi(iy)}{y} = \sup_{y>0} y^2 \int_{\mathbb{R}} \frac{1}{t^2 + y^2} d\sigma(t) = \int_{\mathbb{R}} d\sigma(t).$$

We conclude that φ belongs to $\mathbf{N}_0$.
Step 2. Assume that φ belongs to $\mathbf{N}_0 \subset \mathbf{N}_1$. The function φ has the indicated integral representation, since we can apply proposition F1.

$$y^2 \frac{\operatorname{Im} \varphi(iy)}{y} = \beta y^2 + \int_{\mathbb{R}} \frac{y^2}{t^2 + y^2} d\sigma(t).$$

The supremum of the lefthand side exists, so $\beta = 0$ and also

$$\sup_{y>0} y^2 \frac{\operatorname{Im} \varphi(iy)}{y} = \int_{\mathbb{R}} d\sigma(t).$$

this proves the proposition.

These and related results can be found in [KK]

G FINITE ASYMPTOTIC EXPANSIONS & DIVIDED DIFFERENCES

By

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n),$$

we mean

$$\frac{f(x) - a_0 + a_1 x + \cdots + a_n x^n}{x^n} \to 0, \quad x \to 0.$$

In terms of divided differences this is equivalent to

$$f^{\{0\}}(x) = f(x) = a_0 + o(1),$$

$$f^{\{1\}}(x) = \frac{f(x) - a_0}{x} = a_1 + o(1),$$

$$f^{\{2\}}(x) = \frac{f^{\{1\}}(x) - a_1}{x} = a_2 + o(1),$$

$$\ldots$$

$$f^{\{n\}}(x) = \frac{f^{\{n-1\}}(x) - a_{n-1}}{x} = a_n + o(1).$$

For two functions f, g where

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n),$$

and

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n + o(x^n),$$

we have

(i) the summation rule:

$$f(x) + g(x) = c_0 + c_1 x + \cdots + c_n x^n + o(x^n),$$

where the $c_i = a_i + b_i$, $0 \le i \le n$.

(ii) the multiplication rule:

$$f(x)g(x) = c_0 + c_1 x + \cdots + c_n x^n + o(x^n),$$

where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$, $0 \le i \le n$.

(iii) the division rule:

$$\frac{f(x)}{g(x)} = A(x) + o(x^n),$$

where A is the quotient of P and Q with increasing powers of x, where $P(x) = \sum_{i=0}^{n} a_i x^i$ and $Q(x) = \sum_{i=0}^{n} b_i x^i$.

References

[A] N.I. Achieser, *The classical moment problem and some related questions in analysis*, Moscow, 1961 (Russian) (English translation: Oliver and Boyd, Edinburg, and Hafner, New York, 1965).

[At] F.V. Atkinson, *Discrete and continuous boundary problems*, Academic Press, New York, 1968.

[D] W.F. Donoghue, *Monotone matrix functions and analytic continuation*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[HLS] S. Hassi, H. Langer and H.S.V. de Snoo, "Selfadjoint extensions for a class of symmetric operators with defect numbers (1,1)", 15th OT Conference Proc., (1995), 115-145.

[HS1] S. Hassi and H.S.V. de Snoo, "On some subclasses of Nevanlinna functions", Zeitschrift für Analysis und ihre Anwendungen, 15 (1996), 1-11.

[HS2] S. Hassi and H.S.V. de Snoo, "One-dimensional graph perturbations of selfadjoint relations", Ann. Acad. Sci. Fenn., Series A.I., to appear.

[HSW] S. Hassi, H.S.V. de Snoo and A.D.I. Willemsma, "Smooth rank one perturbations of selfadjoint operators", in preparation.

[KK] I.S. Kac and M.G. Kreĭn, "R-functions-analytic functions mapping the upper halfplane into itself", Supplement I to the Russian edition of F.V. Atkinson, *Discrete and continuous boundary problems*, Mir, Moscow, 1968 (Russian) (English translation: Amer. Math. Soc. Transl., (2) 103 (1974), 1-18).

[LT] H. Langer and B. Textorius, "On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space", Pacific J. Math., 72 (1977), 135-165.

[W] D.V. Widder, *The Laplace transform*, Princeton University Press, Princeton 1946.