A reversible bifurcation analysis of the inverted pendulum

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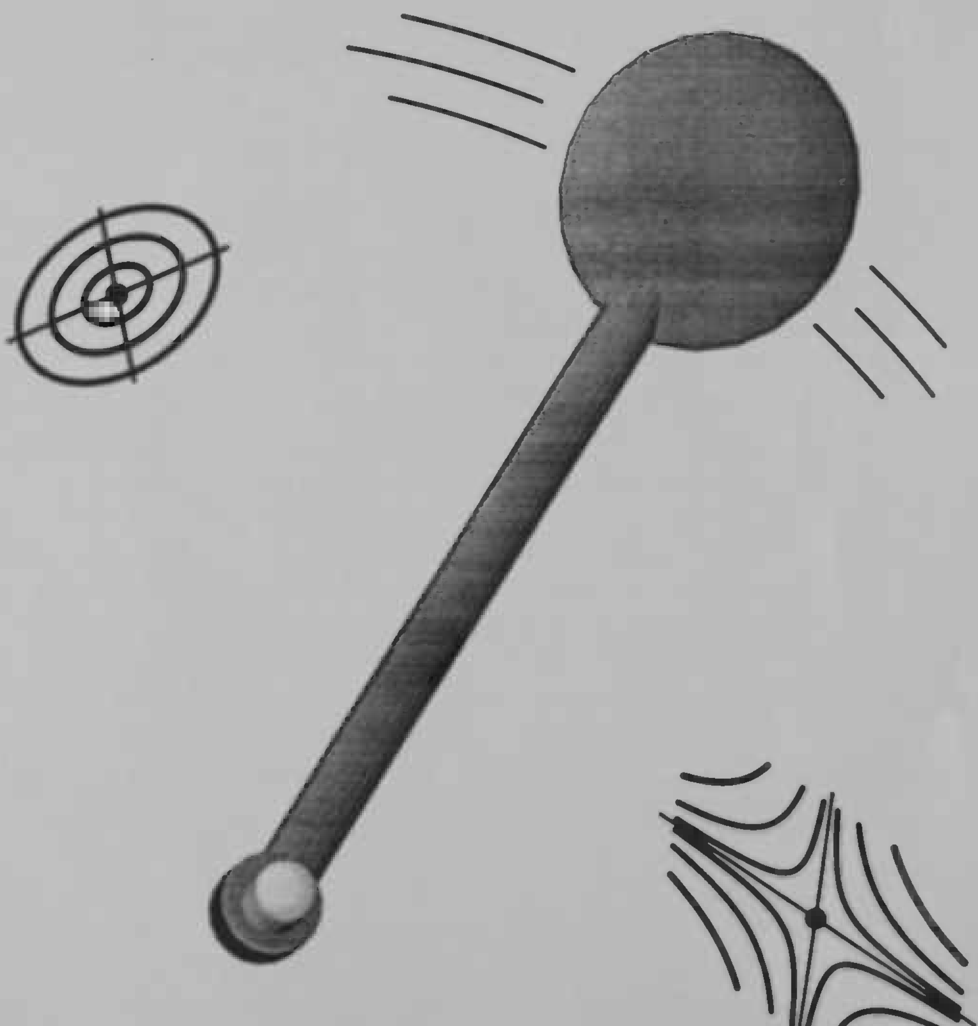

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Abstract

The inverted pendulum with a periodic parametric forcing is considered as a bifurcation problem in the reversible setting. Parameters are given by the size of the forcing and the frequency ratio. Normal form theory provides an integrable approximation of the Poincaré map generated by a planar vector field. Genericity of the model is studied by perturbation analysis, where a spatial symmetry is optional.

1 Introduction

In this paper we study the behaviour of a pendulum near its upper equilibrium. The rod of the pendulum is assumed to be stiff and massless. The upper equilibrium is known to be unstable. Stabilisation can be effected by moving the suspension point periodically up and down in a specific frequency domain, see e.g. [1, 26].

This system is usually referred to as the *inverted* pendulum, which can be modelled by the equation of motion

$$\ddot{x} = (\alpha + \beta\rho(t))V'(x)$$

or alternatively, the extended phase space vector field

$$X = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} + (\alpha + \beta\rho(t))V'(x)\frac{\partial}{\partial y} \tag{1}$$

with corresponding Hamiltonian $H = \frac{1}{2}y^2 - (\alpha + \beta\rho(t))V(x)$. Here $V(x) = 1 - \cos x$, x is the deviation from the upper equilibrium, $y = \dot{x}$ the velocity, and $\rho(t)$ a periodic C^∞ function called the *parametric forcing*, caused by the motion of the suspension point. Further, $\sqrt{\alpha}$ is the ratio of the frequency of the pendulum to that of ρ, and β controls the amplitude of the parametric force. We may assume that ρ is 2π-periodic and has zero average, so we can write $\rho(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k e^{ikt}$, with $\overline{a_k} = a_{-k}$; if necessary we can adjust α to accomplish this. We may also assume that $\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|a_k|^2}{k^2} = 1$.

Finally we assume that $\rho(t)$ is even, implying that X is time-reversible, i.e. $\mathcal{R}_* X = -X$, where $\mathcal{R} : \mathbb{S}\times\mathbb{R}\times\mathbb{S} \longrightarrow \mathbb{S}\times\mathbb{R}\times\mathbb{S}$ is defined by $\mathcal{R}(x, y, t) = (x, -y, -t)$. Consequently H is $\mathcal{R}$-equivariant in the sense that $H \circ \mathcal{R} = H$. If, as in the pendulum case, $V(x)$ is also even, there is another reversing symmetry $\mathcal{S} : (x, y, t) \mapsto (-x, y, -t)$, satisfying $\mathcal{S}_* X = -X$ and $H \circ \mathcal{S} = H$. We distinguish between the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case where X is both $\mathcal{R}$- and $\mathcal{S}$-reversible, and the $\mathbb{Z}_2$-symmetric case where X is only $\mathcal{R}$-reversible.

If V satisfies $V(x) = -V(x + \pi)$, e.g. $V(x) = 1 - \cos x$, there is also a global symmetry $\mathcal{T} : (x, y, t; \alpha, \beta) \mapsto (x + \pi, y, t, -\alpha, -\beta)$, and X, H are $\mathcal{T}$-equivariant.

The stability of the upper equilibrium is determined by the linearised equation. In 1928 such a system was studied by van der Pol & Strutt [26] for two types of parametric forcing:

1. The 'ripple' given by $\ddot{x} - (\alpha + \beta\rho(t))x = 0$, $\rho(t) = \begin{cases} 1\ , t \in [0, \pi) \\ -1\ , t \in [\pi, 2\pi) \end{cases}$

2. Mathieu's equation: $\ddot{x} - (\alpha + \beta \cos t)x = 0$

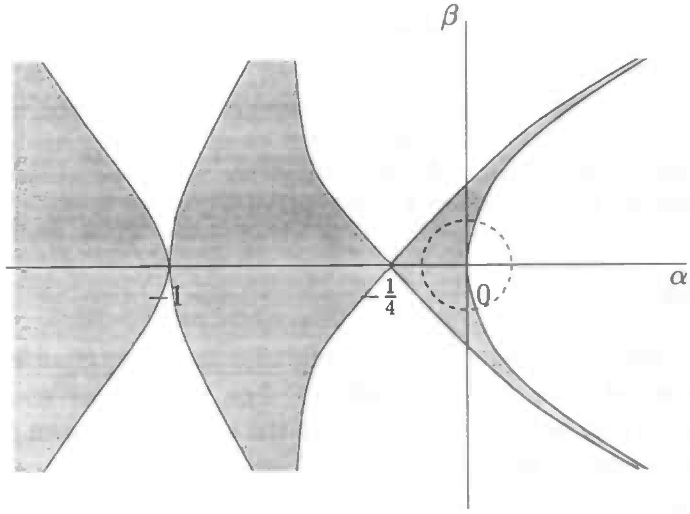

Figure 1: *Stability diagram for Mathieu's equation. Shading indicates stability of* $(x, y) = (0, 0)$. *We only consider a small region around the origin, like indicated by the dashed circle.*

The first example is also treated by Arnol'd [1, pp. 121–122] and Levi and Weckesser [18]. The well-known stability diagram (figure 1) can be computed from Mathieu's equation, cf. Meixner and Schäfke [19]. The behaviour of a parametrically forced pendulum near its lower equilibrium for arbitrary force function and potential is discussed by Broer and Vegter [11]. A topological view on the stability of the inverted pendulum is given by Levi [17].

We study the qualitative behaviour of (1), in particular the subharmonic *bifurcations* as suggested by the linear part. Since we also study the effect of a perturbation of V, we take $V(x)$ to be an arbitrary 2π-periodic C^∞ function satisfying $V'(0) = 0$ — implying that the upper equilibrium $(x, y) = (0, 0)$ is kept — in the sequel. This study is local in the sense that the parameters α and β, as well as the velocity y of the pendulum, are small. The angle x may range over the whole circle. In future work we numerically continue our results to larger values of β.

Method

System (1) lives in a three-dimensional extended phase space. Because of the periodicity in time, one naturally considers the *Poincaré map* — also called *period*, *first return* or *stroboscopic map* — $P : \mathbb{S} \times \mathbb{R} \longrightarrow \mathbb{S} \times \mathbb{R}$, given by

$$X^{2\pi}(x, y, 0) = (P(x, y), 2\pi) ,$$

where X^s denotes the flow of X over time s, also see figure 2.

The Poincaré map inherits the symmetries of X in the following way: let $R, S : \mathbb{S} \times \mathbb{R} \longrightarrow \mathbb{S} \times \mathbb{R}$ be defined by $R(x, y) = (x, -y)$ and $S(x, y) = S(-x, y)$, then P is R- and S-

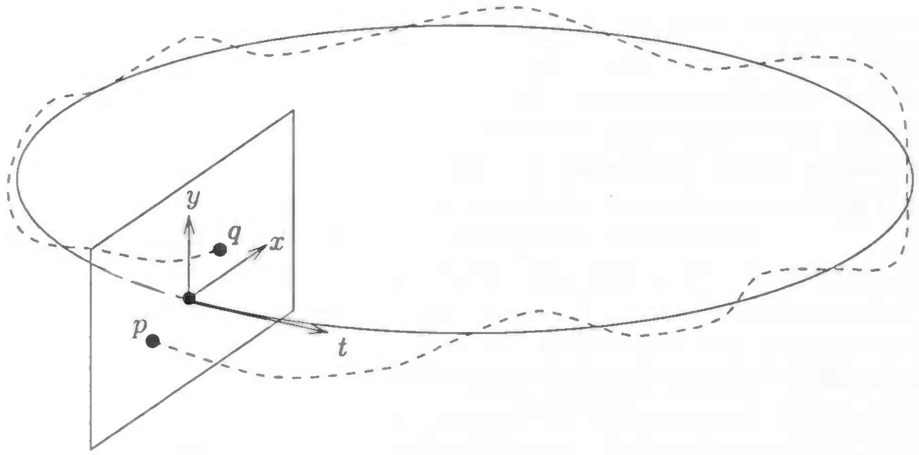

Figure 2: *The Poincaré or first return map P is defined on a section transversal to the flow of the vector field, and maps p to q, being on the same trajectory of X. The transversal section is a subspace of the extended phase space given by $t = 0$.*

reversible whenever X is $\mathcal{R}$- and $\mathcal{S}$-reversible, meaning that $RPR = P^{-1}$ and $SPS = P^{-1}$. Similarly, if X is $\mathcal{T}$-equivariant, then P is T-equivariant in the sense that $TPT = P$, where $T : (x, y; \alpha, \beta) \mapsto (x + \pi, y; -\alpha, -\beta)$.

The top panels of figure 3 show pictures of P, created by `DsTool` by numerical integration, cf. Back et al. [2]. They reveal a geometrical structure underneath some 'noise' at the stable and unstable manifolds of saddle points, and 'pendulum beads' in annuli around stable periodic points. Our goal is to understand this underlying structure in a qualitative sense. We are particularly interested in:

- Fixed points of P, their stability type and, in the case of a saddle point, the local (un-)stable manifolds.
- Cylinders of invariant circles.
- Homo- and heteroclinic connections between saddle points of P.
- Bifurcations of fixed points and their (un-)stable manifolds.

Our study is in Hamiltonian setting, but the Poincaré map is hard to study since it corresponds to a time-dependent vector field. Using a canonical transformation Ψ, provided by *averaging* or *normal form theory*, P can be approximated by an integrable map:

$$\Psi_* P \approx \tilde{X}^{2\pi}, \tag{2}$$

for some planar Hamiltonian vector field $\tilde{X}$. This normal form is valid for $x \in \mathbb{S}$, and $(y; \alpha, \beta)$ small. The transformation commutes with the symmetries R, S and T. The above formula shows that the phase portrait of P is approximated by that of $\tilde{X}$, or, equivalently, by the level curves of the corresponding Hamiltonian $\tilde{H}$.

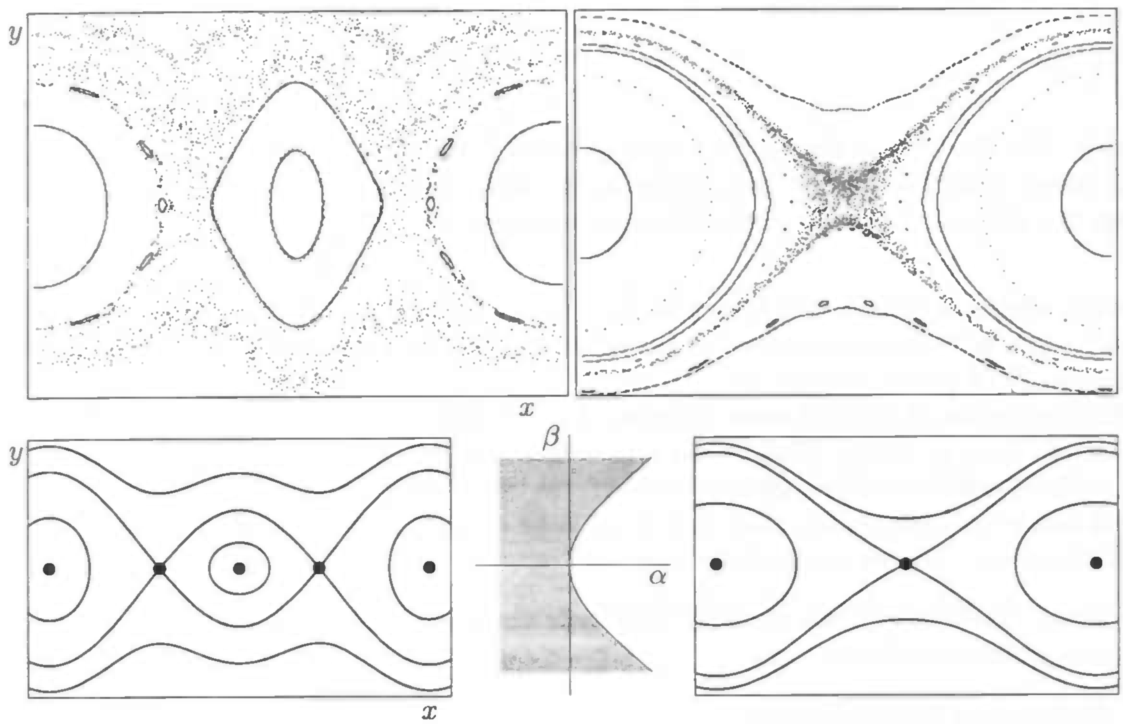

Figure 3: *Top: Numerical simulation of the Poincaré map for two values of* (α, β). *The* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-symmetry is clearly visible. Bottom center: local stability diagram, shading indicates stability of the upper equilibrium. Bottom left and right: corresponding normal form phase portraits. The panels on the left are taken from the stable region, those on the right from the unstable region.*

We further simplify $\tilde{H}$ by applying a non-canonical transformation, which again respects the symmetries. Since we now work in the plane, and hence only need to consider the configuration of the level curves of $\tilde{H}$, non-canonical transformations are allowed, cf. Broer et al. [5, 6]. The resulting Hamiltonian corresponds to a 'potential energy' system and is therefore easy to study. Equivariant singularity theory, applied to the resulting 'potential function', guarantees persistence under perturbations within the class of symmetric systems that can be obtained by normal form truncations.

The planar Hamiltonian yields the pictures in the bottom left and right panels of figure 3. The error in the approximation (2) is a flat time-dependent term. Of the four features listed above, stationary points, bifurcations and Cantor sets of invariant circles are persistent (the latter by KAM theory, cf. e.g. Moser [21]), while homo- and heteroclinic connections generally 'split' into transversal intersections. This causes the noise close to the separatrices of the saddle points, whereas the strings of pendulum beads around stable periodic points are caused by the destruction of periodic (resonant) invariant circles. Both effects are hard to find due to their flatness.

In this paper we focus on the construction of the normalised vector fields generating the integrable Poincaré map. The perturbation analysis is left aside.

Outline of the results

In the case of the pendulum, the upper equilibrium is stable for $\alpha < \beta^2 + O(\beta^3)$. Figure 3 shows the stability diagram, along with normal form phase portraits taken from the stable and the unstable region. The stability boundary coincides with a line of Hamiltonian pitchfork bifurcations. The saddles existing for $\alpha < \beta^2 + O(\beta^3)$ are in heteroclinic connection. This behaviour is persistent in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric context, that is, under even perturbations of ρ and V.

In the $\mathbb{Z}_2$-symmetric case, the heteroclinic connection is lost. The line of pitchfork bifurcations falls apart into a line of transcritical bifurcations at the stability boundary and a line of Hamiltonian saddle-node bifurcations. In between there is a line of heteroclinic bifurcations, cf. figure 4.

Organisation of the paper

In the next section we give a derivation of (1) from physical laws. After this, in section 3, P is approximated by an integrable map, using a normal form transformation. The normal form theory used there is described in the appendix.

In section 4 we simplify further by rescaling the parameters and coordinates, and study the behaviour of the resulting system in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric and $\mathbb{Z}_2$-symmetric cases.

2 Derivation of the model

First we model the free pendulum in upright position. Let ℓ denote the length of the pendulum, and g the acceleration due to gravity. If we take x to be the angle between

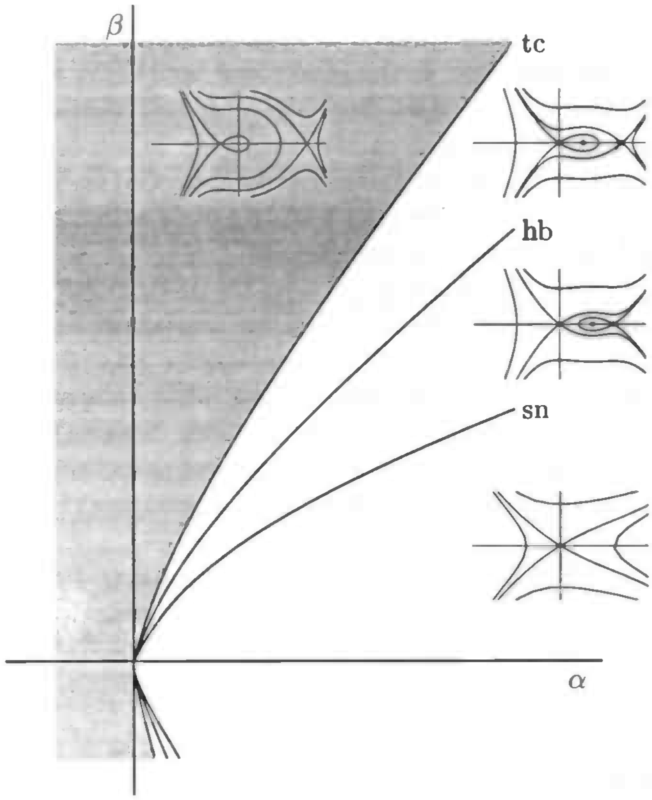

Figure 4: *Bifurcation diagram of the normal form in the $\mathbb{Z}_2$-symmetric context. The coding is explained in section 4.4.*

the position of the pendulum and the upper equilibrium, the equation of motion takes the form

$$\ddot{x} - \omega^2 \sin x = 0 \text{ , with } \omega^2 = \frac{g}{\ell} \tag{3}$$

Let the vertical oscillation of the suspension point be given by a $\frac{2\pi}{\Omega}$-periodic function $\sigma(t)$, for some $\Omega > 0$. The resulting force is $m\ddot{\sigma}(t)$, and the equation of motion is given by:

$$\ddot{x} - (\omega^2 - \frac{\ddot{\sigma}(t)}{\ell}) \sin x = 0 \tag{4}$$

Finally we rescale this to a 2π-periodic system. Define $\bar{t} = \Omega t$, then (4) transforms to

$$\frac{\partial^2}{\partial \bar{t}^2} x - (\frac{\omega^2}{\Omega^2} - \frac{\frac{\partial^2}{\partial \bar{t}^2}\sigma}{\ell}) \sin x$$

Comparing with (1) shows that we should take

$$\alpha = \frac{\omega^2}{\Omega^2} \text{ and } \beta\rho(\bar{t}) = -\frac{1}{\ell}\frac{\partial^2}{\partial \bar{t}^2}\sigma(\Omega^{-1}\bar{t}) \tag{5}$$

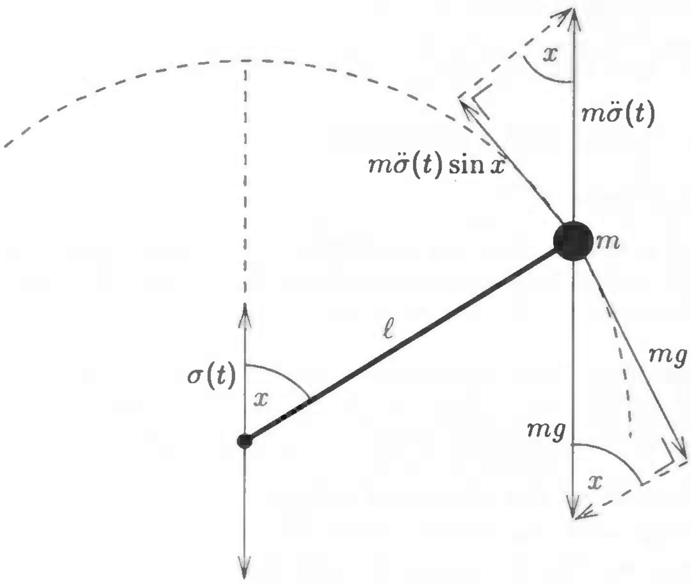

Figure 5: *A driven pendulum. The mass m is concentrated in the free end of the rod, which has length ℓ. The trajectory of the pendulum is determined by two forces: the gravitational force mg, and a force $m\ddot{\sigma}(t)$ caused by the oscillation of the suspension point.*

3 Averaging out the time-dependence

We construct a coordinate transformation pushing the time-dependence of X to a small perturbation term. We use a *normal form* or *averaging* procedure, cf. [3, 10, 11, 12, 14, 15, 23, 25]. The normal form transformation is valid for global x, and respects the symmetries $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ (if present). Moreover, it is canonical. Since ρ and V are C^∞, we may consider the Taylor series of system (1) in (α, β), with coefficients that are periodic in x and t, and formal power series in y. The vector field is now simplified to increasing order in (α, β) by successive coordinate transformations, where 'simple' means time-independent. Normalising to infinite order, and using Borel's theorem, we find a canonical C^∞ transformation which respects the symmetries, and approximates X by a planar vector field, cf. Broer [3] and Broer and Vegter [11].

Theorem 1 (Normalisation of the vector field)
Let the C^∞ vector field X on $\mathbb{S} \times \mathbb{R} \times \mathbb{S} = \{(x, y, t)\}$ have the form (1) with $V'(0) = 0$. Then there exists a C^∞ canonical transformation $\Psi_1 : \mathbb{S} \times \mathbb{R} \times \mathbb{S} \longrightarrow \mathbb{S} \times \mathbb{R} \times \mathbb{S}$, preserving the time t, such that

$$(\Psi_{1\,*}X)(x, y, t; \alpha, \beta) = \frac{\partial}{\partial t} + X_1(x, y; \alpha, \beta) + p_1(x, y, t; \alpha, \beta),$$

where X_1 *has the time-independent form*

$$\begin{aligned} X_1(x,y;\alpha,\beta) &= (y+O(|\alpha,\beta|^2 y))\frac{\partial}{\partial x} - (U'(x;\alpha,\beta)+O(|\alpha,\beta|^3))\frac{\partial}{\partial y} \text{ with} \\ U(x;\alpha,\beta) &= \frac{1}{2}\beta^2(V'(x))^2 - \alpha V(x) \text{ and} \\ p_1(x,y,t;\alpha,\beta) &= O(|y;\alpha,\beta|^\infty). \end{aligned}$$

The remainder $O(|\alpha,\beta|^2 y)$ *is independent of* x *and* $O(|\alpha,\beta|^3))$ *independent of* y. *Moreover, if* X *is* $\mathcal{R}$*-reversible or* $\mathcal{R}$*- and* $\mathcal{S}$*-reversible, then so are* $\Psi_{1*}X$ *and* X_1. *The same holds with respect to* $\mathcal{T}$.

For a proof see the appendix. Next consider the Poincaré map P of X. The normalized system $\Psi_{1*}X$ again is 2π-periodic in t (and x). By theorem 1 we can write a conjugate of P as a small perturbation of the 2π-flow of the planar vector field X_1 :

Corollary 2 (Normalisation of the Poincaré map)
Let P *be the Poincaré map of* X, *as above. Then there exists a* C^∞ *area preserving (symplectic) transformation* $\Phi_1 : \mathbb{S}\times\mathbb{R} \longrightarrow \mathbb{S}\times\mathbb{R}$, *such that*

$$\Phi_1 \circ P \circ \Phi_1^{-1} = X_1^{2\pi} + p_2,$$

where $p_2(x,y;\alpha,\beta) = O(|y;\alpha,\beta|^\infty)$. *Moreover, if* X *is* $\mathcal{R}$*-reversible or* $\mathcal{R}$*- and* $\mathcal{S}$*-reversible, the maps* $\Phi_1 \circ P \circ \Phi_1^{-1}$ *and* $X_1^{2\pi}$ *are* R*-reversible or* R*- and* S*-reversible. The same holds with respect to* $\mathcal{T}$.

Proof: It is easy to see that the Poincaré map of $\Psi_{1*}X$ has the form $\Phi_1 \circ P \circ \Phi_1^{-1}$, where $\Phi_1 := \Psi_1 \mid_{t=0}$. Since Ψ_1 is canonical and t-preserving the map Φ_1 is area preserving. The symmetry properties are direct, also compare Broer and Vegter [11] or Hoveijn and Krauskopf [16].

□

Recall that we are after the understanding of the Poincaré map P. The corollary says that up to the infinitely flat perturbation p_2, small for $(y;\alpha,\beta)$ near $(0;0,0)$, instead of P, we may as well consider the integrable map $X_1^{2\pi}$, which is again area preserving and has all desired symmetries. Hence, we consider the planar vector field X_1, Hamiltonian with respect to a planar function $H_1 = H_1(x,y;\alpha,\beta)$, which by theorem 1 satisfies

$$H_1(x,y;\alpha,\beta) = \frac{1}{2}y^2 + U(x;\alpha,\beta) + O(|\alpha,\beta|^2 y^2) + O(|\alpha,\beta|^3).$$

We simplify further by removing the remainder term $O(|\alpha,\beta|^2 y^2)$. We use a non-symplectic transformation, which is allowed in one degree of freedom.

Indeed, if $\Psi_2 : \mathbb{S}\times\mathbb{R} \to \mathbb{S}\times\mathbb{R}$ is an arbitrary transformation, then $\Psi_{2*}X_1 = (\det D\Psi_2)X_{H_1\circ\Psi_2}$, where $X_{H_1\circ\Psi_2}$ is the vector field with Hamiltonian $H_1 \circ \Psi_2$. So *equivalence* of the vector fields exactly corresponds to *right-equivalence* of the Hamiltonians. This rather qualitative way of looking changes the perspective to the collection of level sets of Hamiltonian functions, compare, e.g., Broer et al. [5, 6]. The next result is a kind of equivariant splitting lemma, compare Golubitsky and Schaeffer [13].

Theorem 3 (Normalisation of the planar Hamiltonian)
Let $H_1 = H_1(x, y; \alpha, \beta)$ be as before. Then there exists a C^∞ coordinate transformation $\Psi_2 : \mathbb{S} \times \mathbb{R} \longrightarrow \mathbb{S} \times \mathbb{R}$, such that

$$(H_1 \circ {\Psi_2}^{-1})(x, y; \alpha, \beta) = H_2(x, y; \alpha, \beta),$$

where

$$H_2(x, y; \alpha, \beta) = \frac{1}{2}y^2 + U(x; \alpha, \beta) + O(|\alpha, \beta|^3).$$

Here the remainder $O(|\alpha, \beta|^3)$ is independent of y. Moreover, if X is $\mathcal{R}$-reversible or $\mathcal{R}$- and $\mathcal{S}$-reversible, H_2 is R-equivariant or R- and S-equivariant. The same holds with respect to $\mathcal{T}$.

Proof: Take $\Psi_2(x, y) = (x, y + y\psi(y; \alpha, \beta))$, for some formal power series ψ such that $\frac{1}{2}(y + y\psi(y; \alpha, \beta))^2 = \frac{1}{2}y^2 + O(|\alpha, \beta|^2 y^2)$, where the remainder is exactly the same as in H_1. Since $\psi(y; \alpha, \beta) = O(|\alpha, \beta|^2)$, Ψ_2 is formally invertible. Borel's theorem then yields a C^∞ transformation with these properties.

□

4 Bifurcation analysis in the plane

4.1 Scaling

We analyse the behaviour of the Hamiltonian $H_2(x, y; \alpha, \beta)$, or rather its truncation $N(x, y; \alpha, \beta) := \frac{1}{2}y^2 + U(x; \alpha, \beta)$. In the sequel we shall see that under a generic condition N is right equivalent to H_2, i.e. the $O(|\alpha, \beta|^3)$-term of H_2 has no qualitative influence.

For similar methods to study bifurcations in Hamiltonian systems, see [13, 11, 5, 6] or [9], and the references quoted there. For background on Hamiltonian bifurcations, compare Meyer [20].

First observe that $U(x; 0, 0) \equiv 0$, expressing a great degeneracy. To overcome this we apply a scaling to the coordinates (x, y) and parameters (α, β), simplifying N to a one-parameter Hamiltonian. Defining

$$\alpha = \bar{\beta}^2 \bar{\alpha}\ ,\ \beta = \bar{\beta}\ ,\ x = \bar{x}\ ,\ y = |\bar{\beta}|\bar{y}\ ,$$

and

$$\begin{aligned} \bar{N}(\bar{x}, \bar{y}; \bar{\alpha}) &= \beta^{-2} N(x, y; \alpha, \beta) \\ \bar{H}_2(\bar{x}, \bar{y}; \bar{\alpha}, \bar{\beta}) &= \beta^{-2} H_2(x, y; \alpha, \beta) \end{aligned}$$

we get

Lemma 4 (scaling)
The Hamiltonian $\bar{N}$ is independent of $\bar{\beta}$, and

$$\begin{aligned} \bar{N}(\bar{x}, \bar{y}; \bar{\alpha}) &= \frac{1}{2}\bar{y}^2 + \bar{U}(\bar{x}; \bar{\alpha}) \\ \textit{where } \bar{U}(\bar{x}; \bar{\alpha}) &= \frac{1}{2}(V'(\bar{x}))^2 - \bar{\alpha} V(\bar{x}) \\ \textit{Moreover, } \bar{H}_2(\bar{x}, \bar{y}; \bar{\alpha}, \bar{\beta}) &= \bar{N}(\bar{x}, \bar{y}; \bar{\alpha}) + O(\bar{\beta}) \end{aligned}$$

where $O(\bar{\beta})$ is independent of $\bar{y}$. The scaling is R-, S- and T-equivariant.

The proof runs by computation. Note that the scaling is well-defined outside an arbitrary small wedge in the (α, β)-plane of the form $|\alpha| > c\beta^2$, for some arbitrary large $c \in \mathbb{R}$. To simplify notation we omit all bars in the sequel.

4.2 The inverted pendulum

Let us return to the case of the inverted pendulum, i.e. we take $V(x) = 1 - \cos x$. Then the vector field X is $\mathcal{R}$- and $\mathcal{S}$-reversible, $\mathcal{T}$-equivariant, and consequently the truncated normalised Hamiltonian N is R-, S- and T-equivariant. It is given by $N = \frac{1}{2}y^2 + U(x;\alpha)$, in rescaled coordinates, with 'potential function'

$$U(x;\alpha) = \frac{1}{2}\sin^2 x + \alpha(\cos x - 1)$$

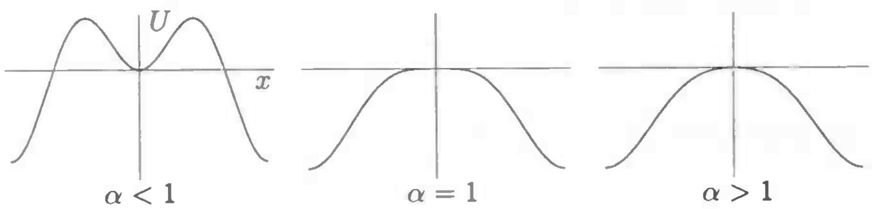

Figure 6: *'Potential energy' U of the inverted pendulum for $x \in \mathbb{S}$. For $\alpha = 1$ a $\mathbb{Z}_2$-equivariant cusp catastrophe occurs at $x = 0$.*

Because of the T-equivariance we only need to consider $\alpha \geq 0$. The 'potential function' is sketched in figure 6. A bifurcation diagram with phaseportraits is given in figure 7. At $(x, y; \alpha) = (0, 0; 1)$ the only bifurcation takes place; it is a Hamiltonian pitchfork bifurcation.

4.3 The $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case

Let us analyse the bifurcation of the pendulum at $(x, y; \alpha) = (0, 0; 1)$ more closely. The Taylor series of $U(x;\alpha)$ at $(x;\alpha) = (0;1)$ is given by

$$U(x;\alpha) = \frac{1}{2}(1-\alpha)x^2 - \frac{1}{8}x^4 + O(|x;\alpha - 1|^5). \qquad (6)$$

In the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric setting a universal model of the *cusp catastrophe* is given by

$$\frac{1}{2}y^2 - \lambda x^2 - x^4 \qquad (7)$$

A comparison of (6) and (7) suggests the following

Theorem 5 (Universal model in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case)
The one parameter family N at $(x, y; \alpha) = (0, 0; 1)$ is a universal $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant unfolding of the cusp singularity, i.e. up to a local equivariant right equivalence and a local reparametrization N is equal to (7).

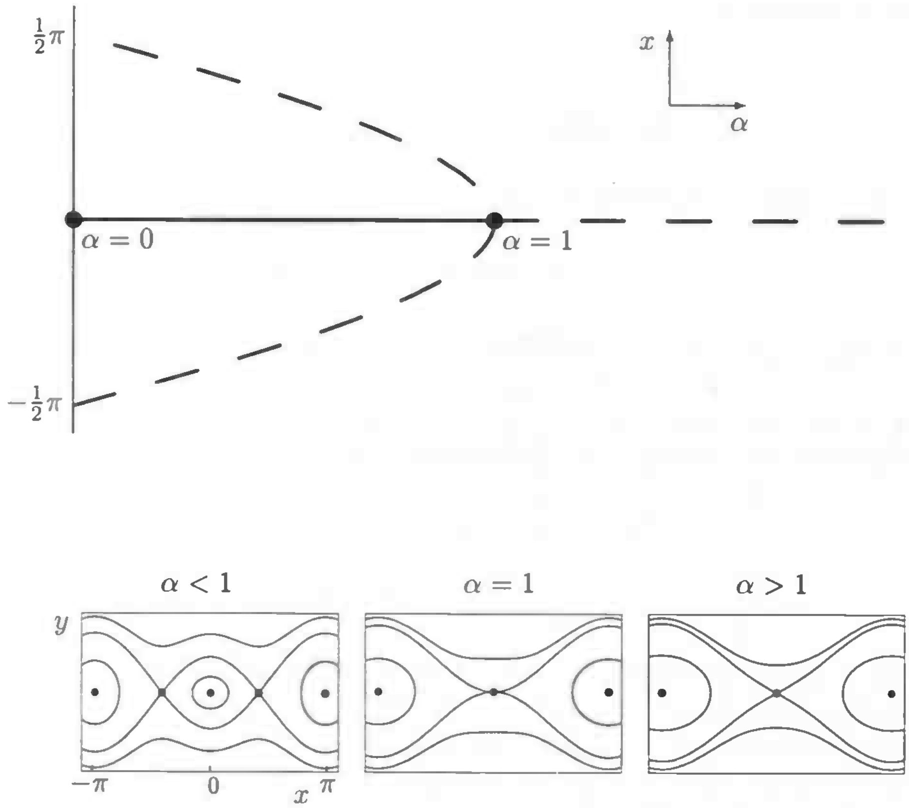

Figure 7: *The Hamiltonian N in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case. Top: bifurcation diagram in the (α, x)-plane. Dashed lines indicate unstable equilibria. Bottom: Corresponding global phase portraits. For $\alpha = 1$ a subcritical Hamiltonian pitchfork bifurcation occurs at $(x, y) = (0, 0)$.*

A proof can be found in Broer et al. [7, 8].

The behaviour of (7) is displayed in figures 8 and 9. Since H_2 is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant unfolding of (7), it has qualitatively the same behaviour. Moreover, the behaviour of (7) is persistent under any small $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant perturbation of V and ρ.

4.4 Breaking the S-symmetry: the $\mathbb{Z}_2$-symmetric case

In this section we discuss the effect of a perturbation of $V(x)$ on the behaviour of N. In general this will destroy the symmetries S and T, while the R-symmetry persists. We demand that $(x, y) = (0, 0)$ is persistent as 'upper equilibrium'. We restrict our attention to a small neighbourhood of $(x, y; \alpha) = (0, 0; 1)$. It turns out that (under generic conditions) the pitchfork bifurcation breaks apart into a *transcritical* and a *Hamiltonian saddle-node* bifurcation.

Let us consider perturbations of the form $V(x; \varepsilon) = V(x) + \varepsilon W(x)$, where $V(x) = 1 - \cos x$ as usual, and $W(x)$ is an arbitrary 2π-periodic function satisfying $W'(0) = 0$ and the generic condition $W'''(0) \neq 0$ — note that this implies that W is not S-equivariant.

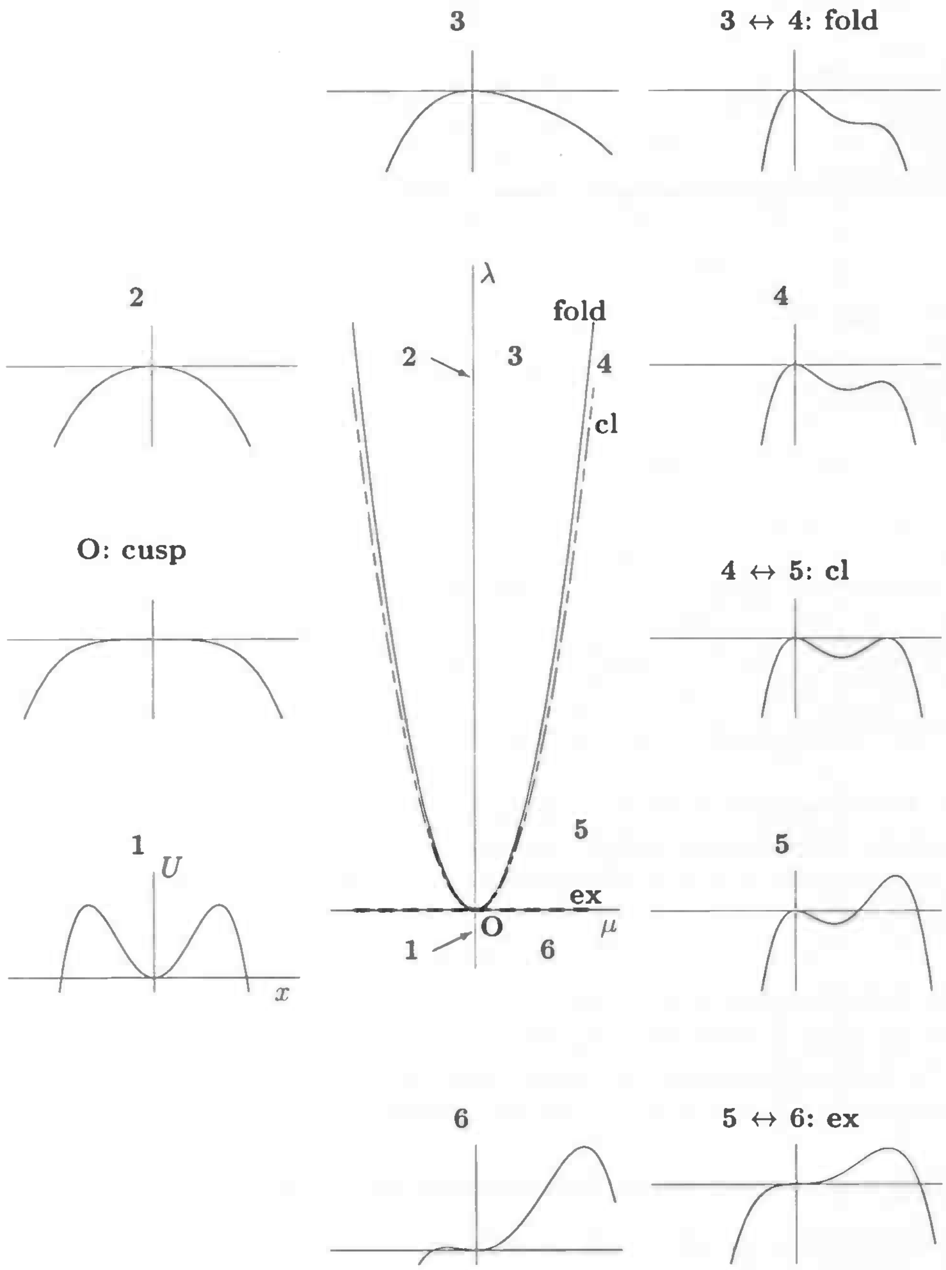

Figure 8: *'Potential energy' of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric (left column, corresponding to $\mu = 0$) and $\mathbb{Z}_2$-symmetric universal unfoldings ($\mu > 0$). In the middle the bifurcation diagram in the (μ, λ)-plane is displayed. The numbering in this diagram correspond to the numbers of the graphs. The coding is explained below.*

Applying the usual transformations to (1) with $V(x)$ replaced by $V(x;\varepsilon)$ yields:

$$\begin{aligned} H_2 &= N(x,y;\alpha,\varepsilon) + O(\beta) \\ \text{where } N(x,y;\alpha,\varepsilon) &= \frac{1}{2}y^2 + U(x;\alpha,\varepsilon) \\ U(x;\alpha,\varepsilon) &= \frac{1}{2}(\sin x + \varepsilon W'(x))^2 + \alpha(\cos x - 1 - \varepsilon W(x)) \end{aligned}$$

Again consider the Taylor expansion of $U(x;\alpha,\varepsilon)$:

$$U(x;\alpha,\varepsilon) = -\frac{1}{2}(1+\varepsilon W''(0))(\alpha-1-\varepsilon W''(0))x^2+\frac{1}{3}\varepsilon W'''(0)x^3-\frac{1}{8}x^4+O(|x;\alpha-1,\varepsilon|^5). \quad (8)$$

Under the condition that $x = 0$ is a critical point of U for all parameter values, a universal model of the *cusp catastrophe* in the $\mathbb{Z}_2$-symmetric setting is given by

$$\frac{1}{2}y^2 - \lambda x^2 + \mu x^3 - x^4 \ , \quad (9)$$

cf. Broer et al. [7, 8]. This leads to

Theorem 6 (Universal model in the $\mathbb{Z}_2$-symmetric case)
Suppose that $W'(0) = 0$ and $W'''(0) \neq 0$. Then, the two parameter family N at $(x,y;\alpha,\varepsilon) = (0,0;1,0)$ is a universal $\mathbb{Z}_2$-equivariant unfolding of the cusp singularity, within the context where $(x,y) = (0,0)$ is kept singular. This means that, up to a local right equivalence, respecting this structure, and a local reparametrization N is equal to (9).

A proof can again be found in [7, 8]. Since N and H_2 are unfoldings of (8), they have the same qualitative behaviour, see figures 8 and 9. Moreover, this behaviour is persistent under any small $\mathbb{Z}_2$-equivariant perturbation of ρ and V that keeps the upper equilibrium at $(x,y) = (0,0)$ and satisfies $W'''(0) \neq 0$.

In figures 8 and 9 we use the following coding:

ex: exchange bifurcation

cl: coinciding levels

tc: transcritical bifurcation

sn: (Hamiltonian) saddle-node bifurcation

pf: (Hamiltonian) pitchfork bifurcation

hb: heteroclinic bifurcation

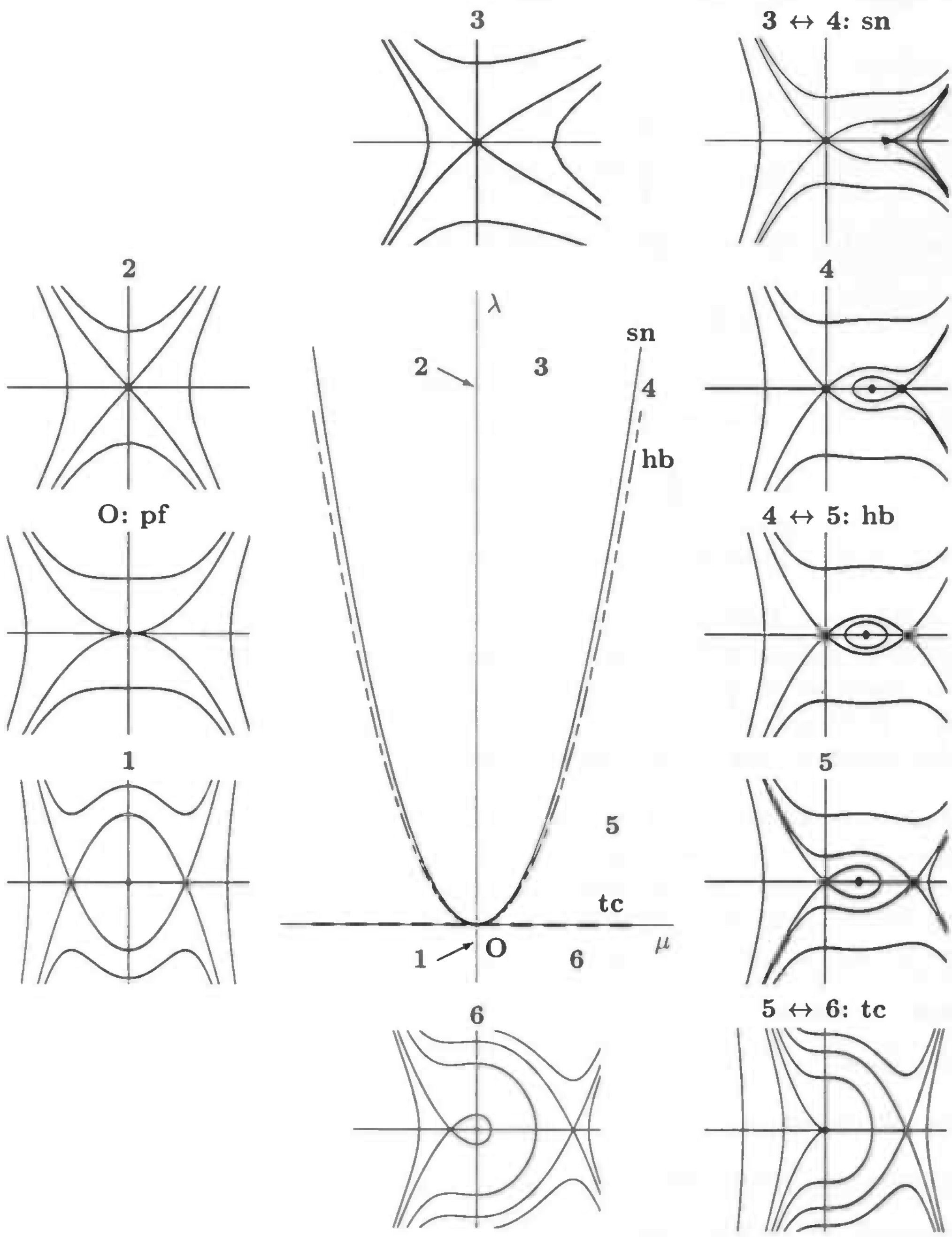

Figure 9: *Phase portraits of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric (left column) and $\mathbb{Z}_2$-symmetric universal unfoldings. In the middle the bifurcation diagram of figure 8 is repeated.*

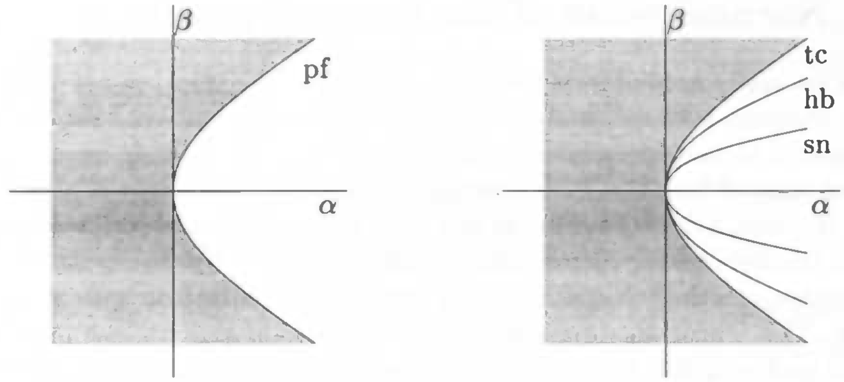

Figure 10: *Local stability diagram in the $\mathbb{Z}_2 \times \mathbb{Z}_2$- (left) and in the $\mathbb{Z}_2$-symmetric context (right). Stability of the origin (upper equilibrium) again is indicated by shading.*

5 Conclusions

Finally, let us rephrase the results of the previous section in terms of the original parameters.

In the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric case, the upper equilibrium is stable for $\alpha < \beta^2 + O(\beta^3, \varepsilon\beta^2)$; the boundary is a line of Hamiltonian pitchfork bifurcations. In the stable region, there are two saddle points going to $x = \pm\frac{1}{2}\pi$ as α goes to 0. By the S-symmetry, the invariant manifolds of these saddles coincide.

In the $\mathbb{Z}_2$-symmetric case, this bifurcation splits, as shown in figure 10; the stability boundary now corresponds to a line of transcritical bifurcations. The heteroclinic connection is lost, but there is a codimension one heteroclinic bifurcation.

Remarks:

1. Using the physical parameters introduced in section 2, the condition $\alpha < \beta^2$ for the origin to be stable can be translated to $\omega\ell < ||\dot{\sigma}||_2$, where $||\dot{\sigma}||_2$ is the $L^2_{2\pi/\Omega}$-norm of $\dot{\sigma}$; we used the Parseval relation here. Recall that σ is a $\frac{2\pi}{\Omega}$-periodic function governing the vertical oscillation of the suspension point, so, if the origin is unstable, it can be stabilised by e.g. increasing the frequency Ω or the amplitude of the parametric force (assuming that ℓ and ω are fixed).

2. If we abandon the condition that $V'(0) = 0$ and $W'(0) = 0$ we recover the 'ordinary' catastrophe with universal model

$$\frac{1}{2}y^2 - \mu x - \lambda x^2 - x^4,$$

(which is $\mathbb{Z}_2$-symmetric by its dependence on y). Also the evenness of ρ can be broken in which case we again expect an ordinary cusp catastrophe as universal model.

Appendix: Normalisation of the inverted pendulum

In this appendix we apply normal form theory to the case of the inverted pendulum. First the normal form theorem is stated, and then it is used to average out the time dependence of (1), for global x.

We consider a class of formal vector fields $\mathcal{H}$, and a given vector field $X \in \mathcal{H}$. Assume that $\mathcal{H} = \prod_{k\geq 0} \mathcal{H}_k$, where the $\mathcal{H}_k$ are linear and form a *graded Lie algebra*[1] with respect to the standard Lie bracket [,] for vector fields, defined by $[X, Y] = \mathrm{D}YX - \mathrm{D}XY$. Normal form theory provides a method to simplify X by transforming it into an appropriate linear subspace $\mathcal{G}$ of $\mathcal{H}$. This transformation can be chosen canonical and symmetry-preserving. The transformed vector field is called the *normalised vector field* or *normal form (of X)*. The subspace $\mathcal{G}$ of normal forms is to some extent determined by the $\mathcal{H}_0$-part of X.

A normal form theorem

For $L \in \mathcal{H}_0$, the *ad-operator* associated to L is the linear map $\mathrm{ad}L : \mathcal{H} \to \mathcal{H}$, $\mathrm{ad}L(Y) := [L, Y]$. Define $\mathrm{ad}_k L := \mathrm{ad}L|_{\mathcal{H}_k} : \mathcal{H}_k \longrightarrow \mathcal{H}_k$. We require the subspace $\mathcal{G}_k$ of $\mathcal{H}_k$ to satisfy $\mathrm{im}\ \mathrm{ad}_k L + \mathcal{G}_k = \mathcal{H}_k$. For details we refer to [4, 11, 24].

Theorem 7 (normal form) *Let $X \in \mathcal{H}$. Write $X = L + \sum_{j\geq 1} X_j$, where $L \in \mathcal{H}_0$ and $X_j \in \mathcal{H}_j$. Then there exists a formal transformation Ψ, and $G_j \in \mathcal{G}_j$, $j = 1, 2, \ldots$, such that*

$$\Psi_* X = L + \sum_{j\geq 1} G_j$$

We can take Ψ to be canonical, and to preserve the symmetries of X.

A proof is given in e.g. [11, 12, 23].

Since any vector field in $L + \prod_{k\geq 1} \mathcal{H}_k$ can be mapped into $\mathcal{G}$, we want to restrict $\mathcal{G}$ as much as possible.

Let the Jordan decomposition of $\mathrm{ad}_k L$ be $\mathrm{ad}_k L = \mathrm{ad}_k L_S + \mathrm{ad}_k L_N$, where $\mathrm{ad}_k L_S$ is semisimple and $\mathrm{ad}_k L_N$ is nilpotent. It is easily seen that $\ker \mathrm{ad}_k L_S + \mathrm{im}\ \mathrm{ad}_k L = \mathcal{H}_k$, so that we can take $\mathcal{G}_k = \ker \mathrm{ad}_k L_S$, but we can do better. The following result can be found in [4, 25].

Lemma 8 *If $\mathcal{G}_k = \ker \mathrm{ad}_k L_S \setminus \mathrm{im}\ \mathrm{ad}_k L_N$, then $\mathrm{im}\ \mathrm{ad}_k L + \mathcal{G}_k = \mathcal{H}_k$.*

In this context "\" should be interpreted as follows: for linear spaces A, B, C we write $C = A \setminus B$ if $(B \cap A) \oplus C = A$. Note that $A \setminus B$ is not uniquely defined.

Application to the inverted pendulum

In this section we sketch the proof of Theorem 1, using normal form theory. We want to take the spaces $\mathcal{H}_k$ such that the normal form is valid for all $x \in \mathbb{S}$. A good choice is

[1] This means that $[\mathcal{H}_k, \mathcal{H}_n] \subseteq \mathcal{H}_{k+n}$ for all $k, n \geq 0$, where [,] is the Lie bracket

to let $\mathcal{H}_k$, $k \in \mathbb{Z}$, $k \geq 0$ be the space of vector fields $Z(x,y,t;\alpha,\beta) = K(x,y,t;\alpha,\beta)\frac{\partial}{\partial t} + M(x,y,t;\alpha,\beta)\frac{\partial}{\partial x} + N(x,y,t;\alpha,\beta)\frac{\partial}{\partial y}$ on $\mathbb{S}\times\mathbb{R}\times\mathbb{S} = \{(x,y,t)\}$ with parameters $(\alpha,\beta) \in \mathbb{R}^2$, satisfying the following conditions:

1. The components K, M and N are homogeneous polynomials of degree k in (α,β) with coefficients that are 2π-periodic in x and t and formal power series in y.

2. $(x,y) = (0,0)$ is a stationary point of Z, i.e. $Z(0,0,t;\alpha,\beta) \equiv 0$.

The spaces $\mathcal{H}_k$ then form a Lie algebra. Let X be given by (1) for a general 2π-periodic V satisfying $V'(0) = 0$, then $X \in \mathcal{H}$ and $L = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x}$. The normalised vector field $\Psi_* X$ associated to X is of the form $L + \sum_{k\geq 1} G_k$, with $G_k \in \mathcal{G}_k$. We can take $\mathcal{G}_k = \ker \mathrm{ad}_k L_S \setminus \mathrm{im}\, \mathrm{ad}_k L_N = \ker \mathrm{ad}_k \frac{\partial}{\partial t} \setminus \mathrm{im}\, \mathrm{ad}_k y\frac{\partial}{\partial x}$. A calculation reveals that

$$\begin{aligned}
\mathcal{G}_k &= \mathrm{span}\left\{\alpha^m\beta^{k-m}h(y)\frac{\partial}{\partial x} : m = 0,1,\dots,k; h(y) = O(y) \text{ a formal power series}\right\} \\
&\oplus \mathrm{span}\left\{\alpha^m\beta^{k-m}e^{inx}\frac{\partial}{\partial y} : m = 0,1,\dots,k; n \in \mathbb{Z}\setminus\{0\}\right\}
\end{aligned}$$

Consequently we can write

$$\Psi_* X = L + yf(y;\alpha,\beta)\frac{\partial}{\partial x} + g(x;\alpha,\beta)\frac{\partial}{\partial y}$$

where f is a formal power series in (y,α,β), and g is a formal power series in (α,β), 2π-periodic in x. Both power series are of first order in (α,β). The transformation Ψ thus obtained is a formal power series in (α,β,y). By Borel's theorem, cf. [3, 11, 22] there exists a C^∞ transformation, defined on a neighbourhood of $(x,y;\alpha,\beta) = 0$, with the same structure-preserving properties. This is exactly the transformation we need in theorem 1.

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