Similarity to contractions

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Preface

This paper has been written as a master thesis to complete my study at the mathematics department of the University of Groningen.

I studied the problem of similarity to contractions, which has been studied before by a lot of mathematicians. So it wasn't difficult to collect enough data about this subject.

In the first chapter I have enumerated some important results of this century followed by a few examples of applying these results. The most important result is Paulsen's theorem about completely polynomially boundedness. That's why I proved this theorem in Chapter 2.

Of course, I supposed that the reader of this essay knows something about Hilbert and Banach spaces but I tried to be as complete as possible.

I wish to thank Prof.dr.ir. A. Dijksma for his enthousiastic supervision and the time he spent on this subject.

I hope you'll enjoy reading this essay.

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Chapter 1

This essay is about similarity to contractions. The problem is as follows:

When is an operator in a Hilbert space similar to a contraction in a Hilbert space?

The question is easy but the answer is quite difficult. There have already been many mathematicians who studied this problem and there have been found some elegant results.

1.1 Results

First we have to explain what we mean by similarity to an operator and what is called a contraction. All operators are considered in the same Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and are bounded.

Definition 1.1: An operator T in $\mathcal{H}$ is called *similar to an operator* T_1 in $\mathcal{H}$ if there exists an invertible operator $S : \mathcal{H} \to \mathcal{H}$ such that $T = ST_1S^{-1}$. By the notation $T \sim T_1$ we will mean that T is similar to the operator T_1.

Definition 1.2: An operator C in $\mathcal{H}$ is called a *contraction* if $\| C \| \leq 1$.

There is an equivalent statement:

Theorem 1.3: Let $T : \mathcal{H} \mapsto \mathcal{H}$ be an operator. The operator T is similar to a contraction iff there is an equivalent Hilbertian norm for which T is a contraction.

Proof: $(\Rightarrow)$ Let $T \sim C$ with C a contraction. Then there exists an invertible operator $S : \mathcal{H} \mapsto \mathcal{H}$ such that $T = S^{-1}CS$. Define $[u, v] = \langle Su, Sv\rangle$. This is an inner product and $[[u]]^2 = \| Su \|^2 \leq \| S \|^2 \| u \|^2$ so $[[u]] \leq \| S \| \| u \|$.
Also follows $\| u \|^2 = \| S^{-1}Su \|^2 \leq \| S^{-1} \|^2 \| Su \|^2 = \| S^{-1} \|^2 [[u]]^2$ so $\| u \| \leq \| S^{-1} \| [[u]]$.
Together these results show that[[ ]] and $\| \; \|$ are equivalent norms and $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space implies that $(\mathcal{H}, [\cdot,\cdot])$ is also a Hilbert space.
It remains to show that T is a contraction on $(\mathcal{H}, [\cdot,\cdot])$. This is easy to see:

$$\begin{aligned} [[Tu]]^2 &= [[S^{-1}CSu]]^2 = [S^{-1}CSu, S^{-1}CSu] \\ &= \langle CSu, CSu\rangle \leq \langle Su, Su\rangle = [u,u] \\ &= [[u]]^2 \end{aligned}$$

$(\Leftarrow)$ $[u, v]$ is an inner produkt on $\mathcal{H}$, continu in both variables:
$| [u, v] |^2 \leq [[u]][[v]] \leq M^2 \| u \| \| v \|$. Riesz Lemma tells us that there is a $G \in B(\mathcal{H})$ such that

$$[u, v] = \langle Gu, v\rangle$$

G is invertible and > 0:
$Gu = 0 \Rightarrow \langle Gu, u\rangle = 0 = [u, u] \Rightarrow u = 0$
$\langle Gu, u\rangle = [u, u] \geq 0$

so G is injective and $G > 0$.
$\langle Gu, v\rangle = [u,v] = \overline{[v,u]} = \overline{\langle Gv, u\rangle} = \langle u, Gv\rangle = \langle G^*u, v\rangle$ so $G = G^*$.
Take $v \perp \operatorname{ran} G$ then : $0 = \langle Gu, v\rangle = [u,v] \quad \forall u$ which implies that $v = 0$ and $\overline{\operatorname{ran}} \, G = \mathcal{H}$.
We claim that if $Gx_n \to y$ then $y \in \operatorname{ran} G$ i.e. $\exists \, x \in \mathcal{H}$ such that $y = Gx$ which means that G is surjective. This is proved as follows:
If $Gx_n \to y$ then Gx_n is Cauchy: $\forall v \ \langle Gx_n - Gx_m, v\rangle \to 0$ if $n, m \to \infty$. But $\langle Gx_n - Gx_m, v\rangle = [x_n - x_m, v]$ and then the theorem about weak convergence says $x_n \rightharpoonup x$ in $\mathcal{H}$ and $Gx_n \rightharpoonup Gx$. Since also $Gx_n \rightharpoonup y$ follows $Gx = y$.
Now we take $S = G^{1/2}$. Given is that T is a contraction with respect to $[\cdot,\cdot]$. Define $C = G^{1/2}TG^{-1/2}$ then $T \sim C$ and C is a contraction on $(\mathcal{H}, \langle\cdot,\cdot\rangle)$:

$$\begin{aligned}\langle Cx, Cx\rangle &= \langle G^{1/2}TG^{-1/2}x, G^{1/2}TG^{-1/2}x\rangle \\ &= \langle GTG^{-1/2}x, TG^{-1/2}x\rangle = [TG^{-1/2}x, TG^{-1/2}x] \\ &\leq [G^{-1/2}x, G^{-1/2}x] = \langle GG^{-1/2}x, G^{-1/2}x\rangle \\ &= \langle G^{-1/2}GG^{-1/2}x, x\rangle = \langle x, x\rangle\end{aligned}$$

In the history three notions play an important role:

Definition 1.4: An operator T is called *power bounded* (p.b.) if $\exists \, M$ such that for all $n \geq 0$

$$\| T^n \| \leq M$$

Definition 1.5: An operator T is called *polynomially bounded* (pol.b.) if $\exists \, M \geq 0$ such that $\forall$ polynomials $p(z)$

$$\| p(T) \| \leq M \sup_{|z|=1} | p(z) | = M \sup_{|z|\leq 1} | p(z) |$$

where the equality follows by the maximum modulus principle.

Definition 1.6: An operator T is called *completely polynomially bounded* (c.pol. b.) if $\exists \, M$ such that $\forall n$ and $\forall n \times n$ matrices $P(z) = (P_{ij})_{i,j=1}^n$ with polynomial entries

$$\| P(T) \|_{B(\mathcal{H}^n)} \leq M \sup_{|z|\leq 1} \| P(z) \|_{B(\mathbb{C}^n)}$$

where $\mathcal{H}^n$ is the Hilbert space $\{x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_i \in \mathcal{H}\}$ with inner product

$\langle x, y\rangle = \begin{pmatrix} \langle x_1, y_1\rangle \\ \vdots \\ \langle x_n, y_n\rangle \end{pmatrix}$ and

$$\| P(T) \|_{B(\mathcal{H}^n)} = \sup_{h \neq 0 \in \mathcal{H}^n} \frac{\| P(T)h \|}{\| h \|}$$

and $\forall \, z \in \mathbb{D} = \{x \mid \ | x | \leq 1\}$, $\| P(z) \|_{B(\mathbb{C}^n)} = \sup_{x \neq 0 \in \mathbb{C}} \frac{\|P(z)x\|_e}{\|x\|_e}$ where $\| \ \|_e$ is the Euclidian norm in $\mathbb{C}^n$.

Remark: Completely polynomially boundedness $\Rightarrow$ polynomially boundedness $\Rightarrow$ power boundedness. Indeed the first implication follows by taking $n = 1$ and the second by considering the polynomials $p(z) = z^n$.

These definitions lead us to three theorems:

Theorem 1.7: If T is similar to a contraction C, then T is p.b..

Theorem 1.8: If T is similar to a contraction C, then T is pol.b..

Theorem 1.9: If T is similar to a contraction C, then T is c.pol.b ..

By the above remark Theorems 1.7 and 1.8 follow from Theorem 1.9, but we shall prove each theorem separately.

Proof of Theorem 1.7: This is easy to see:
$T \sim C$ means there is S such that $T = SCS^{-1}$ hence $T^n = SC^nS^{-1}$ and

$$\begin{aligned} \| T^n \| = \| SC^nS^{-1} \| &\leq \| S \| \| C^n \| \| S^{-1} \| \\ &\leq \| S \| \| S^{-1} \| \| C \|^n \leq \| S \| \| S^{-1} \| \quad \forall n \; = \; 0, 1, 2, \ldots \end{aligned}$$

which means that $\| T^n \| \leq \| S \| \| S^{-1} \| = M \;\; \forall\, n$.

Proof of Theorem 1.8: This is an application of *von Neumann's inequality* which is the following:
if C a contraction in $\mathcal{H}$ then $\forall$ polynomials $p(z)$

$$\| p(C) \| \leq \sup_{|z|=1} | p(z) |$$

The proof is included in Appendix A. T is similar to a contraction C so there is an S such that $T = SCS^{-1}$ hence $p(T) = Sp(C)S^{-1}$ and

$$\begin{aligned} \| p(T) \| &\leq \| S \| \| p(C) \| \| S^{-1} \| \\ &\leq \| S \| \| S^{-1} \| \sup_{|z|=1} | p(z) | = \; M \; \sup_{|z|=1} | p(z) | \end{aligned}$$

with $M = \| S \| \| S^{-1} \|$.

Proof of Theorem 1.9: By the dilation theorem (see Appendix A) there is a unitary operator U on a Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$ such that U is a unitary dilation of T.

Let us denote by $\mathcal{C}$ (resp. $\mathcal{A}(\mathbb{D})$) the space of all continuous functions on $\partial\mathbb{D}$, $\mathcal{C} = \{f : \partial\mathbb{D} \mapsto \mathbb{C} \mid f(z) \text{ cont } \}$ (resp. the closed linear span in $\mathbb{C}$ of the functions $\{e^{int} \mid n \geq 0\}$, $\mathcal{A}(\mathbb{D}) = \text{closure}\{\sum_{n=0}^{k} a_n e^{int} \mid k = 0, 1, 2, \ldots a_n \in \mathbb{C}\}$). We equip $\mathcal{C}$ (or $\mathcal{A}(\mathbb{D})$) with the sup norm which we denote by $\| \; \|_\infty$: $\| f \|_\infty = \sup_{|z|=1} | f(z) |$. Note that $\mathcal{A}(\mathbb{D})$ is a subalgebra of $\mathcal{C}$, it is called the disc algebra.

$\mathcal{C}$ is a C^*-algebra (see Appendix B).
$f \in \mathcal{C}$ can be identified with the multiplication operator $M_f : L^2(\partial\mathbb{D}) \to L^2(\partial\mathbb{D})$, $M_f u \; = \; fu$ and N. Young [11] proved that there holds

Lemma 1: $\| f \|_\infty = \| M_f \|_{B(L^2(\partial\mathbb{D}))}$.

$F \epsilon \mathcal{M}_n(\mathcal{C}) = \{F = (f_{ij})_{i,j=1}^n \mid f_{ij} \epsilon \mathcal{C}\}$ can be interpreted as the linear map $F : (L^2(\partial\mathbb{D}))^n \mapsto (L^2(\partial\mathbb{D}))^n$ given by $(Fu)_i = \sum_{j=1}^n M_{f_{ij}} u_j,\ i = 1, \ldots, n$, where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \epsilon (L^2(\partial\mathbb{D}))^n$. With this interpretation $\mathcal{M}_n(\mathcal{C})$ becomes a C^*-algebra with norm

$$\| F \|_{B((L^2(\partial\mathbb{D}))^n)} = \sup_{u \neq 0\ \epsilon (L^2(\partial\mathbb{D}))^n} \frac{\sqrt{\frac{1}{2\pi}\int_0^{2\pi} \| (F(e^{i\varphi}))u(e^{i\varphi}) \|_e^2 \, d\varphi}}{\sqrt{\frac{1}{2\pi}\int_0^{2\pi} \| u(e^{i\varphi}) \|_e^2 \, d\varphi}}$$

where $\| \, \|_e$ again is the Euclidian norm in $\mathbb{C}^n$ like in Definition 1.6.

Lemma 2: $\| F \|_{B(L^2(\partial\mathbb{D})^n)} \leq \sup_{\varphi \epsilon [0,2\pi]} \| (F(e^{i\varphi})) \|_{B(\mathbb{C}^n)}$
$= \sup_{|z|=1} \| (F(z)) \|_{B(\mathbb{C}^n)}$

Proof: $\| (F(e^{i\varphi}))u(e^{i\varphi}) \|_e^2 \leq \| F(e^{i\varphi}) \|_{B(\mathbb{C}^n)}^2 \| u(e^{i\varphi}) \|_e^2$
$\leq \sup_{|z|=1} \| (F(z)) \|_{B(\mathbb{C}^n)}^2 \| u(e^{i\varphi}) \|_e^2$.

$$\begin{aligned} &\| F \|_{B((L^2(\partial\mathbb{D}))^n)} \\ &\leq \sup_{u \neq 0} \frac{\sup_{|z|=1} \| (F(z)) \|_{B(\mathbb{C}^n)} \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \| u(e^{i\varphi}) \|_e^2 \, d\varphi}}{\sqrt{\frac{1}{2\pi}\int_0^{2\pi} \| u(e^{i\varphi}) \|_e^2 \, d\varphi}} \\ &= \sup_{|z|=1} \| (F(z)) \|_{B(\mathbb{C}^n)} \end{aligned}$$

Let $U \epsilon B(\mathcal{H})$ be unitary. The polynomials $p(z, \bar{z})$ in z and $\bar{z}$ are dense in $\mathcal{C}$ (Stone-Weierstraβ).
$u_U : p(z, \bar{z}) \mapsto p(U, U^*)$ is linear and bounded and we have $u_U(pq) = u_U(p)u_U(q),\ u_U(\bar{p}) = (u_U(p))^*$
Boundedness follows from:

$$(*) \quad \| u_U(p(z, \bar{z})) \| = \| p(U, U^*) \|_{B(\mathcal{H})} \leq \sup_{|z|=1} | p(z, \bar{z}) |$$

(because $U = \int_0^{2\pi} e^{it} dE_t,\ U^n = \int_0^{2\pi} e^{int} dE_t,\ U^{*n} = \int_0^{2\pi} e^{-int} dE_t$, so $\| p(U, U^*) \| = \| \int_0^{2\pi} p(e^{i\varphi}, e^{-i\varphi}) dE_t \| \leq \sup_{|z|=1} | p(z, \bar{z}) |$).
So if $p_n(z, \bar{z}) \to f(z)$ in $\mathcal{C}$ then $p_n(U, U^*)$ is convergent in $B(\mathcal{H})$. Indeed because $(*)\ \| p_n(U, U^*) - p_m(U, U^*) \|_{B(\mathcal{H})} \leq \sup_{|z|=1} | p_n(z, \bar{z}) - p_m(z, \bar{z}) | < \varepsilon \quad \forall\, n, m \geq N(\varepsilon) \quad (p_n \to f)$ so $p_n(U, U^*)$ is Cauchy in $B(\mathcal{H})$ and $B(\mathcal{H})$ is complete so $p_n(U, U^*)$ is convergent. We define

$$(**) \quad f(U) = \lim_{n \to \infty} p_n(U, U^*) \quad \text{in } B(\mathcal{H})$$

We obtain a $*$-representation

$$u_U : \mathcal{C} \mapsto B(\mathcal{H})$$

with $u_U(f) = f(U)$ such that $u_U(\bar{f}) = u_U(f)^*$ and $u_U(fg) = u_U(f)u_U(g)$.
This is checked as follows:
$u_U(\bar{f}) = u_U(\lim_{n\to\infty}\bar{p}) = \lim_{n\to\infty}\overline{p(z,\bar{z})}_{|_{z=U,\bar{z}=U^*}} = \sum \bar{a}_{kj}U^{*k}U^j = (\sum a_{kj}U^kU^{*j})^* = (\lim_{n\to\infty} p(z,\bar{z})_{|_{z=U,\bar{z}=U^*}})^* = (u_U(\lim_{n\to\infty} p))^* = u_U(f)^*$ and
$u_U(fg) = u_U(\lim_{n\to\infty} p_n \lim_{n\to\infty} q_n) = \lim_{n\to\infty} u_U(p_nq_n) = \lim_{n\to\infty}$ $u_U(p_n)u_U(q_n) = u_U(f)u_U(g)$ and this defines a $*$-representation on $\mathcal{C}(\partial\mathbb{D})$ (see Appendix B).
About $*$-representations we have the following Lemma:

Lemma 3: Let $\rho : A \mapsto B(\mathcal{H})$ be a $*$-representation on a C^*-algebra A and assume A has a unit. Then necessarily $\| \rho \| = \sup_{a\neq 0\ \epsilon A} \frac{\|\rho(a)\|_{\mathcal{H}}}{\|a\|} \leq 1$.

For the proof see Appendix B.

Now to matrices.
Let $\tilde{u}_U : \mathcal{M}_n(\mathcal{C}) \mapsto B(\mathcal{H}^n)$ be defined by
$\tilde{u}_U(F(z)) = F(U) = (f_{ij}(U))_{i,j=1}^n \qquad (F(z) = (f_{ij}(z))_{i,j=1}^n)$
We have seen on page 6 that $\mathcal{M}_n(\mathcal{C})$ is a C^*-algebra. $\tilde{u}_U$ is a $*$-representation so $\| \tilde{u}_U \| \leq 1$ or

$$\| F(U) \|_{B(\mathcal{H}^n)} \leq \| F \|_{B((L^2(\partial\mathbb{D}))^n)} \leq \sup_{|z|=1} \| (f_{ij}(z)) \|_{B(\mathbb{C}^n)}$$

$(\| \tilde{u}_U \| = \sup \frac{\|\tilde{u}_U(F)\|}{\|F\|} = \sup \frac{\|F(U)\|}{\|F\|} \leq 1)$.
What we wanted to prove is if $T \,\epsilon\, B(\mathcal{H})$ and $T \sim C$ where C is a contraction then T is completely polynomially bounded (c.pol.b.) i.e.
$\exists\, M$ such that for all n and all $n \times n$ matrices $P = (P_{ij})$ with polynomial entries we have

$$\| P(T) \|_{B(\mathcal{H}^n)} \leq M \sup_{|z|\leq 1} \| P(z) \|_{B(\mathbb{C}^n)}$$

This can be proved as follows:
$P(T) = (P_{ij}(T))_{i,j=1}^n = (P_{ij}(S^{-1}CS))_{i,j=1}^n = \begin{pmatrix} S^{-1} & & \\ & \ddots & \\ & & S^{-1} \end{pmatrix}$
$(P_{ij}(C)) \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix}$ and by the dilation theorem ($C^n = P_{\mathcal{H}}U^n_{|_{\mathcal{H}}}$) this becomes
$= \begin{pmatrix} S^{-1} & & \\ & \ddots & \\ & & S^{-1} \end{pmatrix} \begin{pmatrix} P_{\mathcal{H}} & & \\ & \ddots & \\ & & P_{\mathcal{H}} \end{pmatrix} P(U)\,|_{B(\mathcal{H}^n)} \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix}$.
Then $\| P(T) \| \leq \| S^{-1} \| \cdot 1 \cdot \| P(U)_{|_{B(\mathcal{H}^n)}} \| \| S \| \leq \| S^{-1} \| \| P_{ij}(U) \|$ $\| S \|$. We have proved above $\| F(U) \| \leq \sup_{|z|=1} \| (f_{ij}(z)) \|_{B(\mathbb{C}^n)}$ and we apply this result to $F = P$.
So we get $\| P(T) \| \leq \| S^{-1} \| \| S \| \sup_{|z|=1} \| (P_{ij}(z)) \|_{B(\mathbb{C}^n)}$.
If we define $M := \| S^{-1} \| \| S \|$ we see that T is c.pol.b..

Now we go back to the history of similarity to contractions.

Already in 1946 B. Sz.-Nagy proved the following theorem:

Theorem 1.10: Let T be a linear transformation in Hilbert space $\mathcal{H}$ such that its powers T^n $(n = 0, \pm 1, \pm 2, \ldots)$ are defined everywhere in $\mathcal{H}$ and are uniformly bounded, i.e. $\| T^n \| \leq k$ for some constant k. Then there exists a selfadjoint transformation Q such that

$$\frac{1}{k} I \leq Q \leq kI$$

and QTQ^{-1} is a unitary transformation.

This means that T is similar to a unitary operator U. The question arises:

What remains if only half of the condition holds, T is p.b.?

T is not similar to a unitary operator, because then T^{-1} is similar to a unitary operator which means T and T^{-1} are p.b.. B.Sz.-Nagy proved that if T is p.b. and compact then T is similar to a contraction. So with some extra conditions T is similar to a contraction. However if T only is p.b., it does not hold in general. In 1964 S.R. Foguel gave an example of an operator, in a Hilbert space, with uniformly bounded powers which is not similar to a contraction [3] so the converse of Theorem 1.7 does not hold in general.

Lebow showed that Foguel's example is not polynomially bounded. This lead P.R. Halmos to ask in [2] (problem 6) the following question:

Is every polynomially bounded operator similar to a contraction?

The answer is no. In 1997 G. Pisier gave a very complicated example of a polynomially bounded operator which is not similar to a contraction [6]. So the converse of Theorem 1.8 is not true either.

However the converse of Theorem 1.9 is true. In 1984 V.I. Paulsen was the first who proved this converse [4]. In 1996 G. Pisier gave a different proof [9]. This is included in Chapter 2.

1.2 Examples

Now we go back to Theorem 1.7. There are some interesting cases for which the converse is true. For the first example we recall Theorem 1.10.

Example 1: Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces and $T \in B(\mathcal{H})$. Then $W \in B(\mathcal{G})$ is called a *dilation* of T if
(a) $\mathcal{H} \subset \mathcal{G}$ is a closed subspace
(b) $T^n = P_{\mathcal{H}} W^n_{|_{\mathcal{H}}}$ $\forall\, n \geq 0$. This is equivalent with: there exist 2 Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that

$$W = \begin{pmatrix} W_{11} & * & * \\ 0 & T & * \\ 0 & 0 & W_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \\ \mathcal{H}_2 \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \\ \mathcal{H}_2 \end{pmatrix}$$

and $\mathcal{G} = \mathcal{H}_1 \oplus \mathcal{H} \oplus \mathcal{H}_2$.
Now the following statements are equivalent:
(i) $T \sim C$ with C a contraction

(ii) $\exists$ dilation W of T with W invertible and W and W^{-1} are power bounded.
(i) $\Rightarrow$ (ii) $T \sim C$ means $\exists S$ such that $T = S^{-1}CS$. The dilation theorem in Appendix A tells us that C has a unitary dilation U or in other words

$$C = P_{\mathcal{H}} \begin{pmatrix} U_{11} & * & * \\ 0 & C & * \\ 0 & 0 & U_{22} \end{pmatrix}\Bigg|_{\mathcal{H}} \quad \text{with } U = \begin{pmatrix} U_{11} & * & * \\ 0 & C & * \\ 0 & 0 & U_{22} \end{pmatrix}$$

Then define

$$\begin{aligned} W &= \begin{pmatrix} I & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} U_{11} & * & * \\ 0 & C & * \\ 0 & 0 & U_{22} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} U_{11} & * & * \\ 0 & S^{-1}CS & * \\ 0 & 0 & U_{22} \end{pmatrix} = \begin{pmatrix} U_{11} & * & * \\ 0 & T & * \\ 0 & 0 & U_{22} \end{pmatrix} \end{aligned}$$

so W is a dilation of T.
As you can see W is invertible and

$$W^{\pm n} = \begin{pmatrix} I & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} U^{\pm n} \begin{pmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{pmatrix}.$$

$\| U^{\pm n} \| < M$ so $\| W^{\pm n} \| < N$ which means that W and W^{-1} are power bounded.
(ii) $\Rightarrow$ (i). Let W be a dilation of T with W invertible and W and W^{-1} are power bounded. By Theorem 1.10 there is a selfadjoint operator Q such that $U = QWQ^{-1}$ is a unitary transformation and $\frac{1}{k}I \le Q \le kI$ or in other words W is similar to a unitary operator U on $\mathcal{G}$:

$$W = Q^{-1}UQ$$

W is a dilation of T so there exist 2 Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that

$$W = \begin{pmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \\ \mathcal{H}_2 \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \\ \mathcal{H}_2 \end{pmatrix}$$

and $\mathcal{G} = \mathcal{H}_1 \oplus \mathcal{H} \oplus \mathcal{H}_2$. Then

$$\begin{pmatrix} * & * \\ 0 & T \end{pmatrix} = Q^{-1}UQ_{|\mathcal{H}_1 \oplus \mathcal{H}}$$

We define $Q_1 := Q_{|\mathcal{H}_1 \oplus \mathcal{H}} : \mathcal{H}_1 \oplus \mathcal{H} \mapsto \operatorname{ran} Q_1$. Then U maps $\operatorname{ran} Q_1$ into itself and $Q_1^{-1} := Q^{-1} : \operatorname{ran} Q_1 \mapsto \mathcal{H}_1 \oplus \mathcal{H}$ so we have

$$\begin{pmatrix} * & * \\ 0 & T \end{pmatrix} = Q_1^{-1}U_1Q_1 : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \end{pmatrix}$$

where $U_1 := U_{|_{\operatorname{ran} Q_1}}$ is an isometry. We see that $T = Q_1^{-1}U_1Q_{1|\mathcal{H}}$ hence $T^* = (Q_1^{-1}U_1Q_{1|\mathcal{H}})^* = (Q_1^{-1}U_1Q_1)^*_{|\mathcal{H}} = Q_1^*U_1^*(Q_1^*)^{-1}_{|\mathcal{H}}$. Let $Q_2 = (Q_1^*)^{-1}_{|\mathcal{H}}$:

$\mathcal{H} \mapsto \operatorname{ran} Q_2$ then $Q_2 T^* = U_1^* Q_2$ implies that $T_2 := U^*_{1|_{\operatorname{ran} Q_2}}$ is a contraction from $\operatorname{ran} Q_2$ into itself and we have $T^* = Q_2^{-1} T_2 Q_2$. Finally, let $Q_2 = U_0 \mid Q_2 \mid$ be the polar decomposition of Q_2 where U_0 is unitary and $\mid Q_2 \mid$ acts on $\mathcal{H}$. Then $T^* = \mid Q_2 \mid^{-1} U_0^* T_2 U_0 \mid Q_2 \mid$ and if we set $S = \mid Q_2 \mid^{-1}$ and $T_0 = U_0^* T_2^* U_0$ we see that T_0 is a contration on $\mathcal{H}$ and so $T = S^{-1} T_0 S$ is similar to a contraction.

Example 2: Let T in $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be expansive, i.e. $\| Tx \| \geq \| x \|$ and let C be a contraction. Then $T \sim C \iff T$ is p.b. and C is isometric.
($\Rightarrow$) is always true (see Theorem 1.7).
($\Leftarrow$) $\| x \|^2 \leq \| Tx \|^2 \leq \| T^2 x \|^2 \leq \cdots \leq \| T^n x \|^2 \leq M \| x \|^2$ and $\| T^n x \|$ is an increasing sequence bounded from above so $\lim_{n\to\infty} \| T^n x \|$ exists.
Define $[x,y] = \lim_{n\to\infty} \langle T^n x, T^n y \rangle$. The polarisation formula shows that this limit exists:

$$\langle T^n x, T^n y \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \| T^n (x + i^k y) \|^2 < \infty$$

$[x,y]$ is in fact an inner product and $[[\ ]]$ and $\|\ \|$ are equivalent norms:

$$[[x]]^2 = \lim_{n\to\infty} \| T^n x \|^2 \leq M \| x \|^2$$

and

$$[[x]]^2 = \lim_{n\to\infty} \| T^n x \|^2 \geq \| x \|^2$$

($\| T^n x \|$ is increasing, take $n = 0$).
Also follows $[Tx, Tx] = \lim_{n\to\infty} \| T^n T x \|^2 = \lim_{n\to\infty} \| T^n x \|^2 = [x,x]$ which means that for the norm $[[\ ]]$ T is a contraction and isometric. By Theorem 1.3 it follows that for the norm $\|\ \|$ T is similar to a contraction which we wanted to prove.

Example 3: Let $T \in B(\mathcal{H})$ be a Jordan matrix in $\mathbb{C}^p$.
Then $T \sim C \iff T$ is p.b.
($\Rightarrow$) is always true (see Theorem1.7).
($\Leftarrow$) Let J be a Jordan matrix in $\mathbb{C}^p$ with eigenvalue λ:

$$J = \begin{pmatrix} \lambda & 1 & & \bigcirc \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \bigcirc & & & \lambda \end{pmatrix}$$

Then $J^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & & \bigcirc \\ & \lambda^2 & 2\lambda & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 2\lambda \\ \bigcirc & & & & \lambda^2 \end{pmatrix}$, $J^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & & \cdot \\ & \ddots & \ddots & \\ & & \ddots & 3\lambda^2 \\ \bigcirc & & & \lambda^3 \end{pmatrix}$

and so $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & & \cdot \\ & \ddots & \ddots & \\ & & \ddots & n\lambda^{n-1} \\ \bigcirc & & & \lambda^n \end{pmatrix}$.

Let (e_i) be the usual orthonormal basis.

$$\| J^n e_2 \|^2 = \| \begin{pmatrix} n\lambda^{n-1} \\ \lambda^n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \|^2 = | n\lambda^{n-1} |^2 + | \lambda^{2n} |.$$

We distinguish 4 different cases:
$| \lambda | > 1 : \| J^n e_2 \| \to \infty$ for $n \to \infty$ by $| \lambda^{2n} |$
$| \lambda | = 1$ and $p > 1 : \| J^n e_2 \| \to \infty$ for $n \to \infty$ by $| n\lambda^{n-1} |$
$| \lambda | = 1$ and $p = 1 :$ $J^n = \lambda^n$ and this is bounded
$| \lambda | < 1 : \| J^n \| < M \quad \forall n$
So a Jordan block is p.b. $\Longleftrightarrow | \lambda | < 1, p \geq 1$ or $| \lambda | \leq 1, p = 1$.

If $p = 1$ $J : \mathbb{C} \mapsto \mathbb{C}$, $J = \lambda$ is similar to a contraction because $| \lambda | \leq 1$.
Now for $p > 1$, $J = \lambda I + S \quad | \lambda | < 1$.
Then $J^n = J(\lambda)^n = (\lambda I + S)^n = \sum_{k=0}^{p} (\lambda I)^{n-k} S^k \binom{n}{k}$ where $p = n - 1$ and $\lim \| J(\lambda)^k \|^{1/k} = r(J(\lambda)) \leq 1$ where $r(J(\lambda))$ is the spectral radius: $r(J(\lambda)) = \max | \sigma(J(\lambda)) | = | \lambda | < 1$.
So $\exists k_0$ such that $\forall k \geq k_0 \quad \| J(\lambda)^k \|^{1/k} \leq r < 1$ and $\| J(\lambda)^k \| \leq r^k$.

Define $[x, y] = \sum_{k=0}^{\infty} \langle J(\lambda)^k x, J(\lambda)^k y \rangle$ an inner product op $\mathbb{C}^n$.
Then

$$\begin{aligned}
| \sum_{k=0}^{\infty} \langle J(\lambda)^k x, J(\lambda)^k y \rangle | &\leq \sum_{k=0}^{\infty} \| J(\lambda)^k x \| \| J(\lambda)^k y \| \\
&\leq \sum_{k=0}^{\infty} \| J(\lambda)^k \| \| x \| \| J(\lambda)^k \| \| y \| \\
&\leq \sum_{k=0}^{k_0} \| J(\lambda)^k \|^2 \| x \| \| y \| + \sum_{k=k_0+1}^{\infty} r^{2k} \| x \| \| y \| \\
&\leq K \ \| x \| \| y \|
\end{aligned}$$

so $[[x]] \leq K \| x \|$ and $[[x]]^2 = \sum_{k=0}^{\infty} \| J(\lambda)^k x \|^2 \geq \| J(\lambda)^0 x \|^2 = \| x \|^2$.
This means that $[[\ ]]$ and $\| \ \|$ are equivalent norms.
Also

$$\begin{aligned}
[[J(\lambda)x]]^2 &= \sum_{k=0}^{\infty} \langle J(\lambda)^k J(\lambda) x, J(\lambda)^k J(\lambda) x \rangle \\
&\leq \sum_{k=0}^{\infty} \langle J(\lambda)^k x, J(\lambda)^k x \rangle = [[x]]^2
\end{aligned}$$

which means that for the norm $[[\ ]]$ $J(\lambda)$ is a contraction. By Theorem 1.3 it follows that for the norm $\| \ \|$ $J(\lambda)$ is similar to a contraction and so is T.

We mentioned before B. Sz.-Nagy's example if T is p.b. and compact then T is similar to a contraction, but we are not going to prove this.

There is also an application of Theorem 1.9 by B. Sz.-Nagy and C. Foias [10].

Example 4: Let $T \in B(\mathcal{H})$. Assume $\exists\ \tilde{\mathcal{H}}$ and $U \in B(\tilde{\mathcal{H}})$ unitary and $\exists\ \rho \geq 1$ such that $T^n = \rho\ P_{\mathcal{H}} U^n_{|\mathcal{H}}\quad \forall n$ where $P_{\mathcal{H}}$ is the orthogonal projection of $\tilde{\mathcal{H}}$ onto $\mathcal{H}$. (This is called a ρ-dilation)
Then T is similar to a contraction C.

We will show T is c.pol.b. then by Paulsen's criterion about the converse of Theorem 1.9 which is also true follows that $T\ \sim\ C$.
Let $P(z)$ be a $n \times n$ matrix with polynomial entries. Then $P(T) - P(0) = \rho \begin{pmatrix} P_{\mathcal{H}} & & \\ & \ddots & \\ & & P_{\mathcal{H}} \end{pmatrix} (P(U) - P(0))_{|\mathcal{H}^n}$ and

$$P(T) = \rho \begin{pmatrix} P_{\mathcal{H}} & & \\ & \ddots & \\ & & P_{\mathcal{H}} \end{pmatrix} P(U)_{|\mathcal{H}^n} + (1-\rho) \begin{pmatrix} P_{\mathcal{H}} & & \\ & \ddots & \\ & & P_{\mathcal{H}} \end{pmatrix} P(0)_{|\mathcal{H}^n}.$$

From this follows

$$\begin{aligned} &\| P(T) \|_{B(\mathcal{H}^n)} \\ &\quad \leq \rho\ \| P_{\mathcal{H}} \| \| P(U) \|_{B(\mathcal{H}^n)} + \mid 1-\rho \mid \| P_{\mathcal{H}} \| \| P(0) \|_{B(\mathcal{H}^n)} \\ &\quad \leq \rho\ \| P(U) \|_{B(\tilde{\mathcal{H}}^n)} + \mid 1-\rho \mid \| P(0) \|_{B(\mathcal{H}^n)} \\ &\quad \leq \rho \sup_{|z|\leq 1} \| P(z) \|_e + \mid 1-\rho \mid \| P(0) \|_e \\ &\quad \leq (\rho + \mid 1-\rho \mid) \sup_{|z|\leq 1} \| P(z) \|_e \end{aligned}$$

where $\|\ \|_e$ again is the Euclidian norm in $\mathbb{C}^n$. This means that T is c.pol.b..

Chapter 2

In this chapter we are going to prove that the converse of Theorem 1.9 is also true.

Theorem 2.1: $T \sim C \iff T$ is c.pol.b.

Proof: ($\Rightarrow$) See chapter 1, the proof of Theorem 1.9.
($\Leftarrow$) We will need some theory about completely bounded maps and completely bounded homomorphisms.

2.1 Completely bounded maps

We will start by mentioning the Hahn-Banach theorem:

Theorem 2.2: (Hahn-Banach) Let Λ be a real vector space. Let $\rho : \Lambda \mapsto \mathbb{R}$ be a sublinear map, i.e. a map such that

$$\forall\, x, y \in \Lambda \qquad p(x+y) \leq p(x) + p(y)$$

$$\forall\, x \in \Lambda \;\; \forall\, t \geq 0 \qquad p(tx) = tp(x)$$

Then there is a $\mathbb{R}$-linear functional $f : \Lambda \mapsto \mathbb{R}$ such that

$$\forall\, x \in \Lambda \quad f(x) \leq p(x)$$

Corollary 2.3: Let Λ_+ be a convex cone in a real vector space Λ. Let $q : \Lambda_+ \mapsto \mathbb{R}$ be a superlinear map i.e. a map such that

$$\forall\, x, y \in \Lambda_+ \qquad q(x) + q(y) \leq q(x+y)$$

$$\forall\, x \in \Lambda_+ \;\; \forall\, t \geq 0 \qquad q(tx) = tq(x)$$

Let $p : \Lambda \mapsto \mathbb{R}$ be a sublinear map. If $q(x) \leq p(x)$ for all x in Λ_+ then there is a $\mathbb{R}$-linear functional $f : \Lambda \mapsto \mathbb{R}$ such that

$$\forall\, x \in \Lambda_+ \quad q(x) \leq f(x)$$

$$\forall\, x \in \Lambda \quad f(x) \leq p(x)$$

Proof: Let $r(x) = \inf\{p(x+y) - q(y) \mid y \in \Lambda_+\}$ for $x \in \Lambda$. Then r is sublinear: $r(tx) = \inf\{p(tx+y) - q(y) \mid y \in \Lambda_+\} = \inf\{tp(x+\frac{1}{t}y) - tq(\frac{1}{t}y) \mid y \in \Lambda_+\} = \inf\{tp(x+z) - tq(z) \mid z \in \frac{1}{t}\Lambda_+ = \Lambda_+\} = t \inf\{p(x+z) - q(z) \mid z \in \Lambda_+\} = tr(x) \quad \forall\, t \geq 0$ and
$p(x+y) - q(y) + p(z+v) - q(v) \geq p(x+z+y+v) - q(y+v) = p(x+z+w) - q(w) \geq r(x+z) \quad \forall\, y, v \in \Lambda_+$ and $w = y + v$. Now we can take the infimum on the left side over $y \in \Lambda_+$ and $v \in \Lambda_+$:
$r(x) + r(z) = \inf\{p(x+y) - q(y) \mid y \in \Lambda_+\} + \inf\{p(z+v) - q(v) \mid v \in \Lambda_+\} \geq r(x+z)$.
Also follows $r(x) = \inf\{p(x+y) - q(y) \mid y \in \Lambda_+\} \leq p(x+0) - q(0) = p(x)$ and $-p(-x) = -p(-x) - p(y) + p(y) \leq p(y) - p(-x+y) \leq p(y) - q(-x+y)$ if we take y arbitrary but so that $-x + y \in \Lambda_+$. The inequality holds for

all $-x+y \in \Lambda_+$ so we can take the infumum:
$-p(-x) \leq \inf\{p(y) - q(-x+y) \mid -x+y \in \Lambda_+\} = \inf\{p(x+z) - q(z) \mid z \in \Lambda_+\} = r(x)$
Together these results give:

$$(2.1) \qquad -p(-x) \leq r(x) \leq p(x)$$

which means that $r(x)$ is finite $\forall\, x \in \Lambda$.
$r(-y) = \inf\{p(-y+z) - q(z) \mid z \in \Lambda_+\} \leq p(-y+y) - q(y) = -q(y) \quad \forall\, y \in \Lambda_+$.
By the Hahn-Banach theorem there is a linear functional $f : \Lambda \mapsto \mathbb{R}$ such that $f(x) \leq r(x)$ for all $x \in \Lambda$. By (2.1) follows $f(x) \leq p(x)$ for all $x \in \Lambda$ and $-f(y) = f(-y) \leq r(-y) \leq -q(y)$ for all $y \in \Lambda_+$. This yields the announced result.

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. Let $S \subset B(\mathcal{H})$ be a subspace. For any $n \geq 1$ we denote by $\mathcal{M}_n(S)$ the space of all $n \times n$ matrices (a_{ij}) with coefficients in S with the norm

$$\| (a_{ij}) \|_{\mathcal{M}_n(S)} = \sup \left(\sum_i \| \sum_j a_{ij} x_j \|^2 \right)^{1/2}$$

where the supremum runs over all $x_1, \ldots, x_n$ in $\mathcal{H}$ such that $\sum \| x_j \|^2 \leq 1$.
Let $u : S \mapsto B(\mathcal{K})$ then we define $u_n : \mathcal{M}_n(S) \mapsto \mathcal{M}_n(B(\mathcal{K}))$ by $u_n((a_{ij})) = (u(a_{ij}))$ for $(a_{ij}) \in \mathcal{M}_n(S)$. Then u is called completely bounded (in short c.b.) if there is a constant K such that the maps u_n are uniformly bounded by K i.e. if we have

$$\sup_{n \geq 1} \| u_n \|_{\mathcal{M}_n(S) \mapsto \mathcal{M}_n(B(\mathcal{K}))} \leq K$$

and the c.b. norm $\| u \|_{cb}$ is defined as the smallest constant K for which this holds.
When $\| u \|_{cb} \leq 1$, we say that u is completely contractive (or a complete contraction).

It is quite straightforward to extend the usual definitions to the Banach space case as follows. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. We denote by $B(\mathcal{X}, \mathcal{Y})$ the space of all bounded operators from $\mathcal{X}$ into $\mathcal{Y}$, equipped with the usual operator norm. Let $\mathcal{X}_1$, $\mathcal{Y}_1$ be an other couple of Banach spaces. Let $S \subset B(\mathcal{X}_1, \mathcal{Y}_1)$ be a subspace and let $u : S \mapsto B(\mathcal{X}, \mathcal{Y})$ be a linear map. Let us define $\| (a_{ij}) \|_{\mathcal{M}_n(S)}$ in the same way and $u_n : \mathcal{M}_n(S) \mapsto \mathcal{M}_n(B(\mathcal{X}, \mathcal{Y}))$ by $u_n((a_{ij})) = (u(a_{ij}))$. We will say again that u is c.b. if the maps u_n are uniformly bounded and we define

$$\| u \|_{cb} = \sup_{n \geq 1} \| u_n \|$$

The following theorem is a fundamental factorization of c.b. maps.

Theorem 2.4: Let $\mathcal{H}$ be a Hilbert space and let $S \subset B(\mathcal{H})$ be a subspace. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Let $u : S \mapsto B(\mathcal{X}, \mathcal{Y})$ be a c.b. map. Then there is a Hilbert space $\hat{\mathcal{H}}$, a $*$-representation $\pi : B(\mathcal{H}) \mapsto B(\hat{\mathcal{H}})$ with $\pi(1) = 1$ and operators $V_1 : \mathcal{X} \mapsto \hat{\mathcal{H}}$ and $V_2 : \hat{\mathcal{H}} \mapsto \mathcal{Y}$ with $\| V_1 \| \| V_2 \| \leq \| u \|_{cb}$ such that

$$(2.2) \qquad \forall\, a \in S \quad u(a) = V_2 \pi(a) V_1$$

Conversely, any map of the form (2.2) satisfies

$$\| u \|_{cb} \leq \| V_2 \| \| V_1 \|$$

Formula (2.2) is easier to understand if you look at the following diagram:

$$\begin{array}{ccc} \hat{\mathcal{H}} & \overset{\pi(a)}{\longrightarrow} & \hat{\mathcal{H}} \\ V_1 \uparrow & & \downarrow V_2 \\ \mathcal{X} & \overset{u(a)}{\longrightarrow} & \mathcal{Y} \end{array}$$

We know π has special properties:
(i) π is defined on all of $B(\mathcal{H})$
(ii) π is a $*$-representation
(iii) $\pi(1) = 1$
We can also say: "$u(a)$ looks like a piece of $\pi(a)$".

For the proof of Theorem 2.4 we will introduce some notations. Let $a \in S$ and let I be the space $B(\mathcal{X}, \mathcal{H})$. Let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$, $\mathcal{X}^* = \{\eta : \mathcal{X} \mapsto \mathbb{C} \mid \eta$ linear $\}$ and let $S \otimes \mathcal{X}$ be their algebraic tensor product . If $\sum_{i=1}^n a_i \otimes x_i \in S \otimes \mathcal{X}$ and $\sum_{k=1}^m h_k \otimes \eta_k \in \mathcal{H} \otimes \mathcal{X}^*$ then we define

$$(2.3) \qquad \langle \sum_{i=1}^n a_i \otimes x_i, \sum_{k=1}^m h_k \otimes \eta_k \rangle \overset{\text{def}}{=} \sum_{i,k} \eta_k(x_i) a_i(h_k) \qquad \in \mathcal{H}$$

where $a_i(h_k) \in \mathcal{H}$ and $\eta_k(x_i) \in \mathbb{C}$.

Remark: If $\langle \sum_{i=1}^n a_i \otimes x_i, \sum_{k=1}^m h_k \otimes \eta_k \rangle = 0 \quad \forall\ (\sum h_k \otimes \eta_k)$ then follows $\sum_{i=1}^n a_i \otimes x_i = 0$. Indeed, if $z = \sum_{i=1}^n a_i \otimes x_i$ we may suppose that (x_i) are linearly independent:
Assume $x_1 = b_2 x_2 + \cdots + b_n x_n$ then

$$z = a_1 \otimes x_1 + \sum_{i=2}^n a_i \otimes x_i = \sum_{i=2}^n (a_i + b_i a_1) \otimes x_i$$

so $z = \sum_{i=2}^n c_i \otimes x_i$ with $x_2, \ldots, x_n$ linearly independent.
There exists an $\hat{\eta} \in \mathcal{X}^*$ such that $\hat{\eta}(x_1) = 1$ and $\hat{\eta}(x_i) = 0$ for $i = 2, \ldots, n$ and $0 = \langle \sum_{i=1}^n a_i \otimes x_i, h \otimes \hat{\eta} \rangle = \sum_i \hat{\eta}(x_i) a_i(h) = a_1(h) \quad \forall\ h \in \mathcal{H}$. This implies that $a_1(h) = 0 \ \forall\ h \in \mathcal{H}$ so $a_1 : \mathcal{H} \mapsto \mathcal{H}$ is the 0- operator. We can do the same for $a_2, \ldots, a_n$.
So if $\langle \sum_{i=1}^n a_i \otimes x_i, \sum_{k=1}^m h_k \otimes \eta_k \rangle = 0 \quad \forall (\sum_k h_k \otimes \eta_k)$ then

$$z = \sum_{i=1}^n a_i \otimes x_i = 0 \otimes \sum_{i=1}^n x_i = 0$$

Now for $\xi \in I$ and $z = \sum_{i=1}^n a_i \otimes x_i \in S \otimes \mathcal{X}$ we define $\xi. : S \otimes \mathcal{X} \mapsto \mathcal{H}$ as

$$\xi.z = \sum_{i=1}^n a_i \xi(x_i) \qquad \in \mathcal{H}$$

where $\xi(x_i) \in \mathcal{H}$.

Lemma 2.5: Assume $x_1, \dots, x_n$ are linearly independent in $\mathcal{X}$ and $z \in S \otimes \mathcal{X}$ has the property:
$\xi \in I$ and $\xi(x_i) = 0$ for $i = 1, \dots, n$ implies $\xi.z = 0$
then $\exists\, a_j \in S$ such that

$$z = \sum_{j=1}^{n} a_j \otimes x_j$$

Proof: This is checked as follows:
Take $z = \sum_{k=1}^{m} b_k \otimes u_k \in S \otimes \mathcal{X}$. We are going to prove

$$z' := z - \sum_{j=1}^{n} a_j \otimes x_j = 0$$

Choose $x_j^* \in \mathcal{X}^*$ such that $x_j^*(x_i) = \delta_{ij}$ (i.e. $x_j^*(x_i) = 1$ for $i = j$ and $x_j^*(x_i) = 0$ for $i \neq j$). Define

$$a_j = \sum_{k=1}^{m} b_k x_j^*(u_k) \qquad \in S$$

Then $z' = \sum_{k=1}^{m} b_k \otimes u_k - \sum_{j=1}^{n} a_j \otimes x_j = \sum_{k=1}^{m} b_k \otimes u_k - \sum_{k=1}^{m} \sum_{j=1}^{n} x_j^*(u_k)$ $b_k \otimes x_j$. Choose $\eta' \in \mathcal{X}^*$ and $y \in \mathcal{H}$. Form $\eta = \eta' - \sum_{j=1}^{n} \eta'(x_j) x_j^* \in \mathcal{X}^*$. Define $\xi \in I$ with y in $\mathcal{H}$ arbitrary by

$$\xi(x) = \eta(x) y$$

Then follows $\xi(x_i) = \eta(x_i) y = \left(\eta'(x_i) - \sum \eta'(x_j) x_j^*(x_i)\right) y = (\eta'(x_i) - \eta'(x_i)) y = 0 \cdot y = 0 \quad \forall\, x_i$. This implies $\xi.z = 0$ as we assumed i.e.

$$0 = \xi.z = \sum_{k=1}^{m} b_k \eta(u_k) y = \sum_{k=1}^{m} \eta(u_k) b_k(y)$$

and

$$\begin{aligned}
\langle z', y \otimes \eta' \rangle &= \langle \sum_k b_k \otimes u_k - \sum_k \sum_j x_j^*(u_k) b_k \otimes x_j, y \otimes \eta' \rangle \\
&= \sum_k \eta'(u_k) b_k(y) - \sum_k \sum_j x_j^*(u_k) \eta'(x_j) b_k(y) \\
&= \sum_k \eta(u_k) b_k(y) + \sum_k \sum_j \eta'(x_j) x_j^*(u_k) b_k(y) \\
&\qquad - \sum_k \sum_j x_j^*(u_k) \eta'(x_j) b_k(y) \\
&= \sum_k \eta(u_k) b_k(y) = 0
\end{aligned}$$

And then by the Remark follows $z' = 0$.

Lemma 2.6: Let $(z_i)_{i \leq n}$ be a finite sequence in $S \otimes \mathcal{X}$ and let $(x_i)_{i \leq m}$ be a finite sequence in $\mathcal{X}$. Then

$$\text{(2.4)} \qquad \sum_i \| \xi.z_i \|_{\mathcal{H}}^2 \leq \sum_j \| \xi(x_j) \|_{\mathcal{H}}^2 \qquad \forall\, \xi \in I$$

holds iff there is a matrix (a_{ij}) in $\mathcal{M}_n(S)$ with $\| (a_{ij}) \|_{\mathcal{M}_n(S)} \leq 1$ such that

$$z_i = \sum_{j=1}^{m} a_{ij} \otimes x_j \qquad \forall\, i = 1, 2, \ldots, n$$

Proof: Assume (2.4). If $\xi \in I$ then $\xi(x_i) = 0 \quad \forall\, i = 1, \ldots, n$ implies $\xi . z_i = 0 \quad \forall\, i = 1, \ldots, n$, so we can apply Lemma 2.5: $\exists\, K = (k_{ij}) \in S$ such that

$$z_i = \sum_j k_{ij} \otimes x_j \quad \forall\, i = 1, \ldots, n$$

In general this K does not satisfy $\| K \|_{\mathcal{M}_n(S)} \leq 1$. So we replace K by one that has this property.

Define $E \overset{\text{def}}{=} \{x^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x^*(x_1) \\ \vdots \\ x^*(x_n) \end{pmatrix} \mid x^* \in \mathcal{X}^*\} \subset \mathbb{C}^n$ and let $P = (P_{jk})_{j,k=1}^n$ be the orthogonal projection on E. Then it follows

$$x^*(P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}) = P x^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \forall\, x^*$$

because $x^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in E$ so $P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

If $\sum_j a_j x_j = 0$ then $(a_1 \cdots a_n) P = (0 \cdots 0)$. Indeed, $\sum_j a_j x^*(x_j) = (a_1 \cdots a_n) x^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$ but x^* is arbitrary, hence

$$(a_1 \cdots a_n) P \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \quad \forall\, y_i$$

which implies $(a_1 \cdots a_n) P = (0 \cdots 0)$.

There also holds

$$(0 \cdots 0) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1 \cdots a_n) P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1 \cdots a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \textstyle\sum_i a_i x_i$$

so

$$\sum_i a_i x_i = 0 \iff (a_1 \cdots a_n) P = (0 \cdots 0)$$

Now define $\tilde{E} \overset{\text{def}}{=} \{\xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \xi(x_1) \\ \vdots \\ \xi(x_n) \end{pmatrix} \mid \xi \in I\} \subset \mathcal{H}^n$.

We claim $\tilde{E} = R := \{ \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^n \mid \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \}$.

$P\xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \xi(P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}) = \xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ so $\tilde{E} \subset R$.

Now we claim that also $R \subset \tilde{E}$. Assume $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^n$ and $P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$. We want to construct a $\xi \in I$ such that

$$\xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.$$

Therefore we define $\gamma : span(x_1, \ldots, x_n) \mapsto span(h_1, \ldots, h_n)$ such that $\gamma(\sum_i a_i x_i) = \sum_i a_i h_i$ (especially $\gamma(x_1) = h_1, \ldots, \gamma(x_n) = h_n$).
$\sum_i a_i x_i = 0$ implies $(a_1 \cdots a_n)P = (0 \cdots 0)$ like we have seen before so

$$(a_1 \cdots a_n) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = (a_1 \cdots a_n)P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = 0$$

and this means that γ is well defined ($\gamma(0) = 0$).
From the definition it follows that γ is linear and surjective. Let W be a subspace of $span(x_1, \ldots, x_n)$ such that $span(x_1, \ldots, x_n)$ is the direct sum $span(x_1, \ldots, x_n) = W + \ker\gamma$. Then $\gamma_{|W} : W \mapsto span(h_1, \ldots, h_n)$ is a bijective map.
Choose $(v_1, \ldots, v_m)$ a basis of $span(h_1, \ldots, h_n)$ with $m = \dim W \leq n$ and $w_1, \ldots, w_m$ in W such that $\gamma(w_i) = v_i$. Then is $(w_1, \ldots, w_m)$ a basis of W. Choose $(w_{m+1}, \ldots, w_r)$ a basis of $\ker\gamma$ with $r \leq n - m$ then $(w_1, \ldots, w_m, w_{m+1}, \ldots, w_r)$ is a basis of $span(x_1, \ldots, x_n) \subset \mathcal{X}$.
Take $w_j^* \in \mathcal{X}^*$ such that $w_j^*(w_i) = \delta_{ij}$ and define $\xi \in I$ by

$$\xi(x) = \sum_{j=1}^{m} w_j^*(x) v_j \qquad \in I$$

This means $\xi(w_i) = v_i \;\; \forall\, i = 1, \ldots, m$ and $\xi(w_i) = 0 \;\; \forall\, i = m+1, \ldots, r$ but also $\gamma(w_j) = v_j \;\; \forall\, j = 1, \ldots, m$ and $\gamma(w_j) = 0 \;\; \forall\, j = m+1, \ldots, r$ and ξ and γ are both linear. $(w_1, \ldots, w_r)$ is a basis of $span(x_1, \ldots, x_n)$ so

$$\xi_{|span(x_1, \ldots, x_n)} = \gamma$$

with $\xi(x_i) = \gamma(x_i) = h_i \quad \forall\, i = 1, \ldots, n$ and this proves the above claim.
Take $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in R = \tilde{E}$ then $\exists\, \xi \in I$ such that $P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \xi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.
Now we want to show that $\| A \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \| \leq \| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|$ for an $A = (a_{ij})_{i,j=1}^{n}$ because this implies $\| A \|_{B(\mathcal{H}^n)} \leq 1$.
We have seen before that $z_i = \sum_{j=1}^{n} k_{ij} \otimes x_j$ and because $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ we have

$$\begin{aligned} \sum_{j=1}^{n} k_{ij} \otimes x_j &= \sum_{j=1}^{n} k_{ij} \otimes \sum_{l=1}^{n} P_{jl} x_l \\ &= \sum_{l=1}^{n} \left(\sum_{j=1}^{n} k_{ij} P_{jl} \right) \otimes x_l = \sum_{l=1}^{n} (KP)_{il} \otimes x_l \end{aligned}$$

Define $A = (a_{il})_{i,l=1}^{n} = KP$ then

$$z_i = \sum_{l=1}^{n} a_{il} \otimes x_l$$

We assumed (2.4): $\sum_{i=1}^{n} \| \xi . z_i \|^2 \leq \sum_{l=1}^{n} \| \xi(x_l) \|^2$. This implies

$$\begin{aligned} \| AP \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2 = \| A \begin{pmatrix} \xi(x_1) \\ \vdots \\ \xi(x_n) \end{pmatrix} \|^2 &= \sum_{i=1}^{n} \| \sum_{l=1}^{n} a_{il}\xi(x_l) \|^2 \\ &\leq \sum_{l=1}^{n} \| \xi(x_l) \|^2 = \| \begin{pmatrix} \xi(x_1) \\ \vdots \\ \xi(x_n) \end{pmatrix} \|^2 = \| P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2 \end{aligned}$$

and $AP = KPP = KP^2 = KP = A$ because P is a projection which means

$$\| A \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2 \leq \| P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2$$

$\| P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2 = \langle P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \rangle = \langle P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \rangle \leq \| P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \| \| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|$ so

$$\| P \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \| \leq \| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|$$

Applying this result we get

$$\| A \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2 \leq \| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \|^2$$

which means $\| A \|_{B(\mathcal{H}^n)} \leq 1$.
This shows the "only if" part. The "if" part is easy. If there is a matrix (a_{ij}) in $\mathcal{M}_n(S)$ with $\| (a_{ij}) \|_{\mathcal{M}_n(S)} \leq 1$ such that $\forall\, i = 1, 2, \ldots, n$

$$z_i = \sum_j a_{ij} \otimes x_j$$

then

$$\begin{aligned} \sum_i \| \xi . z_i \|^2 &= \sum_i \| \sum_j a_{ij}\xi(x_j) \|^2 \\ &\leq \| (a_{ij}) \|^2_{\mathcal{M}_n(S)} \sum_j \| \xi(x_j) \|^2 \leq \sum_j \| \xi(x_j) \|^2 \end{aligned}$$

Proof of Theorem 2.4: Let $C = \| u \|_{cb}$ and $\Lambda = \{\phi : I \mapsto \mathbb{R} \mid \exists\, x_1, \ldots, x_n \,\epsilon\, \mathcal{X}$ s.t. $| \phi(\xi) | \leq \sum \| \xi(x_i) \|^2 \quad \forall\, \xi \,\epsilon\, I\}$. Clearly Λ is a real vector space and

Λ is not empty. For example take $x_0 \in \mathcal{X}$ and define ϕ by $\phi(\xi) = \| \xi(x_0) \|^2$. Then $\phi \in \Lambda$.
Let $\Lambda_+ = \{\phi \in \Lambda \mid \phi \geq 0\}$. The preceding example is also suitable for Λ_+ so Λ_+ is not empty either.
We define $\hat{u} : S \otimes \mathcal{X} \mapsto \mathcal{Y}$ as follows:
Let $z = \sum_{i=1}^{n} a_i \otimes x_i \in S \otimes \mathcal{X}$ then

$$\hat{u}(z) = \sum_{i=1}^{n} u(a_i)x_i \quad \in \mathcal{Y}$$

for $u : S \mapsto B(\mathcal{X}, \mathcal{Y})$.
Now we define

$$\forall \phi \in \Lambda \quad p(\phi) = \inf\{C^2 \sum \| x_i \|^2 \mid x_i \in \mathcal{X},\ \phi(\xi) \leq \sum \| \xi(x_i) \|^2,\ \forall \xi \in I\}$$

and

$$\forall \phi \in \Lambda_+ \quad q(\phi) = \sup\{\sum \| \hat{u}(z_i) \|^2 \mid z_i \in S \otimes \mathcal{X},\ \sum \| \xi . z_i \|^2 \leq \phi(\xi),\ \forall \xi \in I\}$$

The set in the definition of p is not empty because we can take the example $\phi(\xi) = \| \xi(x_0) \|^2$ for $x_0 \in \mathcal{X}$ again and $C^2 \sum \| x_i \|^2 \geq 0$ so $p(\phi) \geq 0$. The set in the definition of q is not empty because $z_i = 0 \otimes x_i$ satisfies $\sum \| \xi . z_i \|^2 = \sum \| 0\xi(x_i) \|^2 = 0 \leq \phi(\xi) \quad \forall \xi \in I$ and $\sum \| \hat{u}(z_i) \|^2 = \sum \| u(0)x_i \|^2 = 0$ is an element of this set . $q(\phi) < \infty$ because by Lemma 2.6 we have for $(z_i)_{i=1}^m \in S \otimes \mathcal{X}$ and $(x_j)_{j=1}^n \in \mathcal{X}$

$$\sum_i \| \xi . z_i \|^2 \leq \sum_j \| \xi(x_j) \|^2 \Rightarrow \sum \| \hat{u}(z_i) \|^2 \leq C^2 \sum \| x_j \|^2$$

(if $m < n$ make a n-vector of z by supplying zero's at the end: $(z_1, \ldots, z_m, 0, \ldots, 0)$ and do the same for x if $n < m$).
Indeed if $\sum_i \| \xi . z_i \|^2 \leq \sum_j \| \xi(x_j) \|^2$ then by Lemma 2.6 there is a matrix (a_{ij}) in $\mathcal{M}_n(S)$ with $\| (a_{ij}) \|_{\mathcal{M}_n(S)} \leq 1$ such that

$$z_i = \sum_j a_{ij} \otimes x_j \quad \forall i = 1, 2, \ldots, m$$

and if $u = u_n$ for (a_{ij}) is a $n \times n$ matrix

$$\begin{aligned}
\sum_i \| \hat{u}(z_i) \|^2 &= \sum_i \| \hat{u}(\textstyle\sum_j a_{ij} \otimes x_j) \|^2 = \sum_i \| \sum_j u(a_{ij})x_j \|^2 \\
&= \sum_i \| \textstyle\sum_j u_n(a_{ij})x_j \|^2 = \| u_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \|^2 \\
&\leq \| u_n \|^2 \| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \|^2 \leq \sup_{n \geq 1} \| u_n \|^2 \| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \|^2 \\
&= \| u \|_{cb}^2 \| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \|^2 = C^2 \textstyle\sum_j \| x_j \|^2
\end{aligned}$$

This implies that $q(\phi) < \infty$ and also $q(\phi) \leq p(\phi)$ for all $\phi \in \Lambda_+$.
p is subadditief on Λ:
if $\phi(\xi) \leq \sum \| \xi(x_i) \|^2$ and $\psi(\xi) \leq \sum \| \xi(y_i) \|^2 \quad \forall \xi \in I$ then $(\phi + \psi)\xi =$

$\phi(\xi)+\psi(\xi)\leq\sum\|\ \xi(x_i)\ \|^2+\sum\|\ \xi(y_i)\ \|^2 \quad \forall\,\xi\,\epsilon\, I$ and $p(\phi+\psi)\leq C^2\sum\|\ x_i\ \|^2+C^2\sum\|\ y_i\ \|^2$ so we can take the infimum on the right side and we get:

$$\begin{aligned}
p(\phi+\psi) \ &\leq \ \inf\{C^2\sum\|\ x_i\ \|^2|\ x_i\,\epsilon\,\mathcal{X},\phi(\xi)\leq\sum\|\ \xi(x_i)\ \|^2,\ \forall\,\xi\}\\
&+\inf\{C^2\sum\|\ y_i\ \|^2|\ y_i\,\epsilon\,\mathcal{X},\psi(\xi)\leq\sum\|\ \xi(y_i)\ \|^2,\ \forall\,\xi\}\\
&=p(\phi)+p(\psi)
\end{aligned}$$

Assume $\phi(\xi)\leq\sum\|\ \xi(x_i)\ \|^2 \quad \forall\,\xi$. Then $\forall\, t>0$:
$t\phi(\xi)\leq\sum\|\ \xi(\sqrt{t}x_i)\ \|^2$
and $p(t\phi)\leq C^2\sum\|\ \sqrt{t}x_i\ \|^2=t\ C^2\sum\|\ x_i\ \|^2 \quad \forall\, x_i$ so it also holds for the infimum:

$$p(t\phi)\ \leq\ t\ \inf\{C^2\sum\|\ x_i\ \|^2|\ x_i\,\epsilon\,\mathcal{X},\ \phi(\xi)\leq\sum\|\ \xi(x_i)\ \|^2,\ \forall\,\xi\}\ =\ tp(\phi)$$

On the other hand $\forall\, t>0$:

$$tp(\phi)\ =\ tp(\frac{1}{t}t\phi)\ \leq\ t\frac{1}{t}p(t\phi)\ =\ p(t\phi)$$

Both results give $tp(\phi)\ =\ p(t\phi) \quad \forall\, t>0$.
For $t=0$, $x_i=0\ \ \forall\, i$ satisfies $0\leq\sum\|\ \xi(x_i)\ \|^2 \quad \forall\,\xi\,\epsilon\, I$ so $p(0)=0$ which implies that $p(t\phi)=tp(\phi)$ holds also for $t=0$.
q is superadditief on Λ_+:
if $\sum\|\ \xi.z_i\ \|^2\leq\phi(\xi)$ and $\sum\|\ \xi.w_i\ \|^2\leq\psi(\xi) \quad \forall\,\xi\,\epsilon\, I$ then $(\phi+\psi)\xi=\phi(\xi)+\psi(\xi)\geq\sum\|\ \xi.z_i\ \|^2+\|\ \xi.w_i\ \|^2 \quad \forall\,\xi\,\epsilon\, I$ and $q(\phi+\psi)\geq\sum\|\ \hat{u}.z_i\ \|^2+\sum\|\ \hat{u}.w_i\ \|^2$ so we can take the supremum on the right side and we get:

$$\begin{aligned}
q(\phi+\psi)&\geq\sup\{\sum\|\ \hat{u}(z_i)\ \|^2|\ z_i\,\epsilon\, S\otimes\mathcal{X},\ \sum\|\ \xi.z_i\ \|^2\leq\phi(\xi),\ \forall\,\xi\}\\
&+\sup\{\sum\|\ \hat{u}(w_i)\ \|^2|\ w_i\,\epsilon\, S\otimes\mathcal{X},\ \sum\|\ \xi.w_i\ \|^2\leq\phi(\xi),\ \forall\,\xi\}\\
&=q(\phi)+q(\psi)
\end{aligned}$$

Assume $\sum\|\ \xi.z_i\ \|^2\leq\phi(\xi) \quad \forall\,\xi$. Then $\forall\, t\geq 0$:
$\sum\|\ \xi.\sqrt{t}z_i\ \|^2\leq t\phi(\xi)$
and $q(t\phi)\geq\sum\|\ \hat{u}(\sqrt{t}z_i)\ \|^2=t\sum\|\ \hat{u}(z_i)\ \|^2 \quad \forall\, x_i$
so it also holds for the supremum:

$$q(t\phi)\ \geq\ t\sup\{\sum\|\ \hat{u}(z_i)\ \|^2|\ z_i\,\epsilon\, S\otimes\mathcal{X},\ \sum\|\ \xi.z_i\ \|^2\leq\phi(\xi),\ \forall\,\xi\}\ =\ tq(\phi)$$

On the other side $\forall\, t>0$:

$$tq(\phi)\ =\ tq(\frac{1}{t}t\phi)\ \geq\ t\frac{1}{t}q(t\phi)\ =\ q(t\phi)$$

Both results give $tq(\phi)\ =\ q(t\phi) \quad \forall\, t>0$.
For $t=0$, $\sum\|\ \xi.z_i\ \|^2\leq 0$ implies $z_i=0\ \ \forall\, i$ so $q(0)=0$ which implies that $q(t\phi)=tq(\phi)$ also holds for $t=0$.

Hence by Corollary 2.3 there is a linear form $f:\Lambda\mapsto\mathbb{R}$ such that

$$q(\phi)\ \leq\ f(\phi)\ \leq\ p(\phi) \quad \forall\,\phi\,\epsilon\,\Lambda_+ \tag{2.5}$$

and actually $f(\phi) \leq p(\phi)$ holds $\forall\ \phi \in \Lambda$.
Let us denote by $\Lambda + i\Lambda = \{\lambda + i\mu \mid \lambda,\ \mu \in \Lambda\}$ the complexification of Λ. We can extend f by linearity to a $\mathbb{C}$-linear form on $\Lambda + i\Lambda$ in the following way: $f : \Lambda + i\Lambda \mapsto \mathbb{C},\ f(\lambda + i\mu) = f(\lambda) + if(\mu) \quad \forall\ \lambda,\ \mu \in \Lambda$.
f is $\mathbb{C}$-linear because $f((\lambda+i\mu)+(x+iy)) = f((\lambda+x)+i(\mu+y)) = f(\lambda+x)+if(\mu+y) = f(\lambda)+f(x)+if(\mu)+if(y) = f(\lambda+i\mu)+f(x+iy)\ \forall\,\lambda,\mu,x,y \in \Lambda$ and $f(c(\lambda + i\mu)) = f(c\lambda + ic\mu) = f(c\lambda) + if(c\mu) = c(f(\lambda) + if(\mu)) = cf(\lambda + i\mu)\ \ \forall\ \lambda,\mu \in \Lambda,\ \forall\ c \in \mathbb{C}$ and if $(\lambda + i\mu),\ (x + iy) \in \Lambda + i\Lambda$ then $(\lambda + i\mu)(x + iy) = \lambda x - \mu y + i(\mu x + \lambda y) \in \Lambda + i\Lambda$.

Now we define $\mathcal{K} = \{g : I \mapsto \mathcal{H} \mid \xi \mapsto \| g(\xi) \|^2 \ \ \in \Lambda\}$. This set is not empty. Take for example $x_0 \in \mathcal{X}$ and define $g(\xi) = \xi(x_0) \quad \forall\ \xi \in I$. Then $\phi(\xi) = \| g(\xi) \|^2 = \| \xi(x_0) \|^2$ satisfies $| \phi(\xi) | = \| \xi(x_0) \|^2$ so $\phi \in \Lambda$.
Choose a g and $g' \in \mathcal{K}$ then $\phi : I \mapsto \mathbb{C}$ with $\phi(\xi) = \langle g(\xi), g'(\xi)\rangle$ is in $\Lambda + i\Lambda$. Indeed, by Cauchy-Schwartz

$$\begin{aligned} | \operatorname{Re}\phi | &\leq | \phi(\xi) | = | \langle g(\xi), g'(\xi)\rangle | \leq \| g(\xi) \| \| g'(\xi) \| \\ &\leq \frac{1}{2}\left(\| g(\xi) \|^2 + \| g'(\xi) \|^2\right) \leq \| g(\xi) \|^2 + \| g'(\xi) \|^2 \\ &\leq \sum \| \xi(x_i) \|^2 + \sum \| \xi(y_j) \|^2 \end{aligned}$$

for $x_i, y_j \in X$ and also $| \operatorname{Im}\phi | \leq \sum \| \xi(x_i) \|^2 + \sum \| \xi(y_j) \|^2$. So $\operatorname{Re}\phi$ and $\operatorname{Im}\phi \in \Lambda$ and this implies $\phi \in \Lambda + i\Lambda$. Now we can define

$$\langle g, g'\rangle = f(\phi)$$

with $\phi(\xi) = \langle g(\xi), g'(\xi)\rangle$. This is a semi-inner product on $\mathcal{K}$:
$\langle g_1 + g_2, g'\rangle = f(\langle (g_1 + g_2)(\cdot), g'(\cdot)\rangle) = f(\langle g_1(\cdot) + g_2(\cdot), g'(\cdot)\rangle) = f(\langle g_1(\cdot), g'(\cdot)\rangle + \langle g_2(\cdot), g'(\cdot)\rangle) = f(\langle g_1(\cdot), g'(\cdot)\rangle) + f(\langle g_2(\cdot), g'(\cdot)\rangle) = \langle g_1, g'\rangle + \langle g_2, g'\rangle$
$\langle \alpha g, g'\rangle = f(\langle \alpha g(\cdot), g'(\cdot)\rangle) = f(\alpha\langle g(\cdot), g'(\cdot)\rangle) = \alpha f(\langle g(\cdot), g'(\cdot)\rangle) = \alpha\langle g, g'\rangle$
$\overline{\langle g, g'\rangle} = \overline{f(\langle g(\cdot), g'(\cdot)\rangle)} = f(\overline{\langle g(\cdot), g'(\cdot)\rangle}) = f(\langle g'(\cdot), g(\cdot)\rangle) = \langle g', g\rangle$
(because $\overline{f(\lambda + i\mu)} = \overline{f(\lambda) + if(\mu)} = f(\lambda) - if(\mu) = f(\lambda - i\mu) = f(\overline{\lambda + i\mu})$)
$\langle g, g\rangle = f(\langle g(\cdot), g(\cdot)\rangle) = f(\| g(\cdot) \|^2) = f(\phi) \geq q(\phi) \geq \sum \| \hat{u}(z_i) \|^2 \geq 0$
but $\langle g, g\rangle = 0 \Rightarrow g = 0$ does not hold in general.
The inequality of Cauchy-Schwartz also holds for semi-inner products :

$$| \langle g, h\rangle | \leq \sqrt{\langle g, g\rangle}\sqrt{\langle h, h\rangle}$$

so if $\langle g, g\rangle = 0$ then also $\langle g, h\rangle = 0 \quad \forall\ h \in \mathcal{K}$ and conversely $\langle g, h\rangle = 0 \quad \forall\ h \in \mathcal{K}$ implies $\langle g, g\rangle = 0$ (take $h = g$) $\qquad (*)$

Define $N = \{g \mid \langle g, g\rangle = 0\}$ and $\tilde{\mathcal{K}} = \mathcal{K}/N = \{\tilde{g} \mid \tilde{g} = g + N\}$.
N is a linear space: if $g \in N$ then $\alpha g \in N$ because $\langle \alpha g, \alpha g\rangle = \alpha\bar{\alpha}\langle g, g\rangle = 0$ and if $g_1, g_2 \in N$ then $\langle g_1 + g_2, g_1 + g_2\rangle = \langle g_1, g_1\rangle + \langle g_1, g_2\rangle + \langle g_2, g_1\rangle + \langle g_2, g_2\rangle = 0$ because of $(*)$ so $g_1 + g_2 \in N$.
$\langle \tilde{g}, \tilde{h}\rangle \overset{\text{def}}{=} \langle g, h\rangle$ for a $g \in \tilde{g}$ and a $h \in \tilde{h}$. This definition does not depend on the choice of g and h. This is checked as follows:
Choose also g_1, h_1 such that $\langle \tilde{g}, \tilde{h}\rangle = \langle g_1, h_1\rangle$. Then $g - g_1 = n \in N$ and $h - h_1 = m \in N$ so $\langle g_1, h_1\rangle = \langle g - n, h - m\rangle = \langle g, h\rangle - \langle g, m\rangle - \langle n, h\rangle + \langle n, m\rangle = \langle g, h\rangle$ because of $(*)$.

If $0 = \langle \tilde{g}, \tilde{g} \rangle = \langle g, g \rangle$ then $g \in N$ and $\tilde{g} = g + N = N$ so N is the zero-element of $\tilde{\mathcal{K}}$.
After completing the space $\tilde{\mathcal{K}}$ we obtain a Hilbert space $\hat{\mathcal{H}}$.
For $x \in \mathcal{X}$, let $\tilde{x} \in \tilde{\mathcal{K}}$ be defined by $\tilde{x}(\xi) = \xi(x)$. By the second inequality in (2.5) applied to ϕ with $\phi(\xi) = \| \tilde{x}(\xi) \|^2$ where $\xi \mapsto \phi(\xi) = \| \tilde{x}(\xi) \|^2 =$ $\| \xi(x) \|^2 \quad \epsilon\ \Lambda$ we have

$$\langle \tilde{x}, \tilde{x} \rangle \ = \ f(\phi) \ \leq \ p(\phi) \ \leq \ C^2 \parallel x \parallel^2$$

Let $\hat{x}$ be the equivalent class containing $\tilde{x}$. Then $\{\{x, \hat{x}\} \mid x \in \mathcal{X}\} \subset \mathcal{X}$ $\times \hat{\mathcal{H}}$ is the graph of a linear map $V_1 : \mathcal{X} \mapsto \hat{\mathcal{H}}$ defined by

$$V_1 x \ = \ \hat{x}$$

and $\| V_1 x \| = \| \hat{x} \| = \| \tilde{x} \| \leq C \| x \|$ so $\| V_1 \| \leq C$.
On the other hand, if we take $\phi(\xi) = \| \sum a_i \tilde{x}_i(\xi) \|^2$ then $\forall\ a_i \in S, \forall\ x_i \in \mathcal{X}$

$$\begin{aligned} \phi(\xi) \ &= \| \sum a_i \tilde{x}_i(\xi) \|^2 = \| \sum a_i \xi(x_i) \|^2 \leq \left(\sum \| a_i \| \| \xi(x_i) \| \right)^2 \\ &\leq \sum \| a_i \|^2 \sum \| \xi(x_i) \|^2 = \sum \| \xi(\sqrt{\alpha} x_i) \|^2 \qquad \epsilon\ \Lambda \end{aligned}$$

(where $\alpha = \sum \| a_i \|^2$) and by the first inequality in (2.5) we have

$$\text{(2.6)} \qquad \| \sum u(a_i) x_i \|^2 = \| \hat{u} \left(\sum a_i \otimes x_i \right) \|^2 \leq \ q(\phi) \ \leq \ f(\phi)$$

and we will use this later.
We define

$$\pi : B(\mathcal{H}) \mapsto B(\hat{\mathcal{H}})$$

by setting

$$\pi(a)\hat{g} \ = \ \widehat{ag}$$

for $a \in B(\mathcal{H})$, $\pi(a) \in B(\mathcal{H})$, $g \in \mathcal{K}$ and this is a unit preserving $*$-representation.
Let us check this and see that π is well defined.
If $g \in \mathcal{K}$ then $\hat{g} \in \hat{\mathcal{H}}$ and $ag \in \mathcal{K} \quad \forall\ a \in \mathcal{H}$:
$\xi \mapsto \| ag(\xi) \|^2 \leq \| a \|^2 \| g(\xi) \|^2 \quad \epsilon\ \Lambda$ (because $\| a \|^2 \ \epsilon\ \mathbb{C}$).
Let $g, h \in \mathcal{K}$ and $\hat{g} = g + N = \hat{h} = h + N$. This implies $n = g - h \in N$ and $an = ag - ah$ so $\langle an, k \rangle = \langle n, a^* k \rangle = 0 \quad \forall\ k \in \mathcal{K}$ and $an \in N$. This means $\widehat{ag} = \widehat{ah}$. So if $\hat{g} = \hat{h}$ then $\widehat{ag} = \widehat{ah}$.
π is unit preserving because $\pi(1)\hat{g} = \hat{g} \ \ \forall\ \hat{g} \in \hat{\mathcal{H}}$.
π also is a $*$-representation because
$\pi(st)\hat{g} = \widehat{stg} = \widehat{s(tg)} = \pi(s)\widehat{tg} = \pi(s)\pi(t)\hat{g}$ and
$\langle \pi(a^*)\hat{g}_n, \hat{h}_n \rangle = \langle \widehat{a^* g_n}, \hat{h}_n \rangle = \langle a^* g_n, h_n \rangle = f(\langle a^* g_n(\cdot), h_n(\cdot) \rangle) = f(\langle g_n(\cdot), a$ $h_n(\cdot) \rangle) = \langle g_n, a h_n \rangle = \langle \hat{g}_n, \widehat{a h_n} \rangle = \langle \hat{g}_n, \pi(a)\hat{h}_n \rangle = \langle \pi(a)^* \hat{g}_n, \hat{h}_n \rangle$
which implies $\pi(a^*)\hat{g}_n = \pi(a)^* \hat{g}_n \ \ \forall\ g_n \in \mathcal{K}$ and if $\hat{h}_n \to h$ for $n \to \infty$ and $\hat{g}_n \to g$ then follows $\pi(a^*)\hat{g} = \pi(a)^* \hat{g} \ \ \forall\ g \in \hat{\mathcal{H}}$.
The last thing we have to check is that π is bounded i.e. $\langle \pi(a)\hat{g}_n, \pi(a)\hat{g}_n \rangle \leq$ $const.\langle \hat{g}_n, \hat{g}_n \rangle \ \ \forall\ \hat{g}_n$. Then $\pi(a)$ can be extended by continuity to all of

$\hat{\mathcal{H}}$ and this extension is linear and bounded with the same bound. In this sense $\pi(a) \in B(\hat{\mathcal{H}})$.

$$\begin{aligned}
\langle \pi(a)\hat{g}_n, \pi(a)\hat{g}_n\rangle &= \langle \widehat{ag}_n, \widehat{ag}_n\rangle = \langle ag_n, ag_n\rangle = f(\langle ag_n(\cdot), ag_n(\cdot)\rangle) \\
&= f(\langle a^*ag_n(\cdot), g_n(\cdot)\rangle) = f(\langle \sqrt{a^*a}g_n(\cdot), \sqrt{a^*a}g_n(\cdot)\rangle) \\
&= \| \sqrt{a^*a} \|^2 f(\langle \frac{\sqrt{a^*a}}{\| \sqrt{a^*a} \|} g_n(\cdot), \frac{\sqrt{a^*a}}{\| \sqrt{a^*a} \|} g_n(\cdot)\rangle) \\
&= \| a \|^2 f(\langle bg_n(\cdot), bg_n(\cdot)\rangle) = \| a \|^2 f(\langle g_n(\cdot), g_n(\cdot)\rangle) \\
&- \| a \|^2 f(\langle i\sqrt{1-b^2}g_n(\cdot), i\sqrt{1-b^2}g_n(\cdot)\rangle) = \| a \|^2 \langle g_n, g_n\rangle \\
&- \| a \|^2 \langle i\sqrt{1-b^2}g_n, i\sqrt{1-b^2}g_n\rangle \leq \| a \|^2 \langle g_n, g_n\rangle \\
&= \| a \|^2 \langle \hat{g}_n, \hat{g}_n\rangle
\end{aligned}$$

where $b = \frac{\sqrt{a^*a}}{\|\sqrt{a^*a}\|}$ so $b = b^*$ and $\| b \| = 1$.
Because $a^*a \geq 0$ we can take the squareroot and $\| \sqrt{a^*a} \|^2 = \| a \|^2$ and $\langle bg_n(\xi), bg_n(\xi)\rangle = \langle (b+i\sqrt{1-b^2})g_n(\xi), (b+i\sqrt{1-b^2})g_n(\xi)\rangle - \langle i\sqrt{1-b^2}g_n(\xi), bg_n(\xi)\rangle - \langle bg_n(\xi), i\sqrt{1-b^2}g_n(\xi)\rangle - \langle i\sqrt{1-b^2}g_n(\xi), i\sqrt{1-b^2}g_n(\xi)\rangle = \langle g_n(\xi), g_n(\xi)\rangle - \langle i\sqrt{1-b^2}g_n(\xi), i\sqrt{1-b^2}g_n(\xi)\rangle$ and this last inner product ≥ 0.
If $\hat{g}_n \to \hat{g}$ for $n \to \infty$ then $\langle \pi(a)\hat{g}, \pi(a)\hat{g}\rangle \leq \| a \|^2 \| \hat{g} \|^2$ so $\pi(a) \in B(\mathcal{H})$.

By (2.6) follows $\| \sum u(a_i)x_i \|^2 \leq f(\phi) = f(\| \sum a_i\tilde{x}_i \|^2) = \| \sum a_i\tilde{x}_i \|^2 = \| \sum a_i\hat{x}_i \|^2 = \| \sum \pi(a_i)\tilde{x}_i \|^2 = \| \sum \pi(a_i)V_1x_i \|^2 \quad \forall\, a_i \in S,\ x_i \in \mathcal{X}$ and $\sum \pi(a_i)V_1x_i \in \text{span}(\pi(S)V_1\mathcal{X})$ and $\sum u(a_i)x_i \in \mathcal{Y}$.
This allows us to define a linear map

$$V_2 : \overline{\text{span}}(\pi(S)V_1\mathcal{X}) \mapsto \mathcal{Y}$$

such that

$$\sum u(a_i)x_i = V_2\left(\sum \pi(a_i)V_1x_i\right) \tag{2.7}$$

Finally, we can extend V_2 to an operator $V_2 : \hat{\mathcal{H}} \mapsto \mathcal{Y}$ with norm ≤ 1 by defining $V_2 = 0$ on $(\overline{\text{span}}(\pi(S)V_1\mathcal{X})^\perp = \hat{\mathcal{H}} \ominus \pi(S)V_1\mathcal{X}$.
By omitting the sum and x_i in (2.7) we get the required result (2.1).

The converse is easy:
because π is a $*$-representation follows from the proof of Theorem 1.9, Lemma 3 that $\| \pi \| \leq 1$ and

$$\| \pi \|_{cb} = \sup_{n\geq 1} \| \pi_n \| = \sup_{n\geq 1} \sup_{(a_{ij})\in\mathcal{M}_n(A)} \frac{\| \pi_n((a_{ij})) \|_{B(\mathcal{X}^n)}}{\| (a_{ij}) \|_{B(A^n)}} \leq 1 \tag{2.8}$$

and so

$$\| u \|_{cb} \leq \| V_2 \| \| \pi \|_{cb} \| V_1 \| \leq \| V_2 \| \| V_1 \|$$

2.2 Completely bounded homomorphisms

Let us now go to the study of compressions of homomorphisms.

Let $\mathcal{X}$ be a Banach space, and let $\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{X}$ be closed subspaces. Let $T : \mathcal{X} \mapsto \mathcal{X}$ be a bounded operator and assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are T-invariant i.e. $T(\mathcal{E}_1) \subset \mathcal{E}_1$ and $T(\mathcal{E}_2) \subset \mathcal{E}_2$.
Then $\mathcal{E}_1/\mathcal{E}_2 = \{\tilde{x} \mid \tilde{x} = \{x + \mathcal{E}_2\},\ x \in \mathcal{E}_1\}$ with

$$\| \tilde{x} \| = \inf_{e \in \mathcal{E}_2} \| x + e \|$$

This norm is well defined:
$\| \tilde{x} \| \geq 0$
$\| \tilde{x} \| = 0 = \inf_{e \in \mathcal{E}_2} \| x + e \| \Rightarrow \exists\, e_n \in \mathcal{E}_2$ such that $x + e_n \to 0$ which means $e_n \to -x$ and this implies $x \in \mathcal{E}_2$ so $\tilde{x} = \tilde{0}$
if $c \in \mathbb{C}$, $\tilde{x}, \tilde{y} \in \mathcal{E}_1/\mathcal{E}_2$
$\| c\tilde{x} \| = \inf_{e \in \mathcal{E}_2} \| cx + e \| = | c | \inf_{\frac{e}{c} \in \frac{1}{c}\mathcal{E}_2 = \mathcal{E}_2} \| x + \frac{e}{c} \| = | c | \| \tilde{x} \|$
$\| \tilde{x} + \tilde{y} \| = \| (x+y)^{\sim} \| = \inf_{e \in \mathcal{E}_2} \| x+y+e \| \leq \| x+e'+y+e'' \| \leq \| x+e' \| + \| y+e'' \|$
this holds $\forall e', e'' \in \mathcal{E}_2$ so we can take the infimum, which implies
$\| \tilde{x} + \tilde{y} \| \leq \| \tilde{x} \| + \| \tilde{y} \|$

Let $Q : \mathcal{E}_1 \mapsto \mathcal{E}_1/\mathcal{E}_2$ be the canonical surjection defined by $Q(x) = \tilde{x}$ and let $\tilde{T} \in B(\mathcal{E}_1/\mathcal{E}_2)$ be such that $\tilde{T}Q = QT_{|_{\mathcal{E}_1}}$. Then $\| Q(x) \| = \| \tilde{x} \| = \inf_{e \in \mathcal{E}_2} \| x + e \| \leq \| x \|$ so $\| Q \| \leq 1$ and we can make the following diagram:

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{T|_{\mathcal{E}_1}} & \mathcal{E}_1 \\ Q \downarrow & & \downarrow Q \\ \mathcal{E}_1/\mathcal{E}_2 & \xrightarrow{\tilde{T}} & \mathcal{E}_1/\mathcal{E}_2 \end{array}$$

and $\tilde{T}\tilde{x} = \tilde{T}Qx = QTx = (Tx)^{\sim}\ \forall x \in \mathcal{E}_1$.
Then

$$\begin{aligned} \| \tilde{T}\tilde{x} \| = \| (Tx)^{\sim} \| &= \inf_{e \in \mathcal{E}_2} \| Tx + e \| \leq \inf_{e \in \mathcal{E}_2} \| Tx + Te \| \\ &\leq \inf_{e \in \mathcal{E}_2} \| T \| \| x + e \| = \| T \| \inf_{e \in \mathcal{E}_2} \| x + e \| = \| T \| \| \tilde{x} \| \end{aligned}$$

$\forall\, x \in \mathcal{E}_1$ so $\| \tilde{T} \|_{\mathcal{E}_1/\mathcal{E}_2} \leq \| T \|_{\mathcal{E}_1} \leq \| T \|_{\mathcal{X}}$.
This characterization brings us to the following proposition

Proposition 2.7: Let $\mathcal{A}$ be a Banach algebra and let $u : \mathcal{A} \mapsto B(\mathcal{X})$ be a bounded homomorphism, i.e. u is bounded linear and

$$\forall\, a, b \in A \quad u(ab) = u(a)u(b)$$

Let $\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{X}$ be closed subspaces and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be u-invariant i.e. $\mathcal{E}_1$ and $\mathcal{E}_2$ are $u(a)$-invariant $\forall\, a \in \mathcal{A}$. Then the map $\tilde{u} : \mathcal{A} \mapsto B(\mathcal{E}_1/\mathcal{E}_2)$ defined by $\tilde{u}(a) = (u(a))^{\sim}$ is a homomorphism with $\| \tilde{u} \| \leq \| u \|$. Moreover, if $\mathcal{A}$ is a subalgebra of $B(\mathcal{H})$ (with $\mathcal{H}$ Hilbert) and if u is c.b. then $\tilde{u}$ also is c.b. and $\| \tilde{u} \|_{cb} \leq \| u \|_{cb}$.

Proof: $\forall\, a, b \in \mathcal{A}$ we have

$$\tilde{u}(ab)Q = Qu(ab) = Qu(a)u(b) = \tilde{u}(a)Qu(b) = \tilde{u}(a)\tilde{u}(b)Q$$

which shows that $\tilde{u}$ also is a homomorphism.
We have seen before

$$\| \tilde{u}(a) \|_{B(\mathcal{E}_1/\mathcal{E}_2)} \leq \| u(a) \|_{B(\mathcal{E}_1)} \leq \| u(a) \|_{B(\mathcal{X})}$$

hence $\| \tilde{u} \| \leq \| u \|$.
Define $u_n : \mathcal{A}^n \mapsto B(\mathcal{X}^n)$ as $u_n((A)) = (u(a_{ij}))$ where $A = (a_{ij})_{i,j=1}^n \in \mathcal{A}^n$.
Then

$$\| \tilde{u} \|_{cb} = \sup_{n\geq 1} \| \tilde{u}_n \| = \sup_{n\geq 1} \sup_{(a_{ij})} \frac{\| \tilde{u}_n((a_{ij})) \|_{B(\mathcal{E}_1^n/\mathcal{E}_2^n)}}{\| (a_{ij}) \|_{B(\mathcal{A}^n)}}$$
$$= \sup_{n\geq 1} \sup_{(a_{ij})} \frac{\| (\tilde{u}(a_{ij})) \|_{B(\mathcal{E}_1^n/\mathcal{E}_2^n)}}{\| (a_{ij}) \|_{B(\mathcal{A}^n)}}$$

Now apply the previous result by replacing u by $(u(a_{ij}))$, $\mathcal{A}$ by $\mathcal{A}^n$, $\mathcal{X}$ by $\mathcal{X}^n$, $\mathcal{E}_1$ by $\mathcal{E}_1^n$ and $\mathcal{E}_2$ by $\mathcal{E}_2^n$. This implies $\| \tilde{u}_n((a_{ij})) \| \leq \| u_n((a_{ij})) \|$ $\forall\, (a_{ij})$ $\forall\, n$ and if we take the supremum over (a_{ij}) and $n \geq 1$ we get:

$$\| \tilde{u} \|_{cb} \leq \sup_{n\geq 1} \sup_{(a_{ij})} \frac{\| u_n((a_{ij})) \|_{B(\mathcal{E}_1^n)}}{\| (a_{ij}) \|_{B(\mathcal{A}^n)}}$$
$$\leq \sup_{n\geq 1} \sup_{(a_{ij})} \frac{\| u_n((a_{ij})) \|_{B(\mathcal{X}^n)}}{\| (a_{ij}) \|_{B(\mathcal{A}^n)}} = \| u \|_{cb}$$

$\tilde{u}$ will be called the compression of u to $\mathcal{E}_1/\mathcal{E}_2$.

Remark: If $\mathcal{A} \subset B(\mathcal{H})$ and if $u : \mathcal{A} \mapsto B(\mathcal{G})$ ($\mathcal{G}$ Hilbert) is the restriction to $\mathcal{A}$ of a $*$-representation $\pi : B(\mathcal{H}) \mapsto B(\mathcal{G})$, then we have

$$\| \tilde{u} \|_{cb} \leq \| u \|_{cb} \leq \| \pi \|_{cb} \leq 1$$

Indeed, the first inequality follows by Proposition 2.7. If we define u_n as above and π_n in the same way we get

$$\| u \|_{cb} = \sup_{n\geq 1} \| u_n \| = \sup_{n\geq 1} \sup_{(a_{ij})\in\mathcal{A}^n} \frac{\| u_n((a_{ij})) \|}{\| (a_{ij}) \|}$$
$$\leq \sup_{n\geq 1} \sup_{(a_{ij})\in B(\mathcal{H}^n)} \frac{\| \pi_n((a_{ij})) \|}{\| (a_{ij}) \|} = \| \pi \|_{cb}$$

which explains the second inequality.
We have seen in (2.8) that $\| \pi \|_{cb} \leq 1$.

Proposition 2.8: Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{X}$, $\mathcal{Z}$ be two Banach spaces, let $\pi : \mathcal{A} \mapsto B(\mathcal{Z})$ be a bounded homomorphism, and let $w_1 : \mathcal{X} \mapsto \mathcal{Z}$ and $w_2 : \mathcal{Z} \mapsto \mathcal{X}$ be operators such that $w_2 w_1 = I_{\mathcal{X}}$. Assume that the map $u : \mathcal{A} \mapsto B(\mathcal{X})$ defined by

$$u(a) = w_2 \pi(a) w_1 \quad \forall\, a \in \mathcal{A}$$

is a homomorphism. Then u is similar to a compression of π. More precisely, there are π-invariant subspaces $\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{Z}$ and an isomorphism $S : \mathcal{X} \mapsto \mathcal{E}_1/\mathcal{E}_2$ such that

$$\| S \| \| S^{-1} \| \leq \| w_1 \| \| w_2 \|$$

and such that the compression $\tilde{\pi}$ of π to $\mathcal{E}_1/\mathcal{E}_2$ satisfies

$$u(a) = S^{-1}\tilde{\pi}(a)S \quad \forall \, a \in \mathcal{A}$$

Proof: Let

$$\mathcal{E}_1 = \overline{\text{span}}[w_1(\mathcal{X}), \textstyle\bigcup_{a \in \mathcal{A}} \pi(a) w_1(\mathcal{X})]$$

By definition $\mathcal{E}_1$ is a closed subspace of $\mathcal{Z}$. $\mathcal{E}_1$ also is π-invariant. This is checked as follows :
An element y of $\mathcal{E}_1$ can be written as

$$y = \lim_{n \to \infty} \left(w_1(x_n) + \sum_i \pi(a_{in}) w_1(x_{in}) \right)$$

for some x_n, $x_{in} \in \mathcal{X}$, $a_{in} \in \mathcal{A}$ because $b_1 w_1(x_1) + \cdots + b_n w_n(x_n) = w_1(b_1 x_1 + \cdots + b_n x_n) = w_1(x_n)$ and $\forall \, b \in \mathcal{A}$

$$\begin{aligned} \pi(b)y &= \lim_{n \to \infty} \left(\pi(b) w_1(x_n) + \pi(b) \sum_i \pi(a_{in}) w_1(x_{in}) \right) \\ &= \lim_{n \to \infty} \left(\pi(b) w_1(x_n) + \sum_i \pi(b a_{in}) w_1(x_{in}) \right) \quad \in \mathcal{E}_1 \end{aligned}$$

Let $\mathcal{E}_2 = \mathcal{E}_1 \cap \ker(w_2)$ then $\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{Z}$. We claim that $\mathcal{E}_2$ also is π-invariant. Indeed, consider $z \in \mathcal{E}_1$ such that $w_2(z) = 0$. In the same way as above we can write z as

$$z = \lim_{n \to \infty} \left(w_1(x_n) + \sum_i \pi(a_{in}) w_1(x_{in}) \right)$$

Then because $w_2(z) = 0$, $w_2 w_1 = I_{\mathcal{X}}$ and $u(a) = w_2 \pi(a) w_1$

$$\begin{aligned} 0 = w_2(z) &= \lim_{n \to \infty} \left(w_2 w_1(x_n) + \sum_i w_2 \pi(a_{in}) w_1(x_{in}) \right) \\ &= \lim_{n \to \infty} \left(x_n + \sum_i u(a_{in}) x_{in} \right) \qquad (*) \end{aligned}$$

Hence for all $a \in \mathcal{A}$

$$\begin{aligned} \pi(a)z &= \lim_{n \to \infty} \left(\pi(a) w_1 x_n + \sum_i \pi(a) \pi(a_{in}) w_1(x_{in}) \right) \\ &= \lim_{n \to \infty} \left(\pi(a) w_1 x_n + \sum_i \pi(a a_{in}) w_1(x_{in}) \right) \end{aligned}$$

and so

$$\begin{aligned}
w_2\pi(a)z &= \lim_{n\to\infty}\left(w_2\pi(a)w_1x_n + \sum_i w_2\pi(aa_{in})w_1(x_{in})\right)\\
&= \lim_{n\to\infty}\left(u(a)x_n + \sum_i u(aa_{in})x_{in}\right)\\
&= \lim_{n\to\infty}\left(u(a)x_n + \sum_i u(a)u(a_{in})x_{in}\right)\\
&= \lim_{n\to\infty} u(a)\left(x_n + \sum_i u(a_{in})x_{in}\right) = 0
\end{aligned}$$

because of $(*)$. Since $z \in \mathcal{E}_1$, $\pi(a)z$ also is in $\mathcal{E}_1$ and $w_2\pi(a)z = 0$ which means that $\pi(a)z \in \ker(w_2)$. This implies that $\pi(a)z \in \mathcal{E}_2$ $\forall a$ and proves the claim.
Let $Q : \mathcal{E}_1 \mapsto \mathcal{E}_1/\mathcal{E}_2$ be the canonical surjection. Define $S = Qw_1 : \mathcal{X} \mapsto \mathcal{E}_1/\mathcal{E}_2$ by

$$S(x) = Qw_1(x) \quad \forall x \in \mathcal{X}$$

$w_{2|_{\mathcal{E}_1}} : \mathcal{E}_1 \mapsto \mathcal{X}$ is surjective. Take a $x \in \mathcal{X}$, then $y := w_1(x) \in \mathcal{E}_1$ and since $w_2w_1 = I_{\mathcal{X}}$ $w_2(y) = x$. So for every $x \in \mathcal{X}$ $\exists\, y \in \mathcal{E}_1$ such that $w_2(y) = x$. Now there is a unique isomorphism $R : \mathcal{E}_1/\mathcal{E}_2 \mapsto \mathcal{X}$ with $\| R \| \leq \| w_2 \|$ namely $R(\tilde{x}) = w_2(x+\mathcal{E}_2) = w_2(x+\ker w_2)$ ($\tilde{x} = x+\mathcal{E}_2 \subset x+\ker w_2$) since for $e \in \mathcal{E}_2$ $\| R(\tilde{x}) \| = \| w_2(x+e) \| \leq \| w_2 \| \| x+e \|$ so $\| R\tilde{x} \| \leq \| w_2 \| \| \tilde{x} \|$ such that $RQ = w_{2|_{\mathcal{E}_1}}$. Then we have $RQw_1 = w_2w_1 = I_{\mathcal{X}}$ hence $RS = I_{\mathcal{X}}$. This implies that R is surjective. R also is injective:
$0 = R(\tilde{x}) = w_2(x_0 + ker w_{2|_{\mathcal{E}_1}}) \Longrightarrow x_0 + ker w_{2|_{\mathcal{E}_1}} \in ker w_2$
also $x_0 + ker w_{2|_{\mathcal{E}_1}} \in \mathcal{E}_1$ so $x_0 + ker w_{2|_{\mathcal{E}_1}} \in \mathcal{E}_2$ and this implies $\tilde{x} = \tilde{0}$.
Surjective and injective is the same as invertible and since $RS = I_{\mathcal{X}}$, $R^{-1} = S$. This implies that S also is invertible and $S^{-1} = R$. Moreover we have

$$\| S \| \| S^{-1} \| = \| Qw_1 \| \| R \| \leq \| w_1 \| \| w_2 \|$$

and

$$\begin{aligned}
S^{-1}\tilde{\pi}(a)S &= S^{-1}\tilde{\pi}(a)Qw_1\\
&= RQ\pi(a)w_1\\
&= w_2\pi(a)w_1\\
&= u(a) \qquad \forall a \in \mathcal{A}
\end{aligned}$$

We now come to a theorem which we will need to prove Theorem 2.1

Theorem 2.9: Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. Let $\mathcal{A} \subset B(\mathcal{H})$ be a subalgebra containing a unit 1 and let $u : \mathcal{A} \mapsto B(\mathcal{K})$ be a bounded homomorphism with $u(1) = I_{\mathcal{K}}$. Let K be any constant. The following are equivalent:
(i) The map u is c.b. with $\| u \|_{cb} \leq K$
(ii) There is an isomorphism $R : \mathcal{K} \mapsto \mathcal{K}$ with $\| R \| \| R^{-1} \| \leq K$ such that the map $a \mapsto R^{-1}u(a)R$ is c.b. with c.b. norm ≤ 1.

Proof: (ii) $\Rightarrow$ (i): Let $v(a) = R^{-1}u(a)R$ with $\| R \| \| R^{-1} \| \leq K$ and $\| v \|_{cb} \leq 1$. Then $u(a) = Rv(a)R^{-1}$ and let $v_n : \mathcal{A}^n \mapsto B(\mathcal{K}^n)$ defined by $v_n(A) = (v(a_{ij}))$ for $A = (a_{ij})_{i,j=1}^n \epsilon \mathcal{A}^n$.

Then $u_n(a_{ij}) = \begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix} v_n(a_{ij}) \begin{pmatrix} R^{-1} & & 0 \\ & \ddots & \\ 0 & & R^{-1} \end{pmatrix}$

so $\| u \|_{cb} \leq \sup_{n\geq 1} \sup_{(a_{ij})\epsilon \mathcal{A}^n} \frac{\|R\| \|v_n(a_{ij})\| \|R^{-1}\|}{\|(a_{ij})\|}$
$\leq \| R \| \| v \|_{cb} \| R^{-1} \| \leq K$.

(i) $\Rightarrow$ (ii): Assume (i). By Theorem 2.4 with $S = \mathcal{A}$ and $\mathcal{X} = \mathcal{Y} = \mathcal{K}$ there is a Hilbert space $\hat{\mathcal{H}}$, a $*$-representation $\pi : B(\mathcal{H}) \mapsto B(\hat{\mathcal{H}})$ with $\pi(1) = 1$ and operators $w_1 : \mathcal{K} \mapsto \hat{\mathcal{H}}$ and $w_2 : \hat{\mathcal{H}} \mapsto \mathcal{K}$ with $\| w_1 \| \| w_2 \| \leq \| u \|_{cb}$ such that

$$u(a) = w_1 \pi(a) w_2 \qquad \forall\, a \,\epsilon\, \mathcal{A}$$

By definition of $*$-representations $\pi_{|\mathcal{A}}$ is a homomorphism and this implies $u(a)$ also is a homomorphism. $I_{\mathcal{K}} = u(1) = w_1\pi(1)w_2 = w_1 w_2$ so we can apply the preceding result for $\mathcal{X} = \mathcal{K}$ and $\mathcal{Z} = \hat{\mathcal{H}}$: u is similar to a compression $\tilde{\pi}$ of $\pi_{|\mathcal{A}}$ or in other words

$$u(a) = R\tilde{\pi}(a)R^{-1} \qquad \forall\, a \,\epsilon\, \mathcal{A}$$

and $\| R \| \| R^{-1} \| \leq \| w_1 \| \| w_2 \|$.
But $\| w_1 \| \| w_2 \| \leq \| u \|_{cb} \leq K$ and this implies $\| R \| \| R^{-1} \| \leq K$. By Proposition 2.7 $\| \tilde{\pi} \|_{cb} \leq \| \pi \|_{cb} \leq 1$ and

$$\tilde{\pi}(a) = R^{-1}u(a)R$$

so the map $a \mapsto R^{-1}u(a)R$ is c.b with c.b. norm ≤ 1.

2.3 Proof of Theorem 2.1

We can apply the preceding result to Theorem 2.1 which we wanted to prove. Assume T is c.pol.b. then the homomorphism $P \mapsto P(T)$ where P is a polynomial defines a completely bounded homomorphism u_T ($u_T(P) = P(T)$) from the disc algebra $\mathcal{A}$ into $B(\mathcal{H})$. Indeed, T is c.pol.b. means $\exists\, K$ such that $\forall\, n$ and $\forall\, n \times n$ matrices (P_{ij}) with polynomial entries we have

$$\| (P_{ij}(T)) \|_{B(\mathcal{H}^n)} \leq K \sup_{|z|\leq 1} \| (P_{ij}(z)) \|_{B(\mathbb{C}^n)}$$

Define $u_{Tn} : \mathcal{A}^n \mapsto B(\mathcal{H}^n)$ as $u_{Tn}((P_{ij})) = (u_T(P_{ij}))$ then

$$\begin{aligned}
\| u_T \|_{cb} = \sup_{n\geq 1} \| u_{Tn} \| &= \sup_{n\geq 1} \sup_{(P_{ij})} \frac{\| u_{Tn}((P_{ij})) \|_{B(\mathcal{H}^n)}}{\| (P_{ij}) \|_{\mathcal{A}^n}} \\
&= \sup_{n\geq 1} \sup_{(P_{ij})} \frac{\| (u_T(P_{ij})) \|_{B(\mathcal{H}^n)}}{\| (P_{ij}) \|_{\mathcal{A}^n}} = \sup_{n\geq 1} \sup_{P_{ij}} \frac{\| (P_{ij}(T)) \|_{B(\mathcal{H}^n)}}{\| (P_{ij}) \|_{\mathcal{A}^n}} \\
&\leq \sup_{n\geq 1} \sup_{(P_{ij})} \frac{K \sup_{|z|\leq 1} \| (P_{ij}(z)) \|_{B(\mathbb{C}^n)}}{\| (P_{ij}) \|_{\mathcal{A}^n}} \\
&= \sup_{n\geq 1} \sup_{(P_{ij})} \frac{K \sup_{|z|\leq 1} \| (P_{ij}(z)) \|_{B(\mathbb{C}^n)}}{\sup_{|z|\leq 1} \frac{\|(P_{ij}(z))\|_{B(\mathbb{C}^n)}}{\|z\|_{B(\mathbb{C})}}} \\
&\leq K
\end{aligned}$$

which means that u_T is c.b. with $\| u_T \|_{cb} \leq K$.
By Theorem 2.9 there is an isomorphism $R : \mathcal{K} \mapsto \mathcal{K}$ with $\| R \| \| R^{-1} \| \leq K$ such that the map $P \mapsto R^{-1}u_T(P)R$ is c.b. with $\| R^{-1}u_T R \|_{cb} \leq 1$. Take $P = I$ the identity then $u_T(I) = I(T) = T$ and

$$\| R^{-1}TR \| = \| R^{-1}u_T(I)R \| \leq \| R^{-1}u_T R \|_{cb} \leq 1$$

so T is similar to a contraction.

Appendix A

Dilation theorem: Let $T : \mathcal{H} \mapsto \mathcal{H}$ be a contraction. Then there is a Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ isometrically as a subspace and a unitary operator $U : \tilde{\mathcal{H}} \mapsto \tilde{\mathcal{H}}$ such that
$\forall\, n \geq 0 \quad T^n = P_{\mathcal{H}} U^n_{|\mathcal{H}}$
(where $P_{\mathcal{H}}$ is the projection on $\mathcal{H}$).
When this holds, U is called a *unitary dilation* of T (one also says that U dilates T).

Proof: For any n in Z let $\mathcal{H}_n = \mathcal{H}$, and consider the Hilbertian direct sum $\tilde{\mathcal{H}} = \oplus_{n \epsilon Z}\, \mathcal{H}_n = \begin{pmatrix} \vdots \\ \mathcal{H} \\ \mathcal{H} \\ \vdots \end{pmatrix}$. On $\tilde{\mathcal{H}}$ we introduce the operator $U : \tilde{\mathcal{H}} \mapsto \tilde{\mathcal{H}}$ defined by the following matrix with operator coefficients

$$U = \begin{pmatrix} \ddots & & & & & & \\ \ddots & I & & & & \bigcirc & \\ & 0 & I & & & & \\ & & 0 & D_T & -T^* & & \\ & & & T & D_{T^*} & & \\ \bigcirc & & & & 0 & I & \\ & & & & & 0 & I \\ & & & & & & \ddots \quad \ddots \end{pmatrix}$$

where T stands as the (0,0)-entry and $D_T = (1 - T^*T)^{1/2}$ and $D_{T^*} = (1 - TT^*)^{1/2}$. Equivalent any $(h_n)_{n \epsilon Z}$ is mapped into $U[(h_n)_{n \epsilon Z}]$ $= (h'_n)_{n \epsilon Z}$ with h'_n defined by

$$(*)\ h'_n = \begin{cases} h_{n+1} & \text{if } n \notin \{-1, 0\} \\ D_T h_0 - T^* h_1 & \text{if } n = -1 \\ T h_0 + D_{T^*} h_1 & \text{if } n = 0 \end{cases}$$

We identify $\mathcal{H}$ with $\mathcal{H}_0 \subset \tilde{\mathcal{H}}$ so that we have $P_{\mathcal{H}} U_{|\mathcal{H}} = T$ and more generally $P_{\mathcal{H}} U^n_{|\mathcal{H}} = T^n$ for all $n \geq 0$ (note that U has a triangular form, so the diagonal coefficients of U^n are the obvious ones).
We claim that for all $(h_n)_{n \epsilon Z}$ in $\tilde{\mathcal{H}}$ and $(h'_n)_{n \epsilon Z} = U[(h_n)_{n \epsilon Z}]$ as above we have

$$\| h'_{-1} \|^2 + \| h'_0 \|^2 = \| h_0 \|^2 + \| h_1 \|^2 .$$

Indeed, first note the following identities

$$T^* D_{T^*} = D_T T^* \quad (\text{and } T D_T = D_{T^*} T).$$

Note that $D_{T^*} = f(TT^*)$ and $D_T = f(T^*T)$ with f continuous. By Stone-Weierstraß we can write f as the uniform limit of polynomials P_n : $D_{T^*} = f(TT^*) = \lim P_n(TT^*)$. Then we have
$T^* D_{T^*} = T^* \lim P_n(TT^*) = T^* \lim \sum a_n (TT^*)^n = \lim T^* \sum a_n (TT^*)^n$
$= \lim \sum a_n (T^*T)^n T^* = \lim P_n(T^*T) T^* = D_T T^*$

(and analogous $TD_T = D_{T^*}T$).
Then we can develope $\| h'_{-1} \|^2 + \| h'_0 \|^2$ using (*):
$\| h'_{-1} \|^2 + \| h'_0 \|^2 = \| D_T h_0 - T^* h_1 \|^2 + \| T h_0 + D_{T^*} h_1 \|^2 = \langle D_T h_0 - T^* h_1, D_T h_0 - T^* h_1 \rangle + \langle T h_0 + D_{T^*} h_1, T h_0 + D_{T^*} h_1 \rangle = \langle (1 - T^*T) h_0, h_0 \rangle - \langle D_T T^* h_1, h_0 \rangle - \langle T D_T h_0, h_1 \rangle + \langle T T^* h_1, h_1 \rangle + \langle T^* T h_0, h_0 \rangle + \langle T^* D_{T^*} h_1, h_0 \rangle + \langle D_{T^*} T h_0, h_1 \rangle + \langle (1 - TT^*) h_1, h_1 \rangle = \| h_0 \|^2 + \| h_1 \|^2$.

As a consequence, we find that U is an isometry. Moreover U is surjective since it is easy to invert U. Given $h' = (h'_n)_{n \epsilon Z}$ in $\tilde{\mathcal{H}}$, we have $h' = Uh$ with $h = (h_n)_{n \epsilon Z}$ defined by $h_n = h'_{n-1}$ if $n \notin \{0,1\}$, $h_0 = D_T h'_{-1} + T^* h'_0$ and $h_1 = -T h'_{-1} + D_{T^*} h'_0$. Equivalently it is clear that U is invertible from the following identity for 2×2 matrices with operator entries

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} D_T & -T^* \\ T & D_{T^*} \end{pmatrix} \begin{pmatrix} D_T & T^* \\ -T & D_{T^*} \end{pmatrix}$$
$$= \begin{pmatrix} D_T & T^* \\ -T & D_{T^*} \end{pmatrix} \begin{pmatrix} D_T & -T^* \\ T & D_{T^*} \end{pmatrix}$$

Therefor we conclude that U is a surjective isometry, hence a unitary operator.

Von Neumann's inequality: Let C be a contraction in $\mathcal{H}$. Then

$$\| p(C) \| \leq \sup_{|z|=1} | p(z) |$$

$\forall$ polynomials p.

Proof: First we will prove this for a unitary operator U on K.
$Uf = \int_0^{2\pi} e^{it} \mathrm{d}E(t) f = \lim \sum e^{it_j} (E(t_j) - E(t_{j-1})) f, \quad E(t) : R \mapsto L(K)$
$E(t)$ is a projection so $E^*(t) = E(t)$ and $E(t)^2 = E(t)$. $E(t)E(s) = E(s)$ $E(t) = E_{min}(t, s)$. You can also write $E(t) = \lim_{s \downarrow t} E(s)$. It's easy to see that $E(t) = I$ if $t > 2\pi$ and $E(t) = 0$ if $t < 0$.
Now you can write $p(U)f$ as $\int_0^{2\pi} p(e^{it}) \mathrm{d}E(t) f$ and

$$\| p(U)f \| \leq \int_0^{2\pi} | p(e^{it}) | \mathrm{d}E(t)f \leq \sup_{t \epsilon [0,2\pi]} | p(e^{it}) | \| \int_0^{2\pi} 1 \, \mathrm{d}E(t)f \| =$$
$$= \sup_{t \epsilon [0,2\pi]} | p(e^{it}) | \| E(2\pi)f - E(0)f \| = \sup_{t \epsilon [0,2\pi]} | p(e^{it}) | \| f \| .$$

So $\| p(U) \| \leq \sup_{|z|=1} | p(z) | \quad \forall$ polynomials p.
Now take C a contraction. By the Dilationtheorem there is a Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ isometrically as a subspace and a unitary operator $U : \tilde{\mathcal{H}} \mapsto \tilde{\mathcal{H}}$ such that $\forall\, n \geq 0 \quad C^n = P_{\mathcal{H}} U^n_{|\mathcal{H}}$.
From this follows:

$$\| p(C) \| = \| P_{\mathcal{H}} \, p(U)_{|\mathcal{H}} \| \leq \| p(U) \| \leq \sup_{|z|=1} | p(z) |$$

$\forall$ polynomials p.

Appendix B

Definition: A space A

(a) is called an algebra over $\mathbb{C}$ if
A is a linear space over $\mathbb{C}$
there is a multiplication with properties:
$(xy)z = x(yz)$
$\lambda(xy) = (\lambda x)y = x(\lambda y)$
$x(y+z) = xy + xz; \ (y+z)x = yx + zx \quad \forall\, x,y,z \in A,\ \lambda \in \mathbb{C}.$

(b) is called commutative if $\forall\, x,y \in A \quad xy = yx.$

(c) has a unit if $\exists\, e \in A$ such that $ea = ae = a \quad \forall\, a \in A.$

(d) is normed if there is a norm $\| \ \|$ on A with $\forall\, x,y \in A$
$\| xy \| \leq \| x \| \| y \|.$

(e) is called a Banach algebra if A is an algebra and $(A, \| \ \|)$ is complete.

(f) is called a $*$-algebra if A is an algebra and $\exists\, * : A \mapsto A$ with properties:
$(x^*)^* = x$
$(x+y)^* = x^* + y^*$
$(\lambda x)^* = \bar{\lambda} x^*$
$(xy)^* = y^* x^* \quad \forall\, x,y \in A,\ \lambda \in \mathbb{C}.$

(g) is unitary if A is a $*$-algebra with unit and $\forall\, u \in A \quad u^*u = uu^* = e.$

(h) is selfadjoint if A is a $*$-algebra and $x^* = x \quad \forall\, x \in A.$

(j) is called a Banach*-algebra if
(i) A is a Banach space
(ii) A is a $*$-algebra
(iii) $\forall\, x \in A \quad \| x^* \| = \| x \|.$

(k) is called a C^*-algebra if A is a Banach*-algebra and $\forall\, x \in A$
$\| xx^* \| = \| x \|^2.$

Examples: There are some examples of C^*-algebras which we used in this essay. These are:
$B(\mathcal{H})$, $\mathcal{C}(\partial\mathbb{D})$ and the disc algebra $\mathcal{A}$

Definition: A map $\phi : A \mapsto B$ is called

(a) a homomorphism if
$\phi(x+y) = \phi(x) + \phi(y)$
$\phi(\lambda x) = \lambda\phi(x)$
$\phi(xy) = \phi(x)\phi(y) \quad \forall\, x,y \in A,\ \lambda \in \mathbb{C}.$

(b) a $*$-homomorphism if
(i) ϕ is a homomorphism
(ii) $\phi(x^*) = \phi(x)^* \quad \forall\, x \in A.$

Definition: **(a)** A map $\pi : G \mapsto B(\mathcal{H})$ where G is a group and $\mathcal{H}$ a Hilbert space is called a representation if
$\pi(1) = I$
$\pi(st) = \pi(s)\pi(t)$
and π is unitary if also $\pi(t)^{-1} = \pi(t)^*.$

(b) A map $\rho : A \mapsto B(\mathcal{H})$ where A is a $*$-algebra and $\mathcal{H}$ a Hilbert space is called a $*$-representation if
(i) ρ is linear
(ii) ρ is a representation
(iii) $\rho(a^*) = \rho(a)^*$.

(c) A map $\rho : A \mapsto B(\mathcal{H})$ is called a C^*-algebraic representation if A is a C^*-algebra, $\mathcal{H}$ a Hilbert space and ρ is a $*$-representation.

About $*$-representations we have the following Lemma:

Lemma: Let $\rho : A \mapsto B(\mathcal{H})$ be a $*$-representation on a C^*-algebra A and assume A has a unit. Then necessarily $\| \rho \| = \sup_{a \neq 0\, \epsilon A} \frac{\|\rho(a)\|_{\mathcal{H}}}{\|a\|} \leq 1$.

Proof: Clearly ρ maps unitaries to unitaries:
$\rho(u)\rho(u)^* = \rho(uu^*) = \rho(e) = I = \rho(e) = \rho(u^*u) = \rho(u)^*\rho(u)$ for $u^*u = uu^* = e$.
Hence $\| \rho(u) \| \leq 1$ for any unitary u. Let x be a hermitian element: $x = x^*$ and $\| x \| \leq 1$. Then any $u = x + i\sqrt{1 - x^2}$ is unitary and $x = \mathrm{Re}\, u$.
Also follows $\| \rho(x) \| = \| \rho(Re\ u) \| = \| \rho(\frac{u+u^*}{2}) \| \leq \frac{1}{2} \| \rho(u) \| + \frac{1}{2} \| \rho(u)^* \| \leq \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$.
Hence $\| \rho(x) \| \leq 1$ for any hermitian in the unit ball. Finally, $\| u^*u \| = \| u \|^2$, so that
$\| \rho(x) \|^2 = \| \rho(x)^*\rho(x) \| = \| \rho(x^*x) \| = \| x^*x \| \| \rho(\frac{x^*x}{\|x^*x\|}) \| \leq \| x \|^2$,
and $\frac{\|\rho(x)\|}{\|x\|} \leq 1\ \forall\, x$ which means $\| \rho \| \leq 1$.

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