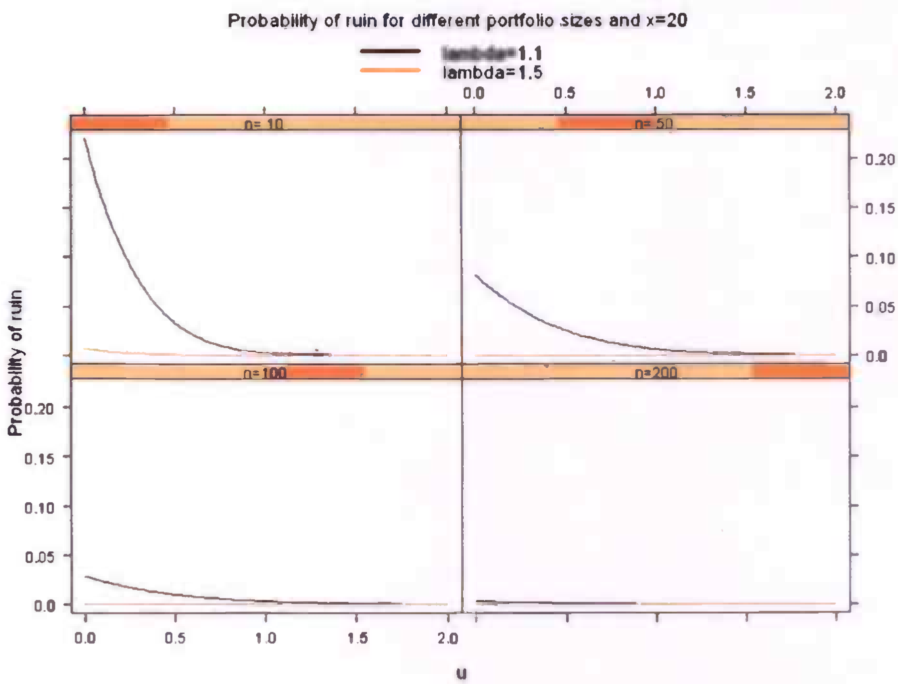
Ruin problem in life insurance

Saskia Peters

24th January 2007



Ruin problem in life insurance

Saskia Peters

24th January 2007

Abstract

We consider an insurer with one portfolio of insureds and we assume the insureds contracted the same life insurance. For different portfolios, we investigate the effect of different factors on the probability of ruin. Therefore, we first have a look at a portfolio consisting of one insured and a portfolio consisting of two insureds. After that, we simulate the probability of ruin for portfolios with other sizes.

Contents

1 Introduction **4**

2 The single premium **6**

3 The life time distribution of a life aged x **8**

4 Portfolio consisting of one insured **9**
4.1 The surplus process 9
4.2 The relation between S and the probability of ruin 12

5 Portfolio consisting of two insureds **16**
5.1 Order statistics 16
5.2 Probability of ruin at time $T_{(1)}$ 17
5.3 Probability of ruin at time $T_{(2)}$ 18
5.4 Probability of ruin in case of two insureds 19
5.5 The effect of age and λ 20

6 Portfolio consisting of more than two insureds **23**
6.1 The simulation 23
6.2 The effect of the value of S 25
6.3 The effect of the size of the portfolio 26
6.4 The effect of the ages of the insureds 28

7 Conclusion **30**

8 Appendix 1: SPLUS script files **33**
8.1 One insured 33
8.2 Two insureds 37

9 Appendix 2: Matlab functions **39**
9.1 Simulation for more than two insureds 39

1 Introduction

Insurers of life insurances have portfolios with insureds. These portfòlios could be very diverse. For example, the insureds shall have different ages, the size of the portfolios can be different and the kind of life insurances can be different. For the insurer, it is interesting to know if his portfolio is risky. By risky, we mean the probability of ruin. We speak of ruin, when the capital of the insurer becomes negative for the first time [2]. The probability of ruin is the probability that ruin occurs within a finite time horizon. So for the insurer, it is interesting to know if there is a high probability of ruin or not with his portfolio. If there is a high probability of ruin, he has to try to change this probability to a lower value. Therefore, it is also interesting to know what the effect is of different factors on this probability, like the age of the insureds or the portfolio size. If an insurer knows what the effect of different factors are, he knows also how he can try to decrease the probability of ruin.

We will consider the ruin problem in life insurance. Therefore, we analyze a portfolio of an insurer. This portfolio consists of N insureds with each contracted a whole life insurance. Every insured pays a single premium. This premium is paid at the beginning of the first year of the contract. For that, the insured receives a positive amount at the end of the year of his death.

The insurer starts with an initial capital. The earnings of the insurer are the premium payments of the insureds and the earnings of interest. The premium that an insured has to pay depends on his age at the moment he enters into the contract, the kind of contract (in this case a whole life insurance) and the rate of interest. We assume the interest rate is constant. On the other hand, we assume that every insured has the same contract. This implies that the amount, receiving at the end of the year of death, is the same for every insured.

To consider the probability of ruin, we will look at the insurer's surplus process. This surplus might become negative and when this first happens, ruin occured [2]. There are different factors that might influence the probability of ruin. Four effects that we will investigate are:

1. Does the probability of ruin depend on the amount of the payment S to the insured at the end of the year of death? And if there is dependence, how does the probability of ruin depend on S?
2. Does the probability of ruin depend on the number of insureds N? And if there is dependence, how does the probability of ruin depend on N?
3. Does the probability of ruin depend on the ages of the insureds? And if there is dependence, how does x affect the probability of ruin?
4. What is the effect of varying λ?

When we consider the behavior of the portfolio under certain circumstances, we first have to define the portfolio. It is realistic that there will be new persons coming into the portfolio. But this event is difficult to put in a model. Therefore, we will consider only portfolios that become empty. So we define a model without new insureds coming into the portfolio and look at how to compute the probability of ruin.

First, we will do different computations. In section 2, we derive formulas to compute the single premiums the insureds have to pay. In section 3, we derive the lifetime distributions of the insured. With these information, we can look at different portfolios. In section 4, we consider a portfolio consisting of one insured. For this portfolio, it is easy to analyze the probability of ruin, because ruin can only occur at one time; when the insured dies. In section 5 we consider a portfolio that consists of two insureds. This becomes a bit more difficult, because there are two times ruin can occur and we do not know which insured will die first. Therefore, we will use order statistics. After that, we consider in section 6 portfolios consisting of more insureds. Because in this case it is difficult to calculate the probability of ruin analytically, we make use of simulation.

2 The single premium

Every insured has to pay a single premium at the beginning of the contract. This premium will depend for example on the age of the insured. Secondly, it depends on the rate of interest. And also the contract, especially the amount of money the insured receives after his death, will influence the premium. Let S be the amount the insured receives at the end of the year of death and let T be the future lifetime of the insured.

We make some assumptions about the time of the different payments. The payment of the premium is at January 1 of the year that the contract starts. Secondly, the insurer earns interest, and this payment to the insurer occcurs at December 31 of every year. At last, the payment S to the insured occurs at December 31 of the year of death.

The variable t defines the time, measured in years. $t = 0$ is at January 1, and defines the time the insured pays the single premium and the contract starts. From $t = 0$ on, year 1 starts. This means that from $t = 0$ up to $t = 1$, we are speaking about year 1. For example, suppose the insured dies in year 5; that is between $t = 4$ and $t = 5$. In that case is T equal to 5. As a result, the age of death is $x + T - 1$, because the insured survives for $T - 1$ years and dies in the next year. For convenience, when we speak about $T = 5$, we mean January the first of the next year. This is because at this time, all the payments of the last year are occured. So when we speak about T, we mean the day after the payment of S and the earnings of interest; January the first of year $T+1$.

To calculate the single premium, we have to look at the present value of S at time T. This is equal to

$$Z = S \cdot \left(\frac{1}{1+i}\right)^T,$$

where Z is the present value of S [1]. We see that Z is a random variable, because T is also a random variable. Therefore, we define

p_x = probability that a life aged x will survive at least one year,
q_x = probability that a life aged x will die within one year,
${}_kp_x$ = probability that a life aged x will survive at least k years.

The last variable can be given by

$${}_kp_x = p_x p_{x+1} p_{x+2} \cdots p_{x+k-1}, \qquad k = 1, 2, 3, \ldots$$

Now we know about Z that

$$\Pr\left(Z = S \cdot \left(\frac{1}{1+i}\right)^k\right) = \Pr(T = k) = {}_{k-1}p_x q_{x+k-1}, \qquad k = 1, 2, 3, \ldots$$

where ${}_0p_x = 1$. The single premium for a life aged x with a contract paying S at the end of the year of death is therefore given by [1]

$$K(x, i, S) = E\left(S \cdot \left(\frac{1}{1+i}\right)^T \middle| x\right) = S \cdot \sum_{k=1}^{\omega - x + 1} \left(\frac{1}{1+i}\right)^k {}_{k-1}p_x q_{x+k-1} \qquad (1)$$

where ω is the highest age, that for example appears in a life table. The probabilities ${}_kp_x$ and q_x can be estimated from a life table. In the next section, we will discuss this issue.

The calculated premium in (1) is an actuarial premium. There is a risk for the insurer, that the insured dies within a small time interval. In that case, there will be a loss for the insurer. The insurer has also costs, other than the amount he has to pay to the insureds, like administration costs. Because of the risk for the insurer and of the other costs, the insurer will ask a higher premium to the insureds than we calculated. Therefote, we multiply $K(x, i, S)$ by a constant $\lambda > 1$. So, the premium the insured has to pay is given by

$$C(x, i, S) = \lambda K(x, i, S).$$

3 The life time distribution of a life aged x

As mentioned in the previous section, we will estimate the probabilities ${}_kp_x$ and q_x from a life table. A life table starts with a number of people l_0 of age 0, for example $l_0 = 100000$. For every age x you can find the corresponding l_x; the number of people who survive to age x. From the life table, we can estimate the probabilities. ${}_kp_x$ is the probability that a life aged x will survive for at least k years, so he will reach at least the age of $x + k$ years. Therefore

$$ {}_kp_x = \frac{l_{x+k}}{l_x}. $$

We know that $p_x + q_x = 1$. Therefore, q_x can be found by

$$ \begin{aligned} q_x &= 1 - p_x \\ &= 1 - \frac{l_{x+1}}{l_x} \\ &= \frac{l_x - l_{x+1}}{l_x}. \end{aligned} $$

We are interested in the future lifetime distribution of a life aged x. Clearly, this distribution depends on x. The probability density function (pdf) of the future lifetime T of a life aged x, can be given by

$$ \begin{aligned} f_T(t,x) &= \Pr(T = t|x) \\ &= {}_{t-1}p_x q_{x+t-1} \\ &= \frac{l_{x+t-1}}{l_x} \frac{l_{x+t-1} - l_{x+t}}{l_{x+t-1}} \\ &= \frac{l_{x+t-1} - l_{x+t}}{l_x}. \end{aligned} \tag{2} $$

The cumulative distribution function (cdf) can be derived from the pdf. We find

$$ \Pr(T \leq t|x) = \sum_{j=1}^{t} {}_{j-1}p_x q_{x+j-1} \tag{3} $$

$$ = \frac{l_x - l_{x+t}}{l_x} $$

$$ = 1 - {}_tp_x. \tag{4} $$

The cdf can now be given by

$$ F_T(t,x) = \begin{cases} \sum_{j=1}^{t} {}_{j-1}p_x q_{x+j-1} & \text{if } t \leq \omega - x + 1 \\ 1 & \text{if } t > \omega - x + 1 \end{cases} $$

We will use the distribution function of T in the next sections.

4 Portfolio consisting of one insured

Consider a portfolio consisting of one insured. We assume there will be no new insureds coming into the portfolio. So when the insured dies, the portfolio is empty. At that time (actually, at the beginning of the next year), we can analyze if ruin occurs. This is also the only time ruin can occur, because after this time, there will be no expenditures.
Define:

x = age of the insured,
T = future lifetime of the insured.

In order to describe the surplus process, we define:

$U(t)$ = the surplus at time t,
u = the initial capital,
i = interest rate,
C = single premium the insured has to pay at the beginning of the contract, where $C(x, i, S) = \lambda K(x, i, S)$ as explained in section 2,
S = the payment of the insurer to insured at the end of the year of death.

4.1 The surplus process

At the end of year T, the year of death of the insured, the insurer has to pay an amount of money and there is a possibility that ruin occurs. Remember, for convenience, when we talk about time $t = T$, we mean the time immediately after the payment to the insured. Because in case of one insured the only time ruin can occur is at time $t = T$, we look at the surplus process at this time:

$$\begin{aligned} U(T) &= u(1+i)^T + C(1+i)^T - S \\ &= (u+C)(1+i)^T - S. \end{aligned}$$

At $t = T$, the initial capital and the premium income have been increased by a factor $(1+i)^T$ due to compound interest. At the same time, capital decreases with an amount of S; the payment to the insured.

We will have a closer look at the surplus process. Three different cases are:

1. $(u+C)(1+i) \geq S$
   Suppose the sum of the inital capital and the premium payment multiplied by $(1+i)$ is greater than or equal to the payment to the insured. $(u+C)(1+i)$ is capital of the insurer after one year. At this time, it is the first possibility that the insurer has to pay S to the insured and also the first possibility of ruin. The value $(u+C)(1+i)$ can only become greater due to compound interest. S will always stay the same. This means that in this case the insurer can always pay an amount of S without a possibility of ruin. So
   $$\psi(u) = 0, \tag{5}$$
   where $\psi(u)$ is the probability of ruin, with inital capital u.

2. $(u+C)(1+i) < S$
   Suppose the sum of u and C multiplied by $(1+i)$ is smaller than S. Again,

$(u + C)(1 + i)$ is capital of the insurer after one year. If the insured dies in year $T = 1$, the insurer has to pay at the end of the year an amount of S. And after this payment, his capital is equal to $(u + C)(1 + i) - S < 0$, so ruin occurs. This means that in case 2, there is always a possibility of ruin:

$$\psi(u) > 0.$$

3. $(u + C)(1 + i)^{\omega - x + 1} < S$
Another extreme for the future lifetime of the insured is that he reaches the maximum age of ω. In that case is the year of death equal to $T = \omega - x + 1$. So the maximum capital is equal to $(u + C)(1 + i)^{\omega - x + 1}$. If this maximum capital is smaller than the payment to the insured, ruin will always occur:

$$\psi(u) = 1. \tag{6}$$

So we have found some boundaries for the probability of ruin. We would like to have a closed formula for this probability. Suppose

$$(u + C)(1 + i) < S \leq (u + C)(1 + i)^{\omega - x + 1}.$$

We make this assumption, because for the other cases, we know already the probability (either zero or one). Ruin occurs when capital becomes negative for the first time, so when $U(T) < 0$. We can write this as

$$\begin{aligned} U(T) &< 0 \\ (u + C)(1 + i)^T - S &< 0 \\ (1 + i)^T &< \frac{S}{u + C} \\ T &< \frac{\log \frac{S}{u+C}}{\log(1 + i)}. \end{aligned} \tag{7}$$

So when T satisfies the inequality in (7), ruin occurs. The probability of ruin can be given by

$$\psi(u) = \Pr\left(T < \frac{\log \frac{S}{u+C}}{\log(1 + i)}\right).$$

With this definition, we can verify the probabilities found in (5) and (6). When $(u + C)(1 + i) \geq S$, then

$$\begin{aligned} \Pr\left(T < \frac{\log \frac{S}{u+C}}{\log(1 + i)}\right) &\leq \Pr\left(T < \frac{\log \frac{(u+C)(1+i)}{u+C}}{\log(1 + i)}\right) \\ &= \Pr\left(T < \frac{\log(1 + i)}{\log(1 + i)}\right) \\ &= \Pr(T < 1) \\ &= 0. \end{aligned}$$

Because $T \geq 1$, we know that $\psi(u) = \Pr(T < 1) = 0$. In the other case, when

$(u+C)(1+i)^{\omega-x} < S$, then

$$\begin{aligned}\Pr\left(T < \frac{\log\frac{S}{u+C}}{\log(1+i)}\right) &> \Pr\left(T < \frac{\log\frac{(u+C)(1+i)^{\omega-x+1}}{u+C}}{\log(1+i)}\right)\\ &= \Pr\left(T < \frac{(\omega-x+1)\log(1+i)}{\log(1+i)}\right)\\ &= \Pr(T<\omega-x+1)\end{aligned}$$

We know that $T \leq \omega - x + 1$, so $\Pr(T \leq \omega - x + 1) = 1$. Therefore, the inequality $\psi(u) > \Pr(T < \omega - x + 1)$ implies $\psi(u) \geq \Pr(T \leq \omega - x + 1) = 1$. So

$$\psi(u) = \Pr\left(T < \frac{\log\frac{S}{u+C}}{\log(1+i)}\right) = 1.$$

We can conclude that the probability of ruin is defined by

$$\psi(u) = \begin{cases} 0 & \text{if } S \leq (u+C)(1+i) \\ \Pr\left(T < \frac{\log\frac{S}{u+C}}{\log(1+i)}\right) & \text{if } (u+C)(1+i) < S \leq (u+C)(1+i)^{\omega-x+1} \\ 1 & \text{if } S > (u+C)(1+i)^{\omega-x+1}. \end{cases} \tag{8}$$

Now we have a closer look at the second case of equation (8), so where $(u+C)(1+i) < S \leq (u+C)(1+i)^{\omega-x+1}$, and at inequality (7). To simplify notation, we define D as the value of $\frac{\log\frac{S}{u+C}}{\log(1+i)}$ rounded up to the next integer. Because T is in years, inequality (7) implies $T < D$. In this case, we know that the probability of ruin is equal to the probability that $T < D$. Because the age of the insured is x at time $t = 0$, and in case of ruin he will die at most in year $D - 1$, it is also the probability that the insured dies before he reaches the age of $x + D - 1$.

With use of the cumulative distribution function given in (3), we can derive the probability of ruin at time T (for the second case in equation (8)):

$$\begin{aligned}\Pr\left(T < \frac{\log\frac{S}{u+C}}{\log(1+i)}\right) &= \Pr(T < D)\\ &= \Pr(T \leq D-1)\\ &= F_T(D-1, x)\\ &= \sum_{i=1}^{D-1} {}_{i-1}p_x q_{x+i-1}.\end{aligned}$$

And these probabilities can be estimated from a life table as explained in section 3. So in this case, we can calculate the probability of ruin analytically. With formule (8), we can investigate the effect of different factors on the probability of ruin in case of one insured.

4.2 The relation between S and the probability of ruin

The probability of ruin depends on D as defined in the previous section, and by definition, this variable depends S, C, u and i. Suppose we only vary S. Remember that $C = C(x, i, S)$ depends also S. So when we change S, this will influence $\psi(u)$ directly, and indirectly by C. We know that

$$C(x, i, S) = \lambda S \cdot \sum_{k=1}^{\omega-x+1} \left(\frac{1}{1+i}\right)^k {}_{k-1}p_x q_{x+k-1}. \tag{9}$$

Suppose that the variables x, i and λ are fixed and we vary S. In that case, we can C write as

$$C = \alpha S,$$

where

$$\alpha = \lambda \sum_{k=1}^{\omega-x+1} \left(\frac{1}{1+i}\right)^k {}_{k-1}p_x q_{x+k-1}. \tag{10}$$

The value α can be seen as the single premium for a life aged x when $S = 1$. Now suppose we multiply S by a constant $\beta > 1$, so S increases. Then also C will be multiplied by β, as can be seen in equation (10). So C is proportional to S for fixed x, i and λ. When we look at

$$\frac{S}{u+C}, \tag{11}$$

which appears in the definition of D, we see the following: when S is multiplied by β, the numerator becomes β times as big. In the denominator, C will in that case also be multiplied by β and consequently the whole denominator is multiplied by less than β. So when S is multiplied by β, the fraction in (11) increases:

$$\frac{\beta S}{u+\beta C} > \frac{\beta S}{\beta u+\beta C} = \frac{S}{u+C}, \quad \beta > 1.$$

As a result, D becomes larger. Therefore, when S increases, the probability of ruin is higher. This will ofcourse only be the case when the probability of ruin is not equal to one. On the other hand, when β is between zero and one, S decreases and the reverse happens: D becomes smaller and the probability of ruin is also smaller (and this only happens when the probability of ruin is not equal to zero).

We will illustrate this relation by a graph. We consider a portfolio of one insured aged 20, so $x = 20$. Furthermore, we assume that $i = 0.03$ and $\lambda = 1.1$ or $\lambda = 1.5$. We choose two different values for λ, so we can also look at the effect of this variable. To calculate the probabilities of dying at a certain age, we use the life table from the '*Actuarieel Genootschap*' [6]. With use of the formulas from section 3, we can calculate the probability density function and use this in the calculations. We vary S to look at the effect of it, and therefore we choose $S = 1$ and $S = 5$. For these values, we calculate the probability of ruin for $u = 0, 0.01, 0.02, \ldots, 2$. We use SPLUS to do these computations [3]. The result is given in figure 1.

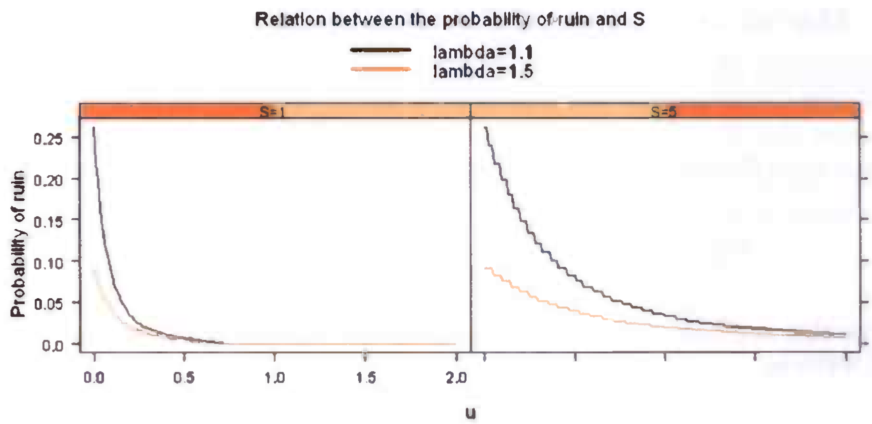


Figure 1: $\psi(u)$ and S

We can see that when u increases, the probability of ruin decreases. This is a logical relation, because when u is bigger, the insurer has more capital and the probability that he cannot pay S to the insured at the end of the year of death is smaller. Furthermore, when we look at the same u for $S = 1$ and $S = 5$, we see that in case of $S = 1$ the probability of ruin is smaller than in case of $S = 5$. This is what we expected on base of the reasoning at the beginning of this section.

When we look at ψ for $u = 0$, we see something remarkable. We see that when $\lambda = 1.1$, $\psi(0) = 0.26$ for $S = 1$ and also for $S = 5$ and when $\lambda = 1.5$, $\psi(0) = 0.09$ for both S. So in case of $u = 0$, ψ has the same value for both S. To see how this can happen, we look at the formula for the probability of ruin. When u is zero, the formula becomes

$$\psi(0) = \Pr\left(\frac{\log \frac{S}{C}}{\log(1+i)}\right).$$

C is the premium, and this one is proportional to S. We saw in (10) that we can write C as αS where α is defined in (10). This results in

$$\begin{aligned}\psi(0) &= \Pr\left(\frac{\log \frac{S}{\alpha S}}{\log(1+i)}\right)\\ &= \Pr\left(\frac{\log \frac{1}{\alpha}}{\log(1+i)}\right)\end{aligned}$$

and so for u is zero, the probability of ruin does not depend on S.

Now suppose u is proportional to S, so $u = \gamma S$. Again we will use that $C = \alpha S$. From the definition of α we know that $\alpha > 0$. Because u and S are positive, we assume also that $\gamma > 0$. It follows that

$$\begin{aligned}
\psi(u) &= \Pr\left(T < \frac{\log \frac{S}{u+C}}{\log(1+i)}\right) \\
&= \Pr\left(T < \frac{\log \frac{S}{\gamma S+\alpha S}}{\log(1+i)}\right) \\
&= \Pr\left(T < \frac{\log \frac{1}{\gamma+\alpha}}{\log(1+i)}\right) \\
&= \Pr\left(T < \frac{-\log(\gamma+\alpha)}{\log(1+i)}\right) \qquad (12)
\end{aligned}$$

In this case, equation (12) does not depend on S anymore. And because α and γ are constants, the fraction in (12) will be constant. That means that fonevery S, the probability of ruin will be constant. We see in the equation a trade-off between initial capital and premium. High initial capital (which reflects in a high γ) permits the insurer to ask the insured a low premium (α is low). On the other hand, when the insurer has low initial capital (γ is low), he has to ask a higher premium to the insured (α has to be higher).

Furthermore, the denominator is always positive, because $1+i > 1$. And if $\gamma+\alpha \geq 1$, the numerator will be negative, and so the whole fraction is negative. In that case is the probability of ruin equal to zero, because the probability that T is smaller than something negative is zero. This result follows also from case 1 in section 4.1: when $\gamma + \alpha \geq 1$, then $u + C = \gamma S + \alpha S \geq S$, and then also $(u+C)(1+i) \geq S$. This implies that $\psi(u) = 0$. On the other hand, if $0 < \gamma+\alpha < 1$ and $\frac{1}{\gamma+\alpha} > 1+i$, then $\frac{-\log(\gamma+\alpha)}{\log(1+i)} \geq 1$ and consequently $\psi(u) > 0$.

We will illustrate these results by an example. Consider a portfolio consisting of one insured aged 20, assume $\lambda = 1.1$ and let u proportional to S, so $u = \gamma S$. This is the same as $S = \frac{1}{\gamma}u$. Let $\gamma = 1$, then $S = u$ and $C = \alpha S = \alpha u$ and remember that α is equal to the single premium when $S = 1$. The result is given in the first graph of figure 2.

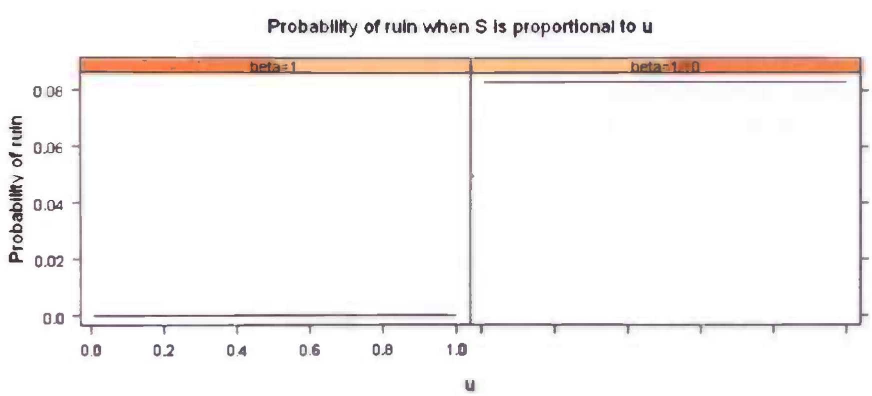


Figure 2: S proportional to u

The premium for a life aged 20, when $S = 1$, is equal to 0.2250054. This means that $\alpha = 0.2250054$ and $\gamma + \alpha = 1.2250054 > 1$. This means that the probability of ruin is equal to zero. You can see in the figure that the probability of ruin is indeed equal to zero.

The second graph gives the result when $\gamma = \frac{1}{10}$. Then $S = 10u$ and $C = \alpha S$ where α is again the premium when $S = 1$. This results in $\gamma + \alpha = 0.3250054 < 1$, so $\psi(u) > 0$. Then

$$\frac{-\log(\gamma + \alpha)}{\log(1 + i)} = \frac{-\log(0.3250054)}{\log(1.03)} = 38.0229710$$

and from the life table, we can derive that

$$\psi(u) = \Pr(T < 39) = \Pr(T \leq 38) = 0.08295849$$

as also can be seen in the graph.

5 Portfolio consisting of two insureds

Now, we consider a portfolio which contains two insureds. To calculate the probability of ruin, we have to define some new symbols:

T_j = future lifetime of insured j, $j = 1, 2$,
x_j = age of the insured j,
C_j = single premium of insured j.

In case of two insureds, there is at more than one time a possibility of ruin. Ruin can occur at $t = T_1$ and at $t = T_2$. We assume that T_1 and T_2 are independent. In this case, we do not know who will die first. Therefore, we will have a look at the order statistics of T_1 and T_2.

5.1 Order statistics

Consider $T_{(1)}$ and $T_{(2)}$ where $T_{(1)} \leq T_{(2)}$. These are the order statistics corresponding to T_1 and T_2, where T_1 and T_2 are ordered from small to big. So $T_{(1)}$ is equal to the smallest one of T_1 and T_2 and $T_{(2)}$ is equal to the largest one of T_1 and T_2 [4]:

$$\begin{aligned} T_{(1)} &= \min(T_1, T_2) \\ T_{(2)} &= \max(T_1, T_2). \end{aligned}$$

The distribution functions

$$F_{T_1}(t) = \Pr(T_1 \leq t)$$

and

$$F_{T_2}(t) = \Pr(T_2 \leq t)$$

can be found from the life table. With this information, we can derive the distributions of the order statistics. We start with the distribution of $T_{(2)}$:

$$\begin{aligned} F_{T_{(2)}}(t) &= \Pr(T_{(2)} \leq t) \\ &= \Pr(T_{(1)} \leq t, T_{(2)} \leq t) \\ &= \Pr(T_1 \leq t, T_2 \leq t) \\ &= \Pr(T_1 \leq t)\Pr(T_2 \leq t) \\ &= F_{T_1}(t)F_{T_2}(t). \end{aligned} \tag{13}$$

So, the distribution function of $T_{(2)}$ can be found by multiplying the distribution functions of T_1 and T_2. The distribution function of $T_{(1)}$ is given by

$$\begin{aligned} F_{T_{(1)}}(t) &= \Pr(T_{(1)} \leq t) \\ &= 1 - \Pr(T_{(1)} > t) \\ &= 1 - \Pr(T_{(1)} > t, T_{(2)} > t) \\ &= 1 - \Pr(T_1 > t, T_2 > t) \\ &= 1 - \Pr(T_1 > t)\Pr(T_2 > t) \\ &= 1 - (1 - \Pr(T_1 \leq t))(1 - \Pr(T_2 \leq t)) \\ &= 1 - (1 - F_{T_1}(t))(1 - F_{T_2}(t)) \\ &= F_{T_1}(t) + F_{T_2}(t) - F_{T_1}(t)F_{T_2}(t). \end{aligned}$$

It can be seen that the distribution function of $T_{(1)}$ is equal to the sum of the distribution functions of T_1 and T_2 minus the product of these two distribution functions.

5.2 Probability of ruin at time $T_{(1)}$

In case of two insureds, there are two times ruin can occur; the first time is $t = T_{(1)}$ and the second time is $t = T_{(2)}$. We will analyze the surplus process at these times.

In year $T_{(1)}$, one of the insureds dies. The other one is still alive. We made the assumption that the insureds contracted the same insurance, which pays an amount of S at the end of the year of death. Therefore, we know that the insurer has to pay S at the end of year $T_{(1)}$. The surplus process at the end of year $T_{(1)}$ can be given by

$$U(T_{(1)}) = (u + C_1 + C_2)(1+i)^{T_{(1)}} - S.$$

The probability of ruin at this time is equal to

$$\begin{aligned} \psi_{T_{(1)}}(u) &= \Pr(U(T_{(1)}) < 0) \\ &= \Pr((u + C_1 + C_2)(1+i)^{T_{(1)}} - S < 0) \\ &= \Pr\left((1+i)^{T_{(1)}} < \frac{S}{u + C_1 + C_2}\right) \\ &= \Pr\left(T_{(1)} < \frac{\log\left(\frac{S}{u+C_1+C_2}\right)}{\log(1+i)}\right) \end{aligned} \tag{14}$$

Define D_1 as the value of $\frac{\log\left(\frac{S}{u+C_1+C_2}\right)}{\log(1+i)}$ rounded up to the next integer. In the same way as in section 4.1, the probability of ruin can be given by

$$\psi_{T_{(1)}}(u) = \begin{cases} 0 & \text{if } D_1 \leq 1 \\ \Pr(T_{(1)} \leq D_1 - 1) & \text{elsewhere.} \end{cases} \tag{15}$$

We know the distribution function of $T_{(1)}$, so we find that the probability of ruin in the second case of equation (15) is equal to

$$\begin{aligned} \psi_{T_{(1)}}(u) &= \Pr(T_{(1)} \leq D_1 - 1) \\ &= F_{T_{(1)}}(D_1 - 1) \\ &= F_{T_1}(D_1 - 1) + F_{T_2}(D_1 - 1) - F_{T_1}(D_1 - 1)F_{T_2}(D_1 - 1). \end{aligned} \tag{16}$$

So even if we do not know which insured will die first, we can compute the probability of ruin at the first time one of the insureds dies, by using the distribution functions of the lifetimes of the insureds.

5.3 Probability of ruin at time $T_{(2)}$

Suppose, at time $t = T_{(1)}$ ruin did occur. In that case, there is also ruin at time $t = T_{(2)}$, because the capital was negative on $t = T_{(1)}$ and after that time, there are no earnings and there is only a payment to the insured. So when we consider the probability of ruin at time $T_{(2)}$, we assume that ruin did not occur at $t = T_{(1)}$. This implies that we know that $T_{(1)} > D_1 - 1$. Because D_1 could be negative, we define a new variable:

$$E_1 = \begin{cases} 1 & \text{if } D_1 < 1 \\ D_1 & \text{elsewhere.} \end{cases}$$

Now we know that when ruin did not occur at time $T_{(1)}$, then $T_{(1)} > E_1 - 1$.

The surplus process at $t = T_{(2)}$ is given by

$$U(T_{(2)}) = U(T_{(1)})(1+i)^{T_{(2)}-T_{(1)}} - S. \tag{17}$$

At time $T_{(1)}$, capital of the insurer is equal to $U(T_{(1)})$. At time $T_{(2)}$, this capital is increased by a factor $(1+i)^{T_{(2)}-T_{(1)}}$. At the same time, capital decreases with an amount of S. The difficulty in the given equation for $U(T_{(2)})$ is the appearance of $T_{(1)}$. We speak of ruin when $U(T_{(2)}) < 0$ and this happens when

$$\begin{aligned} U(T_{(2)}) &< 0 \\ U(T_{(1)})(1+i)^{T_{(2)}-T_{(1)}} - S &< 0 \\ (1+i)^{T_{(2)}-T_{(1)}} &< \frac{S}{U(T_{(1)})} \\ T_{(2)} &< \frac{\log\left(\frac{S}{U(T_{(1)})}\right)}{\log(1+i)} + T_{(1)}. \end{aligned} \tag{18}$$

Again, $T_{(1)}$ appears in this inequality. To handle this difficulty, we will condition on $T_{(1)}$. First we define the function E_2:

$E_2(T_{(1)})$ is equal to $\frac{\log\left(\frac{S}{U(T_{(1)})}\right)}{\log(1+i)} + T_{(1)}$ rounded up to the next integer.
Then, the probability of ruin at time $T_{(2)}$ can be given by

$$\psi_{T_{(2)}}(u) = \begin{cases} 0 & \text{if } E_2(T_{(1)}) \leq 1 \\ \Pr(T_{(2)} \leq E_2(T_{(1)}) - 1 | U(T_{(1)}) > 0) & \text{elsewhere.} \end{cases} \tag{19}$$

When we consider the last equation we find

$$\begin{aligned} & \Pr(T_{(2)} \leq E_2(T_{(1)}) - 1 | U(T_{(1)}) > 0) \\ = \; & \Pr(T_{(2)} \leq E_2(T_{(1)}) - 1 | T_{(1)} > E_1 - 1) \\ = \; & \frac{\Pr(T_{(2)} \leq E_2(T_{(1)}) - 1 \text{ and } T_{(1)} > E_1 - 1)}{\Pr(T_{(1)} > E_1 - 1)} \end{aligned}$$

We would like to know something about the probability that $T_{(2)} \leq E_2(T_{(1)}) - 1$ and $T_{(1)} < E_1 - 1$. But these two events are dependent. Therefore, it is difficult to calculate this probability using the distribution functions of the order statistics. We can derive the probability of ruin in the a different way, where we do not use the order statistics:

$$\begin{aligned}
\psi_{T_{(2)}}(u) &= \Pr(T_2 \leq T_1 \leq E_2(T_2) - 1 \text{ and } T_2 > E_1 - 1) + \\
&\quad \Pr(T_1 \leq T_2 \leq E_2(T_1) - 1 \text{ and } T_1 > E_1 - 1) - \\
&\quad \Pr(T_1 = T_2 \text{ and } T_1 > E_1 - 1 \text{ and } T_2 > E_1 - 1) \\
&= \sum_{j=0}^{k} \Pr(T_2 \leq T_1 \leq E_2(T_2) - 1 \text{ and } T_2 = E_1 + j) + \\
&\quad \Pr(T_1 \leq T_2 \leq E_2(T_1) - 1 \text{ and } T_1 = E_1 + j) - \\
&\quad \Pr(T_1 = E_1 + j \text{ and } E_2 = D_1 + j) \\
&= \sum_{j=0}^{k} \Pr(E_1 + j \leq T_1 \leq E_2(E_1 + j) - 1) \Pr(T_2 = E_1 + j) + \\
&\quad \Pr(E_1 + j \leq T_2 \leq E_2(E_1 + j) - 1) \Pr(T_1 = E_1 + j) - \\
&\quad \Pr(T_1 = E_1 + j) \Pr(T_2 = E_1 + j) \\
&= \sum_{j=0}^{k} (F_{T_1}(E_2(E_1 + j) - 1) - F_{T_1}(E_1 + j - 1)) f_{T_2}(E_1 + j) + \\
&\quad (F_{T_2}(E_2(E_1 + j) - 1) - F_{T_2}(E_1 + j - 1)) f_{T_1}(E_1 + j) - \\
&\quad f_{T_1}(E_1 + j) f_{T_2}(E_1 + j)
\end{aligned}$$

where f and F are given in equations (2) and (4). Furthermore k is defined as the highest integer for which holds that $E_2(E_1 + k) > E_1 + k$. If there is no $k \geq 0$, such that the inequality holds, than the probability of ruin at time $T_{(2)}$ is equal to zero. This calculation is based on the following: first we compute the probability that ruin did not occur at time T_2 and ruin occurs at time T_1 (so we assume that insured 2 dies first), then plus the probability that ruin did not occur at time T_1 and ruin occurs at time T_2 (so we assume that insured 1 dies first), and minus the probability that both insureds dies in the same year.

We will give an explanation in words of the meaning of E_1 and E_2 and the relation between these two variables. E_1 is the lowest value of $T_{(1)}$ for which ruin will not occur. So if $T_{(1)}$ is smaller than E_1, then ruin occurs. E_2 is the lowest value of $T_{(2)}$ for which ruin will not occur. If $T_{(2)}$ is smaller than E_2, then ruin occurs. Now, suppose that $T_{(1)}$ is small, so the first death of an insured occurs soon, and suppose that $T_{(1)} > E_1$. That means that ruin did not occur. But because the first insured dies within a small time interval, there is not much capital. So to be sure that ruin will not occur at time $T_{(2)}$, this time has to be far in the future. On the other hand, when $T_{(1)}$ is much larger than E_1, so ruin did not occur at all, then the insurer has much capital left. So ruin will also not occur if the second insured dies within a small time interval after the death of the first insured. This means that when E_1 increases, E_2 decreases.

5.4 Probability of ruin in case of two insureds

In the previous sections, we derived formulas to compute the probability of ruin at $t = T_{(1)}$ and $t = T_{(2)}$. In the derivation of $\psi_{T_{(2)}}(u)$, we assumed that ruin did not occur at time $T_{(1)}$. Ruin is the first time that capital becomes negative.

So ruin can only occur at time $T_{(2)}$ if it did not occur at time $T_{(1)}$. Therefore, to compute the probability that ruin will ever occur in case of two insureds, we can just sum the two probabilities:

$$\psi(u) = \psi_{T_{(1)}}(u) + \psi_{T_{(2)}}(u).$$

With this formula, we can look at the effect of different factors on the probability of ruin.

5.5 The effect of age and λ

We have derived formulas to calculate the probability of ruin in case of two insureds. To see what the effect is of age and λ on the probability of ruin, we create four different portfolios:

1. $x_1 = 20$ and $x_2 = 20$: young portfolio,
2. $x_1 = 20$ and $x_2 = 30$: young mixed portfolio,
3. $x_1 = 20$ and $x_2 = 50$: mixed portfolio,
4. $x_1 = 50$ and $x_2 = 50$: old portfolio.

With these different portfolios, we can look for the effect of 'young' and 'old' portfolios, where young means a portfolio of relatively young insureds and old means a portfolio of relatively older insureds. We compute the probability of ruin for each portfolio, for $\lambda = 1.1$ and for $\lambda = 1.5$. So we vary x and λ to see the effect of it, and we keep the other variables constant. We choose the following values: $S = 1$ and $i = 0.03$. The result is given in figure 3.

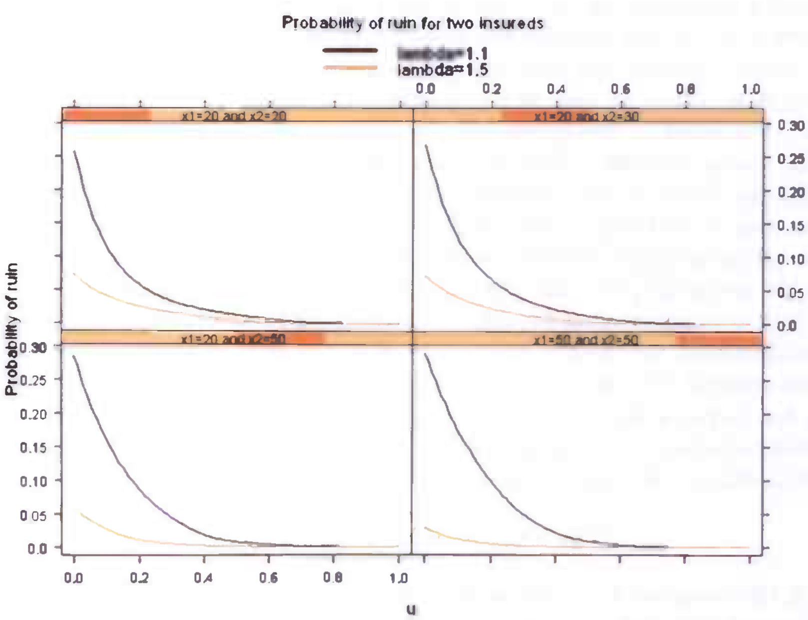


Figure 3: Portfolio of two insureds

A first conclusion based on the graphs, is that for every portfolio, the probability of ruin is higher when $\lambda = 1.1$, than when $\lambda = 1.5$. So a higher λ reflects in a lower probability of ruin. This is a logical result, because a higher λ results only in a higher premium. So the insurer has more income and the expenditures remain the same. Therefore, the probability of ruin will be lower.

For another conclusion, we will have a closer look at the graphs of the 'young' and 'old' portfolios.

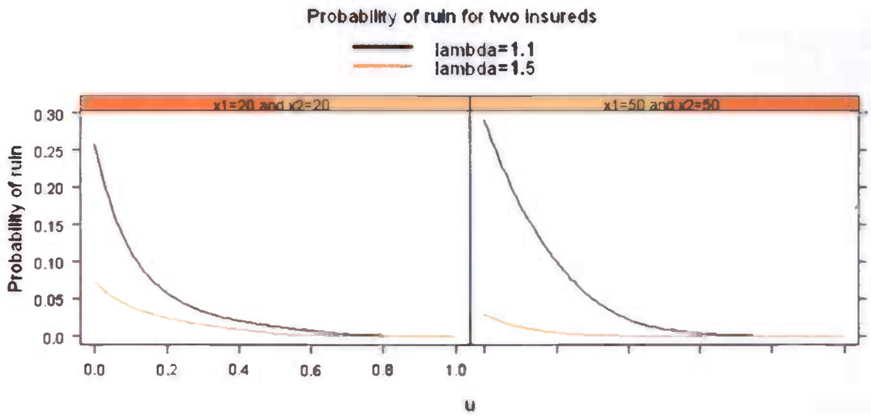


Figure 4: 'Young' and 'old' portfolio

In figure 4 are the probabilities of ruin given for a portfolio of two insureds aged 20 and for a portfolio of two insureds aged 50. First we have a look at the graphs where $\lambda = 1.1$. For small u, the probability of ruin of the portfolio with ages 50 is higher than for the portfolio with ages 20. For larger u, they both approaching zero. When we look at the graphs where $\lambda = 1.5$, we see something remarkable. For small u, the probability of ruin for the old portfolio is lower than for the young portfolio. This is the opposite of the conclusion in case of $\lambda = 1.1$. So the effect of age, depends on the choice of λ. When we vary λ, this reflects only in a change of the single premium. Therefore, we will have a look at the actuarial single premium multiplied by λ. The older the insured is, the higher the actuarial premium will be. This is because if an insured is 'old', there is a high probability that he will die in the near future. This results in a high probability that the insurer has to pay S in the near future and there is a small time interval that he can earn interest. So when an insured is old, say $x = 50$, he has to pay a higher premium than a young insured of age 20. When the actuarial premium is multiplied by a $\lambda > 1$, this will have a greater effect on a higher premium. By this, we mean that

$$\lambda C_{50} - C_{50} > \lambda C_{20} - C_{20},$$

where C_x is the premium for a life aged x. So when the premium is multiplied by λ, the insurer will earn more additional premium in case of an old insured. That results in more capital and therefore, the probability of ruin decreases. The additional premium is greater in case of an old insured, and therefore it will have a greater downwards effect on the probability of ruin when the insured is old, than in case of a young insured. This is also what we see in the figure. When $\lambda = 1.1$ and u is small, the probability of ruin in case of a young portfolio is smaller than in case of an old portfolio. But when $\lambda = 1.5$, the additional premium is greater in case of the old portfolio, and therefore the decrease in the probability of ruin is larger. So it depends on λ, if the probability of ruin for an old portfolio is larger or smaller than for a young portfolio.

6 Portfolio consisting of more than two insureds

In the previous section, we found that $U(T_{(2)})$ can be expressed in terms of $U(T_{(1)})$. We can do the same when there are more insureds in the portfolio. The formula becomes a recursive formula:

$$\begin{aligned} U(T_{(1)}) &= (u + \sum_{m=1}^{N} C_m)(1+i)^{T_{(1)}} - S \\ U(T_{(k)}) &= U(T_{(k-1)})(1+i)^{T_{(k)}-T_{(k-1)}} - S, k = 2, 3, \dots, N \end{aligned} \quad (20)$$

In the previous sections, we succeeded to calculate the probability of ruin analytically. But in case of a portfolio, consisting of more than two insured, it becomes more and more complicated. Therefore, we will use simulation to investigate two questions:

- What is the effect of different values of S on the probability of ruin?
- How does the probability of ruin depends on the number of insureds?
- What is the effect of varying the ages in the portfolio on the probability of ruin?

6.1 The simulation

The idea of the simulation is the following: First we create a portfolio, consisting of N insureds aged x_i, $i = 1, 2, \dots, N$, and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}.$$

So $\mathbf{x}$ is a vector containing the ages of the insureds. For every insured, we will simulate the future lifetime. Therefore, we use the life table [6]. This table lists l_x for $x = 0, 1, \dots, 116$, where l_0 is 10 million and l_{116} is zero. First we create a vector $\mathbf{q}$:

$$\mathbf{q} = \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{115} \end{pmatrix},$$

where $q_x = \frac{l_x - l_{x+1}}{l_x}$ for $x = 0, 1, \dots, 115$. Note that $q_{115} = 1$, because the number of people aged 116 is zero. Then we can compute for every life aged x a vector $\mathbf{p_x}$:

$$\mathbf{p_x} = \begin{pmatrix} {}_0p_x \\ {}_1p_x \\ \vdots \\ {}_{115}p_x \end{pmatrix},$$

where ${}_jp_x = \frac{l_{x+j}}{l_x}$ for $j = 0, 1, \ldots, 115 - x$. With the vector $\mathbf{q}$ and the vectors $\mathbf{p_x}$ for every x, we can find the probability that a life aged x dies at time t by:

$$\mathbf{f_T(t,x)} = \begin{pmatrix} f_T(1,x) \\ f_T(2,x) \\ \vdots \\ f_T(116-x,x) \end{pmatrix},$$

where $f_T(i,x) = \mathbf{p_x}(i-1)\mathbf{q}(x+i-1)$ and $\mathbf{p_x}(i-1)$ means the $(i-1)^{th}$ element of the vector $\mathbf{p_x}$ and $\mathbf{q}(x+i-1)$ means the $(x+i-1)^{th}$ element of $\mathbf{q}$. Now we know the probability density function of the future lifetime for every insured, and we can compute the premium for every insured. Therefore, we fix S and λ and we compute the premium by formula (1).

When we have found the pdf for every life aged x, we can simulate the future lifetime of every insured. For every insured, we look at the age of the insured and the corresponding pdf. We sample for every insured 100000 times a future lifetime, where the future lifetime is sampled from the numbers $1, 2, \ldots, 116 - x$ and the corresponding probabilities are given in the pdf. Then we find the following matrix

$$\mathbf{T_{sample}} = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,100000} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,100000} \\ \vdots & \vdots & \cdots & \vdots \\ T_{N,1} & T_{N,2} & \cdots & T_{N,100000} \end{pmatrix},$$

which contains all values from the simulation. The rows corresponds with 100000 simulations for one insured, and the columns corresponds with one simulation for every insured. Now, we sort the columns from small to large, so we find the ordered future lifetimes. For every simulation (so for every row), we compute capital of the insurer at every time an insured dies. For this computation, we use the recursive formula in (20). When at some moment capital is negative, we know that capital will stay negative. So it is sufficient to look at capital at the last time an insured dies: at time $T_{(N)}$. If this one is negative, ruin is occured.

We do the computation of capital for every simulation, so 100000 times. For every simulation, we look if ruin occured. We sum the times of ruin and divide this sum by the number of simulations. The result is our estimation of the probability of ruin.

We do the simulations in MATLAB. With these simulations, we can look at the effect of different factors on the probability of ruin in case of more than two insureds. Therefore, we will vary one parameter that might influence the probability of ruin and we fix the other parameters.

6.2 The effect of the value of S

First, we will investigate the effect of S on the probability of ruin. Therefore, we vary S and hold the other variables constant. We choose for S the values 1 and 5. In the simulation we use a portfolio with 100 insureds of age 20. Furthermore, we choose two different values for λ: 1.1 and 1.5. We look at the probability of ruin for $u = 0, 0.01, 0.02, \ldots, 2$. The result is given in figure 5.

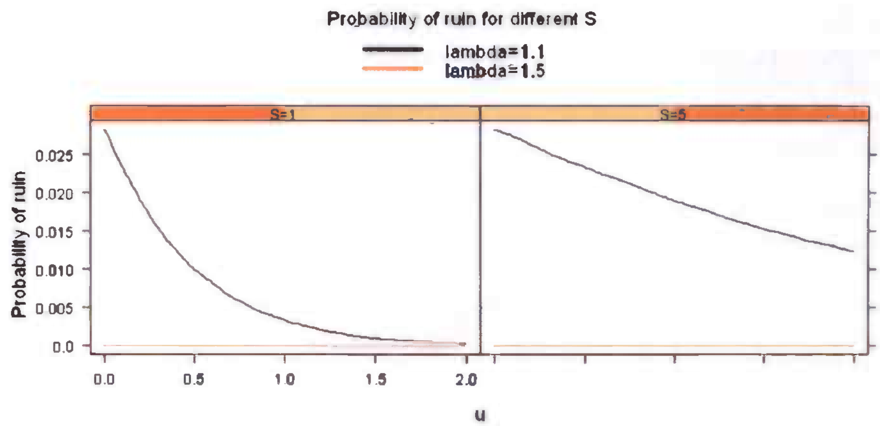


Figure 5: Probability of ruin when $x = 20$ and varying N

We see that when $\lambda = 1.5$, the probability of ruin is equal to zero for every u in both cases. This means that the premiums are high enough to ensure that ruin will not occur. More interesting is the case of $\lambda = 1.1$. When $u = 0$, the probability of ruin is equal for $S = 1$ and $S = 5$ (as was also the case when the portfolio contains one insured). But when u increases, the probability of ruin is higher when $S = 5$ than when $S = 1$. So for the insurer, it is riskier to have a portfolio with insurances that pay a high amount to the insured, than insurances that pay a lower amount to the insured. When $S = 5$, it is not sufficient to have an initial capital of 2, because in that case the probability of ruin is equal to 0.012. This is a relatively high probability. So he has to decrease the probability of ruin. The insurer can do that for example by asking a higher premium to the insureds, or he needs a higher initial capital.

6.3 The effect of the size of the portfolio

To investigate the effect of the size of the portfolio on the probability of ruin, we choose a distribution of the age and varying the size N. First we assume that every insured in the portfolio has age 20, so they all have the same age and consequently the same future lifetime distribution. For N, we choose the values 10, 50, 100 and 200. The result of the simulations for different N is given in figure 6.

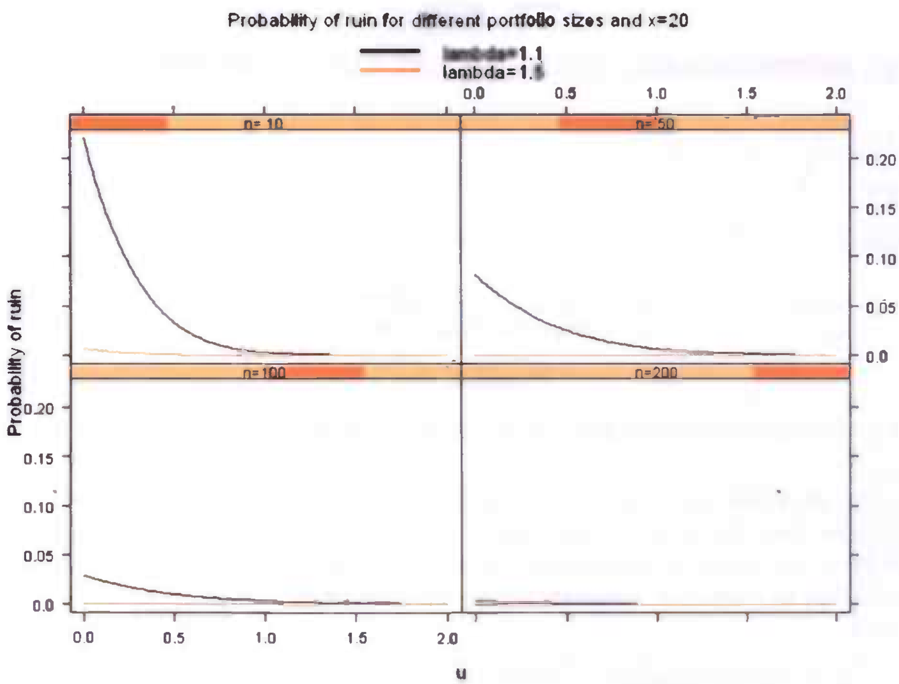


Figure 6: Probability of ruin when $x = 20$ and varying N

When $\lambda = 1.5$, the probability of ruin is in almost all of the cases equal to zero. Only when $N = 10$ and u is very small, there is a small probability that ruin can occur. When $\lambda = 1.1$, we see that there is a clear relation between the size N and the probability of ruin. For $N = 10$ and for small u, the probability of ruin is much larger than for $N = 50$. And in the same way, the probability of ruin for $N = 50$ is larger than $N = 100$. And for $N = 200$, the probability of ruin is almost zero for all u between 0 and 2. So increasing the size of the portfolio has a downwards effect on the probability of ruin.

The example of a portfolio where all insureds have the same age, is not a very realistic example. Therefore, we will have a look at another example, were the insureds represent the distribution of the ages in the Dutch population. We will use the age distribution of the Dutch population in 2005 [5], with ages between 20 and 50 ages. We choose these ages, because this could be a realistic representation of a real portfolio. From the table of the age distribution, we calculate the probability that a person has a certain age, by dividing the number of people of a certain age by the total number of people. With these probabilities we create a portfolio by sampling the ages. We sample ages for two different portfolio sizes: $N = 10$ and $N = 100$. When these ages are sampled, we do simulations for the future lifetimes of the insureds. And with the simulations of the future lifetimes, we can estimate the probability of ruin. The result is given in figure 7.

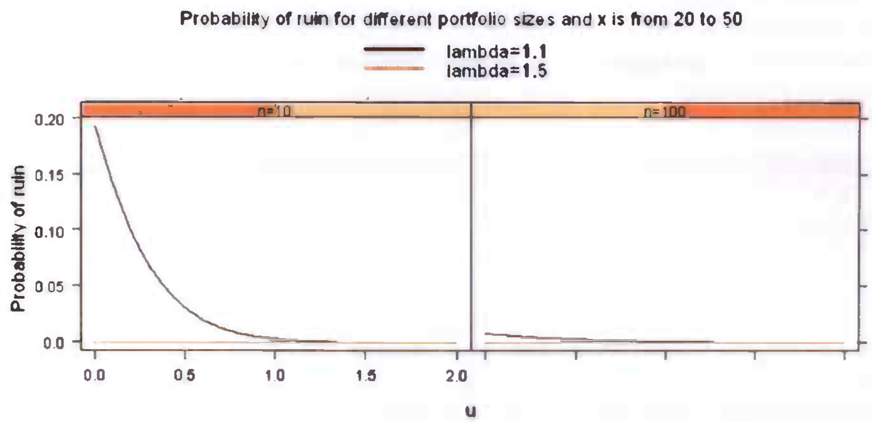


Figure 7: Probability of ruin when x is between 20 and 50 and varying N

Also with this portfolio, you can see that for large N the probability of ruin is smaller than for small N, especially in case of small u. So we can conclude that there is a negative relationship between the size of the portfolio N and the probability of ruin; if N becomes larger, $\psi(u)$ becomes smaller.

6.4 The effect of the ages of the insureds

Another effect we will invesigate is that one of the ages of the insureds. Do the ages influence the probability of ruin? To look at this question, we create different portfolios with the following fixed variables: $N = 100$, $S = 1$ and $\lambda = 1.1$ or $\lambda = 1.5$, and we varying $\mathbf{x}$. The four different portfolios are:

1. all insureds are aged 20,
2. all insureds are aged 50,
3. the insureds come from the age distribution of the Dutch population 2005, with ages from 0 up to and including 95,
4. the insureds come frome the age distribution of the Dutch population 2005, with ages from 20 up to and including 50.

The result of the simulation of the probability of ruin for the different portfolios is given in figure 8.

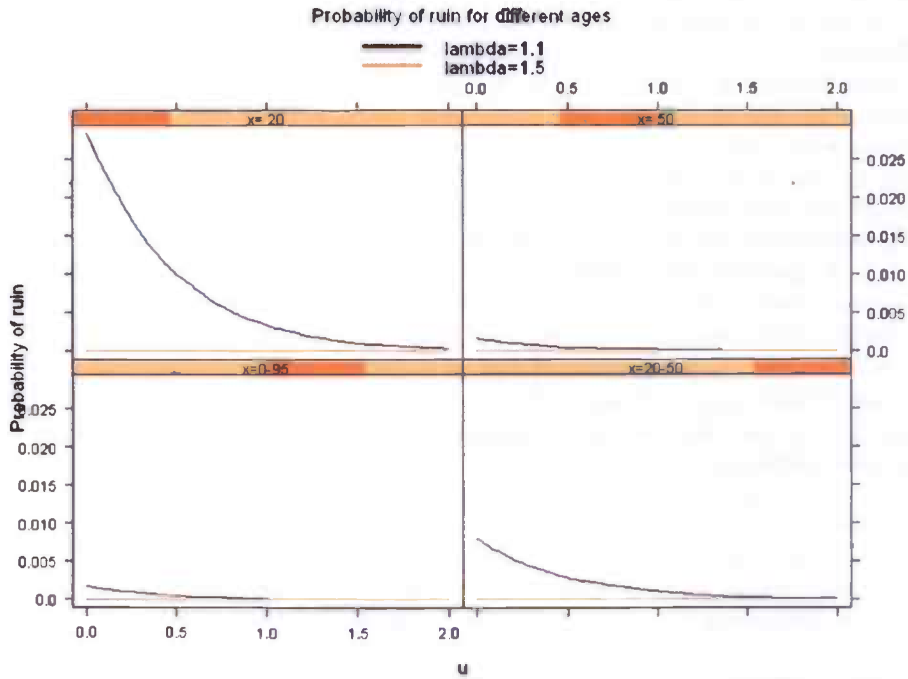


Figure 8: Probability of ruin when $N = 100$ and varying $\mathbf{x}$

Again, in case of $\lambda = 1.5$, the probability of ruin is for all portfolios and for all u equal to zero. So we will look at the cases where $\lambda = 1.1$. We see that the largest probability of ruin appears in the portfolio were the insureds are all age 20, so in a young portfolio. One thing that you may expect is that the probability of ruin is smaller in a mixed portfolio than in a portfolio with the same ages. But when you compare the portfolio with ages 50 with one of the mixed portfolios, you can see that this is not the case. To make a better comparison, we will first compare the portfolios were all ages are equal.

When we compare the portfolio with the ages of 20 with the portfolio with insureds aged 50, we see that when the insureds in the portfolio are older, the probability of ruin is smaller. At first sight, this looks a bit strange. We maybe would expect that older people will die earlier than younger people. So the time the insurer earns interest with a portfolio of older people is shorter. But you have to remember that this uncertainty of time of death is also put in the calculation of the premium, that results in a higher premium for older insureds. With insureds of age 20, you earn less from the premiums than with insureds of age 50. And with insureds of age 20, there is more uncertainty about the time of death. The future lifetime of a life aged 20 is in a range from 0 to 96 and for a life aged 50 is this range from 0 to 66. So this uncertainty in the future lifetime, reflects in a higher probability of ruin with a portfolio with young people, compared with a portfolio with older people.

A second comparison is between the two mixed portfolios. In the first portfolio is more spread between the ages. The ages vary from 0 up to and including 95, where in the second portfolio the ages are from 20 up to and including 50. When we look at the figure, you can see that the probability of ruin is smaller in a portfolio with a greater spread. But we have to be a bit carefull in this conclusion. In case of portfolio with equal ages, we concluded that the probability of ruin with older people is smaller than with younger people in the portfolio. And in case of more spread, there are also more older people in the portfolio. So this can have a downwards effect on the probability of ruin. So it is difficult to give a conclusion about the probability of ruin in case of a mixed portfolio, based on the simulation. Furthermore, we saw in the case of two insured, that when we compare the probability of ruin for different ages, this probability also depends on the choice of λ. This cannot be seen in this simulation, because when $\lambda = 1.5$, the probability of ruin is equal to zero in all cases. But also because this reason, we have to be carefull in making conclusions. It is clear that the age has effect on the probability of ruin, but this effect depends also on other variables.

7 Conclusion

For a portfolio consisting of one insured, we can easily compute the probability of ruin using a life table. The effect of varying S, is that when S increases, the probability of ruin for the same u also increases. Only when $u = 0$, the probability of ruin is the same for all S. This relation does not hold when S is proportional to u. In that case, the probability of ruin is for a fixed S the same for all u.

For a portfolio consisting of two insureds, it is a bit more difficult to compute the probability of ruin. We solved this by using order statistics for the times of death T_1 and T_2. The effect of λ is that when λ increases, the probability of ruin decreases. This is a obvious result, because an increasing λ results only in a higher premium. Therefore, the insurer has more capital and the probability that ruin can occur will be smaller. It turned out that the effect of the ages of the two insureds on the probability of ruin depends on λ. When λ is small, the probability of ruin is higher in case of a portfolio with older people. But this holds only for very small u. When u increases, this effect is the opposite: the probability of ruin is higher in case of a portfolio with younger people. Also when λ becomes larger, the probability of ruin is higher in case of a portfolio with younger people. This is a result of the premium, that is higher for older insureds. The premiums are multiplied by λ and this reflects in a relatively more increasing premium in case of older insureds.

To calculate the probability of ruin in case of more than two insureds is very complicated. Therefore, we used simulation to estimate this probability. The effect of S is the same as we concluded in case of one insured. A larger S results in a higher probability of ruin for the same u and when u is small. Increasing the size of the portfolio reflects in a decreasing probability of ruin. So it is interesting for the insurer to have a large portfolio. When we look at a portfolio consisting of different ages, we see that the probability of ruin is higher in case of a young portfolio. In such a portfolio, the time interval insureds can die is larger. So there is more uncertainty about the time of death, which reflects in a higher probability of ruin. We see that this conclusion is another conclusion than we made in case of two insureds. But we have to be carefull in our conclusion, because we saw in case of two insured that there is dependence between the effect of the ages and the choice of λ. So it is difficult to give a clear relation between the ages of the insureds and the probability of ruin.

So the most clear relation is between N and ψ and also between λ and ψ. Increasing the number of insureds N results in a decreasing probability of ruin. And also an increase of λ reflects in a decrease of the probability of ruin. It is difficult to say something about the effect of the ages, because this is also dependent on λ.

We were interested in the effect of different factors on the probability of ruin. We looked at the effects of S, N and x. For further research, we could also look at the effect of interest. What happens when interest increases or decreases? Or how can we model interest that varies over time and what is the effect of it on the probability of ruin? This is an interesting question, because in real interest also varies. We made also some assumptions. One of these is that the portfolio becomes empty. This is not a very realistic assumption. In real, there are coming new insureds into the portfolio. How can this be modeled and what is the effect of it? A second assumption we made is that the insureds contracted the same insurance, so S is the same for all insureds. This is easy to model, because when an insured dies, we know for sure that the insurer has to pay S. But when S is different for the different insureds, this becomes more difficult. When an insured dies, we do not know which insured dies and so we do not know which amount the insurer has to pay. So how can we model this and what is the effect of it on the probability of ruin? The same holds for the premium. We assumed a single premium. Premium could also be paid annually. But when one insured dies, we do not know which insured it is, and so we do not know which premium will no longer be earned. All these issues could be questions for further research.

References

[1] Hans U. Gerber. *Life Insurance Mathematics.* Springer, third edition, 1997.

[2] Newton L. Bowers Jr., Hans U. Gerber, James C. Hickman, Donald A. Jones, and Cecil J. Nesbitt. *Actuarial Mathematics.* The Society of Actuaries, 1986.

[3] Andreas Krause and Melvin Olson. *The basics of S-PLUS.* Springer, third edition, 2002.

[4] Alexander M. Mood, Franklin A. Graybill, and Buane C. Boes. *Introduction to the theory of statistics.* McGraw-Hill, third edition, 1982.

[5] Centraal Bureau voor de Statistiek (CBS). Population of the netherlands in 2005. http://www.cbs.nl.

[6] Actuarieel Genootschap Woerden. Life table 1995-2000. http://www.ag-ai.nl.

8 Appendix 1: SPLUS script files

8.1 One insured

```
#constants
lambda=1.1
S=1
i=0.03
u1=seq(0,2,0.01)

#vector with lx
lx = matrix(c(10000000,9941605,9936638,9932858,9930395,9928299,
9926414,9924736,9923265,9921860,9920504,9919168,9917771,9916194,
9914263,9911844,9908792,9904835,9900167,9894620,9888451,9881889,
9875261,9868427,9861549,9854693,9847880,9841072,9834321,9827550,
9820644,9813542,9806118,9798345,9790171,9781460,9772213,9762328,
9751704,9740066,9727241,9712954,9697208,9680034,9661050,9640014,
9616734,9591188,9563278,9532469,9498570,9460956,9419676,9374387,
9324240,9268954,9208118,9141509,9068120,8986865,8897261,8797766,
8687264,8564307,8427871,8276947,8111841,7932032,7736728,7525801,
7299388,7055841,6794577,6515414,6218269,5904065,5573590,5229325,
4872535,4505814,4131376,3751316,3371570,2997338,2633796,2285035,
1954190,1645699,1362539,1106917,881226,686946,524709,392822,
287073,203800,140038,92975,59547,36812,22068,12889,7379,4156,
2288,1230,645,329,164,79,37,17,8,3,1,1,0)
,ncol=1,byrow=T)

ages = seq(0,116)
dimnames(lx)=list(ages,"lx")

#vector of probabilities for a life aged x of surviving
# at least k years, for k=0,1,2,...,116-x
kp = function(x)
{
lx[(x+1):117]/rep(lx[(x+1)],length(lx[(x+1):117]))
}

#vector of probabilities of dying within one year
q = ((lx[1:116]-lx[2:117])/lx[1:116])

#vector of probabilities of dying at age x+k, given
# initial age x, k=0,1,2,...,116-x
pdf = function(x)
{
c(kp(x)[1:(116-x)])*c(q[(x+1):116])
}

#calculation of the premium for a life aged x
premium = function(x)
{
```

```
S*lambda*sum(sapply(1:length(pdf(x)), function(j, y)
{(1/(1+i))^j*pdf(y)[j]}, y=x))
}


#Cumulative distribution function F of the future lifetime
# of a life aged x
cdf=function(x)
{
c(cumsum(pdf(x)),rep(1,500))
}

#function for rounding up to the next integer
rounding = function(B)
{
if(round(B)>=B) B1=round(B)
else B1=round(B)+1
}

#calculation of the probability of ruin for one insured
ruin = function(x,B1)
{
if(B1>(116-x))
ruin=1
else
if(B1>=2)
ruin = cdf(x)[(B1-1)]
else ruin = 0
}

ruinu = function(x,u)
{
premium=premium(x)
sapply(1:length(u), function(j,y,z,w)
{ B=log(S/(z[j]+w))/log(1+i)
B1=rounding(B)
ruin(y,B1)
}, y=x, z=u,w=premium)
}

#Calculation when u is proportion to S
ruinuprop = function(x,u,alpha)
{
sapply(1:length(u), function(j,y,z,a)
    { B=log((a*z[j]*S)/(z[j]+a*z[j]*premium(y)))/log(1+i)
B1=rounding(B)
ruin(y,B1)
}, y=x, z=u, a=alpha)
}
```

```
#Plot for the relation between the probability of ruin
# and S (for S=1 and S=5)
S=1
lambda=1.1
S1L11x20=ruinu(20,u1)

S=1
lambda=1.5
S1L15x20=ruinu(20,u1)

S=5
lambda=1.1
S5L11x20=ruinu(20,u1)

S=5
lambda=1.5
S5L15x20=ruinu(20,u1)

Sx20=c(S1L11x20,S1L15x20,S5L11x20,S5L15x20)
m=matrix(c(c(rep(1.1,length(u1)),rep(1.5,length(u1)),
rep(1.1,length(u1)),rep(1.5,length(u1))),c(rep(u1,2),
rep(u1,2)),Sx20),ncol=3,byrow=F)
dimnames(m)=list(NULL,c("lambda","u","psi"))
sis=c(rep("S=1",2*length(u1)),rep("S=5",2*length(u1)))

ruinvarS = data.frame(m,sis)

xyplot(psi~u|sis,groups=lambda,data=ruinvarS,
panel=panel.superpose,as.table=T,col=c(1,5), type="l",xlab="u",
ylab="Probability of ruin",main="Relation between the
probability of ruin and S", key=list(lines=list(col=c(1,5),lwd=3),
text=list(c("lambda=1.1","lambda=1.5"))))

S=1
u=seq(0.01,1,0.01)

#Plot for the relation between the probability of ruin and
# S (when S is proportional)
lambda=1.1
Sprop1L11=ruinuprop(20,u,1)
Sprop10L11=ruinuprop(20,u,10)
r5=c(Sprop1L11,Sprop10L11)

m=matrix(c(rep(u,2),r5),ncol=2,byrow=F)
dimnames(m)=list(NULL,c("u","psi"))
beta=c(rep("beta=1",length(u)),rep("beta=1/10",length(u)))

ruinvarSprop = data.frame(m,beta)
```

```
xyplot(psi~u|beta,data=ruinvarSprop,as.table=T,type="l",col=1,
xlab="u",ylab="Probability of ruin",
main="Probability of ruin when S is proportional to u")


#Plot for the relation between the probability of ruin and
# the age x (x=20 and x=50)
u1=seq(0,2,0.01)
S=1
lambda=1.1
S1L11x20=ruinu(20,u1)

lambda=1.5
S1L15x20=ruinu(20,u1)

lambda=1.1
S1L11x50=ruinu(50,u2)

lambda=1.5
S1L15x50=ruinu(50,u2)

r5=c(S1L11x20,S1L15x20,S1L11x50,S1L15x50)
k=matrix(c(c(rep(1.1,length(u1)),rep(1.5,length(u1)),
rep(1.1,length(u2)),rep(1.5,length(u2))),c(rep(u1,2),rep(u2,2))
,r5),ncol=3,byrow=F)
dimnames(k)=list(NULL,c("lambda","u","psi"))
age=c(rep("x=20",2*length(u1)),rep("x=50",2*length(u2)))

ruinvarx = data.frame(k,age)

xyplot(psi~u|age,groups=lambda,data=ruinvarx,
panel=panel.superpose,as.table=T,col=c(1,5),type="l",
xlab="u",ylab="Probability of ruin",
main="Relation between the probability of ruin and age",
key=list(lines=list(col=c(1,5),lwd=3),
text=list(c("lambda=1.1","lambda=1.5"))))
```

8.2 Two insureds

```
#constants
lambda=1.5
S=1
i=0.03
x=c(50,50)

#vector of probabilities for a life aged x of surviving
# at least k years, for k=0,1,2,...,116-x
kpx1 = c(lx[(x[1]+1):117]/rep(lx[(x[1]+1)],length(lx[(x[1]+1):117])))
kpx2 = c(lx[(x[2]+1):117]/rep(lx[(x[2]+1)],length(lx[(x[2]+1):117])))

#vector of probabilities of dying within one year
q = ((lx[1:116]-lx[2:117])/lx[1:116])

#vector of probabilities of dying at age x+k,
# given initial age x, k=0,2,...,116-x
pdfx1 = c(kpx1[1:(116-x[1])]*q[(x[1]+1):116],rep(0,2000))
pdfx2 = c(kpx2[1:(116-x[2])]*q[(x[2]+1):116],rep(0,2000))

#Cumulative distribution function F of the future lifetime
# of a life aged x
cdfx1 = cumsum(pdfx1)
cdfx2 = cumsum(pdfx2)

#calculation of the premium for a life aged x
premiumx1 = S*lambda*sum(sapply(1:length(pdfx1), function(j)
{(1/(1+i))^j*pdfx1[j]}))
premiumx2 = S*lambda*sum(sapply(1:length(pdfx2), function(j)
{(1/(1+i))^j*pdfx2[j]}))

#function for rounding up to the next integer
rounding = function(B)
{
if(round(B)>=B) B1=round(B)
else B1=round(B)+1
}

#calculation of the probability of ruin for two insureds at
# time T((1)) for one u
ruinT1 = function(B1)
{
if(B1>1)
ruin = cdfx1[(B1-1)]+cdfx2[(B1-1)]-cdfx1[(B1-1)]*cdfx2[(B1-1)]
else
ruin = 0
}

ruinuT1 = function(u)
```

```
{
B=log(S/(u+premiumx1+premiumx2))/log(1+i)
B1=rounding(B)
ruinT1(B1)
}

#Calculation of the probability of ruin for two insureds at
# time T((2)) for one u
ruinT2 = function(u,T1)
{
UT1 = (u+premiumx1+premiumx2)*(1+i)^T1-S
C=log(S/UT1)/log(1+i)+T1
B2=rounding(C)

if(B2<=T1)
ruin=0
else
if (B2<=1)
ruin=0
else
if (T1==1)
ruin=(cdfx1[(B2-1)])*pdfx2[T1]+(cdfx2[(B2-1)])*
pdfx1[T1]-pdfx1[T1]*pdfx2[T1]
else ruin=(cdfx1[(B2-1)]-cdfx1[(T1-1)])*pdfx2[T1]+(cdfx2[(B2-1)]-
cdfx2[(T1-1)])*pdfx1[T1]-pdfx1[T1]*pdfx2[T1]

}

ruinuT2 = function(u)
{
sum(sapply(0:116, function(n,z)
{ B=log(S/(z+premiumx1+premiumx2))/log(1+i)
B1=rounding(B)
if (B1<=0)
B1=1+n
else B1=B1+n
ruinT2(z,B1)
}, z=u))
}


x5050L15=sapply(seq(0,1,0.01),function(k){ruinuT1(k)+ruinuT2(k)})
```

9 Appendix 2: Matlab functions

9.1 Simulation for more than two insureds

```
%sample of ages from the life distribution
sampleagesx2050n10=zeros(10,1);

for j=1:10;
    sampleagesx2050n10(j,:)=randsample(20:50,1,true,bevolking2050);
end;

%sample of future lifetimes

    %use the function trekkingoneage in case of the same ages
function [sample] = trekkingoneage(age,nrinsureds,nrsamples,lx)

sample=zeros(nrinsureds,nrsamples);

q = (lx(1:116)-lx(2:117))./lx(1:116);
kpx = lx((age+1):1:117)/lx((age+1));
pdfx = kpx(1:(length(kpx)-1)).*q((age+1):116);

for j=1:nrinsureds;
sample(j,:)=randsample(1:(116-age),nrsamples,true,pdfx);
end;

sample=sort(sample,1);

    %use trekkingdifferentages in case of different ages.
function [sample] = trekkingdifferentages(x,nrsamples,lx)

sample=zeros(length(x),nrsamples);

q = (lx(1:116)-lx(2:117))./lx(1:116);
for j=1:length(x);
    kpx = lx((x(j)+1):1:117)/lx((x(j)+1));
    pdfx = kpx(1:116-x(j)).*q((x(j)+1):116);
    sample(j,:)=randsample(1:(116-x(j)),nrsamples,true,pdfx);
end;

sample=sort(sample,1);


%calculation of the premium can be done with the function premium
    %use the function premiumoneage in case of the same ages
function [totalpremium]=premiumoneage(age,nrinsureds,lx,S,lambda)

i=0.03;
q = (lx(1:116)-lx(2:117))./lx(1:116);
```

```
kpx = lx((age+1):1:117)/lx((age+1));
pdfx = kpx(1:(116-age)).*q((age+1):116);
premium=0;
  for k=1:length(pdfx);
        premium=premium+(1/(1+i))^k*pdfx(k);
  end;
totalpremium=nrinsureds*S*lambda*premium;

end

    %use the function premiumdifferentages in case of different ages.
function [totalpremium]=premiumdifferentages(x,lx,S,lambda)

i=0.03;
q = (lx(1:116)-lx(2:117))./lx(1:116);

premium=zeros(1,length(x));

for j=1:length(x)
    kpx = lx((x(j)+1):1:117)/lx((x(j)+1));
    pdfx = kpx(1:116-x(j)).*q((x(j)+1):116);
    for k=1:length(pdfx)
        premium(j)=premium(j)+(1/(1+i))^k*pdfx(k);
    end;
    premiums=premium*S*lambda;
end;

totalpremium=sum(premiums);

end

%calculation of the probability of ruin can be done with
% the function simulationninsureds
function [vector] = simulationninsureds(premium,u,sample,i)

nrinsureds=length(sample(:,1));
capital=zeros(nrinsureds,100000);

%loop for the samples
    for j=1:100000;
        capital(1,j)=(u+premium)*(1+i)^sample(1,j)-1;
        %loop for the calculation of the capitals
        for k=2:nrinsureds;
            capital(k,j)=capital(k-1,j)*(1+i)^(sample(k,j)-
            sample(k-1,j))-1;
        end;
    end;

    ruin=zeros(1,100000);
    for m=1:100000
```

```
        ruin(m)=(capital(nrinsureds,m)<0);
    end;

   vector=sum(ruin)/100000;

end

%simulation of the probability of ruin for different u
u= 0: 0.01: 2;
p=zeros(length(u),1);
for k=u;
    p(round(k*100+1),1)=simulationninsureds(premium,k,sample,0.03);
end;
```