The Combined Moment and Interpolation Problem

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Master's thesis

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Preface

In April 1997 I started to think about my Masters thesis to complete my study at the mathematics department of the University of Groningen (R*u*G), the Netherlands.
At the end of May 1997 I stopped by at Prof.dr.ir. A. Dijksma's office and asked him to be my advisor. He was willing to do so and he promised to think about a topic in Operator Theory for my thesis.
In the middle of June I received a telephone call at home from Mr. Dijksma. He asked me to visit him in his office, the next day. The next morning I went to his office and he told me the following:
Mr. Dijksma was invited to visit Western Washington University (WWU) in Bellingham, state of Washington, United States of America, from September through December 1997 to co-operate with Prof. T. Read and Prof. B. Ćurgus. Two of the topics they planned to study were the interpolation problem and the moment problem. Mr. Dijksma and I already decided that my thesis would handle about the combination of these two particular problems, a suggestion from Prof. Heinz Langer from the Technical University of Vienna.
So the reason he asked me to come and see him, actually was to invite me to accompany him to Bellingham. I was flattered and proud and of course I decided to accept this unique and great offer. Although I had to work pretty hard, I also visited a lot of things. It really were incredibly great and unforgettable months and it became a wonderful experience. That's why, once again, I want to thank Mr. Dijksma and WWU for this opportunity they gave me. Also thanks to Mrs. Dijksma for being a kind of 'mother figure' during my visit to the USA and to my parents, who gave me great support to finish this study.
Because I never worked with LaTeX before, it took a while before I finished this thesis; eventhough I hope you'll enjoy reading this report.

Groningen, June 1998
Richard W. Buursema

Contents

Chapter 1

The moment and interpolation problem (MIP)

The problem which will be discussed in this thesis is the combination of two other problems: the interpolation problem and the moment problem.
If there is, the solution to both problems is given by a particular class of functions, the so-called *Nevanlinna functions*, denoted by $\mathbb{N}_0$. These are analytic functions $N(z)$, $N(\bar{z}) = \overline{N(z)}$, $z \in \mathbb{C}^+$, that map $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$. $N(z)$ belongs to $\mathbb{N}_0$ if and only if there exist two real numbers α, β with $\beta \geq 0$ and a nondecreasing function σ with $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$ such that N admits a Herglotz integral representation

$$N(z) = \beta z + \alpha + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t).$$

1.1 Formulation of the MIP

This is the *Nevanlinna-Pick* interpolation problem (IP):
discuss solutions $N \in \mathbb{N}_0$ that satisfy

$$N(z_i) = w_i, \qquad i = 1 \dots n,$$

for given $z_1, \dots, z_n \in \mathbb{C}^+, z_i$ distinct, and given $w_1, \dots, w_n \in \mathbb{C}$. Note that

$$N(z) = \sum_{j=1}^{n} w_j \frac{\prod_{i \neq j}(z - z_i)}{\prod_{i \neq j}(z_j - z_i)}$$

satisfies $N(z_i) = w_i$ but in general is not a Nevanlinna function.
Define the *Pick matrix* (or information matrix) by:

$$\mathbb{P}_I = \left(\frac{w_i - \overline{w}_j}{z_i - \overline{z}_j} \right)_{i,j=1}^{n} = \begin{pmatrix} \frac{w_1 - \overline{w}_1}{z_1 - \overline{z}_1} & \frac{w_1 - \overline{w}_2}{z_1 - \overline{z}_2} & \cdots & \frac{w_1 - \overline{w}_n}{z_1 - \overline{z}_n} \\ \frac{w_2 - \overline{w}_1}{z_2 - \overline{z}_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{w_n - \overline{w}_1}{z_n - \overline{z}_1} & \cdots & \cdots & \frac{w_n - \overline{w}_n}{z_n - \overline{z}_n} \end{pmatrix}.$$

And here is the moment problem (MP):
discuss solutions $N \in \mathbb{N}_0$ that satisfy

$$-\lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}) \leq s_{2m}$$

for given $s_0, \ldots, s_{2m} \in \mathbb{R}$.
The function

$$N(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2m}}{z^{2m+1}}$$

has the right asymptotic behavior but in general is not a Nevanlinna function.
The Pick matrix of this problem is:

$$\mathbb{P}_M = (s_{i+j})_{i,j=0}^{m} = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 & \cdots & s_m \\ s_1 & s_2 & s_3 & \cdots & & \vdots \\ s_2 & s_3 & \cdots & & & \vdots \\ s_3 & \cdots & & & & \vdots \\ \vdots & & & & & \vdots \\ s_m & s_{m+1} & s_{m+2} & \cdots & \cdots & s_{2m} \end{pmatrix}.$$

Now that these two problems are introduced separately, the combined moment and interpolation problem (MIP) is given by:
discuss solutions $N \in \mathbb{N}_0$ that satisfy both the interpolation problem and the moment problem for given $z_i \in \mathbb{C}^+$ distinct, and $w_i \in \mathbb{C}$ $(i = 1, \ldots, n)$ and $s_0, \ldots, s_{2m} \in \mathbb{R}$.

A part of the Pick matrix $\mathbb{P} = (\mathbb{P}_{ij})_{i,j=1}^{n+m+1}$ of the combined problem MIP looks familiar, because it consists of the Pick matrices that belong to the separate problems. The top-left part is the Pick matrix of the IP

$$\mathbb{P}_{ij} = \frac{w_i - \overline{w}_j}{z_i - \overline{z}_j}, \quad 1 \leq i, j \leq n,$$

and the bottom-right part is the Pick matrix of the MP

$$\mathbb{P}_{ij} = s_{i+j-2n-2}, \quad n+1 \leq i, j \leq n+m+1.$$

The other part 'connects' the two separate problems: The top-right part of $\mathbb{P}$ is given by

$$\begin{pmatrix} w_1 & s_0 + w_1 z_1 & s_1 + s_0 z_1 + w_1 z_1^2 & \cdots & s_{m-1} + \cdots + s_0 z_1^{m-1} + w_1 z_1^m \\ w_2 & s_0 + w_2 z_2 & s_1 + s_0 z_2 + w_2 z_2^2 & \cdots & s_{m-1} + \cdots + s_0 z_2^{m-1} + w_2 z_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ w_n & s_0 + w_1 z_n & s_1 + s_0 z_n + w_n z_n^2 & \cdots & s_{m-1} + \cdots + s_0 z_n^{m-1} + w_n z_n^m \end{pmatrix},$$

that is,

$$\mathbb{P}_{ij} = s_{j-n-2} + s_{j-n-3} z_i + \cdots + s_0 z_i^{j-n-2} + w_i z_i^{j-n-1}, \quad 1 \leq i \leq n, \; n+1 \leq j \leq n+m+1.$$

The bottom-left part of $\mathbb{P}$ is the adjoint of the top-right part

$$\mathbb{P}_{ij} = \overline{\mathbb{P}_{ji}}, \quad n+1 \leq i \leq n+m+1, \ 1 \leq j \leq n.$$

Hence the Pick matrix of the combined moment and interpolation problem is:

$$\mathbb{P} = \begin{pmatrix} \frac{w_1-\overline{w}_1}{z_1-\overline{z}_1} & \cdots & \frac{w_1-\overline{w}_n}{z_1-\overline{z}_n} & w_1 & \cdots & s_{m-1}+\cdots+w_1 z_1^m \\ \vdots & & \vdots & w_2 & \cdots & s_{m-1}+\cdots+w_2 z_2^m \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{w_n-\overline{w}_1}{z_n-\overline{z}_1} & \cdots & \frac{w_n-\overline{w}_n}{z_n-\overline{z}_n} & w_n & \cdots & s_{m-1}+\cdots+w_n z_n^m \\ \overline{w}_1 & \cdots & \overline{w}_n & s_0 & \cdots & s_m \\ \overline{s_0+w_1 z_1} & \cdots & \overline{s_0+w_n z_n} & s_1 & \cdots & s_{m+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{s_{m-1}+\cdots+w_1 z_1^m} & \cdots & \overline{s_{m-1}+\cdots+w_n z_n^m} & s_m & \cdots & s_{2m} \end{pmatrix}.$$

Note that the order of $\mathbb{P}$ is $n+m+1$ and if $n=0$ (this means no interpolation), that this matrix is reduced to the Pick matrix of the MP, but if $m=0$ it is NOT the Pick matrix of the IP; in this case it has one extra row and column (the bottom-right entry is s_0!).

1.2 Sufficient condition for existence of a solution

To determine whether or not a solution of the MIP exists, the next theorem is used (see [Ach, page 95]). If there exists exactly one solution the MIP is said to be *determined*, if there is more than one solution *indetermined.*

Theorem 1.1 *For given $s_0, \dots, s_{2m} \in \mathbb{R}$, equivalent are:*
(i) N belongs to $\mathbb{N}_0$ and satisfies

$$\lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\left(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\right) = -s_{2m}.$$

(ii) There exists a nondecreasing function $\sigma(t)$ such that

$$N(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z}, \qquad s_k = \int_{-\infty}^{\infty} t^k \, d\sigma(t), \quad k = 0, \dots, 2m.$$

The proof of this theorem is also in this book.

Because a solution of the MIP must satisfy both the IP and the MP, this theorem can also be applied to the MIP: if N is a solution to the MIP then $N(z_j) = w_j$ and there exists a nondecreasing function $\sigma(t)$:

$$N(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z}, \quad s_k = \int_{-\infty}^{\infty} t^k d\sigma(t), \quad k = 0, \dots, 2m-1,$$

and

$$s'_{2m} := \int_{-\infty}^{\infty} t^{2m} d\sigma(t) \leq s_{2m}.$$

Hence $\int_{-\infty}^{\infty} \frac{1}{t-z_j}\, d\sigma(t) = w_j$ and for $1 \le i, j \le n$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{t-z_i}\frac{1}{t-\overline{z}_j}\, d\sigma(t) &= \\ &= \int_{-\infty}^{\infty} \frac{1}{z_i-\overline{z}_j}\Big(\frac{1}{t-z_i} - \frac{1}{t-\overline{z}_j}\Big)\, d\sigma(t) \\ &= \frac{w_i-\overline{w}_j}{z_i-\overline{z}_j} \\ &= \mathbb{P}_{ij},\end{aligned} \tag{1.1}$$

for $1 \le i \le n,\ n+1 \le j \le n+m+1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{t-z_i} t^{j-n-1}\, d\sigma(t) &\overset{j'=j-n-1}{=} \\ &= \int_{-\infty}^{\infty} \frac{t^{j'}-z_i^{j'}}{t-z_i} + \frac{z_i^{j'}}{t-z_i}\, d\sigma(t) \\ &= \int_{-\infty}^{\infty} \big(t^{j'-1} + t^{j'-2}z_i + \cdots + z_i^{j'-1} + \frac{z_i^{j'}}{t-z_i}\big)\, d\sigma(t) \\ &= s_{j'-1} + s_{j'-2}z_i + \cdots + s_0 z_i^{j'-1} + z_i^{j'} w_i \\ &= \mathbb{P}_{ij},\end{aligned} \tag{1.2}$$

for $n+1 \le i, j \le n+m+1$ except for $i = j = n+m+1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} t^{i-n-1} t^{j-n-1}\, d\sigma(t) &= \\ &= \int_{-\infty}^{\infty} t^{i+j-2n-2}\, d\sigma(t) \\ &= s_{j+i-2n-2} \\ &= \mathbb{P}_{ij},\end{aligned} \tag{1.3}$$

and finally we have

$$\int_{-\infty}^{\infty} t^m t^m\, d\sigma(t) = s'_{2m} \le s_{2m},$$

which shows that

$$\mathbb{P}_{n+m+1,n+m+1} = s_{2m} = s'_{2m} + \underbrace{(s_{2m} - s'_{2m})}_{\ge 0}. \tag{1.4}$$

It follows that

$$\mathbb{P} = \int_{-\infty}^{\infty} \begin{pmatrix} \frac{1}{t-z_1} \\ \vdots \\ \frac{1}{t-z_n} \\ 1 \\ \vdots \\ t^m \end{pmatrix} \begin{pmatrix} \frac{1}{t-\overline{z}_1} & \cdots & \frac{1}{t-\overline{z}_n} & 1 & \cdots & t^m \end{pmatrix} d\sigma(t) + \operatorname{diag}(0, \ldots, 0, s_{2m} - s'_{2m})$$

and hence

$$
\begin{aligned}
\langle \mathbb{P}x, x\rangle \quad = \quad x^*\mathbb{P}x \quad = \\
= \int_{-\infty}^{\infty} \Bigg(\sum_{i=1}^{n} \frac{x_i}{t-\overline{z}_i} \overline{\sum_{i=1}^{n} \frac{x_i}{t-\overline{z}_i}} + \sum_{j=0}^{m} x_{j+n+1}t^j \overline{\sum_{j=0}^{m} x_{j+n+1}t^j} + \sum_{j=0}^{m} x_{j+n+1}t^j \overline{\sum_{i=1}^{n} \frac{x_i}{t-\overline{z}_i}} + \\
+ \sum_{i=1}^{n} \frac{x_i}{t-\overline{z}_i} \overline{\sum_{j=0}^{m} x_{j+n+1}t^j} \Bigg) \, d\sigma(t) \; + \; x^* \operatorname{diag}(0,\dots,0, s_{2m} - s'_{2m})\, x
\end{aligned}
$$

$$
\stackrel{z=iy}{=} \int_{-\infty}^{\infty} \left(\left| \sum_{i=1}^{n} \frac{x_i}{t-\overline{z}_i} \right|^2 + \left| \sum_{j=0}^{m} x_{j+n+1}t^j \right|^2 \right) d\sigma(t) + \underbrace{(s_{2m} - s'_{2m})}_{\geq 0} |x_{n+m+1}|^2 \geq 0\,, \quad \text{so } \mathbb{P} \geq 0.
$$

This result leads to the following theorem:

Theorem 1.2 *If there is a solution of the MIP, then the Pick matrix is nonnegative:* $\mathbb{P} \geq 0$.

Later it turns out that there actually is an 'if-and-only-if'-relation between the two statements in this theorem!
Before we see that, first of all we have to find solutions. For this we need a model.

Chapter 2

The model

To find solutions of the MIP, we will introduce a model. In Section 2.3 we will see some properties of this model and in Chapter 4 we will use these to find the solutions. As a consequence of Theorem 1.2, from now on, we assume that $\mathbb{P} \geq 0$.

2.1 Construction of the model $(\mathcal{H}, S)$

In order to make the model, define the following:

- $\mathcal{L} = \mathbb{C}^{n+m+1}$ with the usual basis $(e_j)_{j=1}^{n+m+1}$, where e_j is the column vector with all zero entries, except for the j^{th} position, which is 1.
- $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, the column vector of length $n+m+1$ with 1 on each position.
- The semi-inner product on the space $\mathcal{L}$, defined by means of the Pick matrix,
$$\langle x, y \rangle_{\mathcal{L}} = y^* \mathbb{P} x = \langle \mathbb{P}x, y \rangle_{\mathbb{C}^{n+m+1}} \qquad x, y \in \mathbb{C}^{n+m+1}.$$

Let

$$\begin{aligned} \mathcal{L}_0 &= \mathcal{L} \cap \mathcal{L}^{\perp} = \\ &= \{x \in \mathcal{L} | \langle x, y \rangle_{\mathcal{L}} = 0 \;\; \forall y \in \mathcal{L}\} \\ &= \{x \in \mathcal{L} | \langle \mathbb{P}x, y \rangle_{\mathbb{C}^{n+m+1}} = 0 \;\; \forall y \in \mathcal{L}\} \\ &= \{x \in \mathcal{L} | \mathbb{P}x = 0\} \\ &= \ker \mathbb{P}. \end{aligned}$$

In the models for the IP and the MP, the diagonal matrix Z and the 'shift-right' matrix S_r are used (see [Dijk] for details):

$$Z = \operatorname{diag}(z_1, \dots, z_n) = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix} \quad ; \quad S_r = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Just as the Pick matrix $\mathbb{P}$ is a composition of $\mathbb{P}_I$ and $\mathbb{P}_M$, the matrix $\mathcal{C}$ (for 'composite' matrix), used to construct the model for the MIP, is a combination of these two matrices Z and S_τ:

$$\mathcal{C} := \left(\begin{array}{c|c} Z^* & 0 \\ \hline e^* & \\ 0 & S_\tau \end{array}\right) = \left(\begin{array}{ccc|ccc} \overline{z}_1 & & 0 & 0 & & 0 \\ & \ddots & & & & \\ 0 & & \overline{z}_n & 0 & & 0 \\ \hline 1 & \cdots & 1 & 0 & & \\ & & & 1 & \ddots & \\ & 0 & & & \ddots & \ddots \\ & & & & 1 & 0 \end{array}\right).$$

With this matrix $\mathcal{C}$, now define the relation

$$S_{\mathcal{L}} = \{\{x, \mathcal{C}x\} | x \in \mathcal{L}, e^*_{n+m+1}x = 0\}.$$

The last condition $e^*_{n+m+1}x = 0$ actually means that the last component of x must be zero. The defined relation $S_{\mathcal{L}}$ is *symmetric*. This property can be proven by calculating the *Lyaponov form* $\mathbb{P}\mathcal{C} - \mathcal{C}^*\mathbb{P}$:

$$(\mathbb{P}\mathcal{C})_{ij} = \overline{z}_j \frac{w_i - \overline{w}_j}{z_i - \overline{z}_j} + w_i \ , \ 1 \le i, j \le n,$$

$$(\mathbb{P}\mathcal{C})_{ij} = \begin{cases} w_i z_i^{j-n} + s_0 z_i^{j-n-1} + s_1 z_i^{j-n-2} + \cdots + s_{j-n-1}, & 1 \le i \le n \ , \ n+1 \le j \le n+m, \\ 0, & 1 \le i \le n \ , \ j = n+m+1, \end{cases}$$

$$(\mathbb{P}\mathcal{C})_{ij} = \overline{w}_j \overline{z}_j^{\,i-n} + s_0 \overline{z}_j^{\,i-n-1} + s_1 \overline{z}_j^{\,i-n-2} + \cdots + s_{i-n-1} \ , \ n+1 \le i \le n+m+1 \ , \ 1 \le j \le n,$$

$$(\mathbb{P}\mathcal{C})_{ij} = \begin{cases} s_{(i-n)+(j-n-1)} = s_{i+j-2n-1}, & n+1 \le i \le n+m+1 \ , \ n+1 \le j \le n+m, \\ 0, & n+1 \le i \le n+m+1 \ , \ j = n+m+1. \end{cases}$$

Notice that $(\mathcal{C}^*\mathbb{P})_{ij} = \overline{(\mathbb{P}\mathcal{C})_{ji}}$ and then obtain that:

$$(\mathcal{C}^*\mathbb{P})_{ij} = z_i \frac{\overline{w}_j - w_i}{\overline{z}_j - z_i} + \overline{w}_j \ , \ 1 \le i, j \le n,$$

$$(\mathcal{C}^*\mathbb{P})_{ij} = w_i z_i^{j-n} + s_0 z_i^{j-n-1} + s_1 z_i^{j-n-2} + \cdots + s_{j-n-1} \ , \ 1 \le i \le n \ , \ n+1 \le j \le n+m+1,$$

$$(\mathcal{C}^*\mathbb{P})_{ij} = \begin{cases} \overline{w}_j \overline{z}_j^{\,i-n} + s_0 \overline{z}_j^{\,i-n-1} + s_1 \overline{z}_j^{\,i-n-2} + \cdots + s_{i-n-1}, & n+1 \le i \le n+m \ , \ 1 \le j \le n, \\ 0, & i = n+m+1 \ , \ 1 \le j \le n, \end{cases}$$

$$(C^*\mathbb{P})_{ij} = \begin{cases} s_{i+j-2n-1}, & n+1 \le i \le n+m\ ,\ n+1 \le j \le n+m+1, \\ 0, & i = n+m+1\ ,\ n+1 \le j \le n+m+1. \end{cases}$$

From these expressions above it follows that:

$$\mathbb{P}C - C^*\mathbb{P} = \underbrace{\left(\begin{array}{ccc|ccc} & & & 0 & \cdots & 0 & -(w_1 z_1^{m+1} + p(z_1)) \\ & 0 & & \vdots & & \vdots & \vdots \\ & & & 0 & \cdots & 0 & -(w_n z_n^{m+1} + p(z_n)) \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & -s_{m+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -s_{2m} \\ \overline{w}_1 \overline{z}_1^{m+1} + p(\overline{z}_1) & \cdots & \overline{w}_n \overline{z}_n^{m+1} + p(\overline{z}_n) & s_{m+1} & \cdots & s_{2m} & 0 \end{array}\right)}_{\text{with } p(z) = s_0 z^m + \cdots + s_{m-1} z + s_m}$$

So it is a matrix with a lot of zeros, except for the last row and column.
For $x, y \in \operatorname{dom} S_{\mathcal{L}}$, use this result to get

$$\begin{aligned} \langle S_{\mathcal{L}} x, y\rangle_{\mathcal{L}} - \langle x, S_{\mathcal{L}} y\rangle_{\mathcal{L}} &= \\ &= y^* \mathbb{P} S_{\mathcal{L}} x - (S_{\mathcal{L}} y)^* \mathbb{P} x \\ &= y^* \mathbb{P} C x - y^* C^* \mathbb{P} x \\ &= y^* (\mathbb{P}C - C^*\mathbb{P}) x \\ &= 0, \end{aligned}$$

because $e_{n+m+1}^* x = e_{n+m+1}^* y = 0$. Hence $S_{\mathcal{L}}$ is symmetric.

The last step is to define the desired model $(\mathcal{H}, S)$:

$$\mathcal{H} := \mathcal{L} \setminus \mathcal{L}_0 = \widehat{\mathcal{L}} = \{\hat{x} | x \in \mathcal{L}\}$$

with $\hat{x} = \{y \in \mathcal{L} | y - x \in \mathcal{L}_0\} = x + \mathcal{L}_0$ and inner product $\langle \hat{x}, \hat{y}\rangle_{\mathcal{H}} = \langle x, y\rangle_{\mathcal{L}}$, and

$$S := \widehat{S}_{\mathcal{L}} = \{\{\hat{x}, \widehat{Cx}\} | \{x, Cx\} \in S_{\mathcal{L}}\}.$$

Note that if $\det \mathbb{P} \neq 0$, $\mathcal{H} = \mathcal{L}$ and $S = S_{\mathcal{L}}$.

In the next two sections we will learn more about S and its properties.

2.2 Linear relations and operators

In this section we will recall some general definitions and results related to linear relations and operators, but apply them directly to our model. At the end of this section we will also show that S in the model in some cases is a *linear relation*, but in other cases it is an *operator*. To see that we will use an example.

First of all we recall the definitions of *linear relation* and *operator* :

A **(closed) linear relation** S in a *Hilbert space* $\mathcal{H}$ is a (closed) linear subset of the direct sum space $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$; a linear relation S is (the graph of) an **operator** if and only if its multivalued part $S(0) = \{g \in \mathcal{H} : \{0, g\} \in S\}$ is the trivial subspace of $\mathcal{H}$: $S(0) = \{0\}$.
The **adjoint** of S is the closed linear relation

$$S^* = \{\{u, v\} \in \mathcal{H} : \langle v, f\rangle_{\mathcal{H}} - \langle u, g\rangle_{\mathcal{H}} = 0 \text{ for all } \{f, g\} \in S\}.$$

S is called **symmetric** if $S \subseteq S^*$ and **selfadjoint** if equality prevails: $S = S^*$.
Now assume S is closed and symmetric; S is called **simple** if

$$\bigcap_{z\in\mathbb{C}\setminus\mathbb{R}} \text{ran}\,(S - z) = \{0\} \Longleftrightarrow \overline{\text{span}}\,\{\ker\,(S^* - z)\,|\, z \in \mathbb{C} \setminus \mathbb{R}\} = \mathcal{H},$$

that is, the eigenelements of S^* corresponding to nonreal eigenvectors of span $\mathcal{H}$.
A selfadjoint linear relation $\tilde{A}$ in a Hilbert space $\tilde{\mathcal{H}}$ is called a **selfadjoint extension** of S in $\mathcal{H}$ if the space $\mathcal{H}$ is a closed subspace of $\tilde{\mathcal{H}}$ and $S \subset \tilde{A}$. The Hilbert space $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is called the **exit space**. $\tilde{A}$ is called a **canonical** extension if $\tilde{\mathcal{H}} = \mathcal{H}$, that is, the exit space consists of the zero element only, otherwise it is called a **noncanonical** extension. Thus in the canonical case, both S and $\tilde{A}$ act in $\mathcal{H}$.
We will be interested in minimal selfadjoint extensions of S. By definition, a selfadjoint extension $\tilde{A}$ in $\tilde{\mathcal{H}}$ of S is called **minimal** if for some (and hence every) $\mu \in \rho(\tilde{A})$,

$$\overline{\text{span}}\{(I + (z - \mu)(\tilde{A} - z)^{-1})\mathcal{H} : z \in \rho(\tilde{A})\} = \tilde{\mathcal{H}}.$$

Since S in our model is symmetric, it admits INFINITELY MANY **noncanonical** minimal selfadjoint extensions. (Here we assume that S is NOT selfadjoint, because in that case the only minimal selfadjoint extension is S itself). But there also are INFINITELY MANY **canonical** minimal selfadjoint extensions of S. To make this clear, we will first introduce the **defect numbers** d^+ and d^- of S: if S is symmetric, then dim $(\ker\,(S^* - z)), z \in \mathbb{C}^{\pm}$, is constant on $\mathbb{C}^{\pm}$. We denote these constants by d^+ and d^- respectively. The **defect index** of a symmetric S is the pair (d^+, d^-).
Since $(\text{ran}\, T)^{\perp} = \ker T^*$, we have for $z \in \mathbb{C}^+$:

$$\text{ran}\,(S - z) = \mathcal{H} \Longleftrightarrow d^- = 0; \quad \text{ran}\,(S - \bar{z}) = \mathcal{H} \Longleftrightarrow d^+ = 0.$$

To find the canonical extensions of S we use the *Cayley transformation* C_μ and its inverse F_μ,

$$C_\mu(S) = \{\{g - \mu f, g - \bar{\mu} f\} | \{f, g\} \in S\}, \quad \mu \in \mathbb{C} \setminus \mathbb{R}.$$

We assume S has defect index $(1,1)$. Then there exists a $f \in (\text{ran}\,(S - \mu))^{\perp} =$ $= \ker\,(S^* - \bar{\mu})$, $\|f\| = 1$ and a $g \in (\text{ran}\,(S - \bar{\mu}))^{\perp} = \ker\,(S^* - \mu)$, $\|g\| = 1$.
If $V = C_\mu(S)$, $\quad V : \text{ran}\,(S - \mu) \longmapsto \text{ran}\,(S - \bar{\mu})$ and we define for $\theta \in [0, 2\pi)$

$$U_\theta h = \begin{cases} Vh & h \in \text{ran}\,(S - \mu) \\ \alpha e^{i\theta} g & h = \alpha f,\ \alpha \in \mathbb{C} \quad (h = \text{span}\,\{f\}), \end{cases}$$

then for $(\text{ran}\,(S - \mu) \oplus \text{span}\,\{f\}) \ni k = k_1 + k_2$ with $k_1 \in \text{ran}\,(S - \mu)$, $k_2 = \alpha f$ holds

$$U_\theta k = V k_1 + \alpha e^{i\theta} g.$$

This U_θ is unitary and a canonical extension of V ($V \subset U_\theta$). We conclude that

$$A_\theta = C_\mu^{-1}(U_\theta) = F_\mu(U_\theta) \supset F_\mu(V) = C_\mu^{-1}(C_\mu(S)) = S,$$

so A_θ is a canonical selfadjoint extension of S. Since the mapping $\theta \in [0, 2\pi) \longmapsto U_\theta$ is injective, we obtain that indeed there are infinitely many canonical (hence minimal) selfadjoint extensions A_θ of S.

Example Take $n = 0$ and $m = 2$, and let $s_0 = s_1 = s_2 = s_3 = 1$.

Then $\mathbb{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & s_4 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. It follows that $\mathbb{P} \geq 0 \Leftrightarrow s_4 \geq 1$.

It is clear that $e_0 = (1\ 0\ 0)^t$ and $e_1 = (0\ 1\ 0)^t$ span the domain of S.
Assume $s_4 > 1$. Then $\{e_0, e_1\}$ and $\{e_1, e_2\} \in S$. Since $\mathbb{P}(e_0 - e_1) = \mathbb{P}(1, -1, 0)^t = 0$ we obtain that $\ker \mathbb{P} = \operatorname{span}\{e_0 - e_1\}$; so for $\{e_0 - e_1, e_1 - e_2\} \in S_{\mathcal{L}}$ we see

$$\{\widehat{e_0 - e_1}, \widehat{e_1 - e_2}\} = \{0, \widehat{e_1 - e_2}\} \in \widehat{S}_{\mathcal{L}} = S.$$

Because $e_1 - e_2$ is not an element of the kernel of $\mathbb{P}$, $\widehat{e_1 - e_2} \neq 0$, so $S = \widehat{S}_{\mathcal{L}}$ is a *relation.*

In the case where $s_4 = 1$, $\ker \mathbb{P} = \operatorname{span}\{e_0 - e_1, e_1 - e_2\}$. But the only elements x in this span, that are also in $\operatorname{dom} S_{\mathcal{L}}$, are multiples of $e_0 - e_1$. In that case $S_{\mathcal{L}} x = Cx$ is a multiple of $e_1 - e_2$, which is also in $\ker \mathbb{P}$, so

$$\{\hat{x}, \widehat{Cx}\} = \{0, 0\}.$$

For every other $x \in \operatorname{dom} S_{\mathcal{L}}$ (it is $x = \alpha e_0 + \beta e_1$, NOT $\alpha = 1, \beta = -1$ or multiples), $S_{\mathcal{L}} x = Cx = \alpha e_1 + \beta e_2 \notin \ker \mathbb{P}$, so

$$\hat{x} \neq 0 \quad \text{and} \quad \widehat{Cx} \neq 0 \ \forall x.$$

Hence $S = \widehat{S}_{\mathcal{L}}$ is an *operator.*

2.3 Properties of the model

It already has been shown in Section 2.1 that $S_{\mathcal{L}}$ is symmetric, but also S has some nice properties – depending on the determinant of the Pick matrix $\mathbb{P}$ – according to the next lemma:

Lemma 2.1 *Assume* $\mathbb{P} \geq 0$.

(1) *If* $\det \mathbb{P} = 0$ *then* S *is selfadjoint and its defect index is (0,0).*

(2) *If* $\det \mathbb{P} \neq 0$, *that is if* $\mathbb{P} > 0$ *then* S *is simple, closed, nondensely defined and symmetric with defect (1,1) and* $\ker(S^* - z)$ *is spanned by*

$$\varphi(z) = \begin{cases} \mathbb{P}^{-1}(C^* - z)^{-1} e_{n+m+1}, & z \neq 0, z_1, \dots, z_n, \\ \mathbb{P}^{-1}(e_{n+1} - \sum_{j=1}^{n} \frac{1}{z_j} e_j), & z = 0, \\ \mathbb{P}^{-1} e_j, & z = z_j,\ j = 1, \dots, n. \end{cases}$$

Proof (1) Since $\det \mathbb{P} = 0$, the kernel of $\mathbb{P}$ is not trivial. Hence $\mathcal{H} = \widehat{\mathcal{L}}$ and $S = \widehat{S}_{\mathcal{L}}$. Let φ be an element in the kernel of $\mathbb{P}$, $0 \neq \varphi \in \ker \mathbb{P}$.
We have

$$x \in \operatorname{ran}(S_{\mathcal{L}} - z) \Leftrightarrow \exists\, y \in \operatorname{dom} S_{\mathcal{L}} : \ (\mathcal{C} - z)y = x \Leftrightarrow$$
$$\overset{z \neq \overline{z}_1, \ldots, \overline{z}_n, 0}{\Longleftrightarrow} y = (\mathcal{C} - z)^{-1}x \in \operatorname{dom} S_{\mathcal{L}} \Leftrightarrow e^*_{n+m+1}(\mathcal{C} - z)^{-1}x = 0.$$

Hence for $g \in \mathcal{L}$:

$$g - \frac{e^*_{n+m+1}(\mathcal{C} - z)^{-1}g}{e^*_{n+m+1}(\mathcal{C} - z)^{-1}\varphi}\varphi \in \operatorname{ran}(S_{\mathcal{L}} - z). \tag{2.1}$$

Since

$$\begin{aligned} u \in \operatorname{ran}(S_{\mathcal{L}} - z) &\Leftrightarrow \\ &\Leftrightarrow u = (S_{\mathcal{L}} - z)f \Leftrightarrow S_{\mathcal{L}}f = u + zf \\ &\Leftrightarrow \{f, u + zf\} \in S_{\mathcal{L}} \Leftrightarrow \{\hat{f}, \hat{u} + z\hat{f}\} \in \widehat{S}_{\mathcal{L}} \\ &\Leftrightarrow \hat{u} \in \operatorname{ran}(\widehat{S}_{\mathcal{L}} - z), \end{aligned}$$

and since $\widehat{\varphi} = 0$ $\ (\hat{x} = \hat{y} \Leftrightarrow x - y \in \mathcal{L}_0 = \ker\ \mathbb{P}\ )$, it follows that $\hat{g} \in \operatorname{ran}(\widehat{S}_{\mathcal{L}} - z)$; of course $g \in \mathcal{L} \Leftrightarrow \hat{g} \in \widehat{\mathcal{L}}$, so this means that

$$\operatorname{ran}(\widehat{S}_{\mathcal{L}} - z) = \widehat{\mathcal{L}} = \mathcal{H}, \quad (z \neq \overline{z}_1, \ldots \overline{z}_n, 0). \tag{2.2}$$

Now let $\varphi, \psi \in \ker\ \mathbb{P}$, then

$$\begin{aligned} \langle \hat{x}, \hat{y} \rangle_{\mathcal{H}} &= \langle x + \varphi, y + \psi \rangle_{\mathcal{L}} = \\ &= \langle x, y \rangle_{\mathcal{L}} + \overline{\langle \psi, x \rangle_{\mathcal{L}}} + \langle \varphi, y \rangle_{\mathcal{L}} = \langle \varphi, \psi \rangle_{\mathcal{L}} \\ &= \langle x, y \rangle_{\mathcal{L}} + \overline{x^*\mathbb{P}\psi} + y^*\mathbb{P}\varphi + \psi^*\mathbb{P}\varphi \\ &= \langle x, y \rangle_{\mathcal{L}}. \end{aligned}$$

Hence, since $S_{\mathcal{L}}$ is symmetric, it follows that $\widehat{S}_{\mathcal{L}}$ is symmetric also. The above equality (2.2), telling that $\operatorname{ran}(\widehat{S}_{\mathcal{L}} - z)$ is equal to the whole space $\mathcal{H}$, now directly shows that the *defect numbers* are zero (see Section 2.2), and hence $\widehat{S}_{\mathcal{L}}$ is *selfadjoint.*

(2) Now assume $\mathbb{P} > 0$ which means that $\det\ \mathbb{P} \neq 0$ so clearly $\mathbb{P}^{-1}$ exists; now the kernel of $\mathbb{P}$ is trivial, $\ker\ \mathbb{P} = \{0\}$, so $\mathcal{H} = \mathcal{L}$ and $S = S_{\mathcal{L}}$. Since $\mathcal{L} = \mathbb{C}^{n+m+1}$ is a Hilbert space, we have in this case that $\mathcal{H}$ is a Hilbert space. Hence S is *closed* and *nondensely defined*, and S is also *symmetric* (this was already found in Section 2.1).

To prove the remaining part, first calculate $(\mathcal{C} - z)^{-1}$ (for $z \neq \overline{z}_1, \ldots, \overline{z}_n, 0$):

(Note: $(S_r - z)^{-1} = -\frac{1}{z}(1 - \frac{1}{z}S_r)^{-1} = -\frac{1}{z}\sum_{n=0}^{m}(\frac{1}{z}S_r)^n = -(\frac{1}{z} + \frac{S_r}{z^2} + \frac{S_r^2}{z^3} + \cdots + \frac{S_r^m}{z^{m+1}})$ because $S_r^{m+1} = 0$, the zero matrix)

$$(\mathcal{C}-z)^{-1} = \left(\begin{array}{c|c} (Z^*-z)^{-1} & 0 \\ \hline -(S_r-z)^{-1}\begin{pmatrix} e^* \\ 0 \end{pmatrix}(Z^*-z)^{-1} & (S_r-z)^{-1} \end{array}\right)$$

$$= \left(\begin{array}{ccc|cccc} \frac{1}{\overline{z}_1-z} & & 0 & 0 & & & 0 \\ & \ddots & & & & & \\ 0 & & \frac{1}{\overline{z}_n-z} & 0 & & & 0 \\ \hline \frac{1}{z}\frac{1}{\overline{z}_1-z} & \cdots & \frac{1}{z}\frac{1}{\overline{z}_n-z} & -\frac{1}{z} & 0 & \cdots & 0 \\ \frac{1}{z^2}\frac{1}{\overline{z}_1-z} & \cdots & \frac{1}{z^2}\frac{1}{\overline{z}_n-z} & -\frac{1}{z^2} & -\frac{1}{z} & 0 & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{1}{z^{m+1}}\frac{1}{\overline{z}_1-z} & \cdots & \frac{1}{z^{m+1}}\frac{1}{\overline{z}_n-z} & -\frac{1}{z^{m+1}} & \cdots & -\frac{1}{z^2} & -\frac{1}{z} \end{array}\right), \quad z \neq \overline{z}_1, \ldots, \overline{z}_n, 0.$$

Because

$$\begin{aligned} &\varphi(z) \in \ker(S^*-z) = \operatorname{ran}(S-\overline{z})^{\perp} \Leftrightarrow \\ &\quad \Leftrightarrow \langle \varphi(z), (\mathcal{C}-\overline{z})x\rangle_{\mathcal{L}} = 0, \quad \forall x : e^*_{n+m+1}x = 0 \\ &\quad \Leftrightarrow x^*(\mathcal{C}-\overline{z})^*\mathbb{P}\varphi(z) = 0, \quad \forall x : e^*_{n+m+1}x = 0 \Leftrightarrow x^* e_{n+m+1} = 0 \\ &\quad \Leftrightarrow (\mathcal{C}-\overline{z})^*\mathbb{P}\varphi(z) = k\cdot e_{n+m+1}, \quad \text{with } k \text{ an arbitrary constant,} \end{aligned}$$

make a distinction between the following cases:

- If $z \neq \overline{z}_1, \ldots, \overline{z}_n, 0$ then $(\mathcal{C}-\overline{z})^{-1}$ exists, so $\varphi(z) \overset{k=1}{=} \mathbb{P}^{-1}(\mathcal{C}^*-z)^{-1}e_{n+m+1}$.
- If $z = 0$ then $\mathcal{C}^*\mathbb{P}\varphi(0) = k\cdot e_{n+m+1}$. Since the last row of $\mathcal{C}^*$ is $(0\cdots 0)$, k must be zero. We obtain that

  $$\mathbb{P}\varphi(0) = \begin{pmatrix} -\frac{1}{z_1} \\ \vdots \\ -\frac{1}{z_n} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_{n+1} - \sum_{j=1}^{n}\frac{1}{z_j}e_j$$

  satisfies the equation with righthand side is zero.
- If $z = \overline{z}_j$, $j = 1, \ldots, n$, then $(\mathcal{C}-\overline{z})^* = \mathcal{C}^* - z$ has a zero $(z_j - z_j)$ on position (j,j). Hence the j^{th} row of $(\mathcal{C}^*-z)$ is e^*_{n+1}. From $(\mathcal{C}^*-z)\mathbb{P}\varphi(z_j) = k\cdot e_{n+m+1}$ we deduce that the $(n+1)^{st}$ entry of $\mathbb{P}\varphi(z_j)$ must be zero. From this it follows that all other entries of $\mathbb{P}\varphi(z_j)$ must be zero also, except for the j^{th} position, which is arbitrary; take it equal to 1 and obtain $\varphi(z) = \mathbb{P}^{-1}e_j$.

Hence, in all three cases, *the kernel of* (S^*-z) *is the span of* $\varphi(z)$.
From this, now also follows directly that $\forall z \in \mathbb{C}^{\pm}$: $\dim(\ker(S^*-z)) = 1$ (of course it is

constant, since S is symmetric), so indeed, the *defect index of S is* $(1,1)$.
We also see that

$$e_1, \dots, e_n \in \overline{\operatorname{span}}\,\{\mathbb{P}\varphi(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}, \tag{2.3}$$

(from the third case), but then also the linear combination $\sum_{j=1}^{n} \frac{1}{z_j} e_j$; since that span also contains $e_{n+1} - \sum_{j=1}^{n} \frac{1}{z_j} e_j$, it follows from the second case that also

$$e_{n+1} \in \overline{\operatorname{span}}\,\{\mathbb{P}\varphi(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}. \tag{2.4}$$

From the first case, that span also contains

$$(\mathcal{C}^* - z)^{-1} e_{n+m+1} = ((\mathcal{C} - \overline{z})^{-1})^* e_{n+m+1} = \frac{1}{z^{m+1}} \begin{pmatrix} \frac{1}{z_1 - z} \\ \vdots \\ \frac{1}{z_n - z} \\ -1 \\ -z \\ \vdots \\ -z^m \end{pmatrix},$$

and because of $e_1, \dots, e_{n+1}$, also all vectors of the form $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ z \\ \vdots \\ z^m \end{pmatrix}$, $z \in \mathbb{C} \setminus \mathbb{R}$,

it is all vectors of the form $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ z^{m-1} \end{pmatrix}$, $z \in \mathbb{C} \setminus \mathbb{R}$ (first $n+1$ entries are zero).

With $z \searrow 0$, we find that

$$e_{n+2} \in \overline{\operatorname{span}}\,\{\mathbb{P}\varphi(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}, \tag{2.5}$$

and so on:

$$e_{n+3}, \dots, e_{n+m+1} \in \overline{\operatorname{span}}\,\{\mathbb{P}\varphi(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}, \tag{2.6}$$

From (2.3), (2.4), (2.5) and (2.6) we conclude that

$$e_1, \dots, e_{n+m+1} \in \overline{\operatorname{span}}\,\{\mathbb{P}\varphi(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}.$$

Hence this span coincides with $\mathcal{H} = \mathbb{C}^{n+m+1}$, and since $\mathbb{P}^{-1}$ exists (so we know that the $n+m+1$ columns of $\mathbb{P}^{-1}$ are linearly independent and form a basis for $\mathbb{C}^{n+m+1}$) the $\mathbb{P}^{-1}\mathbb{P}\varphi(z) = \varphi(z)$ span $\mathcal{H}$ also, so

$$\operatorname{span} \varphi(z) = \ker\,(S^* - z) = \mathcal{H},$$

which implies that S is *simple*. □

Before we use the properties of S – as described in Lemma 2.1 – to find solutions of the MIP, we have to find a kind of relation between S and these solutions. This 'relation' will be introduced in the next chapter.

Chapter 3

Solutions and extensions of the model

3.1 Characterization of solutions via selfadjoint extensions of S in the model

In this section a theorem will be introduced and proved, which will help to find solutions of the MIP, using minimal selfadjoint extensions of the linear relation S in the model. Because, from now on, we only work with this model $(\mathcal{H}, S)$, we will use the notation $\langle \cdot, \cdot \rangle$, without the subscript $\mathcal{L}$, to denote the semi-inner product $\langle x, y \rangle_{\mathcal{L}} \equiv y^* \mathbb{P} x$ as defined in Chapter 2.1.

Theorem 3.1 *The formula*

$$N(z) = \langle (\tilde{A} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1} \rangle$$

establishes a 1-1 correspondence between all solutions $N(z)$ of the MIP and all minimal selfadjoint extensions $\tilde{A}$ of the symmetric relation S in the model.

In the proof of Theorem 3.1 the following theorem is used. Compare it to the theorem in [KL1].

Theorem 3.2 *Equivalent are:*

(a) For $N \in \mathbb{N}_0$ and for some $s'_0, \dots, s'_{2m} \in \mathbb{R}$

$$-\lim_{\substack{z=iy \\ y \to \infty}} z^{2m+1} \left(N(z) + \frac{s'_0}{z} + \cdots + \frac{s'_{2m-1}}{z^{2m}} \right) = s'_{2m}.$$

(b) There exists a selfadjoint operator $\tilde{A}$ in the Hilbert space $\mathcal{H}$ and a $u \in$ dom A^m such that A is u-minimal, that is

$$\mathcal{H} = \overline{span} \, \{ (A - z)^{-1} u \, | z \in \mathbb{C} \backslash \mathbb{R} \},$$

and

$$N(z) = \langle (A - z)^{-1} u, u \rangle.$$

If (a) and (b) hold then

$$s'_k = \begin{cases} \langle A^k u, u\rangle, & 0 \le k \le m, \\ \langle A^m u, A^{k-m} u\rangle, & m+1 \le k \le 2m. \end{cases}$$

The u-minimality in (b) implies uniqueness of the representation up to isomorphism. For the proof of Theorem 3.2 see [KL1].

Proof of Theorem 3.1

Let $S \subset \tilde{A} = \tilde{A}^*$ in $\tilde{\mathcal{H}}$ (so $\tilde{A}$ is a selfadjoint extension of S).
Since dom $S_{\mathcal{L}} = \{x \,|x_{n+m+1} = 0\}$ and $S_{\mathcal{L}} e_{n+j} = \mathcal{C} e_{n+j} = e_{n+j+1}$, $\quad j = 1, \ldots, m$, it follows that

$$\{\hat{e}_{n+1}, \hat{e}_{n+2}\}, \{\hat{e}_{n+2}, \hat{e}_{n+3}\}, \cdots, \{\hat{e}_{n+m-1}, \hat{e}_{n+m}\}, \{\hat{e}_{n+m}, \hat{e}_{n+m+1}\} \in S \subset \tilde{A}.$$

This implies that $\hat{e}_{n+1}, \ldots, \hat{e}_{n+m} \in \operatorname{dom} \tilde{A} = \operatorname{dom} \tilde{A}_s$, where

$$\tilde{A} = \tilde{A}_s \oplus \tilde{A}_\infty \quad \text{and} \quad \tilde{\mathcal{H}} = \mathcal{H}_s \oplus \mathcal{H}_\infty$$

with

$$\tilde{A}_\infty = \{0\} \times \tilde{A}(0) \quad \text{and} \quad \mathcal{H}_\infty = \tilde{A}(0).$$

$\tilde{A}_s$ is a *selfadjoint operator* in $\mathcal{H}_s = \tilde{A}(0)^\perp = \overline{\operatorname{dom}}\, \tilde{A}^* = \overline{\operatorname{dom}}\, \tilde{A} = \overline{\operatorname{dom}}\, \tilde{A}_s$.
P_s is the orthogonal projection in $\tilde{\mathcal{H}}$ onto $\mathcal{H}_s$.
Note that $\hat{e}_{n+m+1}$ does not necessarily belong to $\mathcal{H}_s$ (it does belong to $\mathcal{H}$, the model space, of course).
We find that

$$N(z) := \langle (\tilde{A} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1}\rangle = \langle (\tilde{A}_s - z)^{-1} P_s \hat{e}_{n+1}, P_s \hat{e}_{n+1}\rangle = \langle (\tilde{A}_s - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1}\rangle$$

satisfies (b) of Theorem 3.2:
because of the definition, $\tilde{A}_s$ is a selfadjoint operator in $\mathcal{H}_s = \overline{\operatorname{dom}} \tilde{A}_s$; recall that $S\hat{e}_{n+1} = \hat{e}_{n+2}$ and so on, so

$$S^m \hat{e}_{n+1} = \hat{e}_{n+m+1}.$$

Because dom $S \subset$ dom $\tilde{A}_s$, we also obtain that

$$\tilde{A}_s^m \hat{e}_{n+1} = \hat{e}_{n+m+1}.$$

Since $\tilde{A}_s^{m-1} \hat{e}_{n+1} = \hat{e}_{n+m} \in \operatorname{dom} \tilde{A}_s$, now obtain that $\tilde{A}_s \hat{e}_{n+m} = \tilde{A}_s^m \hat{e}_{n+1}$ 'exists'. Hence,

$$\hat{e}_{n+1} \in \operatorname{dom} \tilde{A}_s^m.$$

Now we showed this, we conclude from the equivalence in Theorem 3.2 that $N(z)$ satisfies a moment problem with a $=$-sign instead of a $\le$-sign; use

$$\langle \hat{e}_j, \hat{e}_k\rangle = \langle e_j, e_k\rangle = e_k^* \mathbb{P} e_j = \mathbb{P}_{kj}$$

to determine what the s'_k in this MP are:

$$s'_k = \begin{cases} \langle \tilde{A}_s^k \hat{e}_{n+1}, \hat{e}_{n+1}\rangle = \langle \hat{e}_{n+k+1}, \hat{e}_{n+1}\rangle = s_k, & 0 \le k < m, \\ \langle \tilde{A}_s^m \hat{e}_{n+1}, \tilde{A}_s^{k-m} \hat{e}_{n+1}\rangle & \\ \qquad = \langle P_s \hat{e}_{n+m+1}, \hat{e}_{n+k-m}\rangle = \langle \hat{e}_{n+m+1}, \hat{e}_{n+1+k-m}\rangle = s_k, & m \le k \le 2m-1, \\ \langle \tilde{A}_s^m \hat{e}_{n+1}, \tilde{A}_s^m \hat{e}_{n+1}\rangle & \\ \qquad = \langle P_s \hat{e}_{n+m+1}, P_s \hat{e}_{n+m+1}\rangle \le \langle \hat{e}_{n+m+1}, \hat{e}_{n+m+1}\rangle = s_{2m}, & k = 2m. \end{cases}$$

Note that this last case is the moment where the $\leq$-sign comes into the original moment problem; the numbers $s_k \in \mathbb{R}$ in the MIP are equal to the s'_k in the MP in part (a) of Theorem 3.2, except for the last one: $s_{2m} \geq s'_{2m}$. (We already saw this in Chapter 1). But this means that

$$N(z) := \langle (\tilde{A} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1} \rangle$$

satisfies

$$- \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\Big(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\Big) \overset{s_k=s'_k}{=}$$
$$= - \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\Big(N(z) + \frac{s'_0}{z} + \cdots + \frac{s'_{2m-1}}{z^{2m}}\Big) = s'_{2m} \leq s_{2m},$$

so it satisfies our MP !
To see that $N(z)$ also satisfies the IP, recall that for $j = 1, \ldots, n$:
$e_j \in \operatorname{dom} S_{\mathcal{L}}$ and

$$(S_{\mathcal{L}} - z)\hat{e}_j = (\mathcal{C} - z)\hat{e}_j = (\overline{z}_j - z)\hat{e}_j + \hat{e}_{n+1},$$

and so

$$\hat{e}_j = (\overline{z}_j - z)(\tilde{A} - z)^{-1}\hat{e}_j + (\tilde{A} - z)^{-1}\hat{e}_{n+1}.$$

It follows that

$$\hat{e}_j + (z - \overline{z}_j)(\tilde{A} - z)^{-1}\hat{e}_j = (\tilde{A} - z)^{-1}\hat{e}_{n+1},$$

so the lefthand side of this equation is independent of j. Hence,

$$\begin{aligned} N(z) = \langle (\tilde{A} - z)^{-1}\hat{e}_{n+1}, \hat{e}_{n+1}\rangle &= \langle \hat{e}_{n+1}, \left((\tilde{A} - z)^{-1}\right)^* \hat{e}_{n+1}\rangle = \\ &= \langle \hat{e}_{n+1}, (\tilde{A} - \bar{z})^{-1}\hat{e}_{n+1}\rangle \\ &= \langle \hat{e}_{n+1}, \hat{e}_j + (\bar{z} - \overline{z}_j)(\tilde{A} - \bar{z})^{-1}\hat{e}_j\rangle \\ &\overset{z=z_j}{=} \langle \hat{e}_{n+1}, \hat{e}_j\rangle \\ &= \mathbb{P}_{j,n+1} = w_j, \end{aligned}$$

so indeed $N(z_j) = w_j, \quad j = 1, \ldots, n$, so $N(z)$ also satisfies the IP !

Now the converse. Let N be a solution of the MIP with moment problem

$$- \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\Big(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\Big) = s'_{2m} \leq s_{2m}.$$

Then, by Theorem 3.2, there is a selfadjoint operator A_1 in $\mathcal{H}_1$ and a $u \in \operatorname{dom} A_1^m$ such that

$$N(z) = \langle (A_1 - z)^{-1}u, u\rangle,$$

and

$$\mathcal{H}_1 = \overline{\operatorname{span}}\,\{(A_1 - z)^{-1}u \,|\, z \in \mathbb{C}\backslash\mathbb{R}\}.$$

Let $e(z) = (A_1 - z)^{-1}u$. By the resolvent identity,

$$\begin{aligned}\langle e(\bar{w}), e(\bar{z})\rangle &= \langle (A_1 - \bar{w})^{-1}u, (A_1 - \bar{z})^{-1}u\rangle = \\ &= \langle (A_1 - z)^{-1}(A_1 - \bar{w})^{-1}u, u\rangle \\ &= \frac{\langle (A_1 - z)^{-1}u, u\rangle - \langle (A_1 - \bar{w})^{-1}u, u\rangle}{z - \bar{w}} \\ &= \frac{N(z) - N(w)^*}{z - \bar{w}}\end{aligned}$$

Hence if we set

$$u_j = e(\bar{z}_j), \quad j = 1, \dots, n,$$

then

$$\langle u_i, u_j\rangle = \frac{N(z_j) - N(z_i)^*}{z_j - \bar{z}_i} = \frac{w_j - \bar{w}_i}{z_j - \bar{z}_i} = \mathbb{P}_{ji}. \tag{3.1}$$

If $s_{2m} - s'_{2m} > 0$ we extend $\mathcal{H}_1$ to

$$\begin{aligned}&\tilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathbb{C}, \\ &\mathbb{C} \perp \mathcal{H}_1, \qquad \langle 1, 1\rangle = s_{2m} - s'_{2m},\end{aligned} \tag{3.2}$$

and we define the selfadjoint relation

$$\tilde{A} = A_1 \oplus \{\{0, \alpha\} | \alpha \in \mathbb{C}\}.$$

($\tilde{A} = \tilde{A}^*$ because $A_1 = A_1^*$ is given and also the second part $\{\{0, \alpha\} | \alpha \in \mathbb{C}\}$ is selfadjoint). Then A_1 is the operator part of $\tilde{A}$:

$$\tilde{A}_s = A_1.$$

If $s_{2m} = s'_{2m}$ we set

$$\tilde{\mathcal{H}} = \mathcal{H}_1 \quad \text{and} \quad \tilde{A} = A_1.$$

In any case, $(A_1 - z)^{-1}u = (\tilde{A} - z)^{-1}u$ and

$$\begin{aligned}N(z) &= \langle (A_1 - z)^{-1}u, u\rangle \\ &= \langle (\tilde{A} - z)^{-1}u, u\rangle.\end{aligned}$$

We set

$$\begin{aligned}&u_{n+1} = u, \\ &u_{n+2} = A_1 u, \\ &\vdots \\ &u_{n+m} = A_1^{m-1}u, \\ &u_{n+m+1} = \begin{cases} A_1^m u + 1, & \text{if } s_{2m} > s'_{2m}, \\ A_1^m u, & \text{if } s_{2m} = s'_{2m}. \end{cases}\end{aligned}$$

Since $A_1^* = A_1$ we also know that $(A_1^m)^* = A_1^m$ so, by Theorem 3.2 (the last part where s'_k is defined) we obtain:

(Use equation (3.2) to obtain $\langle A_1^m u, 1\rangle = 0$)

$$\begin{aligned}
&\langle u_{n+1+i}, u_{n+1+j}\rangle = \\
&\quad = \langle A_1^i u, A_1^j u\rangle \\
&\quad = \begin{cases} \langle A_1^{i+j}u, u\rangle, & 0 \le i+j \le m, \\ \langle A_1^m u, A_1^{i+j-m}u\rangle, & m < i+j \le 2m \end{cases} \\
&\quad = s_{i+j}, \qquad 0 \le i, j \le m,\ i+j \ne 2m \\
&\quad = \mathbb{P}_{n+1+j, n+1+i},
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\langle u_{n+m+1}, u_{n+m+1}\rangle = \\
&\quad = \begin{cases} \langle A_1^m u + 1, A_1^m u + 1\rangle = \langle A_1^m u, A_1^m u\rangle + 2\langle A_1^m u, 1\rangle + \langle 1, 1\rangle, & s_{2m} > s'_{2m}, \\ \langle A_1^m u, A_1^m u\rangle, & s_{2m} = s'_{2m} \end{cases} \\
&\quad = \begin{cases} s'_{2m} + 0 + s_{2m} - s'_{2m}, & s_{2m} > s'_{2m}, \\ s'_{2m}, & s_{2m} = s'_{2m} \end{cases} \\
&\quad = s_{2m} \\
&\quad = \mathbb{P}_{n+m+1, n+m+1}.
\end{aligned} \tag{3.4}$$

Hence we deduce from (3.1), (3.3) and (3.4) that

$$\langle u_i, u_j\rangle = \mathbb{P}_{ji}, \quad i, j = 1, \ldots, n+m+1.$$

Now it follows that the mapping

$$u_j \longmapsto \hat{e}_j \in \mathcal{H} \text{ (the model)}, \quad j = 1, \ldots, n+m+1$$

extends to a unitary mapping

$$W : \operatorname{span}\{u_1, \ldots, u_{n+m+1}\} \longmapsto \mathcal{H}.$$

From

$$\begin{aligned}
A_1 u_j &= A_1(A_1 - \overline{z}_j)^{-1}u = \\
&= \overline{z}_j(A_1 - \overline{z}_j)^{-1}u + u \\
&= \overline{z}_j u_j + u_{n+1}, && \text{for } j = 1, \ldots, n, \\
A_1 u_j &= A_1 A_1^{j-n-1}u = \\
&= u_j, && \text{for } j = n+1, \ldots, n+m,
\end{aligned}$$

and

$$\{u_{n+m}, u_{n+m+1}\} = \{u_{n+m}, A_1 u_{n+m} + 1\} \in \tilde{A},$$

we see that

$$\begin{aligned}
&T = \operatorname{span}\{\{u_i, u_{i+1}\} \mid i = 1, \ldots, n+m\} \subset \tilde{A}, \\
&\operatorname{dom} T = \operatorname{span}\{u_1, \ldots, u_{n+m}\},
\end{aligned}$$

and

$$WTW^{-1} = S.$$

The u-minimality of A_1 in part (b) of Theorem 3.2 implies that $\tilde{A}$ is a *minimal extension* of T. Indeed,

$$\mathcal{K} = \overline{\operatorname{span}}\,\{(I + (z-\mu)(\tilde{A} - z)^{-1})v \mid v \in \operatorname{span}\{u_1, \ldots, u_{n+m+1}\}, z \in \mathbb{C}\setminus\mathbb{R}\}$$

contains

- $u_1, \dots, u_{n+m}$ and $u_{n+m+1} = A_1 u_{n+m} + 1$ (take $z = \mu$)
- $(\tilde{A} - z)^{-1} u_{n+1} = (A_1 - z)^{-1} u_{n+1} = (A_1 - z)^{-1} u$ and hence $\mathcal{H}_1 = \overline{\text{span}}\,\{(A_1 - z)^{-1} u \,|\, z \in \mathbb{C}\backslash\mathbb{R}\}$;
by the u-minimality of A_1, in particular $u_{n+m+1} = A_1 u_{n+m}$, and hence also 1,

and therefore,

$$\mathcal{K} = \tilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathbb{C}.$$

Now recall that W is *unitary*, that is $W^* = W^{-1}$, and finally we have

$$\begin{aligned} N(z) = \langle(\tilde{A} - z)^{-1} u, u\rangle &= \\ &= \langle(\tilde{A} - z)^{-1} u_{n+1}, u_{n+1}\rangle \\ &= \langle(\tilde{A} - z)^{-1} W^{-1} \hat{e}_{n+1}, W^{-1} \hat{e}_{n+1}\rangle \\ &\overset{W^{-1}=W^*}{=} \langle W(\tilde{A} - z)^{-1} W^{-1} \hat{e}_{n+1}, \hat{e}_{n+1}\rangle \\ &= \langle(W\tilde{A}W^{-1} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1}\rangle. \end{aligned} \tag{3.5}$$

Since $T \subset \tilde{A}$ we find $S = WTW^{-1} \subset W\tilde{A}W^{-1}$ so $W\tilde{A}W^{-1}$ is a minimal selfadjoint extension of S. Hence, if we assume that N is a solution of the MIP, we find that it has the form (3.5), which is the form in Theorem 3.1. □

This completes the long proof of Theorem 3.1, but it does give us the main result which leads to all solutions of the MIP, see Corollary 3.3.

Corollary 3.3 *We recall Lemma 2.1 and obtain that, if* $\mathbb{P} \geq 0$,

(i) there is a **UNIQUE** *solution of the MIP if* $\det \mathbb{P} = 0$ *(because the only minimal selfadjoint extension* $\tilde{A}$ *of* $S = S^*$ *is* S *itself);*

(ii) there are **INFINITELY MANY** *solutions of the MIP if* $\mathbb{P} > 0$ *(because there are infinitely many noncanonical and canonical minimal selfadjoint extensions* $\tilde{A}$ *of* S*, as we saw in Section 2.2).*

From Corollary 3.3 we deduce that there really is an 'if-and-only-if'- relation between the two statements in Theorem 1.2, which was already mentioned there.

Corollary 3.4 *The MIP has either no or a unique or infinitely many solutions.*

In the next section we will assume $\mathbb{P} > 0$ and then take a better look at the infinitely many solutions; in the next chapter a representation of the unique solution is given and the *Potapov formula* is introduced for the case $\mathbb{P} > 0$.

3.2 Operator extensions and equality in the MIP

In this section we assume that the MIP has infinitely many solutions. We will prove the next theorem, which implies that there only appears an *equality* sign in the MIP if the solution N corresponds with an *operator extension* of S.

Theorem 3.5 *Suppose the MIP has* infinitely many *solutions, all corresponding to a minimal selfadjoint extension $\tilde{A}$ of S. Let*

$$d_\infty = \inf_{\text{all } N(z)} \; - \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}})$$

where the infimium is taken over all MIP-solutions $N(z)$.

(1) *There exists a unique solution $N_0(z)$ such that*

$$d_\infty = - \lim_{\substack{z=iy \\ y\to\infty}} (N_0(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}) < s_{2m}$$

and this solution corresponds to the selfadjoint canonical nonoperator *extension of S.*

(2) *For all solutions N corresponding to minimal selfadjoint* operator *extensions of S (not only the canonical ones):*

$$- \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}) = s_{2m} \quad \textit{(the maximal value).}$$

(3) *For all solutions N corresponding to minimal selfadjoint* nonoperator *extensions of S, $N \neq N_0$ (so the extensions are all 'noncanonical'):*

$$d_\infty < - \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}) < s_{2m}.$$

Proof Recall $\langle e_j, e_k\rangle = \mathbb{P}_{kj}$ and the notation in the proof of Theorem 3.1. We have

$$\begin{aligned} s_{2m} &= \langle e_{n+m+1}, e_{n+m+1}\rangle = \|e_{n+m+1}\|^2 \\ &\geq \|P_s e_{n+m+1}\|^2 = s'_{2m} = - \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}) \\ &\geq \|P_{\text{dom } S}\, e_{n+m+1}\|^2 =: d_\infty \end{aligned}$$

where

- P_s is the projection in $\tilde{\mathcal{H}}$ onto $\overline{\text{dom}\,\tilde{A}} = \text{dom}\,\tilde{A}_s$, where $N(z) = \langle(\tilde{A}-z)^{-1}e_{n+1}, e_{n+1}\rangle$, and,
- $P_{\text{dom } S}$ is the projection in $\mathcal{H}$ (the model space) onto dom S.

We know $e_{n+m+1} \notin \text{dom}\, S$, so it follows that:

$$P_{\text{dom } S}\, e_{n+m+1} \neq e_{n+m+1} \quad \text{and} \quad \|P_{\text{dom } S}\, e_{n+m+1}\|^2 < \|e_{n+m+1}\|^2.$$

There are 4 possibilities:

(a) $\tilde{A}$ is canonical and a nonoperator extension of S ($\tilde{A}(0) \neq \{0\}$).
Then $\tilde{A}$ is unique:

$$\tilde{A} = S + (0 \times S^*(0)). \tag{3.6}$$

Uniqueness follows from minimality of $\tilde{A}$; we prove $\tilde{A} = \tilde{A}^*$: we show

$$\langle \underbrace{\{x,y\}}_{\in S} + \underbrace{\{0,\lambda u\}}_{u \in S^*(0)}, \underbrace{\{\tilde{x},\tilde{y}\}}_{\in S} + \{0,\lambda u\} \rangle = 0. \tag{3.7}$$

We know S is symmetric, $S \subset S^*$ and $\{0,u\} \in S^*$. The lefthand side of (3.7) is

$$\langle \{x,y\},\{\tilde{x},\tilde{y}\}\rangle + \langle \{x,y\},\{0,\lambda u\}\rangle + \langle \{0,\lambda u\},\{\tilde{x},\tilde{y}\}\rangle + \langle \{0,\lambda u\},\{0,\lambda u\}\rangle,$$

but all these four terms are zero:

- the first one because $\langle x,\tilde{y}\rangle - \langle y,\tilde{x}\rangle = 0$ since $S \subset S^*$, so $\langle Su, v\rangle = \langle u, Sv\rangle$.
- the second one is $\lambda\langle x,u\rangle = \lambda\langle x, S^*(0)\rangle = \lambda\langle Sx, 0\rangle = 0$.
- the third one goes analogous to the second one.
- the last one is $\langle 0,\lambda u\rangle - \langle \lambda u, 0\rangle = 0$.

From (3.7) it follows that $\tilde{A} \subset \tilde{A}^*$ so, with $S \subset S^*$, we obtain $S \subset \tilde{A} \subset \tilde{A}^* \subset S^*$. Since the 'difference' between S and $\tilde{A}$ and between $\tilde{A}^*$ and S^* both are, just like between S and S^*, one dimensional, we deduce that indeed $\tilde{A} = \tilde{A}^*$. □

(3.6) implies dom S = dom $\tilde{A}$, so $P_s = P_{\text{dom } S}$. We obtain

$$d_\infty = \|P_{\text{dom } S}\, e_{n+m+1}\|^2 = \|P_s e_{n+m+1}\|^2 < \|e_{n+m+1}\|^2 = s_{2m}$$

with

$$\|P_s e_{n+m+1}\|^2 = s'_{2m} = -\lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\left(N_0(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\right),$$

if $N_0(z) = \langle (\tilde{A} - z)^{-1} e_{n+1}, e_{n+1}\rangle$.

(b) $\tilde{A}$ is canonical and an operator extension of S ($\tilde{A}(0) = \{0\}$).
Then dom $\tilde{A} = \tilde{\mathcal{H}} = \mathcal{H}$ which implies $P_s = I$, the identity mapping. We obtain

$$P_s e_{n+m+1} = e_{n+m+1} \neq P_{\text{dom } S}\, e_{n+m+1},$$

so

$$s_{2m} = \|e_{n+m+1}\|^2 = \|P_s e_{n+m+1}\|^2 > \|P_{\text{dom } S}\, e_{n+m+1}\|^2 = d_\infty$$

with

$$\|P_s e_{n+m+1}\|^2 = -\lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\left(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\right).$$

(c) $\tilde{A}$ is noncanonical and an operator extension of S ($\tilde{A}(0) = \{0\}$).
Then $\overline{\text{dom}}\,\tilde{A} = \tilde{A}^*(0)^\perp = \tilde{A}(0)^\perp = \{0\}^\perp = \tilde{\mathcal{H}}$ and $P_s = I$. From $e_{n+m+1} \notin \text{dom}\,S$ we obtain

$$P_s e_{n+m+1} = e_{n+m+1} \neq P_{\text{dom}\,S}\, e_{n+m+1},$$

so

$$s_{2m} = \|e_{n+m+1}\|^2 = \|P_s e_{n+m+1}\|^2 > \|P_{\text{dom}\,S}\, e_{n+m+1}\|^2 = d_\infty$$

with

$$\|P_s e_{n+m+1}\|^2 = -\lim_{\substack{z=iy\\ y\to\infty}} z^{2m+1}\left(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\right).$$

(d) $\tilde{A}$ is noncanonical and a nonoperator extension of S ($\tilde{A}(0) \neq \{0\}$).
If we assume $e_{n+m+1} \in \overline{\text{dom}}\,\tilde{A}$ it would imply that the space extension

$$\tilde{\mathcal{H}} = \text{span}\,\{e_{n+m+1}, \text{dom}\,S\} \subset \overline{\text{dom}}\,\tilde{A}.$$

Since $\overline{\text{dom}}\,\tilde{A} = \mathcal{H}_s = \tilde{\mathcal{H}} \ominus \mathcal{H}_\infty = \tilde{\mathcal{H}} \ominus \tilde{A}(0)$, we deduce $\overline{\text{dom}}\,\tilde{A} \perp \tilde{A}(0)$.
But then also

$$\tilde{\mathcal{H}} \perp \tilde{A}(0),$$

which is in contradiction with the condition that $\tilde{A}$ is a minimal extension of S.
Hence $e_{n+m+1} \notin \overline{\text{dom}}\,\tilde{A}$, which implies

$$e_{n+m+1} \neq P_s e_{n+m+1}. \tag{3.8}$$

Since $\overline{\text{dom}}\,\tilde{A} \supset \text{dom}\,S$ it also follows that $P_s u \geq P_{\text{dom}\,S}\, u$ for all u, so we have that **if**

$$\|P_s e_{n+m+1}\| = \|P_{\text{dom}\,S}\, e_{n+m+1}\|, \tag{3.9}$$

then

$$\|P_s e_{n+m+1} - P_{\text{dom}\,S}\, e_{n+m+1}\|^2 = 0$$

or

$$P_s e_{n+m+1} = P_{\text{dom}\,S}\, e_{n+m+1}, \tag{3.10}$$

but this implies

$$\underbrace{e_{n+m+1} - P_s e_{n+m+1}}_{\in\, (\overline{\text{dom}}\,\tilde{A})^\perp = \tilde{A}(0)} \overset{(3.10)}{=} \underbrace{e_{n+m+1} - P_{\text{dom}\,S}\, e_{n+m+1}}_{\in\, (\text{dom}\,S)^\perp \subset S^*(0)} \in \tilde{A}(0) \cap S^*(0). \tag{3.11}$$

If we assume $\tilde{A}(0) \cap S^*(0) \neq \{0\}$ this would imply $S + (0 \times S^*(0)) \subset \tilde{A}$ but then because of the minimality of $\tilde{A}$ we obtain that $\tilde{A} = S + (0 \times S^*(0))$ is canonical, which is in contradiction with the assumption.
Hence

$$\tilde{A}(0) \cap S^*(0) = \{0\}. \tag{3.12}$$

Using (3.8) and (3.12) we now find a contradiction in (3.11):

$$e_{n+m+1} - P_s e_{n+m+1} \neq 0 \in \{0\}.$$

Hence we obtain instead of (3.9) that

$$\|P_s e_{n+m+1}\| > \|P_{\mathrm{dom}\, S}\, e_{n+m+1}\|,$$

so

$$s_{2m} = \|e_{n+m+1}\|^2 > \|P_s e_{n+m+1}\|^2 > \|P_{\mathrm{dom}\, S}\, e_{n+m+1}\|^2 = d_\infty$$

with

$$\|P_s e_{n+m+1}\|^2 = - \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\left(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\right).$$

From this we conclude that possibility (a) proves part (1) in Theorem 3.5, part (2) is proved by possibilities (b) and (c) and possibility (d) proves part (3) of the theorem. □

In the next chapter we will see the analogon of Theorem 3.5 for the MIP with a unique solution.

Chapter 4

Parametrization of solutions

In this chapter we give the unique solution of the degenerate case and a parametrization is given of the nondegenerate case; in the latter case there are infinitely many solutions and we will introduce the *Potapov formula* for these solutions. In Section 4.2 we will see in which case an equality sign appears in the MIP.

4.1 Necessary and sufficient conditions for a unique solution

Recall the model $(\mathcal{H}, S)$ as defined in Chapter 2.1. We already mentioned in Corollary 3.3 that the solution of the MIP is uniquely determined if $\mathbb{P} \geq 0$ and $\det \mathbb{P} = 0$. In this case is $\mathcal{L}_0$ not trivial, that is, the inner product degenerates on $\mathcal{L}$. We form the factor space $\mathcal{H} = \widehat{\mathcal{L}}$ and define the relation $S = \widehat{S}_{\mathcal{L}}$. From Lemma 2.1 part (1) we know that this S is selfadjoint. Hence the only minimal selfadjoint extension $\tilde{A}$ of S – which because of Theorem 3.1 corresponds with a solution of the MIP – is S itself. This leads to the next corollary:

Corollary 4.1 *If the Pick matrix* $\mathbb{P} \geq 0$ *and* $\det \mathbb{P} = 0$, *then the MIP has a unique solution, given by:*

$$N(z) = \frac{\sum_{k=1}^{n} \frac{\overline{w}_k \varphi_k}{\overline{z}_k - z} + \sum_{j=0}^{m} \frac{s_j}{z^{j+1}} \left(\sum_{k=j+1}^{m} \varphi_{n+1+k} z^k \right)}{\sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \varphi_{n+1+k} z^k},$$

where $\varphi \neq 0$ *is an element in the kernel of* $\mathbb{P}$.

Proof In the introduction leading to this corollary, we already mentioned that by Lemma 2.1 and Theorem 3.1 there exists a **unique** solution of the MIP, corresponding with the selfadjoint 'extension' S.
Now let $0 \neq \varphi \in \ker \mathbb{P}$. Then $\widehat{\varphi} = 0$ so for an arbitrary constant λ:

$$(e_{n+1} - \lambda\varphi)^{\wedge} = \hat{e}_{n+1} - \lambda\widehat{\varphi} = \hat{e}_{n+1}.$$

We also know by (2.1) in the proof of Lemma 2.1 (with $g = e_{n+1}$):

$$\left(e_{n+1} - \frac{e^*_{n+m+1}(C - z)^{-1} e_{n+1}}{e^*_{n+m+1}(C - z)^{-1}\varphi} \varphi \right)^{\wedge} \in \operatorname{ran}(S - z) = \operatorname{dom}(S - z)^{-1},$$

and we recall from Section 2.3 that $\langle \hat{x}, \hat{y} \rangle = \langle x, y \rangle$.
Hence by Theorem 3.1, the unique solution is given by the formula

$$\begin{aligned}
N(z) &= \langle (S - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1} \rangle = \\
&= \langle (\widehat{S}_{\mathcal{L}} - z)^{-1} \left[e_{n+1} - \frac{e^*_{n+m+1}(\mathcal{C} - z)^{-1} e_{n+1}}{e^*_{n+m+1}(\mathcal{C} - z)^{-1} \varphi} \varphi \right]^{\wedge}, \hat{e}_{n+1} \rangle \\
&= \langle (S_{\mathcal{L}} - z)^{-1} \left[e_{n+1} - \frac{e^*_{n+m+1}(\mathcal{C} - z)^{-1} e_{n+1}}{e^*_{n+m+1}(\mathcal{C} - z)^{-1} \varphi} \varphi \right], e_{n+1} \rangle \\
&= e^*_{n+1} \mathbb{P} (\mathcal{C} - z)^{-1} \left[e_{n+1} - \frac{e^*_{n+m+1}(\mathcal{C} - z)^{-1} e_{n+1}}{e^*_{n+m+1}(\mathcal{C} - z)^{-1} \varphi} \varphi \right] \\
&= (\overline{w}_1 \cdots \overline{w}_n \; s_0 \cdots s_m) \left[(\mathcal{C} - z)^{-1} e_{n+1} - \frac{e^*_{n+m+1}(\mathcal{C} - z)^{-1} e_{n+1}}{e^*_{n+m+1}(\mathcal{C} - z)^{-1} \varphi} (\mathcal{C} - z)^{-1} \varphi \right] \qquad (4.1)
\end{aligned}$$

In (4.1) we have:

- $(\mathcal{C} - z)^{-1} e_{n+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{z} \\ \vdots \\ -\frac{1}{z^{m+1}} \end{pmatrix}$,

- $e^*_{n+m+1}(\mathcal{C} - z)^{-1} e_{n+1} = -\frac{1}{z^{m+1}}$ and

- $(\mathcal{C} - z)^{-1} \varphi = \begin{pmatrix} \frac{\varphi_1}{\overline{z}_1 - z} \\ \vdots \\ \frac{\varphi_n}{\overline{z}_n - z} \\ \frac{1}{z} \sum_{k=1}^n \frac{\varphi_k}{\overline{z}_k - z} - \frac{\varphi_{n+1}}{z} \\ \frac{1}{z^2} \sum_{k=1}^n \frac{\varphi_k}{\overline{z}_k - z} - \frac{\varphi_{n+1}}{z^2} - \frac{\varphi_{n+2}}{z} \\ \vdots \\ \frac{1}{z^j} \sum_{k=1}^n \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{j-1} \frac{\varphi_{n+1+k}}{z^{j-k}} \\ \vdots \\ \frac{1}{z^{m+1}} \sum_{k=1}^n \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \frac{\varphi_{n+1+k}}{z^{m+1-k}} \end{pmatrix}$.

Also notice that $e^*_{n+m+1}(\mathcal{C} - z)^{-1} \varphi$ is the last row of $(\mathcal{C} - z)^{-1} \varphi$, so we deduce that

$$\begin{aligned}
&\frac{e^*_{n+m+1}(\mathcal{C} - z)^{-1} e_{n+1}}{e^*_{n+m+1}(\mathcal{C} - z)^{-1} \varphi} = \\
&= \frac{-\frac{1}{z^{m+1}}}{\frac{1}{z^{m+1}} \sum_{k=1}^n \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^m \frac{\varphi_{n+1+k}}{z^{m+1-k}}} \\
&= \frac{-1}{\sum_{k=1}^n \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^m \varphi_{n+1+k} z^k}.
\end{aligned}$$

Hence expression (4.1) becomes:

$$(\overline{w}_1 \cdots \overline{w}_n \; s_0 \cdots s_m) \times U$$

with

$$U = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{z} \\ \vdots \\ -\frac{1}{z^{m+1}} \end{pmatrix} + \frac{1}{\sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \varphi_{n+1+k} z^k} \begin{pmatrix} \frac{\varphi_1}{\overline{z}_1 - z} \\ \vdots \\ \frac{\varphi_n}{\overline{z}_n - z} \\ \frac{1}{z} \sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \frac{\varphi_{n+1}}{z} \\ \frac{1}{z^2} \sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \frac{\varphi_{n+1}}{z^2} - \frac{\varphi_{n+2}}{z} \\ \vdots \\ \frac{1}{z^j} \sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{j-1} \frac{\varphi_{n+1+k}}{z^{j-k}} \\ \vdots \\ \frac{1}{z^{m+1}} \sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \frac{\varphi_{n+1+k}}{z^{m+1-k}} \end{pmatrix}.$$

Finally we obtain that

$$N(z) = (\overline{w}_1 \cdots \overline{w}_n \; s_0 \cdots s_m) \times U =$$

$$= -\sum_{j=0}^{m} \frac{s_j}{z^{j+1}} + \frac{\sum_{k=1}^{n} \frac{\overline{w}_k \varphi_k}{\overline{z}_k - z} + \sum_{j=0}^{m} \frac{s_j}{z^{j+1}} \left(\sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{j} \varphi_{n+1+k} z^k \right)}{\sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \varphi_{n+1+k} z^k}.$$

$$= \frac{\sum_{k=1}^{n} \frac{\overline{w}_k \varphi_k}{\overline{z}_k - z} + \sum_{j=0}^{m} \frac{s_j}{z^{j+1}} \left(\sum_{k=j+1}^{m} \varphi_{n+1+k} z^k \right)}{\sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \varphi_{n+1+k} z^k}$$

This is exactly the equation in Corollary 4.1. □

Notice that if the IP or the MP has a unique solution, then of course it is also the unique solution of the MIP. This also follows from the following proposition:

Proposition 4.2 *Let P be a nonnegative square matrix of the form*

$$P = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix},$$

where A and D are also square matrices.
If for some vector $x \neq 0$, $Ax = 0$, then $B^*x = 0$ (which implies $P \begin{pmatrix} x \\ 0 \end{pmatrix} = 0$; hence $\det P = 0$) and similarly if $Dy = 0$ for some vector $y \neq 0$, then $By = 0$ (which implies $P \begin{pmatrix} 0 \\ y \end{pmatrix} = 0$; hence $\det P = 0$).

Proof If $Ax = 0$ then for all y and for all $\lambda \in \mathbb{R}$:

$$0 \leq \langle P \begin{pmatrix} \lambda x \\ y \end{pmatrix}, \begin{pmatrix} \lambda x \\ y \end{pmatrix} \rangle = \lambda \left(\langle By, x \rangle + \langle B^* x, y \rangle \right) + \langle Dy, y \rangle.$$

If we assume $\langle By, x\rangle + \langle B^*x, y\rangle \neq 0$, say it is > 0, then for $\lambda \to -\infty$ (similarly for $\lambda \to \infty$ if it is < 0) the above inequality cannot hold.
Hence

$$\langle By, x\rangle + \langle B^*x, y\rangle = 0. \tag{4.2}$$

Now take $y = B^*x$, then equality (4.2) implies that $B^*x = 0$. □

We will end this section with an example of the unique solution in a particular case, using the formula in Corollary 4.1.

Example In this example we give a MIP in which $\mathbb{P}_I > 0$ and $\mathbb{P}_M > 0$, which implies that the two seperate problems have infinitely many solutions, while the intersection of these solutions sets contains only one function which is the unique solution of the MIP.
Consider the case where $n = 1$ and $m = 1$.
Let

$$\begin{aligned} z_1 &= i \\ w_1 &= i \\ s_0 &= 2 \\ s_1 &= 1 \\ s_2 &= 3. \end{aligned}$$

Hence the Pick matrix is:

$$\mathbb{P} = \begin{pmatrix} 1 & i & 1 \\ -i & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

We see directly that

$$\mathbb{P}_I = \left(\frac{w_1 - \overline{w}_1}{z_1 - \overline{z}_1} \right) = (\,1\,) > 0,$$

and

$$\mathbb{P}_M = \begin{pmatrix} s_0 & s_1 \\ s_1 & s_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} > 0.$$

We want to obtain that the MIP has a unique solution so we want $\mathbb{P} \geq 0$ and $\det \mathbb{P} = 0$.
Since $\mathbb{P} = \mathbb{P}^*$ there is a unitary matrix U such that

$$U^*\mathbb{P}U = D = \mathrm{diag}\,(\lambda_1\ \lambda_2\ \lambda_3),$$

where the λ_i are the eigenvalues of $\mathbb{P}$. It follows that

$$\mathbb{P} \geq 0 \Longleftrightarrow D \geq 0 \quad \text{and} \quad \det\,(U^*\mathbb{P}U) = \det \mathbb{P} = 0 \Longleftrightarrow \det D = 0.$$

This implies that $\mathbb{P}$ should have nonnegative eigenvalues and at least one eigenvalue should be equal to zero.
The eigenvalues are solutions of the equation

$$\det\,(\mathbb{P} - \lambda I) = 0.$$

We find that

$$\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 4.$$

Hence $\mathbb{P} \geq 0$ and $\det \mathbb{P} = 0$, so indeed the combined problem has a unique solution. According to Corollary 4.1 this unique solution of the MIP is given by

$$N(z) = \frac{\frac{\overline{w}_1\varphi_1}{\overline{z}_1 - z} + \frac{s_0}{z}\varphi_{n+2}z}{\frac{\varphi_1}{\overline{z}_1 - z} - (\varphi_{n+1} + \varphi_{n+2}z)} = \frac{\frac{-i\varphi_1}{-i-z} + 2\varphi_3}{\frac{\varphi_1}{-i-z} - (\varphi_2 + \varphi_3 z)}. \tag{4.3}$$

Take

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} 1-3i \\ 2 \\ -1+i \end{pmatrix},$$

then $\mathbb{P}\varphi = 0$, so φ is an element of the kernel of $\mathbb{P}$. We substitute this in (4.3) and we get

$$\begin{aligned} N(z) &= \frac{\frac{-i(1-3i)}{-i-z} + 2(-1+i)}{\frac{1-3i}{-i-z} - (2 + (-1+i)z)} \\ &= -\frac{\frac{i(1-3i)}{i+z} + 2(-1+i)}{\frac{1-3i}{i+z} + (2+(-1+i)z)} \\ &= -\frac{i(1-3i) + 2(-1+i)(i+z)}{1-3i + (2+(-1+i)z)(i+z)} \\ &= -\frac{-i+1+2(i-1)z}{(i-1)z^2 + (1-i)z + (1-i)} \\ &= -\frac{2z-1}{z^2-z-1}. \end{aligned}$$

Hence, using Corollary 4.1 we find that

$$N(z) = -\frac{2z-1}{z^2-z-1} = -\frac{1}{z-\alpha_+} - \frac{1}{z-\alpha_-}, \quad \text{with } \alpha_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{5},$$

should be the unique solution of the MIP for the given data. So let us verify if this solution satisfies both the moment problem and the interpolation problem (Note that the second representation of $N(z)$ directly shows that N is a Nevanlinna function):
Assume $N(z) = -\frac{2z-1}{z^2-z-1}$, then

$$N(z_1) = -\frac{2z_1-1}{z_1^2 - z_1 - 1} = -\frac{2i-1}{i^2-i-1} = \frac{2i-1}{i+2} = \frac{i-2}{i-2}\cdot\frac{2i-1}{i+2} = i = w_1,$$

and

$$-\lim_{\substack{z=iy \\ y\to\infty}} z^3\left(N(z) + \frac{s_0}{z} + \frac{s_1}{z^2}\right) = -\lim_{\substack{z=iy \\ y\to\infty}} \left(\frac{-2z^4+z^3}{z^2-z-1} + 2z^2 + z\right) =$$

$$= -\lim_{\substack{z=iy \\ y\to\infty}} \left(\frac{-2z^4+z^3+2z^4-2z^3-2z^2+z^3-z^2-z}{z^2-z-1}\right)$$

$$= -\lim_{\substack{z=iy \\ y\to\infty}} \left(\frac{-3z^2-z}{z^2-z-1}\right) = -(-3) = 3 \leq s_2 = 3,$$

so indeed $N(z) = -\frac{2z-1}{z^2-z-1}$ satisfies the interpolation problem and the moment problem. Hence it is the unique solution of the MIP !

4.2 Equality in a MIP with unique solution

In the example in the previous section, we found an equality sign in the MP $(-\lim_{\substack{z=iy\\y\to\infty}} z^3(N(z)+\frac{s_0}{z}+\frac{s_1}{z^2}) = s_2)$. Analogous to what we found in Theorem 3.5, where we obtained – for infinitely many solutions – that an equality sign only comes into the MIP if $\tilde{A}$ is an *operator extension* of S, we will show in Theorem 4.3 that a MIP with a unique solution only has an equality sign if S in the model is an *operator*.

Theorem 4.3 *Assume the MIP has a unique solution $N(z) = \langle (S-z)^{-1}\hat{e}_{n+1}, \hat{e}_{n+1}\rangle$, where S is a selfadjoint relation.*
Equivalent are:

(1) $s_{2m} = -\lim_{\substack{z=iy\\y\to\infty}} z^{2m+1}(N(z)+\frac{s_0}{z}+\cdots+\frac{s_{2m-1}}{z^{2m}})$.

(2) S is an operator.

(3) There exists a $\varphi \in \ker \mathbb{P}$ with $\varphi_{n+m+1} \neq 0$ (it is, by scaling, a φ with $\varphi_{n+m+1} = 1$).

Proof Recall the proof of Theorem 3.5. We have

$$s_{2m} = \|\hat{e}_{n+m+1}\|^2 \geq \|P_s\hat{e}_{n+m+1}\|^2 = -\lim_{\substack{z=iy\\y\to\infty}} z^{2m+1}(N(z)+\frac{s_0}{z}+\cdots+\frac{s_{2m-1}}{z^{2m}}), \tag{4.4}$$

where P_s is the projection in $\mathcal{H}$ onto $\operatorname{dom} S$. Note $\operatorname{dom} S$ is closed as $\mathcal{H}$ is finite dimensional. The following equivalences are valid:

$$\begin{aligned}
&S \text{ is an operator } (S(0)=\{0\}) \iff \\
&\quad\iff \operatorname{dom} S = \mathcal{H} \quad (\text{since } (\operatorname{dom} S)^{\perp} = S^*(0) = S(0)\,) \\
&\quad\iff \hat{e}_{n+m+1} \in \operatorname{dom} S \quad (\hat{e}_1,\ldots,\hat{e}_{n+m} \in \mathcal{H} \text{ already}) \\
&\quad\iff \exists\, \varphi \in \ker \mathbb{P}:\ e_{n+m+1} - \varphi \in \operatorname{dom} S_{\mathcal{L}} \\
&\quad\iff \exists\, \varphi \in \ker \mathbb{P}:\ 0 = e^*_{n+m+1}(e_{n+m+1}-\varphi) = 1 - \varphi_{n+m+1}.
\end{aligned}$$

This implies that (2) and (3) are equivalent. Now we show (1) $\iff$ (2):

$$\begin{aligned}
&S \text{ is an operator } \iff \hat{e}_{n+m+1} \in \operatorname{dom} S \quad (\text{seen above}) \iff \\
&\quad\iff P_s\hat{e}_{n+m+1} = \hat{e}_{n+m+1} \\
&\quad\iff \|P_s\hat{e}_{n+m+1}\|^2 = \|\hat{e}_{n+m+1}\|^2 \\
&\quad\iff \text{equality in (4.4)}: s_{2m} = -\lim_{\substack{z=iy\\y\to\infty}} z^{2m+1}(N(z)+\frac{s_0}{z}+\cdots+\frac{s_{2m-1}}{z^{2m}}).
\end{aligned}$$

□

From Theorem 3.5 and Theorem 4.3 we now deduce the following:

Corollary 4.4 *The limit in the MIP attains the maximum value s_{2m} if and only if the solution $N(z)$ – as defined in Theorem 3.1 – corresponds with either a (selfadjoint)* **operator** *or* **operator extension** *(which is minimal), depending on the determinant of $\mathbb{P}$.*

4.3 The Potapov formula: infinitely many solutions

Previously we saw a 1-1 correspondence between solutions N of the MIP and selfadjoint extensions $\tilde{A}$ of S. In this section we consider the case where $\mathbb{P} > 0$, that is, as we obtained before, the case where the MIP has infinitely many solutions.
In this case, we can give a more precise description of all solutions N of the MIP, using a correspondence between these solutions and certain parameters. In order to get this description, we shall apply the description of all u-resolvents of S (see, for example, [KL2]) and Theorem 3.1.

V.P. Potapov gave an explicit formula for all solutions of *the Nevanlinna Pick IP* in terms of a fractional linear transformation. His method was based on the Schwarz Lemma and the so-called Fundamental Matrix Inequality. We use this formula to obtain a similar formula for all solutions of the MIP.
To describe this formula, called – how surprising – *Potapov formula*, first recall the model:

$$\mathcal{H} = \mathcal{L}, \quad S = S_{\mathcal{L}} \quad \text{and} \quad S \text{ has defect index } (1,1).$$

Before we can go on, we need some new theory, concerning *u-resolvents* and *module elements* (see [ADL, pp. 17, 18]):

If $W = (w_{jk})_{j,k=1}^2$ is a 2×2 matrix function and T is a Nevanlinna function, $T \in \tilde{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$, we denote by $W(z)_{T(z)}$ the fractional linear transform

$$W(z)_{T(z)} = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)}.$$

For $T(z) \equiv \infty$ this expression reduces to $w_{11}(z)/w_{21}(z)$.
Given a closed symmetric operator S with defect index (1,1) in a Hilbert space $\mathcal{H}$, an element $u \in \mathcal{H}$ is called a **module** *for S if the set $r_u(S)$ of all $z \in \mathbb{C}$ for which*

$$\mathcal{H} = \text{ran}\,(S - z) \dot{+} \text{span}\,\{u\} = \text{ran}\,(S - z^*) \dot{+} \text{span}\,\{u\}, \quad \textit{direct sum in } \mathcal{H},$$

is not empty. Then each element $f \in \mathcal{H}$ can be decomposed as

$$f = f_z + \lambda_z u,$$

where f_z belongs to $\text{ran}(S - z)$ and λ_z is a complex number. We denote by $\mathsf{P}(z)$ and $\mathsf{Q}(z)$ the linear mappings from the Hilbert space $\mathcal{H}$ to the (one-dimensional Hilbert) space $\mathbb{C}$ defined by

$$\mathsf{P}(z)f = \lambda_z, \qquad \mathsf{Q}(z)f = \langle(\overset{o}{A} - z)^{-1} f_z, u\rangle_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

Here $\overset{o}{A}$ is an arbitrary canonical selfadjoint extension of S. Since $f_z \in \text{ran}\,(S - z)$, $\mathsf{Q}(z)f$ is independent of the special choice of $\overset{o}{A}$. The functionals $\mathsf{P}(z)$ and $\mathsf{Q}(z)$ are holomorphic at all points $z \in r_u(S)$ of regular type of S. Explicit formulas for P and Q can be found in [KL2, pp. 404, 405, formulas (3.3)–(3.5)].

If $\tilde{A}$ is a selfadjoint extension of S in some Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$, then

$$\langle(\tilde{A} - z)^{-1} u, u\rangle_{\tilde{\mathcal{H}}}$$

*is called a u-***resolvent** *of S. The set of all u-resolvents of S can be parametrized over the Nevanlinna functions with parametrization matrix $W(z)$, a so-called u-***resolvent matrix**. *That is, the relation*

$$\langle (\tilde{A}-z)^{-1}u, u\rangle_{\tilde{\mathcal{H}}} = W(z)_{T(z)}$$

establishes a bijective correspondence between all u-resolvents of S and all $T \in \tilde{\mathbb{N}}_0$. If $r_u(S)$ contains a real point a, then the parametrization matrix $W(z)$ can be chosen as the 2×2 matrix function

$$W^a(z) = I - (z-a)\begin{pmatrix} \mathsf{Q}(z) \\ -\mathsf{P}(z) \end{pmatrix}\begin{pmatrix} \mathsf{Q}(a)^* & -\mathsf{P}(a)^* \end{pmatrix} J, \quad \textit{with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The mappings $\mathsf{P}(a)^*$ and $\mathsf{Q}(a)^*$ are the adjoints of $\mathsf{P}(a)$ and $\mathsf{Q}(a)$ and map $\mathbb{C}$ to $\mathcal{H}$, where $\mathbb{C}$ is equipped with the usual inner product.

We now apply these formulas above to the operator $S = S_{\mathcal{L}}$ in our model.
Since

$$\mathcal{H} = \mathcal{L} = \operatorname{ran}(S-z) \dot{+} \operatorname{span}\{e_{n+1}\} \quad \forall z \in \mathbb{C},\ z \neq z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n, 0 \quad ,$$

we obtain that the set

$$r_{e_{n+1}}(S) = \mathbb{C} \setminus \{z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n, 0\}$$

is not empty. Hence e_{n+1} is a module element.
For each element $f \in \mathcal{H}$ we have:

$$f = (\mathcal{C}-z)x + c_f e_{n+1} \in \operatorname{ran}(S-z) \dot{+} \operatorname{span}\{e_{n+1}\}, \tag{4.5}$$

where $\mathbb{C} \ni c_f = \mathsf{P}(z)f$ and $x \in \operatorname{dom} S$, so $e^*_{n+m+1}x = 0$.
From (4.5) it follows that

$$x = (\mathcal{C}-z)^{-1}f - c_f(\mathcal{C}-z)^{-1}e_{n+1}, \qquad z \neq \overline{z}_1, \ldots, \overline{z}_n, 0. \tag{4.6}$$

Now we calculate the formulas for P and Q in our model. Since $0 = e^*_{n+m+1}x$, we deduce from (4.6) that

$$e^*_{n+m+1}(\mathcal{C}-z)^{-1}f = c_f\, e^*_{n+m+1}(\mathcal{C}-z)^{-1}e_{n+1} = c_f \cdot -\frac{1}{z^{m+1}},$$

so

$$\mathsf{P}(z)f = c_f = -z^{m+1}e^*_{n+m+1}(\mathcal{C}-z)^{-1}f. \tag{4.7}$$

In the formula for $\mathsf{Q}(z)$ we have to use the first term in the direct sum decomposition of f as defined in (4.5), it is the part which is an element of $\operatorname{ran}(S-z)$. We rewrite the above decomposition (4.5) of f:

$$(\mathcal{C}-z)x = f - c_f e_{n+1} = f - (\mathsf{P}(z)f)e_{n+1},$$

and substitute this in the formula for $\mathsf{Q}(z)$; Let $\overset{o}{A}$ be an arbitrary canonical selfadjoint extension of S. Recall that $f - (\mathsf{P}(z)f)e_{n+1} \in \operatorname{ran}(S-z) = \operatorname{dom}(S-z)^{-1}$ so

$$(\overset{o}{A} - z)^{-1}(f - (\mathsf{P}(z)f)e_{n+1}) = (S-z)^{-1}(f - (\mathsf{P}(z)f)e_{n+1}).$$

We obtain for $\mathsf{Q}(z)$:

$$\begin{aligned}
\mathsf{Q}(z)f &= \langle(\overset{o}{A} - z)^{-1}(f - (\mathsf{P}(z)f)e_{n+1}), e_{n+1}\rangle_{\mathcal{H}} = \\
&= e_{n+1}^{*}\mathbb{P}\left((\mathcal{C} - z)^{-1}f - c_f(\mathcal{C} - z)^{-1}e_{n+1}\right) \\
&\overset{(4.7)}{=} e_{n+1}^{*}\mathbb{P}\left[(\mathcal{C} - z)^{-1}f + z^{m+1}(\mathcal{C} - z)^{-1}\underbrace{e_{n+1}\,e_{n+m+1}^{*}}_{\text{is a matrix}}(\mathcal{C} - z)^{-1}f\right] \\
&= \left[e_{n+1}^{*}\mathbb{P} + z^{m+1}e_{n+1}^{*}\mathbb{P}\,(\mathcal{C} - z)^{-1}e_{n+1}e_{n+m+1}^{*}\right](\mathcal{C} - z)^{-1}f \\
&= \left[e_{n+1}^{*}\mathbb{P} - z^{m+1}\,(\overline{w}_1 \;\cdots\; \overline{w}_n \; s_0 \;\cdots\; s_m)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{z} \\ \frac{1}{z^2} \\ \vdots \\ \frac{1}{z^{m+1}} \end{pmatrix} e_{n+m+1}^{*}\right](\mathcal{C} - z)^{-1}f \\
&= \left[e_{n+1}^{*}\mathbb{P} - p(z)e_{n+m+1}^{*}\right](\mathcal{C} - z)^{-1}f, \qquad (4.8)
\end{aligned}$$

where $p(z)$ is the same polynomial as defined in Section 2.1: $p(z) = s_0z^m + s_1z^{m-1} + \cdots + s_m$.

Recall that the mappings $\mathsf{P}(z)^*$ and $\mathsf{Q}(z)^*$ – the adjoints of $\mathsf{P}(z)$ and $\mathsf{Q}(z)$ – map $\mathbb{C}$ to $\mathcal{H}$, where $\mathbb{C}$ is equipped with the usual inner product, $\langle x, y\rangle = y^*x$.
Since

$$\langle\varphi, \psi\rangle_{\mathbb{C}} = \psi^*\varphi = \psi^*\mathbb{P}\,\mathbb{P}^{-1}\varphi = \langle\mathbb{P}^{-1}\varphi, \psi\rangle_{\mathcal{H}},$$

it follows that, if $\mathsf{P}(z) = \Phi$, then $\mathsf{P}(z)^* = \mathbb{P}^{-1}\Phi^*$, and analogous for $\mathsf{Q}(z)$.
From (4.7) and (4.8) we have

$$\begin{aligned}
\mathsf{P}(z) &= -z^{m+1}e_{n+m+1}^{*}(\mathcal{C} - z)^{-1}, \\
&(= \quad (\,0\,\ldots 0\; -z^{m+1}\,)\,(\mathcal{C} - z)^{-1}\,) \\
\mathsf{Q}(z) &= \left[e_{n+1}^{*}\mathbb{P} - p(z)e_{n+m+1}^{*}\right](\mathcal{C} - z)^{-1}, \\
&(= [n + 1^{st} \text{ row of } \mathbb{P} - (\,0\,\ldots 0\; p(z)\,)](\mathcal{C} - z)^{-1}\,)
\end{aligned}$$

so (Note: $\mathbb{P} = \mathbb{P}^*$)

$$\begin{aligned}
\mathsf{P}(z)^* &= -\bar{z}^{\,m+1}\,\mathbb{P}^{-1}\,(\mathcal{C} - z)^{-*}e_{n+m+1}, \\
&(= \mathbb{P}^{-1}(\mathcal{C} - z)^{-*}\,(\,0\,\ldots 0\; -\bar{z}^{\,m+1})^t\,) \\
\mathsf{Q}(z)^* &= \mathbb{P}^{-1}\,(\mathcal{C} - z)^{-*}\,[\mathbb{P}\,e_{n+1} - p(\bar{z})e_{n+m+1}] \\
&(= \mathbb{P}^{-1}(\mathcal{C} - z)^{-*}\,[n + 1^{st} \text{ column of } \mathbb{P} - (\,0\,\ldots 0\; p(\bar{z})\,)^t\,]\,).
\end{aligned}$$

From this it follows that

$$\begin{pmatrix} \mathsf{Q}(z) \\ -\mathsf{P}(z) \end{pmatrix} = \begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & s_0 & \cdots & s_{m-1} & s_m - p(z) \\ 0 & \cdots & 0 & 0 & \cdots & 0 & z^{m+1} \end{pmatrix}(\mathcal{C} - z)^{-1},$$

and

$$\begin{pmatrix} \mathsf{Q}(z)^* & -\mathsf{P}(z)^* \end{pmatrix} = \mathbb{P}^{-1}(\mathcal{C}-z)^{-*} \begin{pmatrix} w_1 & 0 \\ \vdots & \vdots \\ w_n & 0 \\ s_0 & 0 \\ \vdots & \vdots \\ s_{m-1} & 0 \\ s_m - p(\bar z) & \bar z^{\,m+1} \end{pmatrix}.$$

If we set

$$L(z) = \begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & s_0 & \cdots & s_{m-1} & s_m - p(z) \\ 0 & \cdots & 0 & 0 & \cdots & 0 & z^{m+1} \end{pmatrix} (\mathcal{C}-z)^{-1},$$

then

$$\mathbb{P}^{-1}(\mathcal{C}-z)^{-*} \begin{pmatrix} w_1 & 0 \\ \vdots & \vdots \\ w_n & 0 \\ s_0 & 0 \\ \vdots & \vdots \\ s_{m-1} & 0 \\ s_m - p(\bar z) & \bar z^{\,m+1} \end{pmatrix} = \mathbb{P}^{-1} L(z)^*.$$

Hence

$$\begin{aligned} W^a(z) &:= I - (z-a) \begin{pmatrix} \mathsf{Q}(z) \\ -\mathsf{P}(z) \end{pmatrix} \begin{pmatrix} \mathsf{Q}(a)^* & -\mathsf{P}(a)^* \end{pmatrix} J \\ &= I - (z-a) L(z) \mathbb{P}^{-1} L(a)^* J. \end{aligned} \tag{4.9}$$

Now we try to find an alternative form for $L(z)$.
Denote $L(z) = (L_{ij})_{i,j}$, $1 \le i \le 2$, $1 \le j \le n+m+1$, so L_{ij} is the element of L on row i and in column j.
We recall the matrix $(\mathcal{C}-z)^{-1}$ in (2) of the proof of Lemma 2.1 and we use the equation

$$\sum_{j=0}^{m} \frac{s_j}{z^{j+1}} \equiv \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots + \frac{s_m}{z^{m+1}} = \frac{p(z)}{z^{m+1}} \tag{4.10}$$

to obtain the first row of $L(z)$:

$$\begin{aligned} L_{11} &= (\,\overline{w}_1 \cdots \overline{w}_n \quad s_0 \cdots s_{m-1} \quad s_m - p(z)\,) \times 1^{st} \text{ column of } (\mathcal{C}-z)^{-1} \\ &= \tfrac{\overline{w}_1}{\bar z_1 - z} + \tfrac{1}{\bar z_1 - z} \textstyle\sum_{j=0}^m \tfrac{s_j}{z^{j+1}} - \tfrac{p(z)}{z^{m+1}} \tfrac{1}{\bar z_1 - z} \overset{(4.10)}{=} \tfrac{\overline{w}_1}{\bar z_1 - z}, \\ L_{12} &= (\,\overline{w}_1 \cdots \overline{w}_n \quad s_0 \cdots s_{m-1} \quad s_m - p(z)\,) \times 2^{nd} \text{ column of } (\mathcal{C}-z)^{-1} \\ &= \tfrac{\overline{w}_2}{\bar z_2 - z} + \tfrac{1}{\bar z_2 - z} \textstyle\sum_{j=0}^m \tfrac{s_j}{z^{j+1}} - \tfrac{p(z)}{z^{m+1}} \tfrac{1}{\bar z_2 - z} \overset{(4.10)}{=} \tfrac{\overline{w}_2}{\bar z_2 - z}, \\ &\vdots \\ L_{1n} &= (\,\overline{w}_1 \cdots \overline{w}_n \quad s_0 \cdots s_{m-1} \quad s_m - p(z)\,) \times n^{th} \text{ column of } (\mathcal{C}-z)^{-1} \\ &= \tfrac{\overline{w}_n}{\bar z_n - z} + \tfrac{1}{\bar z_n - z} \textstyle\sum_{j=0}^m \tfrac{s_j}{z^{j+1}} - \tfrac{p(z)}{z^{m+1}} \tfrac{1}{\bar z_n - z} \overset{(4.10)}{=} \tfrac{\overline{w}_n}{\bar z_n - z}, \end{aligned}$$

$$
\begin{aligned}
L_{1,n+1} &= (\,\overline{w}_1 \;\cdots\; \overline{w}_n \quad s_0 \;\cdots\; s_{m-1} \quad s_m - p(z)\,) \times n{+}1^{st} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= -\textstyle\sum_{j=0}^{m} \frac{s_j}{z^{j+1}} + \frac{p(z)}{z^{m+1}} \stackrel{(4.10)}{=} 0, \\
L_{1,n+2} &= (\,\overline{w}_1 \;\cdots\; \overline{w}_n \quad s_0 \;\cdots\; s_{m-1} \quad s_m - p(z)\,) \times n{+}2^{nd} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= -\textstyle\sum_{j=1}^{m} \frac{s_j}{z^{j}} + \frac{p(z)}{z^{m}} \stackrel{(4.10)\times z}{=} s_0, \\
&\vdots \\
L_{1,n+m-k+2} &= (\,\overline{w}_1 \;\cdots\; \overline{w}_n \quad s_0 \;\cdots\; s_{m-1} \quad s_m - p(z)\,) \times n{+}r^{th} \text{ column of } (\mathcal{C}-z)^{-1} \\
(= L_{n+r}) &= \tfrac{p(z)}{z^k} - \tfrac{s_m}{z^k} - \tfrac{s_{m-1}}{z^{k-1}} - \cdots - \tfrac{s_{m-k+1}}{z} \stackrel{(4.10)\times z^{r-1}}{=} s_0 z^{m-k} + \cdots + s_{m-k}, \\
&\vdots \\
L_{1,n+m+1} &= (\,\overline{w}_1 \;\cdots\; \overline{w}_n \quad s_0 \;\cdots\; s_{m-1} \quad s_m - p(z)\,) \times n{+}m{+}1^{st} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= \tfrac{p(z)-s_m}{z} \stackrel{(4.10)\times z^m}{=} s_0 z^{m-1} + s_1 z^{m-2} + \cdots + s_{m-1}.
\end{aligned}
$$

For the second row of $L(z)$ we directly obtain:

$$
\begin{aligned}
L_{21} &= (\,0 \;\cdots\; 0 \quad z^{m+1}\,) \times 1^{st} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= \tfrac{1}{\overline{z}_1 - z}, \\
&\vdots \\
L_{2n} &= (\,0 \;\cdots\; 0 \quad z^{m+1}\,) \times n^{th} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= \tfrac{1}{\overline{z}_n - z}, \\
L_{2,n+1} &= (\,0 \;\cdots\; 0 \quad z^{m+1}\,) \times n{+}1^{st} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= -1, \\
L_{2,n+2} &= (\,0 \;\cdots\; 0 \quad z^{m+1}\,) \times n{+}2^{nd} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= -z, \\
&\vdots \\
L_{2,n+m+1} &= (\,0 \;\cdots\; 0 \quad z^{m+1}\,) \times n{+}m{+}1^{st} \text{ column of } (\mathcal{C}-z)^{-1} \\
&= -z^m.
\end{aligned}
$$

So the desired alternative form for $L(z)$ is:

$$
L(z) = \begin{pmatrix} \frac{\overline{w}_1}{\overline{z}_1 - z} & \cdots & \frac{\overline{w}_n}{\overline{z}_n - z} & 0 & s_0 & s_0 z + s_1 & \cdots & s_0 z^{m-1} + s_1 z^{m-2} + \cdots + s_{m-1} \\ \frac{1}{\overline{z}_1 - z} & \cdots & \frac{1}{\overline{z}_n - z} & -1 & -z & -z^2 & \cdots & -z^m \end{pmatrix} =
$$

$$
= \begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & 0 & s_0 & \cdots & s_{m-1} \\ 1 & \cdots & 1 & -1 & 0 & \cdots & 0 \end{pmatrix}
\left(\begin{array}{ccc|cccc}
\frac{1}{\overline{z}_1 - z} & & 0 & & & & \\
& \ddots & & & & 0 & \\
0 & & \frac{1}{\overline{z}_n - z} & & & & \\
\hline
& & & 1 & z & \cdots & z^m \\
& 0 & & 0 & \ddots & \ddots & \vdots \\
& & & \vdots & \ddots & \ddots & z \\
& & & 0 & \cdots & 0 & 1
\end{array}\right)
$$

$$= \begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & 0 & s_0 & \cdots & s_{m-1} \\ 1 & \cdots & 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \left(\begin{array}{c|c} (Z^* - z)^{-1} & 0 \\ \hline 0 & (I - zS_r^*)^{-1} \end{array} \right),$$

where Z and S_r are the matrices as defined in Section 2.1.
(To verify the latter expression for the bottom-right block of the second matrix, just multiply it by the previous form and deduce that this product equals the identity matrix).

Using this form of $L(z)$, we can also obtain an expression for the adjoint of $L(a)$, which also appears in equation (4.9) for $W^a(z)$. Note that – according to the theory – W^a is defined for a **real** number a, so we can use for $L(a)^*$ in equation (4.9), that $a = \bar{a}$:

$$L(a)^* = \left(\begin{array}{c|c} (Z^* - a)^{-1} & 0 \\ \hline 0 & (I - aS_r^*)^{-1} \end{array} \right)^* \begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & 0 & s_0 & \cdots & s_{m-1} \\ 1 & \cdots & 1 & -1 & 0 & \cdots & 0 \end{pmatrix}^* =$$

$$\stackrel{a=\bar{a}}{=} \left(\begin{array}{c|c} (Z - a)^{-1} & 0 \\ \hline 0 & (I - aS_r)^{-1} \end{array} \right) \begin{pmatrix} w_1 & 1 \\ \vdots & \vdots \\ w_n & 1 \\ 0 & -1 \\ s_0 & 0 \\ \vdots & \vdots \\ s_{m-1} & 0 \end{pmatrix}.$$

Since there are infinitely many real numbers $a \in r_{e_{n+1}}(S) = \mathbb{C} \setminus \{z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n, 0\}$, the parametrization matrix $W(z)$, which is used (see the theory above) in the fractional linear transform

$$W(z)_{T(z)} = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)},$$

can now – according to the theory – be chosen as $W^a(z)$ for an arbitrary $a \in r_{e_{n+1}}(S) \cap \mathbb{R}$. Hence $W(z)$ is given by the equation in (4.9).

If we now substitute the equations for $L(z)$ and $L(a)^*$ – as computed above – in equation (4.9), we deduce the following formula for the parametrization matrix $W(z)$:

$$W(z) = I - (z-a)\begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & 0 & s_0 & \cdots & s_{m-1} \\ 1 & \cdots & 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \left(\begin{array}{c|c} (Z^* - z)^{-1} & 0 \\ \hline 0 & (I - zS_r^*)^{-1} \end{array}\right) \times$$

$$\times\ \mathbb{P}^{-1} \left(\begin{array}{c|c} (Z - a)^{-1} & 0 \\ \hline 0 & (I - aS_r)^{-1} \end{array}\right) \begin{pmatrix} w_1 & 1 \\ \vdots & \vdots \\ w_n & 1 \\ 0 & -1 \\ s_0 & 0 \\ \vdots & \vdots \\ s_{m-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad (4.11)$$

Note that $W(z)$ is a 2×2 matrix, where all four terms are dependent on z, so that is why we denote it by:

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix}.$$

Finally we recall the last part of the theory at the beginning of this section and use again that e_{n+1} is a module element.
From this we obtain that if $\tilde{A}$ is a selfadjoint extension of S, then the relation

$$\langle (\tilde{A} - z)^{-1} e_{n+1}, e_{n+1} \rangle = W(z)_{T(z)}$$

establishes a 1-1 correspondence between all e_{n+1}-resolvents of S and all $T \in \tilde{\mathbb{N}}_0$.
From Theorem 3.1 we also know that there is 1-1 correspondence between all solutions $N(z)$ of the MIP and all minimal selfadjoint extensions $\tilde{A}$ of S, via the formula

$$N(z) = \langle (\tilde{A} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1} \rangle,$$

but we assumed in this section that $\mathbb{P} > 0$, so $\ker \mathbb{P} = \{0\}$, so we have $\hat{e}_{n+1} = e_{n+1}$. Hence

$$\langle (\tilde{A} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1} \rangle = \langle (\tilde{A} - z)^{-1} e_{n+1}, e_{n+1} \rangle.$$

The combination of these two correspondences above leads to the following theorem:

Theorem 4.5 *Assume $\mathbb{P} > 0$ and let the matrix $W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix}$ be defined by equation (4.11) with arbitrary $a \in \mathbb{C} \setminus \{z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n, 0\} \cap \mathbb{R}$.*
Then,

$$N(z) = \langle (\tilde{A} - z)^{-1} e_{n+1}, e_{n+1} \rangle = W(z)_{T(z)} = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)}$$

establishes a 1-1 correspondence between all solutions $N(z)$ of the MIP and parameters $T(z) \in \tilde{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

Theorem 4.5 actually tells us that if $\mathbb{P} > 0$, we can calculate the matrix $W(z)$ using (4.11) and then take an arbitrary Nevanlinna function $T(z) \in \tilde{\mathbb{N}}_0$ to get a solution of the MIP, which then is given by

$$N(z) = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)}.$$

This formula is called the *Potapov formula* for solutions of the MIP.
(Recall from the theory that if $T(z) \equiv \infty$ the solution reduces to $N(z) = w_{11}(z)/w_{21}(z)$).

We also see from this theorem that if $\mathbb{P} > 0$, indeed there are infinitely many solutions of the MIP, since there are infinitely many Nevanlinna functions (because α and β in the Herglotz integral representation of such a function are arbitrary real, respectively positive numbers).

Chapter 5

Conclusion

At the end of this thesis we will give a brief summary of the results we have found in the previous chapters.

If we are interested in solutions of the combined moment and interpolation problem (MIP), first of all we check if there exist solutions anyway.
For this we get our information from the information matrix $\mathbb{P}$, so we produce that matrix, called Pick matrix, using the given data $z_1, \dots z_n$ and $w_1, \dots w_n$ and $s_0, \dots, s_{2m}$:

$$\mathbb{P} = \begin{pmatrix} \frac{w_1 - \overline{w}_1}{z_1 - \overline{z}_1} & \cdots & \frac{w_1 - \overline{w}_n}{z_1 - \overline{z}_n} & w_1 & \cdots & s_{m-1} + \cdots + w_1 z_1^m \\ \vdots & & \vdots & w_2 & \cdots & s_{m-1} + \cdots + w_2 z_2^m \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{w_n - \overline{w}_1}{z_n - \overline{z}_1} & \cdots & \frac{w_n - \overline{w}_n}{z_n - \overline{z}_n} & w_n & \cdots & s_{m-1} + \cdots + w_n z_n^m \\ \overline{w}_1 & \cdots & \overline{w}_n & s_0 & \cdots & s_m \\ \overline{s_0 + w_1 z_1} & \cdots & \overline{s_0 + w_n z_n} & s_1 & \cdots & s_{m+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{s_{m-1} + \cdots + w_1 z_1^m} & \cdots & \overline{s_{m-1} + \cdots + w_n z_n^m} & s_m & \cdots & s_{2m} \end{pmatrix}.$$

There exists at least one solution of the MIP if and only if $\mathbb{P} \geq 0$ (that is $\langle \mathbb{P}x, x \rangle \geq 0 \quad \forall x$).
If $\mathbb{P} \geq 0$ we continue by defining a model $(\mathcal{H}, S)$:

$\mathcal{H} := \widehat{\mathcal{L}} = \mathbb{C}^{n+m+1} \setminus \ker \mathbb{P}$ with inner product $\langle \hat{x}, \hat{y} \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathcal{L}} = y^* \mathbb{P} x$, and
$S := \widehat{S}_{\mathcal{L}}$ where $S_{\mathcal{L}} x = Cx \; \forall x \in \operatorname{dom} S_{\mathcal{L}} = \{x \in \mathcal{L} | e^*_{n+m+1} x = 0\}$, with C the matrix :

$$C := \left(\begin{array}{c|c} Z^* & 0 \\ \hline e^* & \\ 0 & S_r \end{array} \right) = \left(\begin{array}{ccc|cccc} \overline{z}_1 & & 0 & 0 & & & 0 \\ & \ddots & & & & & \\ 0 & & \overline{z}_n & 0 & & & 0 \\ \hline 1 & \cdots & 1 & 0 & & & \\ & & & 1 & \ddots & & \\ & 0 & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{array} \right).$$

If $\mathbb{P} \geq 0$ and det $\mathbb{P} = 0$, then the MIP has a unique solution, given by:

$$N(z) = \frac{\sum_{k=1}^{n} \frac{\overline{w}_k \varphi_k}{\overline{z}_k - z} + \sum_{j=0}^{m} \frac{s_j}{z^{j+1}} \left(\sum_{k=j+1}^{m} \varphi_{n+1+k} z^k \right)}{\sum_{k=1}^{n} \frac{\varphi_k}{\overline{z}_k - z} - \sum_{k=0}^{m} \varphi_{n+1+k} z^k},$$

with $\varphi \neq 0$ an element in the kernel of $\mathbb{P}$.

If $\mathbb{P} \geq 0$ and det $\mathbb{P} \neq 0$, then the MIP has infinitely many solutions. The general form of a solution is given by $N(z) = \langle (\tilde{A} - z)^{-1} \hat{e}_{n+1}, \hat{e}_{n+1} \rangle$, where $\tilde{A}$ is a minimal selfadjoint extension of S in defined model above.
A form of such a solution by means of the given data is given by the *Potapov formula*:

$$N(z) = \frac{w_{11}(z)T(z) + w_{12}(z)}{w_{21}(z)T(z) + w_{22}(z)} \qquad \text{with:}$$

- $W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix} =$

$$= I - (z-a) \begin{pmatrix} \overline{w}_1 & \cdots & \overline{w}_n & 0 & s_0 & \cdots & s_{m-1} \\ 1 & \cdots & 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \left(\begin{array}{c|c} (Z^* - z)^{-1} & 0 \\ \hline 0 & (I - zS_r^*)^{-1} \end{array} \right) \times$$

$$\times\, \mathbb{P}^{-1} \left(\begin{array}{c|c} (Z - a)^{-1} & 0 \\ \hline 0 & (I - aS_r)^{-1} \end{array} \right) \begin{pmatrix} w_1 & 1 \\ \vdots & \vdots \\ w_n & 1 \\ 0 & -1 \\ s_0 & 0 \\ \vdots & \vdots \\ s_{m-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here a is a real number in the set

$$r_{e_{n+1}}(S) = \{ z \in \mathbb{C} | e_{n+1} \notin \operatorname{ran}(S - z) \cup \operatorname{ran}(S - \bar{z}) \},$$

and Z and S_r are the matrices:

$$Z = \operatorname{diag}(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix} \quad ; \quad S_r = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

- T is a Nevanlinna function: $T(z) \in \tilde{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

This actually leads to one of the most beautiful resluts in this thesis:

The MIP has either no or a unique or infinitely many solutions.

So it is NOT possible that the MIP has, for example, 3 or 5 or 10 solutions !!

We also saw an example in Section 4.1, in which the IP and MP both have infinitely many solutions, but the combined problem MIP has a unique solution.

Another interesting result is in which case the limit in the MP part of the MIP equals s_{2m} and in which case a strict inequality appears:

$$s_{2m} = - \lim_{\substack{z=iy \\ y\to\infty}} z^{2m+1}\left(N(z) + \frac{s_0}{z} + \cdots + \frac{s_{2m-1}}{z^{2m}}\right)$$

$$\Longleftrightarrow$$

If $\det \mathbb{P} = 0$:	If $\det \mathbb{P} \neq 0$:
S in the model is an operator.	$\tilde{A}$ is an operator extension of S.

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