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High rank elliptic surfaces

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Master's thesis

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Abstract

Let $E/\mathbb{Q}(t)$ be an elliptic curve given by the equation $Y^2 = X^3 + g(t^6)$ where g is a polynomial. In the case that g is a quadratic polynomial with simple zeroes $\neq 0$, we show that rank$E(\overline{\mathbb{Q}}(t))$ is either 16 or 18. Moreover we prove that rank 18 really occurs and we give 18 independent points in certain explicit examples. For the subgroup $E(\mathbb{Q}(t))$ we give the theoretical upper bound 9. However, using quadratic polynomials g, we never found more than 4 independent points in $E(\mathbb{Q}(t))$.
Finally we study the case where $\deg(g) = 3$. We show that rank$E(\overline{\mathbb{Q}}(t)) \geq 24$ occurs, by providing an example where one can write down 24 independent points. We finish by studying a case where the Mordell-Weil rank over $\mathbb{Q}(t)$ is at least 5.

1 Introduction

In this paper we present some results concerning elliptic curves and elliptic surfaces. They form a growing part of research in arithmetic. We will recapitalize some basic results which are treated more extensively in [24], [2] and [25].

An elliptic curve over a field $\mathbb{K}$ can be defined by an equation

$$E/\mathbb{K}: \;\; Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6. \tag{1}$$

Throughout this work we will assume that the characteristic of $\mathbb{K}$ is zero. Via a simple transformation we can rewrite (1) in the short form

$$E/\mathbb{K}: \;\; Y^2 = X^3 + c_1X + c_2. \tag{2}$$

The set of solutions of (2) together with an additional point O 'at infinity' forms a group denoted $E(\mathbb{K})$, with the group law described in [24]. The Mordell-Weil theorem is a celebrated theorem about the structure of this group. We will use some special cases of it:

Theorem 1.1 (Mordell,Weil) *[19]. In the cases $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{Q}(t)$ we have*

$$E(\mathbb{K}) \cong E(\mathbb{K})_{tors} \oplus \mathbb{Z}^r;$$

the same is true for $\mathbb{K} = \overline{\mathbb{Q}}(t)$, provided E is not isomorphic over $\overline{\mathbb{Q}}(t)$ to an elliptic curve defined over $\overline{\mathbb{Q}}$.

Here $E(\mathbb{K})_{tors}$ denotes the torsion subgroup of $E(\mathbb{K})$. The number r in this theorem is called the (Mordell-Weil) rank of $E/\mathbb{K}$. Part of the study of elliptic curves is about constructing curves having high Mordell-Weil rank. About the torsion groups that can occur everything is known (see for example [11], [3]). However it is not known whether the rank (r) of an elliptic curve (in general) over $\mathbb{K}$ is bounded or not. In 1996 for example Stéphane Fermigier ([5]) showed the existence of an elliptic curve over $\mathbb{Q}$ of rank ≥ 22. And only in 1997 Martin and McMillen produced rank 23, however they did not publish this result since they are cryptologists at the NSA.

Nowadays the most common method for searching high rank elliptic curves over $\mathbb{Q}$, starts with finding high rank elliptic curves over $\mathbb{Q}(t)$ (or more generally $\mathbb{Q}(V)$ where V is a variety) and then intelligently specializing t. One can use the following specialization theorem([26] and [15]):

Theorem 1.2 (Néron, Silverman) *Let K be a number field and E an elliptic curve defined over the field $K(\mathbb{P}^n)$. Then there are infinitely many points $t \in \mathbb{P}^n(K)$ for which the specialization homomorphism*

$$\sigma_t \;:\; E(K(\mathbb{P}^n)) \;\rightarrow\; E_t(K)$$

is injective.

Here we focus on a simple case, namely an elliptic curve E over $\overline{\mathbb{Q}}(t)$ given by $Y^2 = X^3 + g$. The extra condition in theorem 1.1, E should not be isomorphic over $\overline{\mathbb{Q}}(t)$ to a curve over $\overline{\mathbb{Q}}$, now reduces to the restriction that g should not be a sixth power. We start with the elliptic curve given over $\mathbb{Q}(t)$ by $E : Y^2 = X^3 + f(t^6)$, where $f(t) := a{\cdot}t^2 + b{\cdot}t + c$. Then we find conditions on a, b and c such that the rank of E is as large as possible. We assume $a \neq 0$ and $\Delta_f = b^2 - 4ac \neq 0$, e.g. $f(t)$ has two, distinct zeroes. For rank$E(\overline{\mathbb{Q}}(t))$ we relatively easy obtain an upper bound 18. We can regard E as a surface over $\mathbb{Q}$. It turns out that E is a so-called $K3-$surface. Using the Shioda-Tate formula (21) we see that rank$E(\overline{\mathbb{Q}}(t)) \leq 18$. In fact, David Cox already proved that for some $K3-$surfaces rank 18 occurs and then $E(\mathbb{K})_{tors} = \{0\}$ ([4], theorem 2.1). His proof only gives the existence and in particular no equations of such surfaces. It was made slightly more explicit by Nishiyama ([17]), however still without providing equations. A different proof was given by Kuwata ([9]); it is based on the following result of Inose ([8]). If $f : X \to Y$ is a rational map of finite degree between $K3$-surfaces, then X and Y have the same Néron-Severi rank. Kuwata applies this to the case where Y is given by $(t^3 + ct + d)y^2 = x^3 + ax + b$. Here a, b, c, d are chosen in such a way that $y^2 = x^3 + ax + b$ and $s^2 = t^3 + ct + d$ define non-isomorphic, isogenous, complex multiplication elliptic curves. The surface X is then defined by $(t^3 + ct + d)\eta^6 = x^3 + ax + b$ and the map f by $\eta \mapsto y := \eta^3$. Both X and Y have Néron-Severi rank 20. The elliptic fibration $(x, \eta, t) \mapsto \eta$ has only irreducible fibers and therefore the Mordell-Weil rank is 18. Although this construction yields explicit equations, it does not give 18 independent sections. The method described in this master's thesis actually provides 18 sections as well.

To do this we take the following route:

- we make $E(\overline{\mathbb{Q}}(t))$ into a $\mathbb{Q}(\sqrt{-3})-$linear space V by 'tensoring' it with $\mathbb{Q}$ and using an automorphism of E (section 2);
- we use a linear map Φ on V, of order 6, to decompose V into six eigenspaces $V_{\zeta_6^i}$ (section 2);
- we have $\text{rank}E(\overline{\mathbb{Q}}(t)) = \dim_{\mathbb{Q}} V = \sum_{i=0}^{6} \dim_{\mathbb{Q}} V_{\zeta_6^i}$ and all the $V_{\zeta_6^i}$ correspond to elliptic surfaces. Five of these define rational surfaces (again [6] & [7]). We thereafter find the following equivalence (section 3):

$$\text{rank}E(\overline{\mathbb{Q}}(t)) = 16 \quad \Leftrightarrow \quad \{(x(t), y(t)) \in \overline{\mathbb{Q}}(t)^2 |\ y^2 = x^3 + t^5 f(t)\} = \emptyset \quad (3)$$

$$\text{rank}E(\overline{\mathbb{Q}}(t)) = 18 \quad \Leftrightarrow \quad \{(x(t), y(t)) \in \overline{\mathbb{Q}}(t)^2 |\ y^2 = x^3 + t^5 f(t)\} \neq \emptyset \quad (4)$$

- finally we use MapleV to show that (4) occurs for $(a, b, c) = (-C^5, 11C^6, C^7)$, $C \in \overline{\mathbb{Q}}^*$ (section 4).

We remark here that if Kuwata's method is applied to the elliptic curves given by $y^2 = x^3 + 1$ and $y^2 = x^3 - 15x + 22$ (which are 2-isogenous), then a Weierstraß equation for X is $y^2 = x^3 + 27 \cdot 16((2\eta^3)^4 + 11(2\eta^3)^2 - 1)$. This gives a rank 18 example which is isomorphic (over $\overline{\mathbb{Q}}$) to our example. Hence, in particular, our method provides 18 independent sections for it.

Now that we have proven the rank of $E(\overline{\mathbb{Q}}(t))$ to be at most 18, we have by means of proposition 2.5 bounded the rank of $E(\mathbb{Q}(t))$ by 9. Our next step is to examine $V_1, \cdots, V_{\zeta_6^5}$ separately, or more precisely, to study the elliptic surfaces $E_0, \cdots, E_5$ for which $V_{\zeta_6^i} = E_i(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q}$. The rank of $E_0, \cdots, E_4$ over $\overline{\mathbb{Q}}(t)$ is known because they define rational surfaces. For general elliptic surfaces we have the Shioda-Tate formula

$$MW + 2 + \sum_{\text{bad fibers } \nu} (-1 + \#\text{irred. comp. at } \nu) = NS \quad (5)$$

Here MW and NS denote the ranks of the Mordell-Weil and Néron-Severi groups, respectively. For a rational elliptic surface $NS = 10$. The method described so far actually gives 16 or 18 independent points. Next we want to know how many of these are defined over $\mathbb{Q}$, for a certain choice of $a, b, c \in \mathbb{Q}$. This results in the table found at ((44), section 5), where we summarize all conditions on the coefficients of f. With aid of this table it gets quite easy to investigate to what extend these conditions are compatible. At a glance one already grasps that it is impossible to pick a, b, c in such a way that $\text{rank}E(\mathbb{Q}(t)) = 9$. In fact, we only get to 4. However this is not world shocking news, since we laid great restrictions on the form of our 12^{th}–degree polynomial. We did this to get this sixth-order automorphism, Φ, acting on $E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q}$ and therefore we found the decomposition allowing us to see that we get $\text{rank}E(\overline{\mathbb{Q}}(t)) \geq 16$ for free.

Besides having points (sections) on the $E_i(\mathbb{Q}(t))$, we are of course interested whether or not these are actually generators of the Mordell-Weil group. One thing we already know, by lemma 4.2: the section we find will be of infinite order. Once we know some basics about height functions, we will, aided by theorems of Bremner ([1]) and Silverman ([26], ch. IV), be able to prove what the generators are.

Finally we will discuss some other ways to find high rank elliptic curves over the rationals exploited by other people. We will thoroughly investigate

$$E' : Y^2 = X^3 + t^{18} + 2973t^{12} + 369249t^6 + 11764900$$

which has, over $\mathbb{Q}$, rank ≥ 5. This curve has been constructed by Cam Stewart and Jaap Top ([27]). They did this using a method 'invented' by Mestre ([12]). We have of course special interest in the eigenspace decomposition of $E'(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q}$ and the distribution of the given generators over these subspaces. This might for example hint a way to pick conditions from the table at (44) to get maximal rank on $E(\mathbb{Q}(t))$.

2 Linear algebra as a tool for decomposition

We start with the elliptic curve $E/\bar{\mathbb{Q}}(t)$ defined by

$$Y^2 = X^3 + f(t^6),$$

where f is a non-constant polynomial with $f(0) \neq 0$. We make the group $E(\overline{\mathbb{Q}}(t))$ into a linear space, by "tensoring" it with $\mathbb{Q}$:

$$V := E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q}.$$

For convenience we will write ω for a primitive third root of unity. We will now show that V is actually a $\mathbb{Q}(\sqrt{-3})$-vectorspace. Define the linear map ϕ on V:

$$\phi : E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q} \to E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q} \tag{6}$$

$$\text{as} \quad \phi : (x(t), y(t)) \mapsto -\omega(x(t), y(t)) := (\omega x(t), -y(t)). \tag{7}$$

We see ϕ^6 is the identity and ϕ has order 6. So we have a $\mathbb{Q}(\sqrt{-3})$-structure on V by:

$$(\mathrm{a} - \mathrm{b}\omega) \cdot P := \mathrm{a} \cdot P + \mathrm{b} \cdot \phi(P), \quad \forall \mathrm{a}, \mathrm{b} \in \mathbb{Q}, P \in V.$$

Remark 2.1 *Given a point $P \in E(\overline{\mathbb{Q}}(t))$ of infinite order, the $\mathbb{Q}(\sqrt{-3})$-structure on V implies that P and $\phi(P)$ are linearly independent over $\mathbb{Q}$. Hence we know that if we find one point of infinite order, then the rank is at least 2.*

Now we define Φ as

$$\Phi : \quad \overline{\mathbb{Q}}(t) \to \overline{\mathbb{Q}}(t), \quad t \mapsto -\omega t. \tag{8}$$

This Φ has order 6 and

$$M := \left\{ x \in \overline{\mathbb{Q}}(t) | \Phi(x) = x \right\} = \overline{\mathbb{Q}}(t^6).$$

This follows from Galois theory: $\overline{\mathbb{Q}}(t^6) \subset M$ is trivial; Φ has order 6 $\Rightarrow [\overline{\mathbb{Q}}(t) : M] = 6$ so

$$\underbrace{M \subset \overline{\mathbb{Q}}(t^6) \overset{6}{\subset} \overline{\mathbb{Q}}(t)}_{deg=6} \Rightarrow M = \overline{\mathbb{Q}}(t^6).$$

Next we define Φ on V in the same way:

$$\Phi : \quad E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q} \to E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q} \quad (x(t), y(t)) \mapsto (\Phi(x), \Phi(y)). \tag{9}$$

This Φ is, acting on V, a $\mathbb{Q}(\sqrt{-3})$-linear automorphism of order 6. We now may employ the following lemma:

Lemma 2.2 *Let Ψ be a linear map of finite order m acting on a finite dimensional vector space. Furthermore suppose Ψ has all its eigenvalues rational. Then Ψ is diagonizable.*

Proof: This is a corollary to Maschke's theorem from representation theory:

Theorem 2.3 (Maschke) *If a matrixgroup is reducible, then it is completely reducible. This means if the matrixgroup is conjugated to the matrixgroup in which every matrix has the reduced form $\begin{bmatrix} D_i^{(1)} & X_i \\ 0 & D_i^{(2)} \end{bmatrix}$ then it is conjugated to the matrixgroup obtained by putting $X_i = 0$.*

For a proof of this theorem we refer to [10], page 49.

With our previous notation $\omega = \zeta_3$ we find that the six possible eigenvalues of Φ are $\{(-\omega)^i\}_{0\leq i\leq 5}$, which are all in $\mathbb{Q}(\sqrt{-3})$. So Φ is diagonizable and therefore V is spanned by the eigenspaces: V_1, V_{-1}, V_ω, V_{ω^2}, $V_{-\omega^2}$ and $V_{-\omega}$ (we denote V_α the eigenspace corresponding to the eigenvalue α): $V = \oplus_{i=0}^5 V_{(-\omega)^i}$. In particular, the dimension of V over $\mathbb{Q}$ which is also the rank of $E(\overline{\mathbb{Q}}(t))$ is $\sum_{i=0}^5 \dim_{\mathbb{Q}} V_{(-\omega)^i}$. Now we will focus on these subspaces of V.

Proposition 2.4 *Let E_n be the elliptic curve over $\overline{\mathbb{Q}}(s)$ given by $Y^2 = X^3 + s^n f(s)$. Then*

$$V_{(-\omega)^i} \cong E_i(\overline{\mathbb{Q}}(s)) \otimes \mathbb{Q}.$$

Proof: We will limit ourselves to just two cases, the others are similar.

$i = 3$. We have to consider elements $r \cdot (x(t), y(t)) \in E(\overline{\mathbb{Q}}(t)) \otimes \mathbb{Q}$ with

$$r \cdot (\Phi x(t), \Phi y(t)) = r \cdot \Phi(x(t), y(t)) = r \cdot (-\omega)^3 (x(t), y(t)) = r \cdot (x(t), -y(t)).$$

It is sufficient to consider the elements of $E(\overline{\mathbb{Q}}(t))$. It follows that $x(t)$ and $t^3 y(t) \in \overline{\mathbb{Q}}(t^6)$. If we define $s := t^6$ then we want to construct

$$\{\ P \ \in \ E(\overline{\mathbb{Q}}(t)) \ \mid \ \Phi(P) \ = \ -P\ \} \ \otimes \ \mathbb{Q} \ \xrightarrow{\sim} \ E_3(\overline{\mathbb{Q}}(s)) \ \otimes \ \mathbb{Q}.$$

Now a simple substitution shows that the required isomorphism is given by

$$r \cdot (x(t), y(t)) \mapsto r \cdot (t^6 x(t), t^9 y(t)).$$

Therefore $V_{-1} \cong E_3(\overline{\mathbb{Q}}(s)) \otimes \mathbb{Q}$.

$i = 1$. Here we are looking for elements $P \in E(\overline{\mathbb{Q}}(t))$ such that

$$\Phi(x(t), y(t)) = -\omega(x(t), y(t)) = (\omega x(t), -y(t)).$$

Similar to the previous case it is easily seen that $t^2 x(t), t^3 y(t) \in \overline{\mathbb{Q}}(t^6)$. Since $(t^3 y)^2 = (t^2 x)^3 + t^6 f$ it follows $(t^3 x, t^2 y) \in E_1(\overline{\mathbb{Q}}(s))$. Therefore we have the isomorphism from $V_{-\omega}$ to $E_1(\overline{\mathbb{Q}}(s)) \otimes \mathbb{Q}$ given by

$$r \cdot (x(t), y(t)) \mapsto r \cdot (t^2 x(t), t^3 y(t)).$$

Q.E.D.

So we have

$$\operatorname{rank} E(\overline{\mathbb{Q}}(t)) = \dim_{\mathbb{Q}} V = \sum_{i=0}^{5} \dim_{\mathbb{Q}} V_{(-\omega)^i} = \sum_{i=0}^{5} \operatorname{rank} E_i(\overline{\mathbb{Q}}(s)). \tag{10}$$

We remark all $E_i(\overline{\mathbb{Q}}(s))$ have even rank since the $V_{(-\omega)^i}$ are $\mathbb{Q}(\sqrt{-3})$-vectorspaces.

Proposition 2.5 *Let* $E(\overline{\mathbb{Q}}(t))$ *be as above. Then*

$$\text{rank}E(\overline{\mathbb{Q}}(t)) = 2n \Rightarrow \text{rank}E(\mathbb{Q}(t)) \leq n.$$

Proof: If we take $\sigma : \sqrt{-3} \mapsto -\sqrt{-3}$ being complex conjugation on $L = \mathbb{Q}(\sqrt{-3})$ and $\phi : P \mapsto -\omega P$ our proposition follows from the exactness of

$$0 \to E(\mathbb{Q}(t)) \xrightarrow{\phi} E(L(t)) \xrightarrow{\phi^2+\phi\sigma} E(\mathbb{Q}(t)).$$

Note that $\phi^2 + \phi\sigma = \phi^2 + \sigma\phi^2$, which shows that the image of $\phi^2 + \phi\sigma$ is indeed in $E(\mathbb{Q}(t))$. This is an exact sequence because $\ker(\phi^2 + \phi\sigma) = \phi E(\mathbb{Q}(t))$. To see this, we take an arbitrary, non-zero point $P \in E(L(t))$. One can write $P = (\omega x, -y)$. Then

$$(\phi^2 + \phi\sigma)(P) = 0 \Leftrightarrow \phi^2 P = -\phi\sigma P \quad (11)$$
$$\Leftrightarrow \phi^6 P = P = -\phi^5\sigma P = \phi(\phi\sigma P) \quad (12)$$
$$\Leftrightarrow \phi\phi\sigma(\omega x, -y) = \phi\phi(\overline{\omega x}, -\overline{y}) = \phi(\overline{x}, \overline{y}) = P \quad (13)$$
$$\Leftrightarrow x, y \in \mathbb{Q}(t) \Leftrightarrow P \in \phi E(\mathbb{Q}(t)). \quad (14)$$

Furthermore $(\phi^2 + \phi\sigma)\phi^4 E(\mathbb{Q}(t)) = (1+\sigma)\phi^6 E(\mathbb{Q}(t)) = 2E(\mathbb{Q}(t))$. Hence the image $E(L(t))$ has finite index in $E(\mathbb{Q}(t))$. This implies $\text{rank}E(L(t)) = 2\text{rank}E(\mathbb{Q}(t))$. Note that $\text{rank}E(L(t)) \leq \text{rank}E(\overline{\mathbb{Q}}(t)) = 2n$. Now with $\text{rank}E(\mathbb{Q}(t)) =: m$ one has

$$2m = \text{rank}E(L(t)) \leq 2n.$$

Q.E.D.

We finish this section with one more result on the rank over $\mathbb{Q}(t)$.

Proposition 2.6

$$\text{rank}E(\mathbb{Q}(t)) = \sum_{i=0}^{5} \text{rank}E_i(\mathbb{Q}(t)).$$

Proof: The arguments above imply $\text{rank}E(\mathbb{Q}(\sqrt{-3}, t)) = \sum_{i=0}^{5} \text{rank}E_i(\mathbb{Q}(\sqrt{-3}, t))$. Furthermore, in the proof of proposition 2.5 we saw that $\text{rank}\, E(\mathbb{Q}(\sqrt{-3})(t)) = 2\, \text{rank}\, E(\mathbb{Q}(t))$. This and a similar statement for the E_i complete the proof. **Q.E.D.**

3 Some Geometry [1]

In the next two sections we will explore all the equations more closely, now we will focus on upper bounds for the dimension of each $V_{(-\omega)^i}$. We make some restrictions

[1] For more details about the underlying algebraic geometry you could read [6],[7] or [29].

to our polynomial $f(t)$ we didn't need before: $f(t) := at^2+bt+c$ and $\Delta_f = b^2-4ac \neq 0$, $a \neq 0$. The discriminant and the j-invariant of E are $\Delta_E = -27 * 4 * f^2$ and 0 respectively.

From Néron and Kodaira (see [16],[28]) we have the classification of all possible singular fibers on elliptic surfaces. We know that E_0, E_1, E_2, E_3, E_4 are given by equations : $E_i : y^2 = x^3 + f_i$ over $\mathbb{Q}(t)$ with f_i a polynomial in t with degree ≤ 6 and not a sixth power. It now follows from [20] that they are rational elliptic surfaces. Thus with respect to the first five subspaces we can simply apply the formula due to Shioda and Tate [20] (where the 10 represents the Néron-Severi rank of a rational surface):

$$\text{rank } E_i(\overline{\mathbb{Q}}(t)) = 10 \ - 2 \ - \sum_{singular\ fibers\ \nu} (m_\nu - 1). \tag{15}$$

Here m_ν denotes the number of irreducible components of the singular fiber at ν. The numbers m_ν can be found as follows. For a reducible fibre at some $t = \nu \in \overline{\mathbb{Q}}$ one uses the following table (see for instance [28],[26] page 350+).

Kodaira symbol	II	IV	I_0^*	IV^*	II^*
$\text{ord}_\nu(\Delta_\nu)$	2	4	6	8	10
m_ν	1	3	5	7	9
$E(\overline{\mathbb{Q}}(t))/E_0(\overline{\mathbb{Q}}(t))$	(0)	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	(0)
$\tilde{E}^0(\overline{\mathbb{Q}})$	$\overline{\mathbb{Q}}^+$	$\overline{\mathbb{Q}}^+$	$\overline{\mathbb{Q}}^+$	$\overline{\mathbb{Q}}^+$	$\overline{\mathbb{Q}}^+$

(16)

Note that $E_0(\overline{\mathbb{Q}}(t))$ in this table is in the notation of Tate the subgroup of $E(\overline{\mathbb{Q}}(t))$ consisting of all points which specialize to nonsingular points at $t = \nu$.

We illustrate how to compute m_∞ by an example. Consider E_1, given by $y^2 = x^3 + at^3 + bt^2 + ct$. Here we have singularities at ∞, 0 and at the zeros of f. The substitution

$$\{\eta := Yt^{-3}, \xi := Xt^{-2}, \tau := t^{-1}\} \tag{17}$$

leads to the equation $\eta^2 = \xi^3 + c\tau^5 + b\tau^4 + a\tau^3$, where $t = \infty$ corresponds to $\tau = 0$. Since $\tau = 0$ is a triple zero here, the table above shows $m_\infty = 5$ in the this case. For the other singularities we find $m_0 = m_\alpha = m_\beta = 1$, where α, β denote the (simple) zeros of $f(t)$. Now we determine

$$\sum_{\nu \in \{0,\alpha,\beta,\infty\}} (m_\nu - 1) = 4$$

and substituting this in (15) shows that the rank of $E_1(\overline{\mathbb{Q}}(t))$ is 4. By doing the same computation for the other four E_i defining a rational elliptic surface, we find:

E_i	bad fibers	$\sum(m_\nu - 1)$	rank $E_i = \dim_{\mathbb{Q}} V_{(-\omega)^i}$
E_0	$2 \times II$ & IV^*	6	2
E_1	$2 \times II$ & I_0^*	4	4
E_2	$2 \times II$ & $2 \times IV$	4	4
E_3	$2 \times II$ & I_0^*	4	4
E_4	$2 \times II$ & IV^*	6	2

(18)

Note that the right column sums to 16, from which we conclude that $\text{rank}E(\overline{\mathbb{Q}}(t)) \geq 16$.

Theorem 3.1

$$\dim V_{(-\omega)^5} = 0 \text{ or } 2. \tag{19}$$

and

$$\exists\ a, b, c \in \overline{\mathbb{Q}}, \quad \text{for which} \quad \text{rank}E_5(\overline{\mathbb{Q}}(s)) = 2. \tag{20}$$

Corollary 3.2

$$\text{rank}E(\overline{\mathbb{Q}}(t)) = 16 \text{ or } 18.$$

and

$$\exists\ a, b, c \in \overline{\mathbb{Q}}, \quad \text{for which} \quad \text{rank}E(\overline{\mathbb{Q}}(t)) = 18.$$

Here we prove the first statement of theorem 3.1 in several ways.

Proof: First proof: since $f(t^6)$ has degree 12 and is not divisible by a sixth power E is a $K3$-surface. Hence it follows from the Shioda-Tate formula that $\text{rank}E(\overline{\mathbb{Q}}(t)) \leq 18$:

$$2 + MW + \sum(m_\nu - 1) = NS, \tag{21}$$

where MW, NS denote the ranks of the Mordell-Weil and Néron-Severi groups respectively. For a $K3$-surface one knows $NS \leq 20$. There remains 2 at most for $(MW-)\text{rank}E_5(\overline{\mathbb{Q}}(s))$ and we already remarked that this rank is even.

Another way to look at this is by remarking that E_5 itself is a $K3$-surface. Then Shioda-Tate (21) gives

$$\text{rank}E_5(\overline{\mathbb{Q}}(s)) + \sum(m_\nu - 1) \leq 18.$$

We find 2 fibers of type II^*, hence $\sum(m_\nu - 1) \geq 16$. It follows that 2 is an upper bound for $\dim_{\mathbb{Q}} V_{(-\omega)^5}$.

The third way to prove this, runs as follows. We introduce the curve C over $\mathbb{Q}$ defined as

$$C: \quad t^6 = s^5(as^2 + bs + c).$$

We substitute $u := t/s$ and $v := as - (b - u^6)/2$. In these new variables the equation transforms into

$$(v)^2 - \underbrace{\frac{1}{4}\left(u^{12} - 2bu^6 + b^2 - 4ac\right)}_{g_{12}(u)} = 0.$$

So we have the hyperelliptic curve C given by $v^2 = g_{12}(u)$. This C has genus 5 and the following basis for the space of regular differentials ([6])

$$\frac{du}{v}, \ \frac{u\ du}{v}, \ \frac{u^2\ du}{v}, \ \frac{u^3\ du}{v} \text{ and } \frac{u^4\ du}{v}.$$

Now we construct homomorphisms, similar to [27]:

$$E_5(\overline{\mathbb{Q}}(s)) \xrightarrow{\lambda_1} Mor_{\overline{\mathbb{Q}}}(C, \bar{E}) \xrightarrow{\lambda_2} H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}}) \tag{22}$$

where $\bar{E}$ is given by $Y^2 = X^3 + 1$. The map λ_1 is defined by

$$\begin{gathered} \lambda_1 : E_5(\overline{\mathbb{Q}}(s)) \to Mor_{\overline{\mathbb{Q}}}(C, \bar{E}) \\ \lambda_1(P) = \lambda_1(x(s), y(s)) := \varphi_P, \quad \text{where} \\ Mor_{\overline{\mathbb{Q}}}(C, \bar{E}) \ni \varphi_P : (t, s) \mapsto \left(\tfrac{x(s)}{t^2}, \tfrac{y(s)}{t^3}\right). \end{gathered}$$

Furthermore $\lambda_2 : Mor_{\mathbb{Q}}(C, \bar{E}) \to H^0(C, \Omega^1)$ is given by $\lambda_2(\varphi) := \varphi^* \omega_{\bar{E}}$, i.e. the pullback of the invariant differential of $\bar{E}$ via φ. Now

$$\lambda := \lambda_2 \circ \lambda_1 : E_5(\overline{\mathbb{Q}}(s)) \to H^0\left(C, \Omega^1_{C/\overline{\mathbb{Q}}}\right) \tag{23}$$

is a homomorphism with a finite kernel ([27], Proposition 1). Next we define an automorphism $\Phi' : (s,t) \mapsto (s, \zeta_6 t)$ on C. Then Φ' acts on the differentials of C via the pullback: $\Phi'^* f(s,t)ds = f(s, \zeta_6 t)ds$ for any function f on C. Observe that similar to the Φ defined in (9), we have again a linear action which decomposes a vector-space (this time $H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}})$), the analytic differentials on C) into eigenspaces corresponding to the eigenvalues ζ_6^i:

$$H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}}) = \bigoplus_{i=0}^{5} H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}})(\zeta_6^i).$$

By its definition one finds that the image of λ is in $H^0(C,\Omega^1_{C/\overline{\mathbb{Q}}})(\zeta_6)$. It now follows that ([27])

$$\mathrm{rank}E_5(\overline{\mathbb{Q}}(s)) \leq \dim_{\mathbb{Q}} H^0(C,\Omega^1_{C/\mathbb{Q}(\sqrt{-3})})(\zeta_6). \tag{24}$$

Here $\frac{du}{v}$ generates this eigenspace over $\mathbb{Q}(\sqrt{-3})$. Therefore we find $\dim_{\mathbb{Q}} V_{(-\omega)^5} \leq 2$, hence it is 0 or 2. **Q.E.D.**

4 Rank 18 over $\overline{\mathbb{Q}}(t)$

In this section we will show with the aid of a little computer algebra that $a, b, c \in \mathbb{Q}$ exist such that $E_5 : Y^2 = X^3 + at^7 + bt^6 + ct^5$ has positive rank. This will prove the second part of theorem 3.1.

Let us start by defining our problem in MapleV:

```
> F:= x^3 + t^(n-1)*G;
```

$$F := x^3 + t^{(n-1)}\,G$$

```
> G:=a * t^2 +b*t+c;
```

$$G := a\,t^2 + b\,t + c$$

The easiest solutions of $E : Y^2 = X^3 + t^5G(t)$ to look for are polynomials, we will try to find such solutions. If x and y are elements of $\overline{\mathbb{Q}}[t] \subset \overline{\mathbb{Q}}(t)$, the degree of the left hand side has to be equal the degree of the right hand side. Thus we find $\deg_t(x) \geq 4$ and $\deg_t(y) \geq 6$. We will now look for solutions x and y of minimal degree; both monic and with the constant term in x^3 and y^2 equal. For $d, \cdots, l \in \overline{\mathbb{Q}}$ we define:

```
> x:= t^4 + d * t^3 + e*t^2 + f*t + g^2;
```

$$x := t^4 + d\,t^3 + e\,t^2 + f\,t + g^2$$

```
> y:= t^6 + h*t^5 + i*t^4 + j*t^3 + k*t^2 + l*t + g^3;
```

$$y := t^6 + h\,t^5 + i\,t^4 + j\,t^3 + k\,t^2 + l\,t + g^3$$

```
> left:=expand(y^2);
```

$$\mathit{left} := 2\,t^{11}\,h + 2\,t^{10}\,i + 2\,t^9\,j + 2\,t^8\,k + 2\,t^7\,l + 2\,t^6\,g^3 + h^2\,t^{10} + i^2\,t^8 + j^2\,t^6 + k^2\,t^4 + l^2\,t^2 + g^6 + 2\,h\,t^9\,i + 2\,h\,t^8\,j + 2\,h\,t^7\,k + 2\,h\,t^6\,l + 2\,h\,t^5\,g^3 + 2\,i\,t^7\,j + 2\,i\,t^6\,k + t^{12} + 2\,i\,t^5\,l + 2\,i\,t^4\,g^3 + 2\,j\,t^5\,k + 2\,j\,t^4\,l + 2\,j\,t^3\,g^3 + 2\,k\,t^3\,l + 2\,k\,t^2\,g^3 + 2\,l\,t\,g^3$$

```
> right:=expand(x^3 + t^5 *G);
```

$$\mathit{right} := 6\,d\,t^6\,e\,f + 6\,d\,t^5\,e\,g^2 + 6\,d\,t^4\,f\,g^2 + 6\,e\,t^3\,f\,g^2 + t^{12} + g^6 + 6\,t^9\,d\,e + 6\,t^8\,d\,f + 6\,t^7\,d\,g^2 + 6\,t^7\,e\,f + 6\,t^6\,e\,g^2 + 6\,t^5\,f\,g^2 + 3\,d\,t^3\,g^4 + 3\,d^2\,t^8\,e + 3\,d^2\,t^7\,f + 3\,d^2\,t^6\,g^2 + 3\,d\,t^7\,e^2 + 3\,d\,t^5\,f^2 + 3\,e\,t^2\,g^4 + 3\,e^2\,t^5\,f + 3\,e^2\,t^4\,g^2 + 3\,e\,t^4\,f^2 + 3\,f\,t\,g^4 + 3\,f^2\,t^2\,g^2 + 3\,t^{11}\,d + 3\,t^{10}\,e + 3\,t^9\,f + 3\,t^8\,g^2 + 3\,t^4\,g^4 + 3\,t^{10}\,d^2 + 3\,t^8\,e^2 + 3\,t^6\,f^2 + d^3\,t^9 + e^3\,t^6 + f^3\,t^3 + t^7\,a + t^6\,b + t^5\,c$$

Now we make a list of the coefficients of the left and right hand side.

```
> llist:=[coeff(left,t^12),coeff(left,t^11), coeff(left,t^10),
coeff(left,t^9),coeff(left,t^8),coeff(left,t^7),coeff(left,t^6),
coeff(left,t^5),coeff(left,t^4),coeff(left,t^3),coeff(left,t^2),
coeff(left,t),coeff(left,t,0)];
```

$$llist := [1,\ 2h,\ 2i+h^2,\ 2j+2hi,\ 2k+i^2+2hj,\ 2l+2hk+2ij,\ 2g^3+j^2+2hl+2ik,\ 2hg^3+2il+2jk,\ k^2+2ig^3+2jl,\ 2jg^3+2kl,\ l^2+2kg^3,\ 2lg^3,\ g^6]$$

```
> rlist:=[coeff(right,t^12),coeff(right,t^11), coeff(right,t^10),
coeff(right,t^9),coeff(right,t^8),coeff(right,t^7),
coeff(right,t^6),coeff(right,t^5),coeff(right,t^4),
coeff(right,t^3),coeff(right,t^2), coeff(right,t),
coeff(right,t,0)];
```

$$rlist := [1,\ 3d,\ 3e+3d^2,\ 6de+3f+d^3,\ 6df+3d^2e+3g^2+3e^2,\ 6dg^2+6ef+3d^2f+3de^2+a,\ 6def+6eg^2+3d^2g^2+3f^2+e^3+b,\ 6deg^2+6fg^2+3df^2+3e^2f+c,\ 6dfg^2+3e^2g^2+3ef^2+3g^4,\ 6efg^2+3dg^4+f^3,\ 3eg^4+3f^2g^2,\ 3fg^4,\ g^6]$$

And we look at the difference, which we want to equal zero.

```
> list:=llist-rlist;
```

$$list := [0,\ -3d+2h,\ -3e-3d^2+2i+h^2,\ -6de-3f-d^3+2j+2hi,\ -6df-3d^2e-3g^2-3e^2+2k+i^2+2hj,\ -6dg^2-6ef-3d^2f-3de^2-a+2l+2hk+2ij,\ -6def-6eg^2-3d^2g^2-3f^2-e^3-b+2g^3+j^2+2hl+2ik,\ -6deg^2-6fg^2-3df^2-3e^2f-c+2hg^3+2il+2jk,\ -6dfg^2-3e^2g^2-3ef^2-3g^4+k^2+2ig^3+2jl,\ -6efg^2-3dg^4-f^3+2jg^3+2kl,\ -3eg^4-3f^2g^2+l^2+2kg^3,\ -3fg^4+2lg^3,\ 0]$$

Next we take out the "easy" ones:

```
> h:= 3/2 *d;
```

$$h := \frac{3}{2}d$$

```
> simplify(list);
```

$$[0,\ 0,\ -3e-\frac{3}{4}d^2+2i,\ -6de-3f-d^3+2j+3di,\ -6df-3d^2e-3g^2-3e^2+2k+i^2+3dj,\ -6dg^2-6ef-3d^2f-3de^2-a+2l+3dk+2ij,\ -6def-6eg^2-3d^2g^2-3f^2-e^3-b+2g^3+j^2+3dl+2ik,\ -6deg^2-6fg^2-3df^2-3e^2f-c+3dg^3+2il+2jk,$$

$-6\,d\,f\,g^2-3\,e^2\,g^2-3\,e\,f^2-3\,g^4+k^2+2\,i\,g^3+2\,j\,l,$
$-6\,e\,f\,g^2-3\,d\,g^4-f^3+2\,j\,g^3+2\,k\,l,\ -3\,e\,g^4-3\,f^2\,g^2+l^2+2\,k\,g^3,$
$-3\,f\,g^4+2\,l\,g^3,\ 0]$

```
> g:=(2*l)/(3*f);
```

$$g:=\frac{2}{3}\frac{l}{f}$$

```
> list;
```

$$[0,\ 0,\ -3\,e-\frac{3}{4}\,d^2+2\,i,\ -6\,d\,e-3\,f-d^3+2\,j+3\,d\,i,$$
$$-6\,d\,f-3\,d^2\,e-\frac{4}{3}\frac{l^2}{f^2}-3\,e^2+2\,k+i^2+3\,d\,j,$$
$$-\frac{8}{3}\frac{d\,l^2}{f^2}-6\,e\,f-3\,d^2\,f-3\,d\,e^2-a+2\,l+3\,d\,k+2\,i\,j,$$
$$-6\,d\,e\,f-\frac{8}{3}\frac{e\,l^2}{f^2}-\frac{4}{3}\frac{d^2\,l^2}{f^2}-3\,f^2-e^3-b+\frac{16}{27}\frac{l^3}{f^3}+j^2+3\,d\,l+2\,i\,k,$$
$$-\frac{8}{3}\frac{d\,e\,l^2}{f^2}-\frac{8}{3}\frac{l^2}{f}-3\,d\,f^2-3\,e^2\,f-c+\frac{8}{9}\frac{d\,l^3}{f^3}+2\,i\,l+2\,j\,k,$$
$$-\frac{8}{3}\frac{d\,l^2}{f}-\frac{4}{3}\frac{e^2\,l^2}{f^2}-3\,e\,f^2-\frac{16}{27}\frac{l^4}{f^4}+k^2+\frac{16}{27}\frac{i\,l^3}{f^3}+2\,j\,l,$$
$$-\frac{8}{3}\frac{e\,l^2}{f}-\frac{16}{27}\frac{d\,l^4}{f^4}-f^3+\frac{16}{27}\frac{j\,l^3}{f^3}+2\,k\,l,\ -\frac{16}{27}\frac{e\,l^4}{f^4}-\frac{1}{3}\,l^2+\frac{16}{27}\frac{k\,l^3}{f^3},\ 0,\ 0]$$

```
> list:=[list[3],list[4],list[5],list[6],list[7], list[8],
> list[9],list[10],list[11]];
```

$$list:=[-3\,e-\frac{3}{4}\,d^2+2\,i,\ -6\,d\,e-3\,f-d^3+2\,j+3\,d\,i,$$
$$-6\,d\,f-3\,d^2\,e-\frac{4}{3}\frac{l^2}{f^2}-3\,e^2+2\,k+i^2+3\,d\,j,$$
$$-\frac{8}{3}\frac{d\,l^2}{f^2}-6\,e\,f-3\,d^2\,f-3\,d\,e^2-a+2\,l+3\,d\,k+2\,i\,j,$$
$$-6\,d\,e\,f-\frac{8}{3}\frac{e\,l^2}{f^2}-\frac{4}{3}\frac{d^2\,l^2}{f^2}-3\,f^2-e^3-b+\frac{16}{27}\frac{l^3}{f^3}+j^2+3\,d\,l+2\,i\,k,$$
$$-\frac{8}{3}\frac{d\,e\,l^2}{f^2}-\frac{8}{3}\frac{l^2}{f}-3\,d\,f^2-3\,e^2\,f-c+\frac{8}{9}\frac{d\,l^3}{f^3}+2\,i\,l+2\,j\,k,$$
$$-\frac{8}{3}\frac{d\,l^2}{f}-\frac{4}{3}\frac{e^2\,l^2}{f^2}-3\,e\,f^2-\frac{16}{27}\frac{l^4}{f^4}+k^2+\frac{16}{27}\frac{i\,l^3}{f^3}+2\,j\,l,$$
$$-\frac{8}{3}\frac{e\,l^2}{f}-\frac{16}{27}\frac{d\,l^4}{f^4}-f^3+\frac{16}{27}\frac{j\,l^3}{f^3}+2\,k\,l,\ -\frac{16}{27}\frac{e\,l^4}{f^4}-\frac{1}{3}\,l^2+\frac{16}{27}\frac{k\,l^3}{f^3}]$$

We see as well that

$$
\begin{aligned}
a &= -\frac{8}{3}\frac{d\,l^2}{f^2} - 6\,e\,f - 3\,d^2\,f - 3\,d\,e^2 + 2\,l + 3\,d\,k + 2\,i\,j && (25)\\
b &= -6\,d\,e\,f - \frac{8}{3}\frac{e\,l^2}{f^2} - \frac{4}{3}\frac{d^2\,l^2}{f^2} - 3\,f^2 - e^3 + \frac{16}{27}\frac{l^3}{f^3} + j^2 + 3\,d\,l + 2\,i\,k && (26)\\
c &= -\frac{8}{3}\frac{d\,e\,l^2}{f^2} - \frac{8}{3}\frac{l^2}{f} - 3\,d\,f^2 - 3\,e^2\,f + \frac{8}{9}\frac{d\,l^3}{f^3} + 2\,i\,l + 2\,j\,k && (27)\\
i &= \frac{3}{2}\,e + \frac{3}{8}\,d^2 && (28)\\
j &= \frac{3}{4}\,d\,e + \frac{3}{2}\,f - \frac{1}{16}\,d^3 && (29)\\
e &= -\frac{9}{16}\frac{f^4}{l^2} + \frac{k\,f}{l} && (30)
\end{aligned}
$$

Substituting them in this order and simplifying the remaining list gives:

$$
\begin{aligned}
list := [&-\frac{3}{2}\,f^3\,d - \frac{27}{128}\frac{d^2\,f^6}{l^2} + \frac{3}{8}\frac{d^2\,f^3\,k}{l} - \frac{4}{3}\,l^2 - \frac{243}{1024}\frac{f^{10}}{l^4} + \frac{27}{32}\frac{f^7\,k}{l^3}\\
&-\frac{3}{4}\frac{f^4\,k^2}{l^2} + 2\,f^2\,k - \frac{3}{64}\,f^2\,d^4,\\
&-\frac{8}{3}\,f^3\,d\,l^2 + \frac{81}{64}\frac{f^{10}}{l^2} - \frac{3}{2}\frac{f^7\,k}{l} - \frac{1}{3}\,f^4\,k^2 - \frac{16}{27}\,l^4 + \frac{5}{2}\,l\,f^5 + \frac{8}{9}\,f^2\,l^2\,k\\
&+\frac{2}{9}\,f\,l^3\,d^2 - \frac{27}{32}\frac{f^8\,d}{l}\\
&+\frac{3}{2}\,f^5\,d\,k - \frac{1}{8}\,f^4\,l\,d^3,\ \frac{1}{2}\,f^7 - \frac{2}{3}\,f^4\,k\,l - \frac{16}{27}\,d\,l^4 - \frac{1}{4}\,f^5\,l\,d + \frac{4}{9}\,d\,f^2\,l^2\,k\\
&+\frac{8}{9}\,f^2\,l^3 - \frac{1}{27}\,d^3\,f\,l^3]
\end{aligned}
$$

Finally we can solve $list = 0$:

```
> aap:=solve({%[1],%[2],%[3]},{d,k,l});
```

$$
aap := \{k = \frac{1}{24}\frac{32\,l^3 + 27\,f^5}{f^2\,l},\ l = l,\ d = \frac{3}{2}\frac{f^2}{l}\}, \qquad (31)
$$

$$
\begin{aligned}
&\{l = \frac{1}{4}\,\mathrm{RootOf}(2\,\_Z^3 - 3\,f^2)\,f,\\
&d = -4\,\frac{\mathrm{RootOf}(2\,\_Z^3 - 3\,f^2)^2}{f},\\
&k = \frac{25}{12}\,\mathrm{RootOf}(2\,\_Z^3 - 3\,f^2)^2\} \qquad (32)
\end{aligned}
$$

The first solution (31) is the trivial one, leading to $a = b = c = 0$. We therefore focus on the second one

```
> assign(aap[2]);
> alias(alpha=RootOf(2*X^3=3*f^2,X));
```

$$I, \alpha$$

```
> expand({d,e,g,h,i,j,k,l});
```

$$\{\frac{25}{12}\alpha^2,\ -9\frac{f^2}{\alpha^2}+\frac{25}{3}\alpha,\ \frac{1}{6}\alpha,\ -6\frac{\alpha^2}{f},\ -\frac{27}{2}\frac{f^2}{\alpha^2}+\frac{25}{2}\alpha+6\frac{\alpha^4}{f^2},$$
$$\frac{57}{2}f-25\frac{\alpha^3}{f}+4\frac{\alpha^6}{f^3},\ -4\frac{\alpha^2}{f},\ \frac{1}{4}\alpha f\} \tag{33}$$

```
> jaapa:=simplify(llist[6]-rlist[6]);
```

$$jaapa := -24\,\alpha\, f - a$$

```
> jaapb:=simplify(llist[7]-rlist[7]);
```

$$jaapb := 132\, f^2 - b$$

```
> jaapc:=simplify(llist[8]-rlist[8]);
```

$$jaapc := 4\, f\, \alpha^2 - c$$

By now we have expressed all unknowns in rational expressions in f and $\alpha = (\frac{3f^2}{2})^{1/3}\zeta_3^i$. In this way we have described solutions for $Y^2 = X^3 + at^7 + bt^6 + ct^5$. Note that if we take f such that $2Z^3 - 3f^2 = 0$ has a rational root, then we can find a solution in $\mathbb{Q}(t)$. With $\alpha = RootOf(2Z^3 - 3f^2)$ we find

$$\begin{aligned} a &= -24f\alpha \\ b &= 132f^2 \\ c &= 4f\alpha^2. \end{aligned}$$

Since we are looking for solutions $a, b, c \in \mathbb{Q}$, α has to be rational. This implies $f = 12\beta^3$ for any $\beta \in \mathbb{Q}$. The rational α now equals $6\beta^2$ and

$$\begin{aligned} a &= -24\, f\, \alpha = -24\,(12\,\beta^3)\,(6\,\beta^2) = -1728\,\beta^5 = -2^6\,3^3\,\beta^5 \\ b &= 132\, f^2 = 132\,(12\,\beta^3)^2 = 19008\,\beta^6 = 2^6\,3^3\,11\,\beta^6 \\ c &= 4\, f\, \alpha^2 = 4\,(12\,\beta^3)\,(6\,\beta^2)^2 = 2^6\,3^3\,\beta^7. \end{aligned} \tag{34}$$

To simplify the equations we find, one uses

Lemma 4.1 *Let $E_d/\mathbb{K}$ be the elliptic curve given by $Y^2 = X^3 + d$. Then for any $k \in \mathbb{K}^*$ the curve E_d is isomorphic over $\mathbb{K}$ to E_{dk^6}. Moreover E_d is 3-isogenous over $\mathbb{K}$ to E_{-27d}.*

Proof: The isomorphism is defined as

$$(X,Y) \mapsto (X\,k^{-2}, Y\,k^{-3}), \tag{35}$$

and the isogeny as

$$(X,Y) \mapsto ((Y^2+3d)X^{-2}, Y(X^3-8d)X^{-3}). \tag{36}$$

Q.E.D.

Using (35) of this lemma we can simplify (34), for any $\beta \in \mathbb{Q}$, to:

$$a = -27\beta^5 \quad b = 297\beta^6 \quad c = 27\beta^7.$$

This leads to

$$\begin{aligned} x &:= \frac{1}{4}\,t^4 - 3\,\beta\,t^3 + \frac{7}{2}\,\beta^2\,t^2 + 3\,\beta^3\,t + \frac{1}{4}\,\beta^4, \\ y &:= \frac{1}{8}\,t^6 - \frac{9}{4}\,\beta\,t^5 + \frac{75}{8}\,\beta^2\,t^4 + \frac{75}{8}\,\beta^4\,t^2 + \frac{9}{4}\,\beta^5\,t + \frac{1}{8}\,\beta^6 \end{aligned} \tag{37}$$

as a solution of:

$$E_5 : Y^2 = X^3 - 27\,(\beta^5\,t^7 - 11\,\beta^6\,t^6 - \beta^7\,t^5).$$

Furthermore we have an isogeny from

$$E_5 : Y^2 = X^3 - 27(\beta^5 t^7 - 11\beta^6 t^6 - \beta^7 t^5) \;\text{ to }\; E_5' : \eta^2 = \xi^3 + \beta^5 t^7 - 11\beta^6 t^6 - \beta^7 t^5.$$

Using (36) and MapleV we find that $(x,y) \in E_5(\overline{\mathbb{Q}}(t))$ transforms into $(\xi,\eta) \in E_5'(\overline{\mathbb{Q}}(t))$, given by:

```
> xi:=simplify((y^2+3*t^5*G)/x^2);
eta:=simplify(y*(x^3-8*t^5*G)/x^3);
```

$$\begin{aligned} \xi := \frac{1}{36}(&t^{12} + 7848\,\beta^7\,t^5 + \beta^{12} + 474\,t^{10}\,\beta^2 + 5775\,t^8\,\beta^4 - 36\,t^{11}\,\beta + 67628\,t^6\,\beta^6 \\ &- 7848\,\beta^5\,t^7 - 2700\,\beta^3\,t^9 + 5775\,\beta^8\,t^4 + 2700\,\beta^9\,t^3 + 474\,\beta^{10}\,t^2 + 36\,\beta^{11}\,t)/ \\ &(t^4 - 12\,\beta\,t^3 + 14\,\beta^2\,t^2 + 12\,\beta^3\,t + \beta^4)^2 \end{aligned}$$

$$\begin{aligned} \eta := \frac{1}{216}(&(t^6 - 18\,\beta\,t^5 + 75\,\beta^2\,t^4 + 75\,\beta^4\,t^2 + 18\,\beta^5\,t + \beta^6)(t^{12} - 12888\,\beta^7\,t^5 \\ &+ \beta^{12} + 474\,t^{10}\,\beta^2 + 5775\,t^8\,\beta^4 - 36\,t^{11}\,\beta - 160468\,t^6\,\beta^6 + 12888\,\beta^5\,t^7 \\ &- 2700\,\beta^3\,t^9 + 5775\,\beta^8\,t^4 + 2700\,\beta^9\,t^3 + 474\,\beta^{10}\,t^2 + 36\,\beta^{11}\,t))/ \\ &(t^4 - 12\,\beta\,t^3 + 14\,\beta^2\,t^2 + 12\,\beta^3\,t + \beta^4)^3 \end{aligned}$$

```
> simplify(eta^2-xi^3);
```

$$t^5\,\beta^5\,(t^2 - 11\,t\,\beta - \beta^2)$$

We already conjecture that this $(\xi, \eta) \in E_5'(\overline{\mathbb{Q}}(t))$ is a non-torsion point, since computing $[m]P$ for increasing m shows that the numerator and denominator increase rapidly (this is a somewhat instinctive notion of height). However we still have to prove the following.

Lemma 4.2 *An elliptic surface (i.e. an elliptic curve over $K(t)$, K a field of characteristic 0) with a bad fiber of type II has no torsion.*

Remark 4.3 *The same holds for a curve with a bad fiber of type II**

Proof: To see this we first return to the table at (16), in particular to the last two rows. Here $\tilde{E}(\overline{\mathbb{Q}})$ is the image of $E(\overline{\mathbb{Q}}(t))$ after specializing at a bad fiber $t = \nu$ and $\tilde{E}^0(\overline{\mathbb{Q}})$ denotes the non-singular (smooth) part of $\tilde{E}(\overline{\mathbb{Q}})$. Furthermore the $E_0(\overline{\mathbb{Q}}(t))$ is derived from

$$\begin{array}{ccc} E(\overline{\mathbb{Q}}(t)) & \underset{t=\nu}{\longrightarrow} & \tilde{E}(\overline{\mathbb{Q}}) \\ \cup & & \cup \\ E_0(\overline{\mathbb{Q}}(t)) & \underset{\psi}{\longrightarrow} & \tilde{E}^0(\overline{\mathbb{Q}}). \end{array}$$

The specialization homomorphism ψ in the diagram is 1-1 on torsion. We can read off $\tilde{E}^0(\overline{\mathbb{Q}})$ in the table at (16), it is the additive group $\overline{\mathbb{Q}}^+$. Thus $\tilde{E}^0(\overline{\mathbb{Q}})$ and hence $E_0(\overline{\mathbb{Q}}(t))$ is torsion free. If in the diagram above, we pick ν such that we have a fiber of type II at $t = \nu$ and find that $E(\overline{\mathbb{Q}}(t))/E_0(\overline{\mathbb{Q}}(t)) = (0)$. Therefore $E(\overline{\mathbb{Q}}(t))$ is torsion free if it has a bad fiber of type II. **Q.E.D.**

All the curves we study have according to table 18 and the second part of the proof of theorem 3.1 bad fibers of type II; in every case at the roots of $at^2 + bt + c = 0$. So there is no torsion on any of the E_i.

Proposition 4.4 *The section $(X, Y) \in E_5(\overline{\mathbb{Q}}(t))$ we found at (37), is actually a generator of its Mordell-Weil group.*

Proof: This is easily checked with aid of theorem 3.1 from [1], which implies that either the section we have in (37) is the generator, or the generator has an x-component of degree 1, which is impossible. **Q.E.D.**

However, this does not imply that (ξ, η) is a generator of E_5'. In fact (we can denote the Mordell-Weil lattice for E_5 as $P \cdot \mathbb{Z}[\omega]$) it is α times the generator, for $\alpha \in \mathbb{Z}[\sqrt{-3}]$ an element with norm 3.

5 Sections over $\mathbb{Q}(t)$

In this section we will study the other five elliptic 'curves' defining an eigenspace of E. We recall the definition of $E_i/\mathbb{Q}(s)$:

$$E_i : Y^2 = X^3 + s^i(as^2 + bs + c),\ 0 \leq i \leq 5.$$

Since we have already determined the rank over $\overline{\mathbb{Q}}(s)$ in section 3 with aid of the Shioda-Tate formula, we are more interested in rank$E_i(\mathbb{Q}(s))$ now.

We have an elliptic curve E_k over $\mathbb{Q}(t)$ of the form $Y^2 = X^3 + k$, with k a polynomial in t. There are three 'tools' to modify k without essentially changing E, being:

- translation: we can translate $t \mapsto (t - k_1/(n \cdot k_0))$ in order to loose the term with t^{n-1} in $k = k_0 t^n + k_1 t^{n-1} + \cdots + k_n$;
- scaling: we can make k_0 n^{th}-power free via $t \mapsto \alpha t$ for a suitable $\alpha \in \mathbb{Q}^*$;
- replace k by $\beta^6 k$ for any $\beta \in \mathbb{Q}^*$.

We use the notation $k \sim k'$ if k and k' are related via a sequence of these three transformations.

We cite theorem 1.5 from [1]:

Theorem 5.1 *For k of degree 2, the rank h of the group of $\mathbb{Q}(t)$-rational points on $Y^2 = X^3 + k$ over $\mathbb{Q}(t)$ is determined as follows:*

1. *$h = 1 \Leftrightarrow \exists \lambda \in \mathbb{Q}^*$ such that one of the following holds*
   - *(a) $k \sim t^2 + \lambda^3$, and if so the Mordell-Weil group is generated by $(-\lambda, t)$;*
   - *(b) $k \sim -3t^2 + \lambda^3$, and in this case the generator [2] is $((\frac{2t}{\lambda})^2 - \lambda, ((\frac{2t}{\lambda})^3 - 3t)$.*
2. *$h = 0 \Leftrightarrow h \neq 1$.*

Remark 5.2 *The cases (1a) and (1b) are connected via the 3-isogeny (36):*

$$-3t^2 + \lambda^3 \quad \sim \quad -27t^2 + \lambda^3,$$

$$Y^2 = X^3 - 27t^2 + \lambda^3 \quad \mapsto \quad \eta^2 = \xi^3 + t^2 - \lambda'^3.$$

For the proof we refer to Bremner ([1]). It relies on the following fact, which is a special case of a result of Manin, proven by Shioda in [20]:

[2] If P generates the Mordell-Weil group $-P$ does it as well, nonetheless we'll call P *the* generator.

Fact 5.3 *If $E/\overline{\mathbb{Q}}(t) : Y^2 = X^3 + f(t)$, with f a polynomial of degree ≤ 6 which is not a sixth power, then the polynomial solutions with $\deg(x) \leq 2$ span the Mordell-Weil group over $\overline{\mathbb{Q}}(t)$.*

We will now look at the consequences for our a, b and c. In the original definition of $f(t)$ at the start of section 3 the only restrictions we laid upon the coefficients were: $a \neq 0$ and $b^2 - 4ac \neq 0$. After transforming f to Bremner's standard form we have, with a' the square free part of a:

$$E_0 : Y^2 = X^3 + a't^2 + c - \frac{b^2}{4a}.$$

Therefore theorem 5.1 implies that $(a' = 1 \ \vee \ a' = -3\ )$ and $c - \frac{b^2}{4a} \in \mathbb{Q}^3$ gives $\text{rank}E_0(\mathbb{Q}(t)) = 1$. We found these restrictions on a, b, c via direct computation in MapleV as well.

We now continue with $E_1(\mathbb{Q}(t))$, this promises to get a little more exciting, because we saw before that the Mordell-Weil group over $\overline{\mathbb{Q}}(t)$ has rank 4 and therefore proposition 2.5 suggests we might find two independent points in $E_1(\mathbb{Q}(t))$. We start again with a theorem from Bremner ([1], theorem 1.1).

Theorem 5.4 *For k of degree 3, the rank h of the group of $\mathbb{Q}(t)$-rational points on $Y^2 = X^3 + k$ over $\mathbb{Q}(t)$ is determined as follows:*

1. *$h = 2 \Leftrightarrow$ one of the following:*
   (a) *$k \sim -t^3 + 16$, and if so the Mordell-Weil group is generated by $(t, 4)$ and $(\frac{1}{4}t^2, \frac{1}{8}t^3 - 4)$;*
   (b) *$\exists \lambda \in \mathbb{Q}$ such that $k \sim -t^3 + 3\left(\lambda^3 + 1\right)t + \left(\frac{1}{4}\lambda^6 + 5\lambda^3 - 2\right)$, and in this case the generators are $(t + 3, 3t + \lambda + 5)$ and $\frac{1}{\lambda^3}\left(\lambda(t^2 - 2t + 1 - 2\lambda^3), t^3 - 3t^2 + 3(1 - \lambda^2)t - (\frac{1}{2}\lambda^6 - 3\lambda^3 + 1)\right)$.*

2. *$h = 1 \Leftrightarrow \exists \lambda \in \mathbb{Q}$ such that one of the following (for the corresponding generators you'd better read [1]):*
   (a) *$k \sim -t^3 + \lambda^2, 2\lambda \notin \mathbb{Q}^{*3}$*
   (b) *$k \sim -t^3 + 3(2\lambda + 1)t + (\lambda^2 + 10\lambda - 2), 2\lambda \notin \mathbb{Q}^{*3}$*
   (c) *$k \sim \lambda t^3 - \frac{1}{3}\lambda^2(\lambda + 1)t + \frac{\lambda^2}{108}(8\lambda^2 - 20\lambda - 1), \lambda \notin \mathbb{Q}^{*3}$*
   (d) *$k \sim \lambda t^3 - 3\lambda^2(\lambda + 1)t + \frac{\lambda^2}{4}(8\lambda^2 - 20\lambda - 1), \lambda \notin \mathbb{Q}^{*3}$*
   (e) *$k \sim -t^3 + \frac{1}{3}(2\lambda + 1)t - \frac{1}{27}(\lambda^2 + 10\lambda - 2), 2\lambda \notin \mathbb{Q}^{*3}$*
   (f) *$k \sim \lambda t^3 - 432\lambda^2, \lambda \notin \mathbb{Q}^{*3}$*

(g) $k \sim \lambda t^3 + 16\lambda^2, \lambda \notin \mathbb{Q}^{*3}$

(h) $k \sim -t^3 - 27\lambda^2, 2\lambda \notin \mathbb{Q}^{*3}$

3. $h = 0$ *otherwise.*

Remark 5.5 *With the* $3-$*isogeny from lemma 4.1 we easily see (with* $\approx$ *denoting isogeny): 2a* $\approx$ *2h, 2b* $\approx$ *2e, 2c* $\approx$ *2d, 2f* $\approx$ *2g. Furthermore we could substitute* $\lambda = -1$ *in 1b; this yields* $-t^3 - \frac{27}{4}$ *and so 1a and 1b* $|_{\lambda=-1}$ *are isogenous as well.*

To apply this to our situation note that $tf(t) = at^3 + bt^2 + ct$ is equivalent to

$$\frac{a}{a^{*3}}\, t^3 \;+\; \frac{3ac - b^2}{3aa^*}\, t \;+\; \frac{2b^3 - 9abc}{27a^2} \tag{38}$$

in which a^* is such that a/a^{*3} is cubefree. We are especially interested in the cases for which $\mathrm{rank} E_1(\mathbb{Q}(t)) = 2$. Since we then need $a \in \mathbb{Q}^{*3}$ we find with (38):

$$tf(t) \;=\; t^3 \;-\; \frac{b^2 - 3ac}{3a^{4/3}}\, t \;+\; \frac{2b^3 - 9abc}{27a^2}.$$

Using the $3-$isogeny (36) we find:

$$E_1/\mathbb{Q}(t) \;:\; Y^2 \;=\; X^3 - \;t^3 \;+\; 3(b^2 - 3ac)\;t\; - (2b^3 - 9abc) \quad \text{has rank 2} \;\Leftrightarrow$$
$$\exists \lambda \in \mathbb{Q}\;\; E_1 \;\;\approx\;\; Y^2 \;=\; X^3 - t^3 + 3\left(\lambda^3 + 1\right)t + \left(\frac{1}{4}\lambda^6 + 5\lambda^3 - 2\right).$$

In particular $\mathrm{rank} E_1(\mathbb{Q}(t)) = 2$ in case $\exists\, \lambda \in \mathbb{Q}$ s.t. the following conditions hold:

$$a \in \mathbb{Q}^{*3}; \;\; (b^2 - 3ac) = \left(\lambda^3 + 1\right); \quad -\; (2b^3 - 9abc) = \left(\frac{1}{4}\lambda^6 + 5\lambda^3 - 2\right).$$

This is equivalent to

$$a \in \mathbb{Q}^{*3}\;,\;\; b = \tfrac{1}{2}\alpha_1 + 2\frac{\lambda^3+1}{\alpha_1}\;,\;\; c = \tfrac{1}{36}\frac{-8+8b^3+20\lambda^3+\lambda^6}{ab}, \;\; \text{where } \; \lambda \in \mathbb{Q}$$
$$\text{s.t. } \;\alpha_1 := \left(-8 + 20\lambda^3 + \lambda^6 + \sqrt{\lambda^3(\lambda - 8)^3}\right)^{1/3} \in \mathbb{Q}$$

For example, $\lambda = -2$ and $a \neq 0$ a cube, comply with these conditions, and then $b = -1$ and $c = 8/(3a)$. When we look at the rank 1 cases we do not find much nicer conditions. If we want to force $a, b, c \in \mathbb{Q}$ in the form of 2b we find, with $2\lambda \notin \mathbb{Q}^{*3}$ such that $\alpha_2 := (-8 + 4\lambda^2 + 40\lambda + 4\sqrt{\lambda(\lambda - 4)^3})^{1/3}$ is rational:

$$a \in \mathbb{Q}^{*3}, \quad b = \frac{1}{2}\alpha_2 - 2\frac{-2\lambda - 1}{\alpha_2}, \quad c = -\frac{1}{3}\frac{-b^2 + 2\lambda + 1}{a}.$$

In the cases 2d, 2a and 2g we have, with λ satisfying the corresponding conditions in theorem 5.4 and for which a, b, c are rational,

$$a \in \lambda \mathbb{Q}^{*3}, \quad b = \frac{1}{2}\alpha_3 - 2\frac{\lambda^3+\lambda^2}{\alpha_3}, \quad c = \frac{1}{3}\frac{b}{a}$$
$$a \in \mathbb{Q}^{*3}, \quad b = \lambda^{2/3}, \quad c = \frac{1}{3}\frac{\lambda^{4/3}}{a}$$
$$a \in \lambda \mathbb{Q}^{*3}, \quad b = 2(2\lambda^2)^{1/3}, \quad c = \frac{4}{3}\frac{(2\lambda^2)^{2/3}}{a}$$

respectively.
Here $\alpha_3 := \left(8\lambda^4 - 20\lambda^3 - \lambda^2 + \sqrt{64\lambda^9 + 256\lambda^8 - 128\lambda^7 + 448\lambda^6 + 40\lambda^5 + \lambda^4}\right)^{1/3}$.

We will now continue with the cases where $4 \leq \deg_t k \leq 6$, which Bremner doesn't treat. From Schwartz [18] we have the following helpful lemma:

Lemma 5.6 *The sixth degree polynomial*

$$\alpha t^6 + \beta t^5 + \gamma t^4 + \delta t^3 + \epsilon t^2 + \zeta t + \eta, \quad \alpha \neq 0$$

is a perfect square (in $\mathbb{C}(t)$) if and only if these three polynomial equations in the coefficients are satisfied:

$$64\alpha^3\epsilon = 4\beta\left(8\alpha^2\delta - \beta\left(4\alpha\gamma - \beta^2\right)\right) + \left(4\alpha\gamma - \beta^2\right)^2 \tag{39}$$
$$64\alpha^4\zeta = \left(8\alpha^2\delta - \beta\left(4\alpha\gamma - \beta^2\right)\right)\left(4\alpha\gamma - \beta^2\right) \tag{40}$$
$$256\alpha^5\eta = \left(8\alpha^2\delta - \beta\left(4\alpha\gamma - \beta^2\right)\right)^2 \tag{41}$$

We will be looking for polynomials $x(t)$ of degree 2 such that

$$x(t)^3 + t^i\left(at^2 + bt + c\right), 2 \leq i \leq 4$$

is a perfect square in $\mathbb{Q}[t]$. This comes down to checking the conditions in the lemma above. Using MapleV again we can quickly express the coefficients of $x(t)$ in a, b, c. From these expressions we can deduce restrictions on our a, b, c for which the actual solutions are in $\mathbb{Q}(t)$. The condition that $x(t) \in \mathbb{Q}[t]$ may look like a restriction. However a conjecture of Schwartz in [18] claims it is not.

Lemma 5.7 *Let $E_2/\mathbb{Q}(t)$ be given by $Y^2 = X^3 + at^4 + bt^3 + ct^2$. Then we know from proposition 2.5 and table 18 that the Mordell-Weil rank is at most 2. We give a sufficient condition for obtaining rank one over $\mathbb{Q}(t)$:*

$$b - 2\sqrt{ac} \in \mathbb{Q}^{*3} \quad \wedge \quad a, c \in -3\mathbb{Q}^{*2}.$$

In other words, for any triple $(a, b, c) \in \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*$ the elliptic curve given by $y^2 = x^3 - 3a^2t^4 + (b^3 - 6ac)t^3 - 3c^2t^2$ has Mordell-Weil rank 1. A supplementary, sufficient condition for obtaining a $\mathbb{Q}(t)-$rank of 2 is $12ac - b^3 \in \mathbb{Q}^{*3}$ when E_2 is given by $y^2 = x^3 - 3a^2t^4 + (b^3 - 6ac)t^3 - 3c^2t^2$.

Proof: In this proof we assume that E_2 is given by $Y^2 = X^3 - 3a^2t^4 + (b^3 - 6ac)t^3 - 3c^2t^2$. Then is the case the rank is 1 we have a generator P in $\mathbb{Q}(t)$ with $x-$coordinate

$$x[P] = 4\frac{a^2}{b^2}t^2 + \frac{8ac - b^3}{b^2}t + 4\frac{c^2}{b^2}.$$

If the second condition is satisfied as well (e.g. $\text{rank}(E_2(\mathbb{Q}(t)) = 2)$ we have another point, Q

$$x[Q] = \frac{(2a)^2}{((12ac - b^3)\alpha^2)^2}t^2 + \frac{4ac - b^3}{(12ac - b^3)\alpha}t + (2\alpha)^2,$$

with $\alpha := (12ac - b^3)^{-1/3}$. It remains to prove the independence of P and Q; the elementary things like checking whether the polynomial f still has 2 distinct, non-zero roots we leave as an exercise. Computing the determinant of

$$\begin{pmatrix} < P, P > & < P, Q > \\ < P, Q > & < Q, Q > \end{pmatrix}$$

comes down to determining the intersection of the sections P and Q at the bad fibers of E_2 since we have the following formulae ([22]), theorem 8.6:

$$< P, P > \;=\; 2 - 2(PO) - \sum_{\nu} \text{contr}_{\nu}(P) \qquad (42)$$

$$< P, Q > \;=\; 1 + (PO) + (QO) - (PQ) - \sum_{\nu} \text{contr}_{\nu}(P, Q). \qquad (43)$$

Here (PO) is the intersection number of the sections (P) and (O), and $\text{contr}_{\nu}(P, Q)$ is the local contribution at ν. In our case, when the $x-$coordinates are given as above $(PO) = (QO) = 0$. This is easily seen using the transformations at (17) and looking at $\tau = 0$. Furthermore the local contributions at the roots of f are zero, since the bad fibers there are of type II. Since the leading coefficients and constant terms of both $x[P]$ and $x[Q]$ are all non-zero we have $\text{contr}_0(P) = \text{contr}_{\infty}(P) = \text{contr}_0(Q) = \text{contr}_{\infty}(Q) = 0$. Similarly one finds $\text{contr}_0(P, Q) = \text{contr}_{\infty}(P, Q) = 0$. It remains to compute the determinant of

$$\begin{pmatrix} 2 & 1 - (PQ) \\ 1 - (PQ) & 2 \end{pmatrix}$$

which is non-zero, because $0 \leq (PQ) \leq 2$. Hence P and Q are independent. **Q.E.D.**

We now explain a trick which allows us to avoid similar calculations in the other cases (E_3&E_4). The substitutions $\xi = x/t^2, \eta = y/t^3, s = 1/t$ define an isomorphism between the curves given by $y^2 = x^3 + t^n(at^2 + bt + c)$ and $\eta^2 = \xi^3 + s^{4-n}(cs^2 + bs + a)$. In this way the cases E_3 and E_4 reduce to the work we already did for E_1 and E_0.

5.1 Rank at least 4 over $\mathbb{Q}(t)$

In this subsection we will summarize the calculations we did in the last one and a half section in a table. With the aid of this table we hope to be able to see what rank over $\mathbb{Q}(t)$ we can achieve. We also give the results one finds for E_3 and E_4 by following the indicated method. As explained above the conditions on a, b, c presented here are sufficient, but not in all cases necessary. If all conditions in one row are meeted, then $\mathbb{Q}[t] \ni f(t) \sim t^i(a\,t^2 + b\,t + c) \Rightarrow E : Y^2 = X^3 + f$ has the indicated rank.

	rank$E_i(\mathbb{Q}(t))$	a	b	c
E_0	1	k_1^2		$\left(\frac{b}{2k_1}\right)^2 - k_2^3$
		$-3k_1^2$		$-\frac{1}{3}\left(\frac{b}{2k_1}\right)^2 - k_2^3$
E_1	2	k^3	$\frac{1}{2}\alpha_1 + 2\frac{\lambda^3+1}{\alpha_1}$	$\frac{1}{36}\frac{-8+8b^3+20\lambda^3+\lambda^6}{ab}$
	1	λk^3	$2(2\lambda^2)^{1/3}$	$\frac{4}{3}\frac{(2\lambda^2)^{2/3}}{a}$
	1	k^3	$\lambda^{2/3}$	$\frac{1}{3}\frac{\lambda^{4/3}}{a}$
	1	k^3	$\frac{1}{2}\alpha_2 - 2\frac{-2\lambda-1}{\alpha_2}$	$-\frac{1}{3}\frac{-b^2+2\lambda+1}{a}$
	1	λk^3	$\frac{1}{2}\alpha_3 - 2\frac{\lambda^3+\lambda^2}{\alpha_3}$	$\frac{1}{3}\frac{b}{a}$
E_2	1	$-3k_1^2$	$b = k^3 - 6k_1k_2$	$-3k_2^2$
	2	as for rank 1 and		$12k_1k_2 - k^3 \in \mathbb{Q}^{*3}$
E_3	2	$\frac{1}{36}\frac{-8+8b^3+20\lambda^3+\lambda^6}{ab}$	$\frac{1}{2}\alpha_1 + 2\frac{\lambda^3+1}{\alpha_1}$	k^3
E_4	1	$\left(\frac{b}{2k_1}\right)^2 - k_2^3$		k_1^2
		$-\frac{1}{3}\left(\frac{b}{2k_1}\right)^2 - k_2^3$		$-3k_1^2$
E_5	1	$-k^5$	$11k^6$	k^7

(44)

In this table, one should choose $\lambda \in \mathbb{Q}^*$ for which the a, b, c and the α_i's given below are rational. Furthermore, in the cases where $a = \lambda k^3$ and $a = k^3$, λ resp. 2λ should not be a cube.

$$
\begin{aligned}
\alpha_1 &= \left(-8 + 20\lambda^3 + \lambda^6 + \sqrt{(\lambda^3 - 8)^3\lambda^3}\right)^{1/3} \\
\alpha_2 &= (-8 + 4\lambda^2 + 40\lambda + 4\sqrt{\lambda(\lambda - 4)^3})^{1/3} \\
\alpha_3 &= \left(8\lambda^4 - 20\lambda^3 - \lambda^2 + \sqrt{64\lambda^9 + 256\lambda^8 - 128\lambda^7 + 448\lambda^6 + 40\lambda^5 + \lambda^4}\right)^{1/3}
\end{aligned}
$$

With this table we hoped to construct an elliptic curve over $\mathbb{Q}(t)$ with rank 4. There were two obvious possibilities. We tried to meet the conditions for rank$E_1(\mathbb{Q}(t)) = 2$ and attemped to pick λ such that the polynomial $at^2 + bt + c$ is reciprocal. Since this did not work we took a reciprocal polynomial satisfying rank$E_2(\mathbb{Q}(t)) = 2$ and tried to force it in the form required for rank$E_0(\mathbb{Q}(t)) = 1$. The maximum rank we obtained with these efforts is 3. Take $a = c = -27648$, $b = 55296$ and you will find three rational points on E_0, E_2 and E_4. Nonetheless we managed to produce a rank 4 curve over $\mathbb{Q}$ having the desired form.

Theorem 5.8 *Let $E/\mathbb{Q}(t)$ be defined by $Y^2 = X^3 + t^{12} - 26\,t^6 - 343$, then rank $E(\mathbb{Q}(t)) \geq 4$.*

Remark 5.9 *We found this curve with a construction method similar to the one described in section 6.1. We take*

$$F(X,Y) = Y^3 + (X-Y)(X-2Y)(X - Y(1-t^3)/2).$$

Now the curve defined by $F(X,Y) = 1$ contains five rational points (given by $Y = 1, 0$ or $X = 0$).

Proof: The polynomial $t^{12} - 26\,t^6 - 343$ does actually comply with enough conditions from table 44 to find rank 4 over $\mathbb{Q}$. In E_0 one takes $k_1 = 1$ and $k_2 = 8$. In E_1 we find two independent solutions since substituting $k = 1$ and $\lambda = 4/11$ gives $a, b, c \in \mathbb{Q}$ and $a\,t^3 + b\,t^2 + c\,t \sim t^3 - 26\,t^2 - 343t$. In fact $a\,t^3 + \alpha^2\,b\,t^2 + \alpha^4 c\,t = t^3 - 26\,t^2 - 343t$ if $\alpha = 11/3$. In E_3 we find one solution (in the table this is the case where α_2 appears): take $k = 1$ and $\lambda = 32/1331$. Then the following points are, defined over $E(\mathbb{Q}(t))$, independent:

$$\begin{aligned}
(8, t-13) &\in E_0(\mathbb{Q}(t)) \\
(-t+49, -11\,t+343) &\in E_1(\mathbb{Q}(t)) \\
\left(\frac{9}{16}\,t^2 + \frac{43}{8}\,t + \frac{49}{16}, \frac{1}{64}(27\,t^3 + 387\,t^2 + 1145\,t - 343)\right) &\in E_1(\mathbb{Q}(t)) \\
(t^2 + 7\,t, t^3 + 11\,t^2) &\in E_3(\mathbb{Q}(t))
\end{aligned}$$

Q.E.D.

Defined as points on $E/\mathbb{Q}(t)$ they transform to

$(8, t^6 + 13)$	$(\frac{-t^6+49}{t^2}, \frac{-11\,t^6+343}{t^3})$
$(t^6 + 7, t^9 + 11\,t^3)$	$\frac{1}{64}(36\,t^{10} + 344\,t^4 + \frac{196}{t^2}, 27\,t^{15} + 387\,t^9 + 1145\,t^3 - \frac{343}{t^3})$.

With aid of Apecs it is easily checked that (for instance specializing $t = \frac{1}{2}$) these are independent.

6 Higher ranks

The method we exploited during this exercise is based on the fact that we have automorphisms on $Y^2 = X^3 + at^{12} + bt^6 + c$. In this way we decomposed our original 'solution-space' into six 'easier' to handle subspaces. For five out of six we found, using the Shioda-Tate formula (21), the rank over $\overline{\mathbb{Q}}(t)$ already sums up to 16. In the remaining situation we found a solution as well. This allowed us to conclude that there are choices for a, b, c possible such that $\text{rank}E(\overline{\mathbb{Q}}(t)) = 18$. We will now consider the question whether we can do the same trick for $f(t^6)$ with $\deg_{t^6} f \geq 3$ and f squarefree.

Take $f(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$, for example. Our method yields for the Mordell-Weil rank over $\overline{\mathbb{Q}}(t)$ of

$$E/\overline{\mathbb{Q}}(t): \; Y^2 = X^3 + f(t^6)$$

a lower bound 20. Namely, using the same eigenspace decomposition as before we use the Shioda-Tate formula for the four rational elliptic surfaces corresponding to eigenspaces. In all four cases we have three or four bad fibers of type II, combined with one fiber of type IV in two cases and one of type I_0^* in the other two cases. Using table (16), we see that these four already deliver us rank 20. The two remaining eigenspaces correspond to $E_i/\overline{\mathbb{Q}}(t) \; : \; Y^2 = X^3 + t^i f(t)$ with $i = 4, 5$. These two are related using the same trick as in the previous section on page 23, moreover, if $f(t) = \alpha t^3 + \beta t^2 + \beta t + \alpha$, then $E_4 \cong E_5$. Hence in this case one non-trivial point in $E_4(\overline{\mathbb{Q}}(t))$ implies that $\text{rank}E(\overline{\mathbb{Q}}(t))$ is at least 24. We will show that this actually occurs:

Theorem 6.1 *Let $E/\overline{\mathbb{Q}}(t)$ be given by $y^2 = x^3 + \alpha t^{18} + \beta t^{12} + \beta t^6 + \alpha$. There are choices $\alpha, \beta \in \overline{\mathbb{Q}}$ for which $\text{rank}E(\overline{\mathbb{Q}}(t)) \geq 24$.*

Proof: Take α and β in the reciprocal polynomial f, as follows

$\alpha := -\frac{2374607306925}{312257184832}\chi - \frac{41850914247}{312257184832} + \frac{248479785}{19516074052}\chi^4 - \frac{3720916521}{19516074052}\chi^3 + \frac{138267287253}{78064296208}\chi^2$

$\beta := -\frac{4820672097}{1080474688} - \frac{68601659589}{270118672}\chi + \frac{2300589}{33764834}\chi^4 - \frac{256283919}{67529668}\chi^3 + \frac{14712536385}{270118672}\chi^2$, where χ is a root of $16X^5 - 272X^4 + 6192X^3 - 105760X^2 + 558468X + 9829$. These values were found using a computation similar to the one presented in section 4. In the present case we wanted a degree 4 polynomial x such that $x^3 + t^4(\alpha t^3 + \beta t^2 + \beta t + \alpha)$ is a square. **Q.E.D.**

We tried to use polynomials f of higher degree in a similar way, but this does not seem to lead to even higher ranks in an obvious way. However, it turns out that the current record for the rank of an elliptic curve over $\overline{\mathbb{Q}}(t)$ is obtained for an elliptic curve of the form considered here

Proposition 6.2 (Shioda) *Take $f(t) = t^{360} + 1$, then $E : y^2 = x^3 + f(t)$ has $rank E(\overline{\mathbb{Q}}(t)) = 68$.*

We remark that it is possible to decompose the corresponding vectorspace twice, with respect to Φ. In this way one sees that one of the eigenspace is associated with the equation $y^2 = x^3 + t^{10} + 1$.

Proof: This is stated as a remark in [23]. It can be proven using the method explained in [21]. **Q.E.D.**

6.1 An example

Proposition 6.3 *Let $E/\mathbb{Q}(t)$ be given by*

$$Y^2 = X^3 + t^{18} + 2973\,t^{12} + 369249\,t^6 + 11764900.$$

The $\mathbb{Q}(t)-$rank of E is at least 5.

Proof: Cam Stewart and Jaap Top give this curve (in [27]) as an example of the construction of high rank sextic twists of $y^2 = x^3 + k$ via a construction due to Mestre ([12]). Mestre actually found $\mathbb{Q}(t)-$ rank ≥ 7 employing this method ([13]), however his examples are not of the form $y^2 = x^3 + f(t^6)$. We will now summarize their procedure:

Start with $x_1 = 1, x_2 = 2, x_3 = -3, x_4 = 0, x_5 = t$ and $x_6 = -t$. Remark that $(x_1 + x_2 + x_3 + x_4 + x_5) = -x_6$. Now define:

$$p(X) := (X - x_1)(X - x_2)\cdots(X - x_6) = X^6 - (t^2 + 7)X^4 + 6X^3 + 7t^2X^2 - 6t^2X,$$

$$g(X) := X^2 - \frac{t^2 + 7}{3},$$

$$r(X) := 6X^3 + (7t^2 - \frac{(t^2 + 7)^2}{3})X^2 + 6t^2X - \left(\frac{t^2 + 7}{3}\right)^3$$

and

$$F(X, Y) := 6X^3 + (7t^2 - \frac{(t^2 + 7)^2}{3})X^2Y + 6t^2XY^2 - \left(\frac{t^2 + 7}{3}\right)^3 Y^3.$$

On the curve $E_F : F(X, Y) = -1$ one now clearly has the points $P_i := \left(\frac{x_i}{g(x_i)}, \frac{1}{g(x_i)}\right)$, $1 \leq i \leq 6$. The following fact comes from Mordell [14]:

$$(4G)^2 = (4H)^3 - 27 \cdot 16 \cdot D_0 \cdot F^2, \qquad (45)$$

where $H(X,Y)$ and $G(X,Y)$ are the quadratic and cubic covariant of $F(X,Y)$ respectively and D_0 is the discriminant of F:

$$H = \frac{1}{4}\begin{vmatrix} \left(\frac{\partial}{\partial x}\right)^2 F & \frac{\partial^2}{\partial x \partial y} F \\ \frac{\partial^2}{\partial x \partial y} F & \left(\frac{\partial}{\partial y}\right)^2 F \end{vmatrix} \quad , \quad G = \begin{vmatrix} \frac{\partial}{\partial x} F & \frac{\partial}{\partial y} F \\ \frac{\partial}{\partial x} H & \frac{\partial}{\partial y} H \end{vmatrix}$$

$$\text{and } \; D_0 = \frac{4}{729}\,(-369249\,t^6 + 11764900 + 2973\,t^{12} + t^{18}).$$

Therefore one has a morphism from E_F to $E_D : y^2 = x^3 - 27 \cdot 16 \cdot D_0$, given by $E_F \ni (X,Y) \mapsto (4\,H(X,Y), 4\,G(X,Y)) \in E_D$. One computes that the images of the P_i generate a group of rank 5. With lemma 4.1 we can simplify E_D to the form above:

$$E_D/\mathbb{Q}(t) \quad : \quad y^2 = x^3 - \frac{64}{27}\,(t^{18} + 2973\,t^{12} - 369249\,t^6 + 11764900) \;\approx \tag{46}$$

$$Y^2 = X^3 + t^{18} + 2973t^{12} + 369249t^6 + 11764900. \tag{47}$$

Q.E.D.

Using proposition 2.6, there exist five independent points in $\oplus_i E_i(\mathbb{Q}(t))$. We will now describe how to find them. The following points are the (independent) images of the $P_1, \cdots, P_5$ above, they are points $Q_1, \cdots, Q_5$ on (46).

$$\{\frac{4}{3}\,\frac{t^{10} + 17\,t^8 + 7\,t^6 + 244\,t^4 + 368\,t^2 + 5488}{(4+t^2)^2}, \tag{48}$$

$$8\,\frac{t^{14} + 12\,t^{12} - 33\,t^{10} - 400\,t^8 - 3918\,t^6 - 6792\,t^4 - 10976\,t^2 + 65856}{(4+t^2)^3}\}$$

$$\{\frac{4}{3}\,\frac{t^{10} + 26\,t^8 - 119\,t^6 + 55\,t^4 - 2350\,t^2 + 8575}{(-5+t^2)^2}, \tag{49}$$

$$8\,\frac{2\,t^{14} + 12\,t^{12} - 48\,t^{10} - 800\,t^8 + 1815\,t^6 + 825\,t^4 + 42875\,t^2 - 128625}{(-5+t^2)^3}\}$$

$$\{\frac{4}{3}\,\frac{t^{10} + 41\,t^8 - 329\,t^6 + 9820\,t^4 + 28400\,t^2 + 137200}{(-20+t^2)^2}, \tag{50}$$

$$-24\,(t^{14} - 4\,t^{12} + 351\,t^{10} - 400\,t^8 + 44230\,t^6 + 300600\,t^4 + 1372000\,t^2$$
$$+2744000)/(-20+t^2)^3\}$$

$$\{\frac{4}{3}\,\frac{t^{10} + 14\,t^8 + 49\,t^6 + 1315\,t^4 + 4802\,t^2 + 16807}{(t^2+7)^2}, \tag{51}$$

$$24\,\frac{2\,t^{12} - 588\,t^8 - 3887\,t^6 - 21609\,t^4 - 50421\,t^2 - 117649}{(t^2+7)^3}\}$$

$$\{\frac{4}{3}\frac{4\,t^{10}-28\,t^8+490\,t^6-1512\,t^5+2173\,t^4-5292\,t^3+12005\,t^2-18522\,t+16807}{(2\,t^2-7)^2},$$
$$-8(28t^{13}-48\,t^{12}-294\,t^{11}+3087\,t^9-7056\,t^8+17612\,t^7-54888\,t^6+145677\,t^5$$
$$-259308\,t^4+333396\,t^3-605052\,t^2+823543\,t-352947)/(2\,t^2-7)^3\}$$

If we transform these sections to solutions of (47) they get very messy, so we will leave them in this form. In the present notation the E_i are given by $Y^2 = X^3 - \frac{64}{27}\,t^i\,(t^3+2973\,t^2-369249\,t+11764900)$. Naively starting to find points in the $E_i(\mathbb{Q}(t))$ using MapleV will quickly give $(x(t),y(t)) = \left(-t^6-19,\,54\,t^6-3429\right)$ as a solution to (47). Points in the other eigenspaces (different from E_0) are not as easily found. Therefore we will try to determine the solutions in the eigenspaces via 'projections' of the five points given above. Recall that

$$\Phi: \quad (x(t),y(t)) \mapsto (x(-\omega t),y(-\omega t))$$

has the properties $\Phi|_{E(\overline{\mathbb{Q}}(t^6))}$ is the identity and $\Phi^6 - Id$ is the zero map.

Lemma 6.4 *The following endomorphisms of $E(\overline{\mathbb{Q}}(t))$ give, after tensoring with $\mathbb{Q}$, the exact sequences:*

$$\begin{array}{lll}
0 \to \oplus_{i=1}^{5}\,V_{(-\omega)^i} & \longrightarrow V \xrightarrow{\Phi^5+\Phi^4+\Phi^3+\Phi^2+\Phi+Id} V_1 \to 0 & \\
0 \to \oplus_{i\neq 3}\,V_{(-\omega)^i} & \longrightarrow V \xrightarrow{\Phi^5-\Phi^4+\Phi^3-\Phi^2+\Phi-Id} V_{-1} \to 0 & \\
0 \to \oplus_{i\neq 2,4} V_{(-\omega)^i} & \longrightarrow V \xrightarrow{\Phi^4-\Phi^3+\Phi-Id} V_{\omega^2} \oplus V_{\omega^4} \to 0 & (52)\\
0 \to \oplus_{i\neq 1,5} V_{(-\omega)^i} & \longrightarrow V \xrightarrow{\Phi^4+\Phi^3-\Phi-Id} V_{-\omega} \oplus V_{(-\omega)^5} \to 0\,. &
\end{array}$$

Proof: The endomorphism $\Phi^5 + \cdots + Id$ maps an element of V onto six times its V_1−component, since the other eigenspaces have eigenvalues which are zeroes of $X^5+X^4+X^3+X^2+X+1$. The other cases are similar. **Q.E.D.**

The question now arises, inspired by proposition 2.6, whether we will find exactly five points in the $E_i(\mathbb{Q}(t))$ using the 'projections' above. At first we already see that when we have a rational point on E, mapping them to E_0 or E_3 gives 0 or a rational point as well. This is the case, because, when σ is complex conjugation, we have the following identities on $E(\mathbb{Q}(t))$: $\Phi^n = \sigma\,\Phi^{6-n}$. This makes the first two endomorphisms invariant under σ. We will now use the solutions from Jaap Top and Cam Stewart to find solutions lying in the eigenspaces. Since Q_1 given by (48) is an even function, $Id+\Phi+\Phi^2$ already maps it on the eigenspace isomorphic to $E_0(\mathbb{Q}(t))\otimes\mathbb{Q}$. We find:

$$(\Phi^2+\Phi+Id)(Q_1) = \left\{\frac{1}{3}\frac{4t^{18}+15780t^{12}-1970772t^6+62736988}{(t^6+19)^2},\right. \qquad (53)$$

$$\left.\frac{-144t^{24}-372312t^{18}+71469000t^{12}-4506172488t^{6}+95632075944}{(t^{6}+19)^{3}}\right\}.$$

The points Q_2, Q_3 and Q_4 map similarly to this section and $\Phi^5+\Phi^4+\Phi^3+\Phi^2+\Phi+Id$ maps Q_5 on two times this point. Furthermore we find

$$(\Phi - Id)(Q_1) = \left\{\frac{4}{27}\frac{(t^{24}+2353t^{18}-65280t^{12}-11856128t^{6}+481890304)\,\zeta_6}{(t^{6}-176)^{2}t^{4}},\right.$$
$$\frac{2\,\zeta_6-1}{243}(8t^{36}+31152t^{30}+10592976t^{24}-1843837184t^{18}+98403373056t^{12}$$
$$\left.-3123188662272t^{6}+84627647627264/((t^{6}-176)^{3}t^{6})\right\}\in V_{\omega^2};$$

$$(\Phi - Id)(Q_2) = \left\{\frac{1}{27}\frac{(t^{24}-5396t^{18}+1119750t^{12}-78147500t^{6}+1838265625)\,\zeta_6}{(t^{6}-50)^{2}t^{4}},\right.$$
$$\frac{2\,\zeta_6-1}{243}(t^{36}+15234t^{30}-8326785t^{24}+1504055000t^{18}-125482603125t^{12}$$
$$\left.+5025861093750t^{6}-78815638671875)/((t^{6}-50)^{3}t^{6})\right\}\in V_{\omega^2}\ \text{and}$$

$$(\Phi - Id)(Q_4) = \left\{\frac{1-\zeta_6}{107163}\frac{t^{24}+3316t^{18}+983886t^{12}-296004884t^{6}+13841287201}{t^{8}},\right.$$
$$\frac{2\,\zeta_6-1}{60761421}(t^{36}+4974t^{30}-5599275t^{24}+275974000t^{18}-452910414075t^{12}$$
$$\left.-52237017896574t^{6}-1628413597910449/(t^{12})\right\}\in V_{\omega^4}.$$

These points are all lying on $y^2 = x^3 - 16\cdot 27\cdot D_0$ but are not rational. Since $\zeta_6^3 = -1$ (as is $(1-\zeta_6)^3 = \zeta^{15} = -1$) and $2\zeta_6 - 1 = \sqrt{-3}$, a point on E_D given by $(\zeta_6 f(t), (2\zeta_6-1)g(t))$ provides a point $(-f(t), g(t))$ on the quadratic twist $E_D^{(-3)}$, which is 3-isogenous to E_D (36). In this way we do get rational points. We find that $(Id-\Phi^3)(Q_5)$ is a rational point in $V_{-\omega}$:

$$\left\{\frac{256\,t^{24}+585472\,t^{18}-71939136\,t^{12}+2043327832\,t^{6}+13841287201}{441\,t^{2}(64\,t^{12}-5488\,t^{6}+117649)},\right.$$
$$\frac{1}{9261}(4096\,t^{36}+14051328\,t^{30}-198211584\,t^{24}-337672600000\,t^{18}$$
$$31193763629784\,t^{12}-835294000005948\,t^{6}+1628413597910449)/$$
$$\left.t^{3}\,(512\,t^{18}-65856\,t^{12}+2823576\,t^{6}-40353607)\right\}.$$

Summarizing this, we have bounds for the ranks of the E_i: $1\le \mathrm{rank}E_0(\mathbb{Q}(s))\le 2$, $1\le \mathrm{rank}E_1(\mathbb{Q}(s))\le 3$, $2\le \mathrm{rank}E_2(\mathbb{Q}(s))\le 3$ and $1\le \mathrm{rank}E_4(\mathbb{Q}(s))\le 2$. The lower bounds are obvious from the sections given above, the upper bounds are derived from the Shioda-Tate formula. Furthermore we have, since $\Phi^2-\Phi+Id=0$,

$$<\{Q_i\},\{\Phi Q_i\}> = <\Phi>\cdot <Q_1,Q_2,Q_3,Q_4,Q_5> \in E(L(t)).$$

Because $< \{Q_i\}, \{\Phi Q_i\} >$ contains at most 10 independent sections over $\mathbb{Q}(\sqrt{-3})$, hence at most 5 over $\mathbb{Q}$.

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