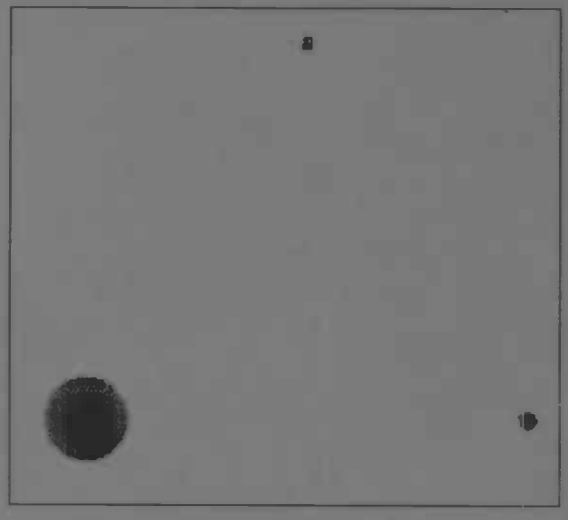
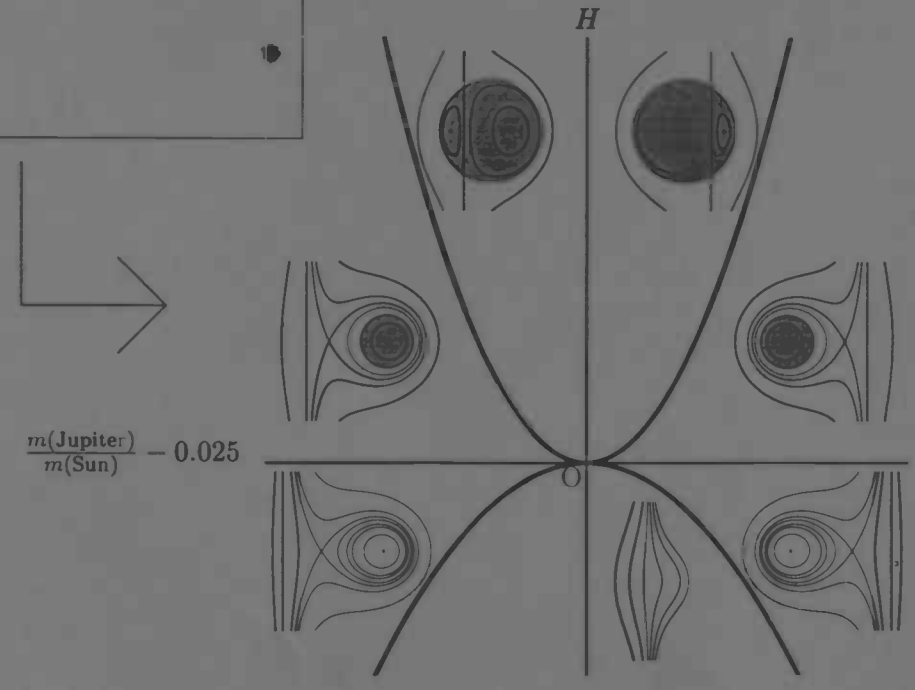
Analysis of Resonances in the Three Body Problem using Planar Reduction

Bart Oldeman





$\frac{m(\text{Jupiter})}{m(\text{Sun})} = 0.025$

Department of
Mathematics

RuG

Master's thesis

Analysis of Resonances in the Three Body Problem using Planar Reduction

Bart Oldeman

University of Groningen
Department of Mathematics
P.O. Box 800
9700 AV Groningen

August 1998

Supervisors:	Prof. dr. H.W. Broer Dr. G. Vegter
in cooperation with:	Drs. G.A. Lunter

Preface

This thesis work started in September 1997, when Gerton Lunter proposed to consider the three body problem next to the spring-pendulum in his own research. I would like to thank him for proposing this interesting subject, help, cooperation, pictures and `Mathematica` files. Thanks also go to my supervisors, Henk Broer and Gert Vegter, for many global comments and teaching how to write a report well.

The rest of the thanks go to Igor Hoveijn and Bernd Krauskopf for a helpful discussion we had in Göttingen, to Rutger Kock and Florian Wagener, without whom I probably would not be in dynamical systems, to the other people in Groningen and Göttingen for providing a nice stay here and there, and to my friends and my family for their support.

Contents

Abstract

The planar (circular restricted) three body problem in resonance is a model system for formal reduction to one degree of freedom. It is investigated in the neighborhood of the Lagrangian equilateral triangular solution. The reduction can be handled by equivariant singularity theory with one or two distinguished parameters, yielding an integrable approximation of the Poincaré map. This makes a concise description possible of certain bifurcations, which can be checked by numerical simulations.

Chapter 1

Introduction

A quote:

> The three body problem, which was described by Whittaker as *"the most celebrated of all dynamical systems"* and which fulfilled for Hilbert the necessary criteria for a good mathematical problem, can be simply stated: three particles move in space under their mutual gravitational attraction; given their initial conditions, determine their subsequent motion. Like many mathematical problems, the simplicity of its statement belies the complexity of its solution. For although the one and two body problems can be solved in closed form by means of elementary functions, the three body problem is a complicated nonlinear problem, and no similar type of solution exists. (Barrow-Green [3]).

The three body problem amounts to a system of 18 first order differential equations. It can also be seen as an autonomous Hamiltonian system with 9 degrees of freedom. To handle such a system it is desirable to find as many as possible independent integrals. Integrals are quantities which are constant along the solutions of the differential equations. An integral enables us to reduce the order or the number of degrees of freedom of the system, by setting it to a constant value. If we were so lucky as to find 9 independent integrals and symmetries, holding the former fixed would define a solution curve, and we have integrated the problem. However, in the three body problem this is not the case, as was shown by Poincaré.

There exist 10 classical integration constants. Using these and a device due to Jacobi the system can be reduced to an autonomous Hamiltonian system with 4 degrees of freedom. This can be done in the following way: we can identify configurations which differ by a translation and/or a rotation about the angular momentum vector. The latter is classically called "elimination of the node", being the device due to Jacobi. Then we can get rid of 4 position variables. Using E. Noether's theorem (symmetries imply integrals) we can fix the 4 corresponding momenta, which are the linear momentum (3) and the length of the angular momentum vector (1). Fixing the other 2 components of the angular momentum reduces to a system with 4 degrees of freedom. This

reduction is now classical, and described in many text books, e.g. Arnold [2], Meyer&Hall[15], Whittaker[23] and Wintner[24]

If all motion is restricted to a plane, we have the *planar three body problem.* In that case, 3 degrees of freedom are left. Xia [25] showed that no additional real analytical integrals exist in this 3 d.o.f. system. If the mass of one particle is so small that it does not influence the motion of the other two, but is affected by the other two in the usual way, the resulting 2 degrees of freedom system is called the *planar restricted three body problem.* If the other two move around their center of mass in circles, we have the *planar circular restricted three body problem.*

There are no equilibrium solutions for the 3-body problem, but there are periodic solutions called *relative equilibria.* There all masses have the same and *constant* angular velocity with respect to the center of mass, and move in periodic orbits. Two types of relative equilibria exist: the *Euler-Moulton collinear solutions*, where the masses are all on a uniformly rotating straight line, and the *Lagrangian equilateral triangle solutions*, where the masses are at the vertices of a uniformly rotating equilateral triangle of fixed size. These relative equilibria exist for all mass ratios, not just in the case where all masses are equal.

In a rotating coordinate system the relative equilibria become *actual* equilibria. Then the above mentioned line and triangle are fixed. We shall consider the *planar* (circular restricted) three body problem in the neighbourhood of a Lagrangian equilibrium. To do this, we change coordinates such that this equilibrium corresponds to the origin. To first order in the smallest mass μ, the full problem decouples to a restricted problem and the linearized equations at the circular solution of the heavy masses, as $\mu \downarrow 0$. In this way we are able to compare and to see whether and in what sense the "full" results are generalisations of the "restricted" ones. Indeed we will consider the restricted problem as a special case of the full problem.

The systems (both full and restricted) can be normalized using Birkhoff's procedure. For almost all mass ratios this *formally* transformed system can be integrated, and the dynamics reduce to quasi-periodic motion on a 3-torus. However, in the case of a resonance, i.e., if the eigenvalues of the linearized system have a *rational* ratio, this cannot be done. The normalized system reduces to two degrees of freedom if there is an overall resonance, i.e., all eigenvalues have a rational ratio. The only integral then present is the quadratic part of the Hamiltonian. The normalized system reduces to one degree of freedom when only two pairs of eigenvalues are in resonance. This gives rise to one or two formal integrals. These integrals become parameters in the reduced system, but by their special nature they are called *distinguished* parameters.

We consider the latter case. Analysis of the 1:1-resonance gives problems, because then the linearized system is not semisimple. This case has been investigated by Schmidt [18].

We restrict to other resonances, giving rise to an additional rotational symmetry in the normalized system. This enables us to reduce to a one degree of freedom system. Its flow gives an integrable approximation of the Poincaré map in the two degree of freedom system, either over the long or short period.

These periods correspond to the smallest and largest frequency of the harmonic truncations of the constituting oscillators, respectively.

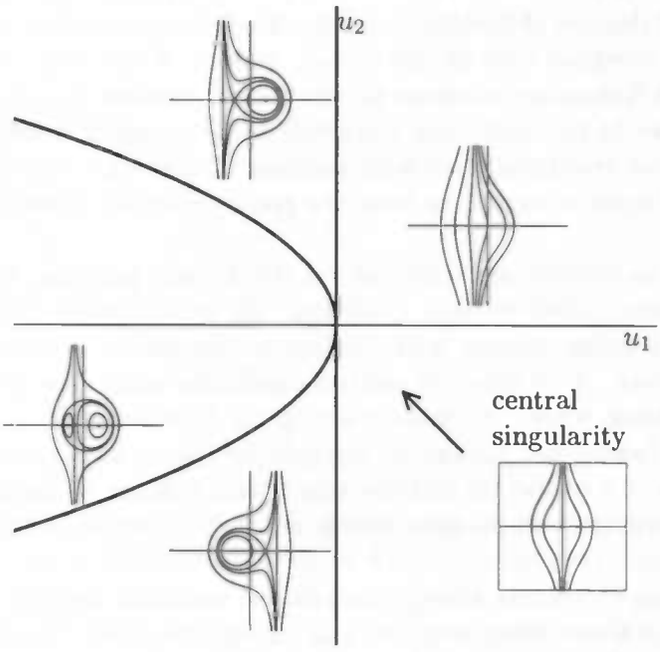


Figure 1.1: Bifurcation diagram of $x(x^2+y^2)+u_1x+u_2y^2$. Across the bifurcation lines saddle-center bifurcations occur. Across the parabola $u_1+3u_2^2=0$ Hamiltonian pitchfork bifurcations occur due to $\mathbb{Z}_2$-symmetry in this versal deformation.

It can be studied by looking at the configuration of the level sets. To do this we perform further coordinate transformations using singularity theory. This is possible in the case of a 1:2 resonance and gives rise to a versal deformation of the hyperbolic umbilic $x(x^2+y^2)$ for the long period. For the short one there appears to be a connection with the family $x^4+2ax^2y^2-y^4$ for $a<0$. This helps to understand certain bifurcations, as depicted in figure 1.1 and 1.2.

	planar, circular RTBP			planar, full TBP		
	d.o.f.	p	symmetry	d.o.f.	p	symmetry
Original	2	1	–	6	2	transl.$\times\mathbb{S}_1$
Identify translations				4	2	$\mathbb{S}_1$
Identify rotations				3	2	–
Birkhoff normal form	2	1	$\mathbb{S}_1$	3	2	$\mathbb{S}_1$
Planar reduction	1	2	–	1	3	–
Symmetric normal form	1	2	$\mathbb{Z}_2$	1	3	$\mathbb{Z}_2$
Versal deformation	1	2	$\mathbb{Z}_2$	1	2	$\mathbb{Z}_2$

Table 1.1: Overview of reductions and normalizations. Everything is viewed locally with respect to the equilateral triangular relative equilibria. The number of parameters, including distinguished ones, is denoted by p.

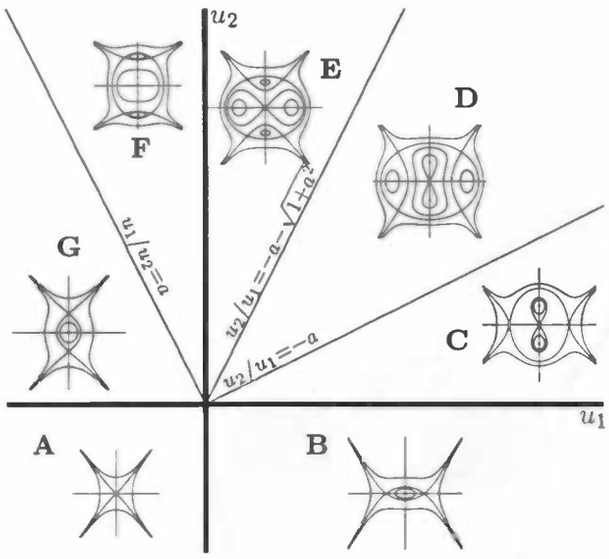


Figure 1.2: Bifurcation diagram of $x^4 + 2ax^2y^2 - y^4 - 2u_1x^2 + 2u_2y^2$ for $a < 0$, see also BLV [8]. The central singularity qualitatively looks like diagram A.

With the help of computer algebra, all transformations can be calculated explicitly, and, up to a small perturbation, we can predict what happens in the original system. We will compare these results with numerical simulations (figures 1.3, 1.4 and 1.6). A partial connection with figures 1.1 and 1.2 is given by figures 1.5 and 1.7: a portion of the phase space is not actually visited by the system.

This paper is organized as follows. In the next chapter some preliminaries are given about Hamiltonian and celestial mechanics. In chapter 3 the restricted and full three body problems are investigated. In chapter 4 the normalization using Birkhoff's procedure, reduction to one degree of freedom, and further normalization using singularity theory are investigated for a general Hamiltonian system in resonance, where the linearized system is semisimple. In chapter 5 this theory is applied to the three body problem near the Lagrangian relative equilibrium. Chapter 6 is devoted to a numerical comparison. Conclusions can be found in the final chapter.

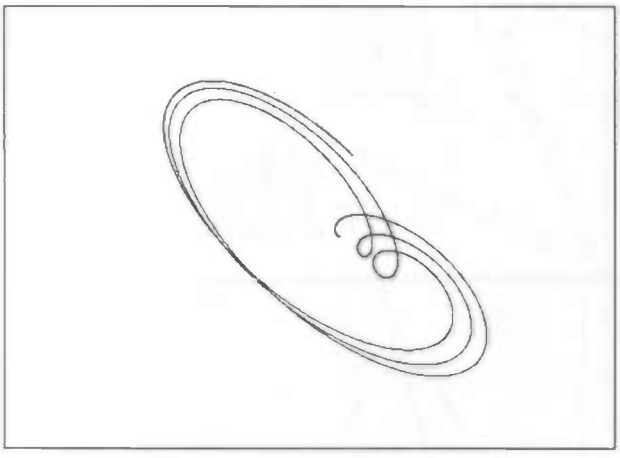


Figure 1.3: A quasi-periodic orbit near a triangular relative equilibrium in configuration space. This corresponds to a circle in figures 1.4 and 1.6.

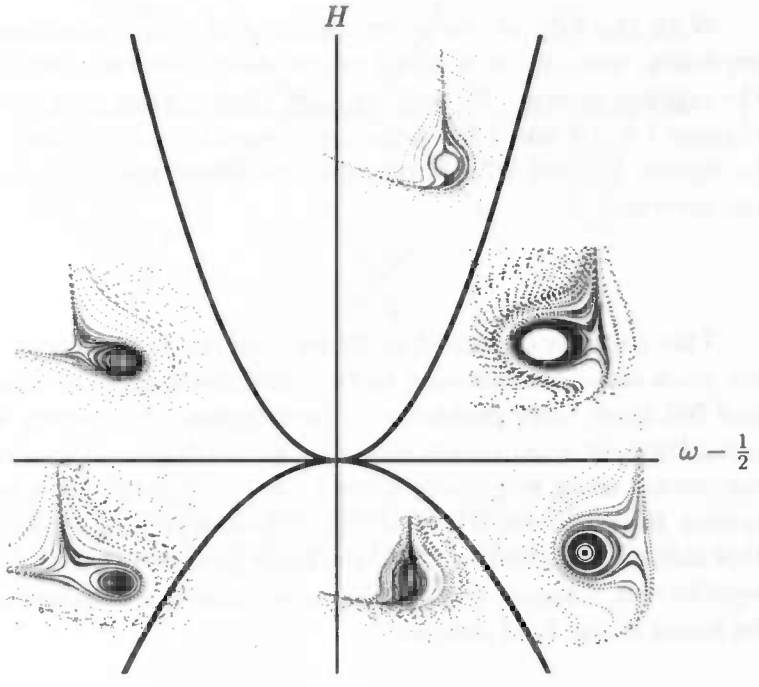


Figure 1.4: Isoenergetic Poincaré maps over the long period with $H = \pm 10^{-7}$ (fixed) and eigenvalue ratios ω of 0.495, 0.5 and 0.505, respectively. The parabolas are approximations here.

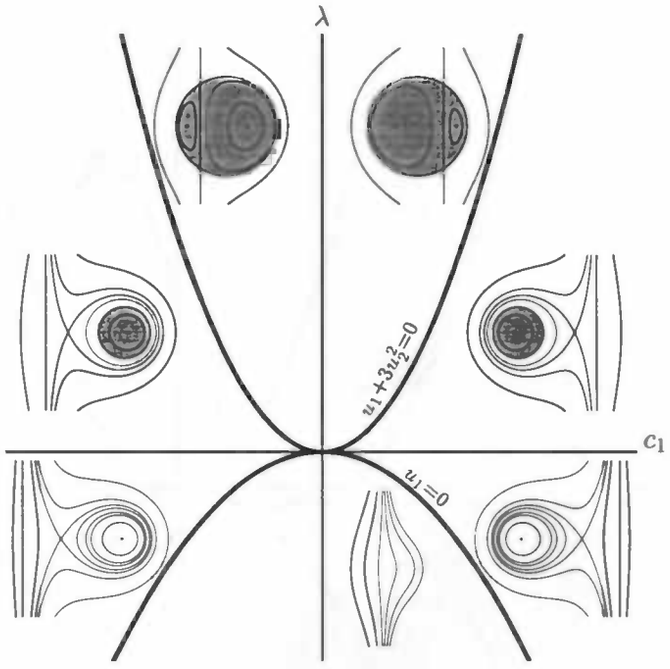


Figure 1.5: Bifurcation diagram: distinguished parameter λ versus detuning c_1 from 1:2-resonance of the planar reduced system H^r (4.12). Grey areas denote portions of phase space that do not correspond to phase points of the original system.

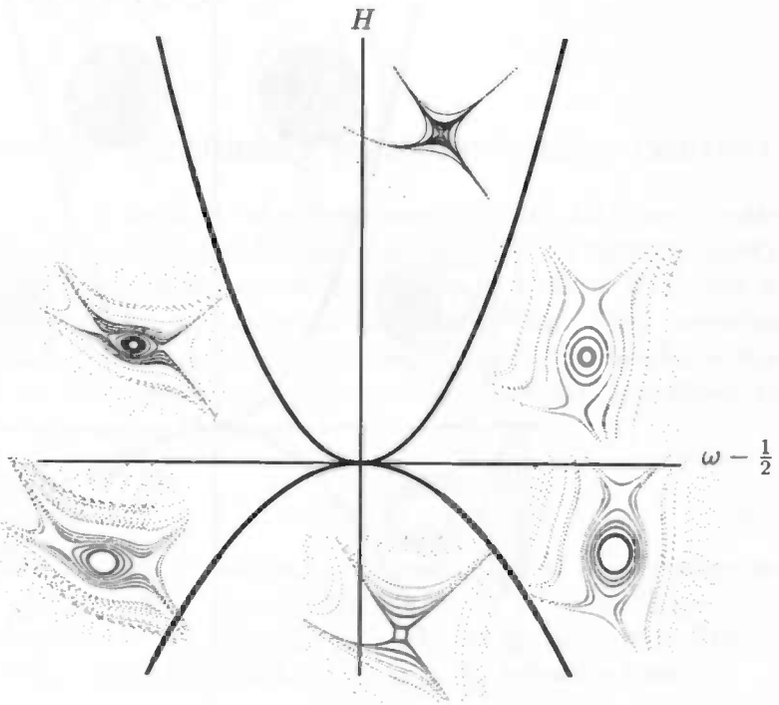


Figure 1.6: Isoenergetic Poincaré maps over the short period with $H = \pm 10^{-7}$ and eigenvalue ratios ω of 0.495, 0.5 and 0.505, respectively, as in figure 1.4.

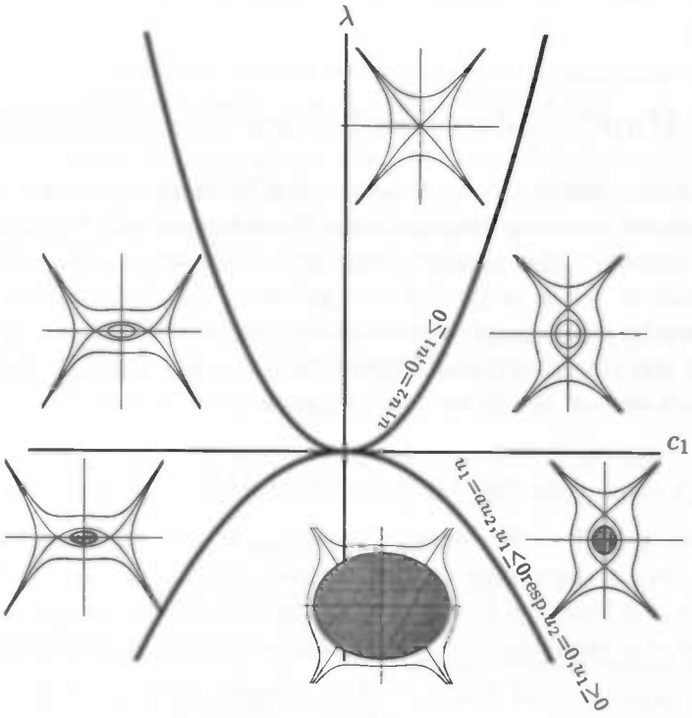


Figure 1.7: Bifurcation diagram as in figure 1.5 for the short period.

Chapter 2

Preliminaries

2.1 Hamiltonian mechanics in a Newtonian context

The n-body problem is an example of a Newtonian system. All dynamics are then defined by a single function, the *Hamiltonian*, which just is the total energy of the system. The corresponding dynamical system is a vector field that is expressed in terms of partial derivatives of the Hamiltonian. This *canonical* form will be maintained when making coordinate changes. The results in this section are all taken from Meyer&Hall[15]; see also Arnold[1] and Broer[4]. Newton's second law in vector form says

$$F(q) = M\ddot{q} \tag{2.1}$$

where $q(t) \in \mathbb{R}^n$ is the vector of positions of the particles, M is a nonsingular $n \times n$ matrix depending on the masses, and $F : \mathbb{R}^n \to \mathbb{R}^n$ is the force field. If there is a function $U : \mathbb{R}^n \to \mathbb{R}$, called *potential energy*, such that $F(q) = -\frac{\partial U}{\partial q}(q) = -\operatorname{grad} U(q)$, then (2.1) can be written in the following form:

$$M\ddot{q} = -\frac{\partial U(q)}{\partial q}, \tag{2.2}$$

or as a first order system,

$$\dot{p} = -\frac{\partial U(q,t)}{\partial q}, \quad p = M\dot{q}, \tag{2.3}$$

where $p(t) \in \mathbb{R}^n$ is the vector of *(conjugate) momenta*. Then $\dot{q} = M^{-1}p = \frac{\partial T}{\partial p}$, where $T = \frac{1}{2}p^T M^{-1} p$ is called the *kinetic energy*. Consider the *Hamiltonian* $H = T + U$, the total energy of the system. It is easily seen that the first order system in (2.3) can be written in the *canonical form*

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{2.4}$$

A first direct consequence of this form is the conservation of energy. Indeed,

$$\dot{H} = \frac{\partial H}{\partial q} \cdot \dot{q} + \frac{\partial H}{\partial p} \cdot \dot{p} + \frac{\partial H}{\partial t} = 0,$$

since H is independent of t. Let $x = (q, p) \in \mathbb{R}^{2n}$, then (2.4) can also be written as

$$\dot{x} = J\frac{\partial H(x,t)}{\partial x}, \tag{2.5}$$

where $J \in gl(2n, \mathbb{R})$ is given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.6}$$

When performing a coordinate change in a Hamiltonian system we want to preserve the special form defined by (2.4). Such a coordinate change is called *symplectic* or *canonical*.

Definition 1. *A time independent change of coordinates is symplectic if and only if it preserves the symplectic form* $\omega = dp \wedge dq = \sum dp_j \wedge dq_j$

In $\mathbb{R}^2$, $dp \wedge dq$ denotes the area form, that is, $(dp \wedge dq)(\theta, \eta)$ gives the area of the parallelogram spanned by the vectors $\theta, \eta \in T_{(q,p)}(\mathbb{R}^2)$. So in $\mathbb{R}^2$, symplectic is the same as area preserving. In higher dimensions symplectic transformations also preserve the (hyper)volume. However, in that case they preserve much more structure than just that.

In general being symplectic can be conveniently expressed in terms of the matrix J, since $(\frac{\partial F}{\partial x})^T J \frac{\partial H}{\partial x} = \omega(X_F, X_H)$, where X_F and X_H denote the vector fields corresponding to F and H, respectively.

Theorem 2. *Let* $\Xi : O \mapsto \mathbb{R}^{2n} : x \mapsto \xi = \Xi(x)$ *be a smooth map, where* O *is some open, bounded set in* $\mathbb{R}^{2n}$; Ξ *is symplectic if the following holds for the Jacobian matrix* $\frac{\partial \Xi}{\partial x}$*:*

$$(\frac{\partial \Xi}{\partial x})^T J \frac{\partial \Xi}{\partial x} = J \tag{2.7}$$

Theorem 3. *Given a Hamiltonian* $H : \mathbb{R}^{2n} \to \mathbb{R}$ *and the corresponding vector field* X *of the canonical form* (2.4). *Also let* $\Xi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be symplectic. Then the transformed vector field* $\Xi_* X$ *is canonical with respect to the Hamiltonian* $H \circ \Xi^{-1}$.

A proof can for instance be found in [4]. The relevance of this theorem is that it states that for symplectic transformations Ξ only the Hamiltonian H has to be transformed. The transformed vector field can then be computed easily and directly.

Let $\phi_{t_0,x_0} : \mathbb{R} \to \mathbb{R}^{2n}$ be a solution of (2.4). Often we are interested in the derivative with respect to the time of a function along $\phi = \phi_{t_0,x_0}$. So, define $F(t) := F(\phi(t))$. Then define Poisson brackets $\{,\}$ by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} = (\frac{\partial F}{\partial x})^T J \frac{\partial H}{\partial x} := \{F, H\}$$

The Poisson bracket is skew symmetric, bilinear and satisfies Jacobi's identity: for each F, G, H: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

An *integral* for (2.4) is a function which is constant along the solutions of (2.4). The following holds

Theorem 4. *Let H be the Hamiltonian as before, and let F, G and H be autonomous. Then*

1. *F is an integral for (2.4) if and only if $\{F, H\} = 0$.*
2. *H is an integral for (2.4).*
3. *If F and G are integrals for (2.4), then so is $\{F, G\}$.*
4. *A symplectic change of coordinates preserves Poisson brackets.*

2.1.1 E.Noether's theorem and reduction

An often used technique applied in this paper applies the well known fact that symmetries give rise to integrals, enabling to reduce the number of degrees of freedom. This is possible due to Noether's theorem:

Theorem 5. *Let X_F^t be the flow of the vector field generated by F and let it be a* symplectic symmetry *for a Hamiltonian H, i.e., $H(X_F^t(x)) = H(x)$ for all $x \in \mathbb{R}^{2n}$ and all $t \in \mathbb{R}$. Then F is an integral for the system with Hamiltonian H.*

Proof. (See also [15] and [1]). Differentiate $H(X_F^t(x)) = H(x)$ with respect to t. Then

$$0 = \frac{\partial H(X_F^t(x))}{\partial x}\frac{\partial X_F^t(x)}{\partial t} = \frac{\partial H(X_F^t(x))}{\partial x} J \frac{\partial F(X_F^t(x))}{\partial x} = \{H, F\}(X_F^t(x))$$

Substituting $t = 0$ and using theorem 4 finishes the proof. □

If F is a phase variable then its conjugate ϕ is called *cyclic* and the symmetry is equal to $(\phi, F) \mapsto (\phi + t, F)$, that is, H does not depend on ϕ. When F is fixed the number of degrees of freedom is then reduced by 1 and F can be regarded as a parameter. Because F stems from the phase variables in the original system it is called a *distinguished* parameter.

Using this theorem it is easy to see the conservation of H when autonomous Hamiltonians are used: then t is a cyclic variable.

2.1.2 Poincaré sections and maps

Consider a general system $\dot{x} = f(x)$ and suppose it has a periodic orbit with period T. Then one can take a certain point q on that orbit and define a *Poincaré section* transverse to $f'(q)$ of codimension 1. The *Poincaré map* maps a point p from a section to the point where the flow starting at p hits the section again. It is clear that q is a fixed point of such a map. The behaviour of points nearby q under this map gives information about orbits in the neighborhood of the periodic one. Of course there are many ways to define a Poincaré section. Much used are the following:

- Simply use the time-T map with a section $t = 0(\mod T)$. This is a stroboscopic mapping and well-defined if f is time-independent or T-periodic in t.

- Define a section where exactly one of the phase variables is equal to some value, that is, section={var=value} of codimension 1. This will be done in this paper. If time is reparametrized such that it is equal to an angular phase variable we see the connection with the stroboscopic mapping.

2.2 Celestial Mechanics

Celestial mechanics deals with particles moving in space attracting each other with gravitation given by the *inverse square law*. This means that the force F_{ij}, that particle i exerts on particle j, is given by

$$F_{ij} = G\frac{m_i m_j e_{q_{ij}}}{||q_{ij}||^2} := G\frac{m_i m_j}{||q_j - q_i||^3}(q_j - q_i)$$

where the universal gravitational constant $G = 1$ [1] , m_i and m_j denote the masses and q_i and q_j denote the positions of particles i and j, respectively.

Some problems in celestial mechanics are:

- The Kepler problem (central force field): describe the motion of a particle which is attracted to a fixed center by a force proportional to the mass and depending only on the distance between the particle and the center. Examples are forces due to the inverse square law, or due to an attraction proportional to the distance.

- The n-body problem: describe the motion of n particles attracting each other in $\mathbb{R}^3$, usually with the inverse square law.

- The restricted n-body problem: suppose the motion of $n-1$ particles is known. Describe the motion of a particle with negligible mass which does not influence the motion of the others, but is influenced by them.

Some important quantities are:

$P = \sum m_i \dot{q}_i$: The total (linear) momentum vector

$c = \sum m_i (q_i \times \dot{q}_i)$: The total angular momentum vector

$U = -G\sum (m_i m_j)/||q_i - q_j||$: The potential energy, a scalar

$T = \sum m_i ||\dot{q}_i||^2/2$: The kinetic energy, a scalar

$h = T + U$: Total energy; $h = H(\dots)$ when used as a Hamiltonian

$I = \sum m_i ||q_i||^2/2$: The total moment of inertia, a scalar

$\xi = (\sum m_i q_i)/(\sum m_i)$: The center of mass or barycenter vector

Loosely speaking, for the system as a whole P measures its velocity, c its rotational velocity, and I its size. In a *closed* system $\sum F_i = 0$, that is, there are no external forces working on the system.

[1] The units are chosen such that $G = 1$. In reality, $G = 6.6732 \times 10^{-11} \mathrm{m}^3/\mathrm{s}^2\mathrm{kg}$

Theorem 6. *In a closed, conservative system the following properties hold:*

- *The center of mass moves uniformly and rectilinearly ($\ddot{\xi} = 0$, so the total linear momentum P is preserved).*
- *The total energy h is preserved.*
- *The total angular momentum c is preserved.*
- $\ddot{I} = 4h - 2U = 4T + 2U$ *(Lagrange).*

The last statement implies that equilibria do not exist. Indeed, in an equilibrium point one has both $\ddot{I} = 0$ and $T = 0$. This cannot occur since $U < 0$. See for instance Pollard [16].

Chapter 3

The three body problem

In this chapter the circular restricted and full three body problem are investigated. The former can easily be written as a 2 degrees of freedom system, while for the latter some reductions have to be made to go from 6 to 3 degrees of freedom. In the last section these two problems are compared.

At first sight equilateral triangular solutions may seem to be a mathematical curiosity. Indeed, they were so for Lagrange. However, in the beginning of the 20th century two clouds of asteroids were found at equal distances from Jupiter and the sun. Eternally they are either following or being chased by Jupiter and are called the Trojan and Greek asteroids.

3.1 The circular restricted three body problem

So we start with the circular restricted three body problem. In this case two particles move around their common center of mass in circles and another particle with infinitesimally small mass is attracted by the other two, but does not influence them. We also assume that all motion takes place in a constant plane.

Suppose the center of mass is at the origin. The first mass, denoted by S for "sun", has distance μ to the origin; the other mass, denoted by J for "Jupiter", has distance $1-\mu$. If we put their total mass equal to 1, S must have mass $1-\mu$ and J mass μ.

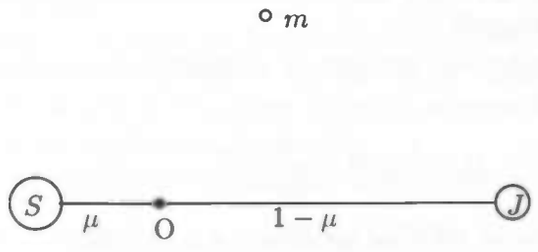


Figure 3.1: The RTBP

Following Newton's second law and the inverse square gravitation law the third body is attracted according to the following formula, where $x \in \mathbb{R}^2$ is the

position and m the mass of the third body:

$$m\ddot{x} = \frac{mm_S}{||x-S||^3}(S-x) + \frac{mm_J}{||x-J||^3}(J-x) \tag{3.1}$$

Observe that m can be cancelled. Nothing changes in (3.1) upon identification of $\mathbb{R}^2$ with $\mathbb{C}$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with i. However, the motions of S and J can then easily be described by $S = -\mu e^{it}$ and $J = (1-\mu)e^{it}$. Substitution in (3.1) then yields

$$\ddot{x} = \frac{1-\mu}{|x-\mu e^{it}|^3}(-\mu e^{it} - x) + \frac{\mu}{|x-(1-\mu)e^{it}|^3}((1-\mu)e^{it} - x). \tag{3.2}$$

To eliminate the time-dependence, we change to rotating coordinates: $x = ze^{it}$, which gives

$$e^{it}(\ddot{z} + 2i\dot{z} - z) = \frac{1-\mu}{|z-\mu|^3}(-\mu e^{it} - ze^{it}) + \frac{\mu}{|z-(1-\mu)|^3}((1-\mu)e^{it} - ze^{it}).$$

Note that the factor e^{it} can be cancelled here. This gives

$$\ddot{z} + 2i\dot{z} - z = \frac{1-\mu}{|z-\mu|^3}(-\mu - z) + \frac{\mu}{|z-(1-\mu)|^3}((1-\mu) - z). \tag{3.3}$$

A relative equilibrium with respect to the rotating coordinates is now determined by the equations $\dot{z} = \ddot{z} = 0$. This amounts to the equation

$$z(-1 + \frac{1-\mu}{||z-\mu||^3} + \frac{\mu}{||z-(1-\mu)||^3}) = -\mu(1-\mu)(\frac{1}{||z-\mu||^3} - \frac{1}{||z-(1-\mu)||^3}, \tag{3.4}$$

so either z is real, that is we have a collinear solution, or both sides vanish. The latter occurs if and only if $||z-\mu|| = ||z-(1-\mu)|| = 1$, that is, we have an equilateral triangle. Note that in the original coordinates these relative equilibria are 2π-periodic solutions.

There are three collinear points, one for each order of masses on the line, and two equilateral triangular solutions, one for each side of the line JS. Classically these points are denoted by L_1, L_2, L_3, L_4 and L_5. For a rotating picture, see the web site:
`http://www.geom.umn.edu/~megraw/CR3BP_html/cr3bp_fixed.html`, related to Thurman and Worfolk [21].

A Hamiltonian for this problem in non-rotating coordinates can easily be derived from (3.1):

$$H = \frac{1}{2}||y||^2 - \frac{1-\mu}{||x-S||} - \frac{\mu}{||x-J||}$$

The transformation to rotating coordinates is symplectic. A proof of this is given by Lunter [13] and Meyer&Hall [15]. Because this transformation is time dependent, the integral invariant of Poincaré-Cartan (see [1]) must be used. In rotating coordinates the Hamiltonian becomes:

$$H = \frac{1}{2}(y_1^2 + y_2^2) + x_2y_1 - x_1y_2 - \frac{1-\mu}{\sqrt{(x_1+\mu)^2 + x_2^2}} - \frac{\mu}{\sqrt{(x_1+\mu-1)^2 + x_2^2}},$$

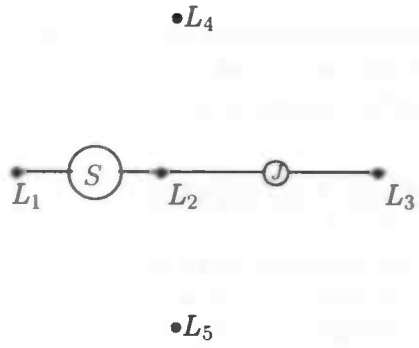


Figure 3.2: The relative equilibria of the RTBP

where H is not equal to the total energy, since we have cancelled m in 3.1. However, H is an integral, and depends on an integral called the *Jacobi constant.*

Up to a translation the origin corresponds to an equilibrium, and up to a well chosen additive constant the constant and linear parts of this Hamiltonian vanish. The quadratic part H_2 of the Hamiltonian can then be written as $\frac{1}{2}x^T Sx$, where $x \in \mathbb{R}^4, S \in gl(4,\mathbb{R})$. Define $A = JS$, with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

then

$$\dot{x} = J\frac{\partial H}{\partial x} = JSx = Ax.$$

A is called Hamiltonian or infinitesimally symplectic, which can be expressed by $A^T J + JA = 0$. A has some complicated entries, but the characteristic polynomial is rather simple: it is equal to

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu), \tag{3.5}$$

in the equilateral equilibria, and

$$\lambda^4 + (1-\mu(1-\mu))\lambda^2 - (2\mu^2(1-\mu)^2 + 3\mu(1-\mu)) \tag{3.6}$$

in the collinear equilibria.

In the collinear case two roots are real and two are purely imaginary for all values of μ, so there are two eigenvalues $\pm a$ and two of the form $\pm ib$, where a and b are real. So for all values of μ there exists an eigenvalue with a positive real part, and hence the equilibrium is unstable.

However, in the equilateral case, (3.5) has zeros satisfying:

$$\lambda^2 = \frac{1}{2}(-1 \pm \sqrt{1 - 27\mu(1-\mu)}),$$

so λ is purely imaginary only if $27\mu(1-\mu) \leq 1$, or, if $\mu \leq 1/2$, $\mu \leq \mu_1 = 1/2\sqrt{1-\sqrt{69}/9}$, where μ_1 is called Routh's critical mass ratio. When the parameter μ passes through μ_1, two pairs of eigenvalues coincide and become complex with non-vanishing real part: $\pm a \pm bi$, where $a, b \in \mathbb{R}$. This phenomenon

has been studied extensively and is called the Hamiltonian Hopf bifurcation, see Van der Meer [22]. We are interested in linearly stable relative equilibria and, therefore, in the sequel, we restrict to the equilateral case with $0 < \mu < \mu_1$.

3.2 Three bodies, the system

In the general three body problem three particles with masses m_1, m_2 and m_3 attract one another according to the inverse square law, i.e., according to (2.3) with $q_i, p_i \in \mathbb{R}^3, i = 1, 2, 3$, with

$$U = -G\left(\frac{m_1 m_2}{||q_1 - q_2||} + \frac{m_2 m_3}{||q_2 - q_3||} + \frac{m_3 m_1}{||q_3 - q_1||}\right) \tag{3.7}$$

and

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{3.8}$$

For simplicity, we change the units of time and mass such that both $G = 1$ and $m_1 + m_2 + m_3 = 1$. The unit of distance will be redefined later on. Now the Hamiltonian, i.e., the total energy, is

$$H = \frac{||p_1||^2}{2m_1} + \frac{||p_2||^2}{2m_2} + \frac{||p_3||^2}{2m_3} - \frac{m_1 m_2}{||q_1 - q_2||} - \frac{m_2 m_3}{||q_2 - q_3||} - \frac{m_3 m_1}{||q_3 - q_1||} \tag{3.9}$$

Observe that the potential energy only depends on the mutual distances of the particles. The number of d.o.f. in this system equals 3*3=9. It can be reduced to 4 using the integrals and symmetries mentioned in the introduction. A symmetric way of describing the reduced Hamiltonian makes use of the mutual distances and the angle between the plane spanned by the masses and the invariant Laplace plane orthogonal to the constant total angular momentum vector. The exact formula can be found in Arnold [2] and Wintner [24]. It can be expressed as

$$H = H_{|c|}(Q_1, Q_2, Q_3, \theta, P_1, P_2, P_3, \Theta) =$$
$$\text{complicated expression for the kinetic energy} - \left(\frac{m_1 m_2}{Q_1} + \frac{m_2 m_3}{Q_2} + \frac{m_1 m_3}{Q_3}\right) \tag{3.10}$$

where the Q_i denote the mutual distances, θ is the angle between the above mentioned planes, Θ is the conjugate momentum of θ and $|c|$ is the length of the total angular momentum vector c.

This Hamiltonian has the following scaling property: for any $a \neq 0$,

$$H_{a|c|}(q_1, q_2, q_3, \theta, p_1, p_2, p_3, \Theta) = \frac{1}{a^2} H_{|c|}\left(\frac{q_1}{a^2}, \frac{q_2}{a^2}, \frac{q_3}{a^2}, \theta, ap_1, ap_2, ap_3, \frac{\Theta}{a}\right), \tag{3.11}$$

So it doesn't matter which value of the angular momentum is chosen, provided it is nonzero.

When fixing θ at zero in (3.10), all motion takes place in the Laplace plane and we have the 3 d.o.f. *planar three body problem*, which is the main problem of the paper. Additional restrictions can be made to further restrict to 2 d.o.f.

1. The *restricted* planar three body problem: putting $m_3 = \epsilon^2$, to first order in ϵ, the problem decouples into a 1 d.o.f. Kepler problem and a 2 d.o.f. restricted problem, see [15];

2. Two masses are equal and the distances between those masses and the third one (q_2 and q_3) are always equal. Then they form, except when collinear, an isosceles triangle. This 2 d.o.f. system is called the planar isosceles three body problem, see Simó and Martínez [20].

3.2.1 An explicit reduction of the planar three body problem

An explicit reduction, that is with help of explicit transformations, is necessary for symbolic and numerical computations. The planar three body problem, i.e. the Hamiltonian (3.9) with all variables in $\mathbb{R}^2$ has a quadratic part that turned out to be too complicated, so we use other coordinates. Here we first restrict to the planar problem, reducing from 9 to 6 d.o.f. Fixing linear and angular momentum then enables us to reduce to 3 d.o.f. This reduction is taken from Meyer&Hall [15].

Eliminating the center of mass

One way of fixing the center of mass is by changing to Jacobi-coordinates (u_0, u_1, u_2). These position variables are defined by $u_0 = m_1q_1 + m_2q_2 + m_3q_3$, $u_1 = q_2 - q_1$ and $u_2 = q_3 - \frac{m_1q_1+m_2q_2}{m_1+m_2}$, such that u_0 is the center of mass, u_1 points from m_1 to m_2, and u_2 points from the center of mass of m_1 and m_2 to m_3. The corresponding momentum variables (v_0, v_1, v_2) are given by $v_0 = p_1 + p_2 + p_3$, $v_1 = \frac{m_1p_2-m_2p_1}{m_1+m_2}$, $v_2 = (m_1 + m_2)p_3 - m_3(p_1 + p_2)$. Then

$$H = \frac{||v_0||^2}{2} + \frac{||v_1||^2}{2M_1} + \frac{||v_2||^2}{2M_2} - \frac{m_1m_2}{||u_1||} - \frac{m_1m_2}{||u_2 + \alpha_0u_1||} - \frac{m_2m_3}{||u_2 - \alpha_1u_1||} \quad (3.12)$$

where

$$M_1 = \frac{m_1m_2}{m_1 + m_2}, M_2 = m_3(m_1 + m_2), \alpha_0 = \frac{m_2}{m_1 + m_2}, \alpha_1 = \frac{m_1}{m_1 + m_2}$$

An advantage of Jacobi-coordinates is that the kinetic energy has the same form as before. Since H does not depend on the center of mass u_0, this variable is cyclic and v_0, the total linear momentum is an integral. Since equations (2.4) do not change when adding a constant to H, we discard the first term, and the Hamiltonian only depends on $u_1, u_2, v_1, v_2 \in \mathbb{R}^2$ thereby reducing to a 4 d.o.f. system.

Eliminating the angular momentum

To see the rotational symmetry we can change to polar coordinates. A symplectic way of introducing these is:

$$u_i = (r_i \cos\theta_i, r_i \sin\theta_i), R_i = m_i \dot{r_i}, \Theta_i = m_i r_i^2 \dot{\theta_i}$$

The Hamiltonian in these coordinates is given by

$$\begin{aligned} H = &\frac{R_1^2 + \frac{\Theta_1^2}{r_1^2}}{2M_1} + \frac{R_2^2 + \frac{\Theta_2^2}{r_2^2}}{2M_2} - \frac{m_2 m_1}{r_1} - \\ &\frac{m_1 m_3}{\sqrt{r_2^2 + \alpha_0^2 r_1^2 + 2\alpha_0 r_1 r_2 \cos(\theta_2 - \theta_1)}} - \\ &\frac{m_2 m_3}{\sqrt{r_2^2 + \alpha_1^2 r_1^2 - 2\alpha_1 r_1 r_2 \cos(\theta_2 - \theta_1)}}, \end{aligned} \tag{3.13}$$

where the capitals R_i, Θ_i are the conjugate momenta of the lowercase configuration variables r_i, θ_i.

Since H only depends on $\theta_2 - \theta_1$, we introduce symplectic coordinates

$$\phi_1 = \theta_2 - \theta_1, \phi_2 = \theta_2, \Phi_1 = -\Theta_1, \Phi_2 = \Theta_1 + \Theta_2.$$

Then

$$\begin{aligned} H = &\frac{R_1^2 + \frac{\Phi_1^2}{r_1^2}}{2M_1} + \frac{R_2^2 + \frac{(\Phi_1 + \Phi_2)^2}{r_2^2}}{2M_2} - \frac{m_2 m_1}{r_1} - \\ &\frac{m_1 m_3}{\sqrt{r_2^2 + \alpha_0^2 r_1^2 + 2\alpha_0 r_1 r_2 \cos(\phi_1)}} - \\ &\frac{m_2 m_3}{\sqrt{r_2^2 + \alpha_1^2 r_1^2 - 2\alpha_1 r_1 r_2 \cos(\phi_1)}} \end{aligned} \tag{3.14}$$

In these new coordinates ϕ_2 is cyclic, so Φ_2, the total angular momentum, is an integral. Fixing Φ_2 reduces the system to 3 d.o.f. . Now consider Φ_2 as a distinguished parameter, and put $H = H_{\Phi_2}(r_1, r_2, \phi_1, R_1, R_2, \Phi_1)$. Next observe the following scaling property:

$$H_{a\Phi_2}(r_1, r_2, \phi_1, R_1, R_2, \Phi_1) = \frac{1}{a^2} H_{\Phi_2}(\frac{r_1}{a^2}, \frac{r_2}{a^2}, \phi_1, aR_1, aR_2, \frac{\Phi_1}{a}),$$

similar to (3.11). This property enables us to set the length of the sides of the equilateral triangle to 1.

By discarding the angle ϕ_2 configurations which differ by a rotation have been identified. So configurations that only rotate become equilibria here. The variables can now be visualized using figure 3.2.1.

Remark: it is possible to reduce it even further to 2 d.o.f. by eliminating time and total energy. However, the so obtained Hamiltonian is non-autonomous and does not stand for the total energy:

$$R_1 = K_{\Phi_2}(r_1, r_2, \phi_1, R_2, \Phi_1, H);$$

$$\frac{\mathrm{d}r_2}{\mathrm{d}r_1} = \frac{\partial K}{\partial R_2}, \frac{\mathrm{d}R_2}{\mathrm{d}r_1} = -\frac{\partial K}{\partial r_2}, \frac{\mathrm{d}\phi_1}{\mathrm{d}r_1} = \frac{\partial K}{\partial \Phi_1}, \frac{\mathrm{d}\Phi_1}{\mathrm{d}r_1} = -\frac{\partial K}{\partial \phi_1}.$$

This remark fits in the story of reductions, but it will not be use further because this Hamiltonian is non-autonomous.

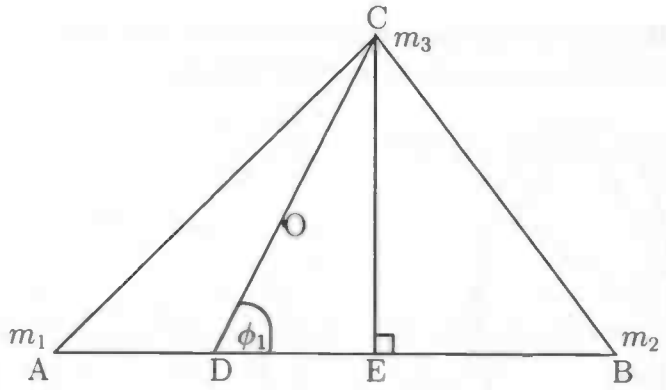


Figure 3.3:

$$O = \text{the center of mass}$$

$$D = \text{the center of mass of } m_1 \text{ and } m_2$$

$$r_1 = AB, \quad r_2 = DC$$

$$AC = \sqrt{r_2^2 + \alpha_0^2 r_1^2 + 2\alpha_0 r_1 r_2 \cos\phi_1}$$

$$BC = \sqrt{r_2^2 + \alpha_1^2 r_1^2 - 2\alpha_1 r_1 r_2 \cos\phi_1}$$

$$\alpha_0 r_1 = AD = \frac{m_2}{m_1 + m_2} r_1, \quad \alpha_1 r_1 = DB = \frac{m_1}{m_1 + m_2} r_1$$

$$R_1 = M_1 \dot{r_1} = \text{the linear momentum of } D$$

$$R_2 = M_2 \dot{r_2} = \text{the linear momentum of } m_3 \text{ with respect to } D$$

$$\dot{\theta_i} = \text{angular velocity of } r_i \text{ w.r.t. } O$$

$$\Phi_2 = M_1 r_1^2 \dot{\theta_1} + M_2 r_2^2 \dot{\theta_2} = \text{the total angular momentum: an integral}$$

$$\Phi_1 = -M_1 r_1^2 \dot{\theta_1} = -\text{the angular momentum of } D$$

3.2.2 Central configurations and equilibria

Whereas in the original (9 d.o.f.) system no equilibria occur, in the reduced system they do. In these *relative* equilibria the particles rotate with constant angular velocity around the center of mass. These equilibria are special cases of central configurations (c.c.), where the configuration during the whole motion only changes by a (non-constant) translation, rotation, or dilation. In these motions the angles of the triangle remain invariant, that is $r_2 = vr_1, \phi_1 = constant$. Then for the three variables q_i from (3.10) the following holds

$$\ddot{q_i} = \frac{c^2}{q_i^3} + \frac{\text{const}}{q_i^2}.$$

This is the differential equation for the Kepler problem with inverse square law; so in a c.c. all particles move in ellipses around each other. In a relative

equilibrium there is no dilation, so q_i is constant, and the particles rotate in circles with the same and constant angular velocity.

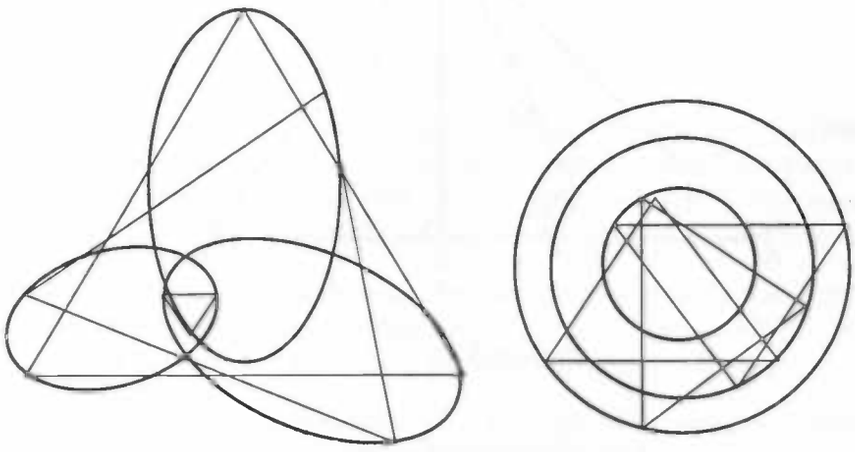


Figure 3.4: A central configuration and relative equilibrium.

This also means that then 4 times the total moment of inertia, $4I = \sum m_j m_k q_i^2$, is equal to $c/\omega = c = -U = m_1m_2 + m_2m_3 + m_1m_3$ (see Meyer and Hall [15]). So if the total angular momentum c is zero, I would be equal to 0, so all particles are at the same point. Therefore we may assume that c is nonzero.

By computation two types of c.c. can be found:

1. If $\phi_1 = 0$, that is, the three bodies are on one line. These are the Euler-Moulton collinear solutions. There are $3!/2 = 3$ of these solutions, one for each order of the masses on the line. However, even the linearized system is not stable at this solution.

2. If $\phi_1 \neq 0$, then the three sides of the triangle must be equal: the Lagrangian equilateral triangle solution. By a rescaling of distances we may put r_1 in the corresponding equilibrium to 1. Then by the formula $r_1^3 = G(m_1 + m_2 + m_3)/\omega^2$ from e.g. Pollard [16], where ω is the angular velocity: $r_1 = 1, r_2 = v = \sqrt{1 - \alpha_0 + \alpha_0^2}, R_1 = 0, R_2 = 0, \phi_1 = c = \arccos(\frac{\alpha_1 - \alpha_2}{2v}), \Phi_1 = -M_1, \Phi_2 = m_1m_2 + m_2m_3 + m_1m_3, \omega = 1$.

The equilateral triangular equilibrium corresponds to the following in figure 3.2.1:

$$AE = \frac{1}{2}, EC = \frac{1}{2}\sqrt{3}, \gamma := DE = AE - AD = \frac{1}{2} - \frac{m_2}{m_1 + m_2} = \frac{m_1 - m_2}{2(m_1 + m_2)}$$

$$\nu := \bar{r_2} = \sqrt{\gamma^2 + (\frac{1}{2}\sqrt{3})^2} = \sqrt{\gamma^2 + \frac{3}{4}}$$

$$\cos\bar{\phi_1} = \frac{\gamma}{\nu} \Rightarrow \bar{\phi_1} = \arccos\frac{\gamma}{\nu}$$

$$\bar{R_1} = 0, \bar{R_2} = 0, \bar{\theta_1} = \bar{\theta_2} = 1, \bar{\Phi_1} = -M_1, \bar{\Phi_2} = M_1 + M_2\nu^2$$

A translation is carried out such that this equilibrium corresponds to $(r_1, r_2, \phi_1, R_1, R_2, \Phi_1) = 0$ in Hamiltonian (3.14):

$$H = \frac{R_1^2 + \frac{(\Phi_1 - M_1)^2}{(r_1+1)^2}}{2M_1} + \frac{R_2^2 + \frac{(\Phi_1 + M_2\nu^2)^2}{(r_2+\nu)^2}}{2M_2} - \frac{m_2 m_1}{r_1 + 1} - \frac{m_1 m_3}{AC} - \frac{m_2 m_3}{BC} - H_0 \tag{3.15}$$

where

$$AC = \sqrt{(r_2+\nu)^2 + \alpha_0^2(r_1+1)^2 + 2\alpha_0(r_1+1)(r_2+\nu)\cos(\arccos(\frac{\gamma}{\nu}) + \phi_1)},$$

$$BC = \sqrt{(r_2+\nu)^2 + \alpha_1^2(r_1+1)^2 - 2\alpha_1(r_1+1)(r_2+\nu)\cos(\arccos(\frac{\gamma}{\nu}) + \phi_1)}$$

and H_0 has been chosen such that at the equilibrium $H = 0$.

Now, again, the linear terms vanish, and we can take a look at the second order part. This time the characteristic polynomial of the corresponding matrix is equal to

$$(\lambda^2 + 1)(\lambda^4 + \lambda^2 + \frac{27}{4}\sigma), \text{ where } \sigma = m_1 m_2 + m_2 m_3 + m_3 m_1.$$

This is clearly a generalization of the restricted case (3.5). See also Wintner [24] and Siegel and Moser [19].[1] For the Euler points it is equal to $(\lambda^2+1)(\lambda^4+(1-\sigma)\lambda^2 - (2\sigma^2+3\sigma)$.[2] For the latter a computation yields that there is always at least one zero with non-vanishing real part. For the former, its zeroes are purely imaginary for $27\sigma \leq 1$. This approximately means that one of the particles has to be at least 25 times heavier than the other two together; see section 5.3.

Look at the Lagrange equilibrium, and suppose that $27\sigma \leq 1$. The six eigenvalues are then $\pm i, \pm i\omega_1, \pm i\omega_2$. The real numbers ω_1 and ω_2 then satisfy

$$\omega_{1,2} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \sqrt{1 - 27\sigma}} \tag{3.16}$$

They are uniquely determined by

$$0 < \omega_2 \leq \frac{1}{\sqrt{2}} \leq \omega_1 < \omega_0 = 1$$

$$\omega_1^2 + \omega_2^2 = 1$$

$$\omega_1^2 \omega_2^2 = \frac{27}{4}\sigma$$

So, because σ is nonzero, all eigenvalues must be that.

3.3 Relationship between the restricted and the full problem

As can be expected, when one of the masses is very small, the restricted and full problems approximately coincide. This has been made precise by Meyer &

[1] According to [24] this polynomial has been found by G. Gascheau in 1843.

[2] discovered by J. Liouville in 1842.

Hall[15] as follows. Let m_3 be the small mass, and define H as the Hamiltonian of the full problem, H_r as the Hamiltonian of the restricted problem, and $m_3 = \epsilon^2$. Then

$$H = H^r + H^h + O(\epsilon),$$

where $H^h = (x^2 + y^2)/2$ is the Hamiltonian of the harmonic oscillator, which is independent of H^r. This corresponds to the linearized equations about the circular solution of the sun and Jupiter, and gives rise to the eigenvalues $\pm i$ found above. Using this the following theorem can be proven (see [15]):

Theorem 7. *(Hadjidemetriou). Any elementary periodic solution of the planar restricted three body problem whose period is not a multiple of 2π can be continued into the full three body problem with one small mass.*

If the above mentioned period is a multiple of 2π there is a resonance with respect to the eigenvalues $\pm i$, which is not present in the restricted problem. Then Lyapunov's continuation theorem (see chapter 5) cannot be applied.

Chapter 4

Normalization of a Hamiltonian near a resonant equilibrium

Here we normalize a general Hamiltonian where the matrix of the linearized system is semisimple in the neighbourhood of a certain resonance. Then the ratio of one eigenvalue to another is in the neighbourhood of a fixed rational number. For systems with more than two degrees of freedom we assume that all other eigenvalues are not in a rational ratio with respect to each other and the first two.

First the standard Birkhoff normalization is applied. The so obtained system contains an additional rotational (formal) symmetry, which enables to reduce to one degree of freedom. Then additional symplectic transformations are applied yielding a system containing a $\mathbb{Z}_2$-symmetry. This symmetry is formal, so in the original system an approximate symmetry is expected.

Next we pay attention to the 1:2-resonance. We will apply further non-symplectic coordinate transformations in the same way as in BHLV[7], to eventually reach the system $H(x,y) = x(x^2+y^2) + u_1 x + u_2 y^2$. The bifurcations of this final system are studied. In the next chapter these results are applied to the three body problem.

4.1 Normalizing the quadratic part H_2

To carry out Birkhoff's normalization procedure using `Mathematica` in a reasonable time first the matrix A corresponding to the vector field of the quadratic part of the Hamiltonian must be put on diagonal form.

Assume A is semisimple. Then A can be brought into normal form D using standard linear algebra: let T be a matrix of eigenvectors of A which has been scaled such that $T^T J T = 1/(2i)J$. T is by definition symplectic with multiplier $1/(2i)$.

Then $TAT^{-1} = D$, or $TA = DT$ so carrying out a coordinate change $z = Tx$

gives

$$\dot{z} = T\dot{x} = TAx = DTx = Dz$$

After multiplying with $2i$ the new H_2 becomes:

$$H_2 = i\sum_{j=1}^{n} \omega_j z_j \bar{z}_j = \sum_{j=1}^{n} \omega_j L_j, \tag{4.1}$$

where both complex and action-angle variables are used.

4.2 Birkhoff normalization

Consider a Hamiltonian H, with an equilibrium point at the origin, and assume that $H(0) = 0$. Then H has a Taylor series expansion of the form

$$H(x) = \sum_{i=2}^{\infty} H_i(x), \tag{4.2}$$

where H_i is a homogeneous polynomial of degree i in x. Let H's linearization be of the form

$$\dot{x} = Ax = J\frac{\partial H_0}{\partial x}.$$

Theorem 8. *Let A be semisimple. Then there exists a formal, symplectic change of coordinates,*

$$x = \phi(y) = y + \dots$$

which transforms the Hamiltonian to

$$\tilde{H}(y) = \sum_{i=2}^{\infty} G_i(y),$$

where G_i is a "good" homogeneous Hamiltonian of degree i in y such that

$$\{G_i, H_2\} = 0$$

for all i.

Proof. (See for instance Meyer and Hall [15], Broer [5] or Lunter [13].) The proof runs by induction. A transformation has to be found normalizing H from degree $m-1$ to degree m. In the proof H_i is subject to change with notice.

For $m = 2$, the first step, nothing has to be done. Suppose H is normalized up to degree $m-1$, where $m \geq 3$, i.e.:

$$H(x) = \sum_{i=2}^{m-1} G_i(x) + H_m(x) + H_{m+1}(x) + \dots$$

Let F_m be a homogeneous polynomial of degree m in x and let $X^1_{F_m}$ be the time-1 flow of the vector field with Hamiltonian F_m. Then

$$H(y) := H(X^1_{F_m}(x)) = \sum_{i=2}^{m-1} G_i(y) + H_m(y) + \{H_2, F\}(y) + h.o.t.$$

So the *homological* equation,

$$H_m + \{H_2, F_m\} = G_m,$$

has to be solved. Thus we can transform away all "bad" elements of the image of $ad^m_{H_2}(x) := \{x, H_2\}$, restricted to homogeneous polynomials of degree m. Because A is semisimple, ad_{H_2} is also semisimple (see Broer[5]). So a complement of its image is its kernel. Therefore we can take away all elements except for those in the kernel of ad_{H_2}, meaning that $\{H^{\#}_m, H_2\} = 0$.

The composition of all time-1 flows yields the normalized Hamiltonian. In general this composition does not converge, but if we restrict to normalizing up to a certain order and forget the higher order terms the composition is finite and Taylor's formula applies. □

In general, however, the above obtained normal form is not unique (see [17] and [15]). We can take advantage of this to normalize further. This will be postponed until we have 1 d.o.f. system.

Suppose we have a general n d.o.f. Hamiltonian system as before with two eigenvalues of the quadratic part near a rational ratio. After normalizing the quadratic part is as follows

$$H_2 = i\omega_1 z_1 \bar{z}_1 + i(d + \frac{p}{q})\omega_1 z_2 \bar{z}_2 + i\sum_{j=3}^{n} \omega_j z_j \bar{z}_j,$$

where the parameter d is a resonance *detuning*, measuring the deviation from the exact resonance. We assume that p is a positive integer and q an integer that can also be negative. Now apply Birkhoff's procedure as if $d = 0$. Since it doesn't affect the quadratic part, the detuning doesn't disappear.

We divide H by ω_1, which amounts to a time-reparametrisation.

Proposition 9. *Suppose $H_2 = iz_1\bar{z}_1 + i(d + \frac{p}{q})z_2\bar{z}_2 + i\sum_{j=3}^{n}\omega_j z_j \bar{z}_j$, then the set of elements of $ker ad_{H_2}$ is either given by*

$$\mathbb{R}[[z_1\bar{z}_1, \ldots, z_n\bar{z}_n, z_1^p \bar{z}_2^q, \bar{z}_1^p z_2^q]]$$

if q is positive, or

$$\mathbb{R}[[z_1\bar{z}_1, \ldots, z_n\bar{z}_n, z_1^p z_2^{|q|}, \bar{z}_1^p \bar{z}_2^{|q|}]]$$

if q is negative. In action-angle coordinates this same set is given by

$$\mathbb{R}[[L_1, \ldots, L_n, L_1^{p/2} L_2^{|q|/2} \cos(p\phi_1 - q\phi_2), L_1^{p/2} L_2^{|q|/2} \sin(p\phi_1 - q\phi_2)]]$$

Here $\mathbb{R}[[a_1, \ldots, a_n]]$ denotes the ring of formal power series in the a_i with coefficients in $\mathbb{R}$.

Proof. If q is positive, for a general monomial $f = z_1^{m_1}\bar{z}_1^{n_1}\dots$ we have $\{f, H_2\} = f((n_1-m_1)+\frac{p}{q}(n_2-m_2)+\omega_3(n_3-m_3)+\dots)$. This expression is zero if $n_i = m_i$ for all i, or if $n_1 - \frac{p}{q}m_2 = 0$, $n_2 - \frac{q}{p}m_1 = 0$ and the other exponents are zero. Linear combinations of these expressions are also zero, so kerad_{H_2} is the algebra as described above. For negative q the proof is similar. □

So the normalized Hamiltonian H^n can take the following form in action-angle coordinates. Near $(p : q)$-resonance, i.e. $|\omega|$ near $p/|q| < 1$, with $\gcd(p,q)=1$ (ω and q may be negative) the following holds:

$$H^n(L,\phi) = L_1 + \omega L_2 + \omega_3 L_3 + \dots + \omega_n L_n + h.o.t.(L_1, L_2, \dots, L_n, L_1^{p/2}L_2^{|q|/2}\cos(p\phi_1 - q\phi_2), L_1^{p/2}L_2^{|q|/2}\sin(p\phi_1 - q\phi_2)). \quad (4.3)$$

So the angles $\phi_3, \dots, \phi_n$ are now cyclic and the corresponding actions are integrals. From now on these integrals will be considered as distinguished parameters. The terms of H^n only depending on these actions will be disgarded, because they are dynamically irrelevant (like constants).

4.3 Reduction to one degree of freedom

Let r, s be integers such that $pr - qs = 1$. After a symplectic coordinate change of the form

$$\begin{pmatrix}\tilde{L}_1\\ \tilde{L}_2\end{pmatrix} = \begin{pmatrix}r & s\\ q & p\end{pmatrix}\begin{pmatrix}L_1\\ L_2\end{pmatrix}, \quad \begin{pmatrix}\tilde{\phi}_1\\ \tilde{\phi}_2\end{pmatrix} = \begin{pmatrix}p & -q\\ -s & r\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad (4.4)$$

and dropping the tildes the system reads:

$$H^n(L,\phi) = \frac{1}{q}L_2 + b_1 L_1 + h.o.t.(L_1, L_2, (pL_1 - sL_2)^{p/2}(-qL_1 + rL_2)^{|q|/2}\cos\phi_1, (pL_1 - sL_2)^{p/2}(-qL_1 + rL_2)^{|q|/2}\sin\phi_1). \quad (4.5)$$

Therefore ϕ_2 is cyclic, so L_2 is conserved. This enables us to define $\lambda = L_2/q$ as a distinguished parameter. Because all other terms are small, λ is an approximation of H^n. The coefficient b_1 now serves as a detuning parameter. Finally, the constant term λ is dropped and the symplectic translation $L_1 \mapsto L_1 + \frac{sq}{p}\lambda$ is applied. Then the to 1 d.o.f. reduced Hamiltonian H^r is of the form ($L = L_1, \phi = \phi_1$):

$$H^r(L,\phi) = b_1 L + h.o.t.(L, \lambda, L^{p/2}(q(\frac{\lambda}{p} - L))^{|q|/2}\cos\phi, L^{p/2}(q(\frac{\lambda}{p} - L))^{|q|/2}\sin\phi). \quad (4.6)$$

Remark: The original action variables are non-negative. This translates to the condition that $q(\frac{\lambda}{p} - L) \geq 0$, that is, $L \leq \lambda/p$ if q is positive and $L \geq \lambda/p$ if

q is negative. This means that for the phase space relating to (4.6) either the part inside or outside a singular circle $L = (x^2 + y^2)/2 = \lambda/p$ corresponds to the phase space of the original system.

4.4 Further symplectic transformations

As mentioned before, the normal form (4.6) is not unique. This fact can be exploited to apply extra symplectic transformations to obtain the following result, a generalisation of Henrard[12] and Sanders and Van der Meer[17].

Theorem 10. *There exists a symplectic change of coordinates bringing* H^r (4.6) *into the following form*

$$H^R = b_1 L + b_2 R_1 + F_1(L, \lambda) + R_1 F_2(L, \lambda) \tag{4.7}$$

where $R_1 = L^{p/2}(\frac{\lambda}{p} - L)^{|q|/2} \cos\phi$ *and* $R_2 = L^{p/2}(\frac{\lambda}{p} - L)^{|q|/2} \sin\phi$. *We can choose* F_i *such that*

$$\frac{\partial^{p+|q|-1} F_i}{\partial L^{p+|q|-1}}(0,0) = 0 \text{ for } i = 1, 2$$

Proof. The first H_i (compare (4.2)) with cos and sin terms of H^r is

$$H_{p+|q|} = F(L, \lambda) + (aR_1 + bR_2),$$

where F, if $p + |q|$ is even, is a homogeneous polynomial of degree $(p + |q|)/2$ in L and λ.

A suitable constant angle can be added to ϕ (a symplectic coordinate transformation), such that the sine term, i.e., bR_2 in in $H_{p+|q|}$ disappears. Now we have $H_{p+|q|} = F(L, \lambda) + b_2 R_1$. Because $R_1^2 + R_2^2 = L^p(\frac{\lambda}{p} - L)^{|q|}$, H^r can now be written in the following form:

$$H^r = b_1 L + b_2 R_1 + f_1(L, \lambda, R_1) + R_2 f_2(L, \lambda, R_1)$$

Let F_m be a homogeneous polynomial of degree $m - p - |q| - 2$ in the corresponding complex or cartesian coordinates, such that $\{F_m, H_2\} = 0$. As with Birkhoff's procedure the variables are changed with $y = X^1_{F_m}(x)$:

$$H^r \circ X^1_{F_m} = H_2 + H_3 + \cdots + H_m + \{H_{p+|q|}, F_m\} + H_{m+1} + \ldots$$

and

$$\{H_{p+|q|}, F_m\} = \{R_1 + f(L, \lambda), F_m\} = \{R_1, F_m\} + \tilde{H}_m.$$

That is, we can carry out a similar inductive procedure as Birkhoff's and can now transform away all elements of $im(ad_{R_1})$ with domain $kerad_{H_2}$. So

take a monomial in $kerad_{H_2}$ and see what $ad_{H_{R_1}}$ does to it. A calculation yields the following:

$$ad_{R_1}(\lambda^m L^n R_1^k) = -n\lambda^m L^{n-1} R_1^k R_2 \quad (n \geq 1). \tag{4.8}$$

$$\begin{aligned} ad_{R_1}(\lambda^m L^n R_1^k R_2) = \lambda^m L^{n-1} R_1^k (nR_1^2 - nL^p(\frac{\lambda}{p} - L)^{|q|} - \\ L(\frac{1}{2}L^{p-1}(\frac{\lambda}{p} - L)^{|q|-1}(\lambda - L(p+|q|)))) \quad (n \geq 1). \end{aligned} \tag{4.9}$$

$$ad_{R_1}(\lambda^m R_1^k R_2) = \lambda^m R_1^k (\frac{1}{2}L^{p-1}(\frac{\lambda}{p} - L)^{|q|-1}(\lambda - L(p+|q|))). \tag{4.10}$$

Equation (4.8) enables us to get rid of all terms with R_2. Then (4.9) allows for the elimination of all terms with R_1^k, where $k \geq 2$. Finally (4.10) can remove $L^{p+|q|-1}$.

In the case of 1:2-resonance this normal form is also unique, see [17], but we will not use nor prove that fact. □

Observe that H^r has a $\mathbb{Z}_2$-symmetry $\phi \mapsto -\phi$. This is a formal symmetry, since it is only visible in the normalized system, or in a normal form truncation. We expect to observe an approximate symmetry in the non-normalized system. This symmetry will be used in the next section.

4.5 Normalization using singularity theory

Returning to cartesian coordinates in the normal form (4.7) for H^R we get

$$\begin{aligned} H^R(x,y) = b_1\frac{x^2+y^2}{2} + O(|x^2+y^2,\lambda|^2) + \\ (b_2 + O(|x^2+y^2,\lambda|)(x^2+y^2)^{\frac{p-1}{2}}(\frac{\lambda}{p} - \frac{x^2+y^2}{2})^{|q|/2}x), \end{aligned} \tag{4.11}$$

with restrictions as in theorem 10. For $p = 1$ and $q = \pm 2$, H^R can be expanded as follows:

$$H^R(x,y) = b_1\frac{x^2+y^2}{2} + b_2x(\lambda - \frac{x^2+y^2}{2}) + O(|x,y,\lambda|^4), \tag{4.12}$$

In this section the configuration of the level curves of (4.11) with $p = 1$ and $q = \pm 2$ is considered. In that case the flow of the reduced system gives an approximation for the Poincaré map with section $\phi_2 = 0$ in the original system. For this general, not necessarily symplectic morphism will be used.

First H^R is divided by $-b_2/2$ yielding

$$H^c(x,y) = x(x^2+y^2) - 2\lambda x + c_1(x^2+y^2) + O(|x,y,\lambda|^4),$$

where $c_1 = -b_1/b_2$. This is a deformation of the *central singularity* $x(x^2+y^2)$, the hyperbolic umbilic, by the parameters λ, c_i. We can apply a coordinate transformation (see BHLV[7]) such that this deformation becomes

$$H^u(x,y,u_1,u_2) = x(x^2+y^2) + u_1x + u_2y^2,$$

so that there exist φ and ρ such that

$$H^c = H^u(\varphi(x, y, c_i, \lambda), \rho_1(c_i, \lambda), \rho_2(c_i, \lambda)))$$

A calculation using BHLV [7] yielded that

$$u_1 = \rho_1(c_i, \lambda) = -\frac{c_1^2}{3} - 2\lambda + O(|\lambda, c_i|^3) \tag{4.13}$$

$$u_2 = \rho_2(c_i, \lambda) = \frac{2}{3}c_1 + O(|\lambda, c_i|^2) \tag{4.14}$$

A posteriori we see that u_1 and u_2 are small, because λ and c_1 are small.
Remark 1: another form of the versal deformation is $H(x, y) = x(\frac{1}{3}x^2 + y^2) + u_1 x + u_2(y^2 - x^2)$, also compare Hanßmann [11] and BCKV [6]. The deformations are connected with each other using a linear affine coordinate change in x en y.
Remark 2: Using a calculation in `Mathematica` it was also found that if we do not do further symplectic transformations as in section 4.4, the $\mathbb{Z}_2$-symmetry appears in the now general versal deformation: in $x(x^2+y^2)+u_1x+u_2y^2+u_3y$, u_3 appeared to be zero up to arbitrary high order in λ and the c_i.

4.6 Another way of reduction

The reduction as applied in section 4.3 can be done using another coordinate change. This section describes how this leads to another versal deformation for the case $p = 1, q = -2$. After a symplectic coordinate change of the form

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{4.15}$$

and dropping the tildes the system reads:

$$H^n(L,\phi) = L_1 + b_1 L_2 + h.o.t.(L_1, L_2, (L_1 + L_2/2)^{1/2} L_2 \cos(2\phi_2), (L_1 + L_2/2)^{1/2} L_2 \sin(2\phi_2)). \tag{4.16}$$

Therefore ϕ_1 is cyclic, so L_1 is conserved, and will be considered as a distinguished parameter λ. This system relates to the original one with Poincaré section $\phi_1 = 0$. The coefficient b_1 serves again as a detuning parameter. Then the to 1 d.o.f. reduced Hamiltonian H^r is of the form ($L = L_2, \phi = \phi_2$):

$$H^r(L,\phi) = b_1 L + h.o.t.(L, \lambda, (\lambda + \frac{L}{2})^{1/2} L \cos(2\phi), (\lambda + \frac{L}{2})^{1/2} L \sin(2\phi)). \tag{4.17}$$

As in section 4.4 further normalization can be applied to eliminate the sine terms in H^r. Returning to cartesian coordinates then gives

$$H^r(x, y) = b_1 \frac{x^2 + y^2}{2} + h.o.t.(x^2 + y^2, \lambda, (\lambda + \frac{x^2 + y^2}{4})^{1/2}(x^2 - y^2)).$$

We put $z = (\lambda + \frac{x^2+y^2}{4})^{1/2}$:

$$H^r(x,y) = 2b_1(z^2 - \lambda) + h.o.t.(z^2 - \lambda, \lambda, 2z(2z^2 - 2\lambda - y^2)).$$

Using the same technique as in section 4.5 gives a versal deformation of the form

$$H^u(z,y,u_1,u_2) = z(4z^2 - 2y^2 - 4\lambda) + u_1 z + u_2 y^2,$$

and so

$$H^u(x,y,u_1,u_2) = (\lambda + \frac{x^2+y^2}{4})^{1/2}(x^2+y^2) + u_1(\lambda + \frac{x^2+y^2}{4})^{1/2} + u_2 y^2$$

This is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric normal form reminding of figure 1.2. The actual connection with this picture is a research problem, due to the square roots occurring in H^u.

4.7 Bifurcation analysis

We consider for which parameters H^u has a degenerate fixed point, so the determinant of the Hessian matrix must be zero. This gives three conditions:

$$\begin{aligned} xy + u_2 y &= 0 \\ 3x^2 + y^2 + u_1 &= 0 \\ 3x(x+u_2) - y^2 &= 0 \end{aligned}$$

This gives a set given by two parameters x and y. Elimination of these gives the following curves (see BHLV [7]): $u_1 + 3u_2^2 = 0$ and $u_1 = 0$. A bifurcation diagram is given by figure 1.1. The connection with (4.13) is given by the following proposition, proven by an easy calculation:

Proposition 11. *If we solve λ from $u_1 + 3u_2^2 = 0$, $u_1 = 0$ and (4.13) we get*

$$\lambda = -\frac{c_1^2}{6} + O(c_1^3) \text{ and } \lambda = \frac{c_1^2}{2} + O(c_1^3),$$

respectively.

These approximate parabolas are sketched in figure 1.5. They must be the same in figure 1.7. This is a qualitative result, which will be made more precise in the next chapter.

Chapter 5

The normalization procedure applied to the three body problem

In this chapter the normalization procedure of the previous chapter will be applied to the three body problem. First we investigate what types of resonance can occur. Then we inspect what kind of normal form form the previous chapter we get, and relate the bifurcations to differences in the original system. Finally the stability, that is, whether orbits do not tend too far from their initial conditions, is investigated. An important theorem used here is:

Theorem 12. (The Lyapunov Center Theorem) Assume that a system with a non-degenerate integral has an equilibrium point with exponents $\pm\omega i, \lambda_3, \ldots, \lambda_m$, where $i\omega \neq 0$ is pure imaginary. If $\lambda_j/i\omega$ is never an integer for $j = 3, \ldots, m$, then there exists a one-parameter family of periodic orbits emanating from the equilibrium point. Moreover, when approaching the equilibrium point along the family, the periods tend to $2\pi/\omega$.

Proof. See Meyer & Hall [15]. The proof uses of the fact that in this case the periodic orbit can be continued from the linearized to the full system, a result following from the implicit function theorem. For this, see also Broer [5]. □

This theorem will be used to investigate the existence of long and short period families near a 1:2-resonance. They are the near 4π and near 2π periodic solutions corresponding to eigenvalues of about $\pm i/2$ and i found and used in chapters 3 and 4.

5.1 Normalization

5.1.1 Eigenvalues and resonances

Suppose the eigenvalues of the linearized system near the Lagrangian equilibrium are given by (3.16):

$$\omega_{1,2} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \sqrt{1 - 27\sigma}}$$

Then three types of resonance can occur, the first in the restricted and in the full problem, the last two only in the full problem:

1. Between ω_1 and ω_2: $(\omega_1 : \omega_2) = (a : b)$, where none of ω_1 and ω_2 is rational, then

$$\omega_1^2 + \omega_2^2 = \omega_1^2(1 + (\frac{b}{a})^2) = \omega_1^2(\frac{a^2 + b^2}{a^2}) = 1 \Rightarrow$$
$$\omega_1 = \frac{a}{\sqrt{a^2 + b^2}}, \quad \omega_2 = \frac{b}{\sqrt{a^2 + b^2}}$$
$$\omega_1^2\omega_2^2 = (\frac{ab}{a^2 + b^2})^2 = \frac{27}{4}\sigma \Rightarrow \sigma = \frac{4}{27}(\frac{ab}{a^2 + b^2})^2$$

   The 1:1-resonance corresponds to $27\sigma = 1$. This implies loss of stability in the linearized system, see section 5.3.

2. When exactly one of the ω_i's is rational ($\omega_i = a/b$) and $a < b$, then the other is equal to

$$\sqrt{1 - (\frac{a}{b})^2} = \frac{\sqrt{b^2 - a^2}}{b} \quad \text{and, hence,}$$
$$\omega_1^2\omega_2^2 = \frac{(b^2 - a^2)a^2}{b^4} = \frac{27}{4}\sigma \Rightarrow \sigma = \frac{4}{27}\frac{b^4}{(b^2 - a^2)a^2},$$

   so $\sqrt{b^2 - a^2}$ must be irrational.

3. An overall resonance occurs if all the ω_i's are rational. Then from case 1. we can conclude that $\sqrt{a^2 + b^2}$ must be an integer. These are the famous Pythagorean triples (3:4:5), (5:12:13) etc.

5.1.2 Normalizing H_2

The 4×4 matrix normalizing H_2 for the restricted three body problem has been calculated in Meyer&Hall [15]. For the full problem, a 6×6 matrix is needed. These matrices can be found using linear algebra techniques discussed in the previous chapter.

The normalized H_2, if not in 1:1-resonance, has the following form in action-angle variables:

$$H_2 = L_0 + \omega_1 L_1 - \omega_2 L_2, \tag{5.1}$$

where L_0 may be dropped for the restricted three body problem.

Because ω_1 and ω_2 are positive and there is a minus sign in 5.1, the system is indefinite. At this point (5.1) can be compared with (4.1) and all further normalization and reduction discussed in chapter 4 can be applied.

5.1.3 Birkhoff normalization

After Birkhoff normalization, reduction to one d.o.f. and further symplectic transformations all have been applied to the system corresponding to (5.1), we arrive at the following form, following (4.11):

$$H^R = b_1 \frac{x^2+y^2}{2} + O(|x^2+y^2,\lambda|^2) + \\ (b_2 + O(|x^2+y^2,\lambda|)(x^2+y^2)^{\frac{p-1}{2}}(\frac{\lambda}{p} - \frac{x^2+y^2}{2})^{-q/2}x), \quad (5.2)$$

where now q is negative. In the restricted problem the coefficients b_i only depend on the mass ratio μ, whereas in the full problem they depend on three parameters: for instance m_1, m_2, and the non-resonant action, a distinguished parameter. The exact expression for the restricted problem is as follows:

Proposition 13. *In the restricted problem, near 1:2-resonance,*

$$b_1 = \omega - \frac{1}{2} = \frac{\omega_2}{\omega_1} - \frac{1}{2} = \frac{\sqrt{1-\sqrt{1-27\mu(1-\mu)}}}{\sqrt{1+\sqrt{1-27\mu(1-\mu)}}} - \frac{1}{2}$$

$$b_2 = \sqrt{\frac{176060831}{61}}\frac{5^{1/4}}{3528} + O(\omega - \frac{1}{2}) \approx 0.72 + O(1-2\omega)$$

5.2 Bifurcations

If the size of the masses is fixed we can investigate what happens if the total angular momentum c and energy h vary. Then bifurcations occur at the coordinate-axis $c = 0$ and $h = 0$ and the values corresponding to the relative equilibria of Euler-Moulton and Lagrange (see Arnold[2]). These are not considered here, since c is kept constant.

We can also study what happens when c is kept constant, and the masses vary. At $m_1m_2 + m_1m_3 + m_2m_3 = 1/27$ a Hamiltonian Hopf bifurcation occurs, compare [22] and section 3.1. Here the eigenvalues of the linear part go off the imaginary axis and turn into a 'symplectic quadruple'. This occurs both in the restricted and the full three body problem. We consider the neighbourhood of the (1:2) resonance at the normalized system and then study how this can be coupled to the real system.

5.2.1 Bifurcations of the versal deformation in (1:2)-resonance

For studying these bifurcations we consider H^R (4.11). In the (1:2)-resonance it can be written up to 3th order as

$$H^R = x(2\lambda - x^2 + y^2),$$

with a $\mathbb{Z}_2$ symmetry. So here the central singularity is the same as in [7]. This has a versal deformation of the form

$$H^u := x(x^2 + y^2) + u_1 x + u_2 y^2.$$

The analysis has been made in chapter 4. The main point here is that the system lives outside the singular circle, as explained in section 4.3. This circle is visible in figures 1.1 and 1.2.

Outside resonance, Lyapunov's theorem assures the existence of a one parameter family of short, near 2π periodic solutions and a family of long, near 4π periodic solutions. Now we can consider figure 1.5. Actually, this is a projection of a hyperboloid on the plane, which can be explained as follows. In cartesian coordinates, the Hamiltonian corresponding to (4.3) can be expressed by

$$H = x_1^2 + y_1^2 - \omega(x_2^2 + y_2^2) + h.o.t..$$

Restricting to a Poincaré section defined by $y_2 = 0$ and $x_2 > 0$ gives an approximate hyperboloid

$$H = x_1^2 + y_1^2 - \omega x_2^2 + h.o.t. = \text{constant},$$

which is projected on the x_1, y_1 plane, see figure 5.1.

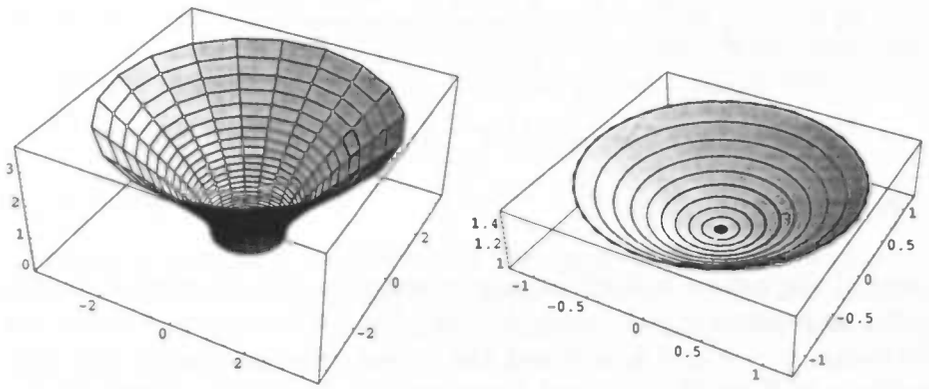


Figure 5.1: The upper half of a hyperboloid. The pictures left and right denote the situation where H is positive and negative, respectively.

The flow of the system defined by H^u gives an approximation of the isoenergetic Poincaré map over the long period: the section is such that long-period orbits correspond to equilibrium points in the Poincaré map. Now assume the detuning is nonzero. If travelling from the bottom of figure 1.5 or 1.7 at first there are only unbounded orbits. Then across the parabola $\lambda = -c_1^2/3(u_1 = 0)$ a Hamiltonian saddle-centre bifurcation occurs. This yields both a stable center and an unstable saddle in figure 1.5, both corresponding to long period orbits parametrized by λ. In figure 1.7 the singular circle corresponds to the center and the two saddles to the saddle in figure 1.5.

At the c_1 axis the long period orbit contracts to the equilibrium, and above this axis the singular circle appears in figure 1.5. On this circle, the original

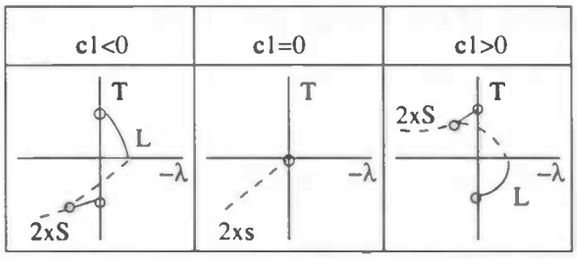


Figure 5.2: Period curves for the manifold of long period orbits (from Henrard [12]). Dotted lines denote unstable orbits.

$L_2 = 0$, so we have short period orbits here, corresponding to the center in 1.7. Other circles around the center are quasi-periodic orbits in the original system.

At the upper parabola a pitchfork bifurcation takes place: the saddle splits into two saddles, connected with two heteroclinic connections along the singular circle, due to the $\mathbb{Z}_2$-symmetry. Since these two new saddles are on the singular circle, they also correspond to short period orbits, and we have long is exactly two times short here. So the pitchfork bifurcation also translates to a period doubling here. In figure 1.7 it can be seen that the stable center merges with the two saddles and forms a new saddle: the stable short period orbit branches with the unstable long period orbit on an unstable short period orbit traveled twice. In this picture the connection with the Möbius strip, compare [9], is also clear.

At the exact resonance, Lyapunov's theorem assures the existence of the short period family, but not of the long one. When looking at the diagrams we see that the long period orbits contract between the two parabolas. Below the c_1-axis, only unstable orbits exist. At the origin we now have an unstable equilibrium, and above this equilibrium becomes the short period orbit traveled twice.

Observe that only between the two parabolas stable orbits exist, so these are also boundaries for stability. The boundaries can be calculated explicitly, by combining propositions 11 and 13:

Proposition 14. *The above mentioned parabolas can be computed by:*

$$\lambda = \frac{-(1-2\omega)^2}{24.88} + O(1-2\omega)^3$$

and

$$\lambda = \frac{(1-2\omega)^2}{8.29} + O(1-2\omega)^3,$$

respectively.

5.2.2 Bifurcations of the versal deformation in other resonances

Again look at H^r (4.11). In the (1:3)-resonance it can be written up to 4th order as

$$H^r = x(\lambda - \frac{1}{3}x^2 + y^2)^{3/2} + \gamma(x^2 + y^2)^2.$$

When only the terms up to fourth order are taken, the bifurcations are investigated in [2]. However since we have a broken exponent, we do not have a polynomial expansion in x and y, so an analysis using a versal deformation appears to be much more difficult. When using a technique as in section 4.6, and also for other resonances, the versal deformation turns out to have infinite codimension. However, other methods do exist, and make use of a technique by Duistermaat [10]; it will not be used in this paper, but it will be used in Lunter [14].

5.3 Stability

"Is the universe stable" is almost a theological question. Here it appears more like "is the solar system stable". Indeed, as we will see, one of the three masses must be more than about 25 times heavier than the other two together, and can be seen as a "sun".

If $\phi(t, \xi)$ is a solution flow, so $\phi(0, \xi) = \xi$, the point 0 is said to be stable (in the sense of Lyapunov) if for every $\epsilon > 0$ there is a $\delta > 0$ such that $||\phi(t, \xi)|| < \epsilon$ for all t whenever $||\xi|| < \delta$

5.3.1 Linear stability

A necessary condition for stability is that none of the eigenvalues of the linearized system have a positive real part. If the system is Hamiltonian, this can only happen if all eigenvalues are are purely imaginary, and this occurs here when $m_1m_2 + m_1m_3 + m_2m_3 < 4/27$ (recall that $m_1 + m_2 + m_3 = 1$). Let $m_1 \geq m_2 \geq m_3$, then a necessary condition for this to happen is that $m_2 + m_3 \geq (1 - \sqrt{69}/9)/2 = 0.03852$, that is, m_1 must be about 24.96 times heavier than the other two together. In the restricted problem, ($m_3 = 0$) simplifies to Routh's critical mass ratio, as mentioned in section 3.1.

If we set $u = (m_1 + m_2)/\sqrt{2}$ and $v = (m_1 - m_2)/\sqrt{2}$ the original condition becomes $(u - \sqrt{2}/3)^2 + (v/\sqrt{3})^2 > (4/9)^2$, so m_1 and m_2 are outside an ellipse rotated with 45 degrees with its major axis equal to $4/9\sqrt{3}$, its minor axis equal to 4/9 and center $(1/3, 1/3)$. Furthermore the m_i's must lie in a triangle where the m_i's are positive and $m_1 + m_2 \leq 1$. See figure 5.3, where the linearized system is stable inside the three little "triangles".

It is sufficient to take m_3 as the greatest mass, so look at the lower left "triangle". The points where the curve cuts the axis correspond to Routh's critical mass ratio, $(1 - \sqrt{69}/9)/2$, as can be expected because in the restricted problem one of the masses is taken equal to zero. The curve isn't far from a straight line here; the greatest distance occurs when $m_1 = m_2$: then $m_1 =$

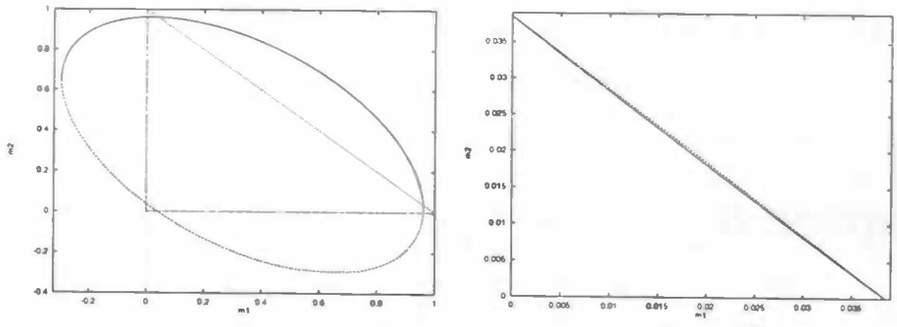


Figure 5.3: Linear stability in three "triangles" and a zoom of a "triangle".

$(3 - 2\sqrt{2})/9 = 0.019063\ldots$ on the curve whereas $m_1 = (1 - \sqrt{69}/9)/4 = 0.019260\ldots$ on the straight line.

If 0 is a local minimum or maximum of H, then 0 is stable. However, it is not in this system, see (5.1): H is indefinite.

5.3.2 Stability in the sense of Lyapunov

For further investigations KAM theory is necessary. In a two d.o.f. system, it can help proving stability; flows are being encapsulated by a two-torus in a three-dimensional manifold. For instance the Lagrange- equilibrium for the restricted three body problem for purely imaginary characteristic exponents has been proven to be stable except at the 1:2 and 1:3 resonances, by considering singularities in the normal form up to 6th order. Also compare figure 1.5.

In this case, at three degrees of freedom there is a 3-torus in a 5D-manifold like a point in a plane or a line in space, and does not encapsulate anything. In this case the following hypothesis applies, see Arnold [2]:

Hypothesis 15. *For this situation the typical case is* topological *instability: through an arbitrarily small neighbourhood of any point there pass phase trajectories along which the slow variables drift away the initial value by a quantity of order 1.*

Mostly this goes exponentially slow, and one speaks of Arnold diffusion. However, following KAM-theory there is metric stability: stability for most initial conditions. A fast evolution is possible only at resonances. (see also [2])

For instance, when $\omega_1 = 2\omega_2$ with help of the function $W = -I_1 I_2^{1/2} \sin(2\phi_1 + \phi_2)$, defined on the flow, instability can be proven for the normalized and the real system as well, see Meyer and Hall[15]. We can also look at figure 1.5 and see that all stable orbits vanish by contraction when $c_1 = 0$. In a similar way the system is also unstable in a (1:3) resonance.

So, assuming the linearized system is stable, the following holds: for most initial conditions and mass ratios the whole system is stable, for other conditions it is a bit unstable, and exceptionally very unstable.

Chapter 6

Numerical comparison

A simple simulation has been written for an overall look. Further, `DSTool` is used to compare the original system with the normalized system.

6.1 Some numerical experiments

Using a little Pascal program orbits are calculated numerically and plotted on the screen. After some seconds the orbits often quit the equilibrium circles, due to discretisation/rounding errors, which serve as a perturbation as well.

In figure 6.1 two bodies finally are going one way while describing ellipses around each other, and the other goes the other way:

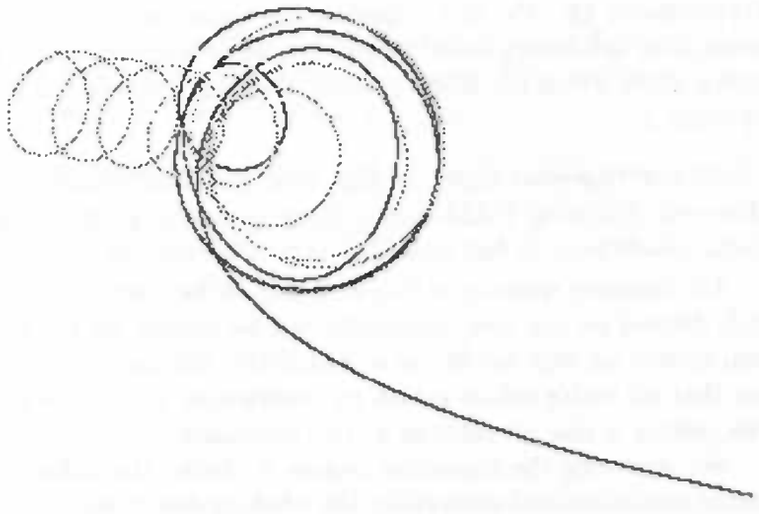


Figure 6.1: Orbits of the three masses in configuration space, mass ratio 1:3:4, initially starting in circular orbits, but perturbed due to linear instability.

6.2 Numerical comparison between the versal deformation and the real system

The (reduced) planar three body system lives on a six dimensional phase space. It is customary to restrict to energy level sets, thereby reducing the dimension to 5. A Poincaré section subsequently reduces the associated vector field to a 4 dimensional map. For the restricted three body problem, the same argument leads to a 2 dimensional map. According to [6], the above obtained normal form is an integrable approximation of the corresponding Poincaré map. Whereas it is possible to plot the 'restricted' 2D, a 4D map is very difficult to plot. However, we can plot only the 2 "resonant" coordinates and check that the other behave like a harmonic oscillator. It is also possible to plot the orbits of an approximate Poincaré map where the two slow variables are assumed to be constant, what of course must be and can be easily checked. However, due to the many technicalities involved, we restricted to plots of the 2D map.

Some of these were made using `DSTool`, using the fourth order Runge-Kutta quality control method. They are visible in figures 1.4 and 1.6. There is a clear correspondence between these and figures 1.5 and 1.7. One can observe unstable and quasi-periodic orbits, the singular circle, higher order and symmetry perturbations. In figure 1.3 these quasi-periodic orbits are plotted in configuration space: here we see (in rotating coordinates!) how a particle can move in the neighbourhood of the relative equilibrium L_4.

The boundaries as computed in proposition 14 have been checked using `DSTool` obtaining the results as in table 6.1. For $\lambda_{\text{measured}}$ we have taken $\frac{\sqrt{5}}{2}H \approx$

H	ω	$\lambda_{\text{measured}}$	$\lambda_{\text{predicted}}$
10^{-7}	0.49935	$1.22 \cdot 10^{-7}$	$1.736 \cdot 10^{-7}$
10^{-7}	0.50055	$1.22 \cdot 10^{-7}$	$1.46 \cdot 10^{-7}$
-10^{-7}	0.49913	$-1.22 \cdot 10^{-7}$	$-1.11 \cdot 10^{-7}$
-10^{-7}	0.50087	$-1.22 \cdot 10^{-7}$	$-1.11 \cdot 10^{-7}$

Table 6.1: Comparison of bifurcation values, found numerically and analytically

$1.22H$ because we divided H by approximately $\sqrt{5}/2$ in section 5.1.2, in the case of a 1:2-resonance. The predictions are not very bad, but to find the exact boundary numerically is difficult due to the existence of unstable orbits on both sides.

Chapter 7

Conclusions

We have seen that the method treated in chapter 4 can do a very good job in the prediction of what occurs in the three body problem near the Lagrangian equilibrium near 1:2-resonance. Not only can we predict how the Poincaré map behaves, but also for which parameters in the original system bifurcations occur. The phenomena treated here have been investigated by others, see for instance Arnold[2] and Henrard[12]. But they used a system that was cut off at fourth order.

The power of the versal deformation used here is that we did not have to cut off, but could predict what happens using several formal coordinate transformations. However, unfortunately this technique seems to work only for the 1:2-resonance.

7.1 Further research

Still, there remains a lot that can be investigated in much more detail:

- The extension of all results from the restricted to the full three body problem. This is also a technical problem, because of the existence of one extra parameter and one more degree of freedom.
- The exact relation between figures 1.2 and 1.7, as sketched in section 4.6.
- An extension to other resonances, if possible.

Bibliography

[1] V.I. Arnold. *Mathematical models of classical mechanics.* Springer-Verlag, 1978.

[2] V.I. Arnold, editor. *Dynamical Systems III.* Springer-Verlag, 1988.

[3] June Barrow-Green. *Poincaré and the Three Body Problem.* AMS/LMS, 1997.

[4] H.W. Broer. Conservative dynamical systems. In *Structures in Dynamics*, volume 2 of *Studies in Mathematical Physics*, pages 267–302. North-Holland, 1991.

[5] H.W. Broer. *Notes on perturbation theory.* University of Groningen, 1992.

[6] H.W. Broer, S.-N. Chow, Y. Kim, and G. Vegter. A normally elliptic hamiltonian bifurcation. *ZAMP*, 44:389–432, 1993.

[7] H.W. Broer, I. Hoveijn, G.A. Lunter, and G. Vegter. Resonances in the spring pendulum: algorithms for equivariant singularity theory. *Nonlinearity*, 1998. Submitted.

[8] H.W. Broer, G.A. Lunter, and G. Vegter. Equivariant singularity theory with distinguished parameters: Two case studies of resonant hamiltonian systems. *Physica D*, 112:64–80, 1998.

[9] H.W. Broer and G. Vegter. Bifurcational aspects of parametric resonance. In *Dynamics Reported, New Series*, volume 1, pages 1–51. Springer-Verlag, 1992.

[10] J.J. Duistermaat. Bifurcations of periodic solutions near equilibrium points of hamiltonian systems. In *Bifurcation Theory and Applications*, volume 1057 of *LNM*, pages 57–105. Springer-Verlag, 1984.

[11] Heinz Hanßmann. The reversible umbilic bifurcation. *Physica D*, 112:81–94, 1998.

[12] Jacques Henrard. Periodic orbits from a resonant equilibrium. *Celestial Mechanics*, 1:437–466, 1969.

[13] G.A. Lunter. The implementation of a normal form theorem. Master's thesis, University of Groningen, 1994.

[14] G.A. Lunter. *In preparation.* PhD thesis, University of Groningen, 1998.

[15] K.R. Meyer and G.R. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-Body problem.* Springer, 1992.

[16] Harry Pollard. *Mathematical Introduction to Celestial Mechanics.* Prentice-Hall, 1966.

[17] J.A. Sanders and J.-C. van der Meer. Unique normal form of the hamiltonian 1:2-resonance. In H.W. Broer and F. Takens, editors, *Geometry and analysis in nonlinear dynamics*, volume 222 of *Pitman Research Notes in Mathematics*, pages 56–69. Longman Scientific & Technical, 1992.

[18] D.S. Schmidt. Transformation to versal normal form. In K.R. Meyer and D.S. Schmidt, editors, *Computer Aided Proofs in Analysis*, volume 28 of *IMA*. Springer-Verlag, 1990.

[19] C.L. Siegel and J.K.Moser. *Lectures on Celestial Mechanics.* Springer-Verlag, 1971.

[20] Carles Simó and Regina Martínez. Qualitative study of the planar isosceles three-body problem. *Celestial Mechanics*, 41:179–251, 1988.

[21] R. Thurman and P.A. Worfolk. The geometry of halo orbits in the circular restricted three-body problem. Geometry Center Research Report GCG95, University of Minnesota, 1996.

[22] J.-C. van der Meer. *The Hamiltonian Hopf Bifurcation*, volume 1160 of *LNM*. Springer-Verlag, 1985.

[23] E.T. Whittaker. *A treatise on the analytical dynamics of particles and rigid bodies.* Cambridge University Press, 1917.

[24] Aurel Wintner. *The analytical foundations of celestial mechanics.* Princeton University Press, 1947.

[25] Z. Xia. Arnold diffusion and oscillatory solutions in the planar three-body problem. *Journal of Differential Equations*, 110:289–321, 1994.