A Class of Singular Linear Partial Differential Equations

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Contents

Introduction

Analytic ordinary differential equations can often be reduced to a normal form by means of normalizing transformations. For example, L. Stolovitch has shown in [8] that the system

$$z^2 \frac{\mathrm{d}x_i}{\mathrm{d}z} = x_i(\lambda_i + \alpha_i z) + z f_i(z,x), \quad i = 1,2,\dots,n$$

can be linearized to

$$z^2 \frac{\mathrm{d}y_i}{\mathrm{d}z} = y_i(\lambda_i + \alpha_i z), \quad i = 1,2,\dots,n,$$

by means of a transformation $x_i = y_i + \varphi_i(z,y)$, $i = 1,2,\dots,n$, if certain conditions are satisfied. The main condition is $\mathcal{I}m\ \lambda_i \geq 0$, $i = 1,2,\dots,n$. The transformation functions φ_i are a solution of a system of singular non-linear partial differential equations.

In this report we study a related class of linear partial differential equations with a singularity at the origin:

$$z^2 \frac{\partial \Phi}{\partial z} + \sum_{j=1}^{n} (\lambda_j + \alpha_j z) y_j \frac{\partial \Phi}{\partial y_j} + A(z,y)\Phi = z a(z,y), \tag{0.1}$$

where $z \in \mathbb{C}$, $y \in \mathbb{C}^n$, $\Phi = \Phi(z,y) \in \mathbb{C}^n$, $(\lambda_j, \alpha_j) \in \mathbb{C} \times \mathbb{C}$, $\mathcal{I}m\ \lambda_j \geq 0$, $A(z,y)$ an $n \times n$-matrix analytic at $(0,0)$, $A(z,y) = A_0 + A_1 z + O(z^2)$ as $z \to 0$, with A_0 and A_1 diagonal matrices, and $a(z,y)$ an n-vector analytic at $(0,0)$.

Contents of the chapters.

This report starts with a concise review of notations, definitions and results. Most important are the definitions of the Laplace and Borel transformation and some of their properties. Most of these properties are proven in the appendix.

In chapter 2 we treat a system of singular linear differential equations that we encountered while we were proving the existence of a formal power series solution of (0.1). The problem is treated with the aid of the method in [2], section 2.

Chapter 3 can be divided into two parts. The first section lists a few properties of analytic functions in several complex variables, while the other section deals with interchanging Borel transformation and summation.

In chapter 4 we prove the existence of a formal power series solution of (0.1) if certain extra assumptions are fulfilled. We actually prove the existence of a formal series $\sum_{Q\in\mathbb{N}^n} \Phi_Q(z)y^Q$, where all coefficients Φ_Q are analytic in a sector S of opening arbitrary close to 2π. The sector S is independent of the multi-index Q. Moreover, all the coefficients Φ_Q can be written as Laplace integrals (cf. theorem 4.2.1).

In chapter 5 and chapter 6 we make the assumption of no resonance. In that case we can and shall make the assumption that all α_j are equal to zero. Moreover we assume $A_1 = 0$ (i.e. all the diagonal elements of A_1 are equal to zero).

In chapter 5 we consider the case that the eigenvalues λ_j also satisfy a Siegel condition. Using the contraction mapping principle we see that the formal solution with analytic coefficients, found in chapter 4, can be lifted to a solution $\Phi(z, y)$ of (0.1), which is analytic in a neighbourhood of $(0,0)$ with z restricted to S (cf. theorem 5.6.2).

In chapter 6 we give a different proof of almost the same result as in the preceding chapter, but this proof doesn't use the Siegel condition (cf. theorem 6.6.2). The proof is on the analogy of the method in [2], section 4.

Chapter 7 deals with a special case if resonance relations are allowed. Under some additional hypotheses, the proof of the existence of an analytic solution of (0.1) is quite similar to the proof given in chapter 5 (cf. theorem 7.1.2).

Chapter 1

Notations and definitions

In this chapter we give a concise review of notations and definitions that we will use in this report.

1.1 Some notations

When we write $\mathbb{N}$ we mean the set of natural numbers *inclusive* 0.

We use the notation $\mathbb{C}[[z]]$ for the set of all formal power series with complex coefficients, while, for $n \in \mathbb{N}_{\geq 1}$, $\mathsf{M}_n(\mathbb{C})$ is the set of all $n \times n$-matrices with complex coefficients.

For $\rho > 0$ we define the disc $\Delta_1(\rho)$ by

$$\Delta_1(\rho) := \left\{ z \in \mathbb{C} \;\middle|\; |z| < \rho \right\}.$$

If $\rho > 0$ and $n \in \mathbb{N}_{\geq 2}$, then we define the polydisc $\Delta_n(\rho)$ by

$$\Delta_n(\rho) := \left\{ z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n \;\middle|\; |z_j| < \rho, \;\; \forall j = 1, 2, \dots, n \right\}.$$

A sector (on the Riemann surface of the Logarithm) is defined to be a set of the form

$$S(d, \alpha, \rho) := \left\{ z = re^{i\varphi} \in \mathbb{C} \;\middle|\; 0 < r < \rho, \;\; d - \frac{\alpha}{2} < \varphi < d + \frac{\alpha}{2} \right\},$$

where $d \in \mathbb{R}$, $\alpha > 0$ and ρ either a positive real number or $+\infty$. We shall refer to d, resp. α, resp. ρ as *the bisecting direction*, resp. *the opening*, resp. *the radius* of $S(d, \alpha, \rho)$. In case $\rho = \infty$ we mostly write $S(d, \alpha)$ instead of $S(d, \alpha, \infty)$.

A closed subsector $\overline{S}_1 \subset S(d, \alpha)$ is a set of the form

$$\overline{S}_1 = \{ z \in S(d, \alpha) \mid \beta_1 \leq \arg z \leq \beta_2 \},$$

where $d - \alpha/2 < \beta_1 \leq \beta_2 < d + \alpha/2$.

A bounded closed subsector $\overline{S}_1 \subset S(d, \alpha, \rho)$ (where $\rho \leq \infty$) is a closed

subsector of $S(d,\alpha)$ with finite radius smaller than ρ.
Note that the name "*closed*" is a bit confusing, because a (bounded) closed subsector is **not** a closed set in the usual sense (the origin does not belong to such a sector).
A neighbourhood of 0 in $S(d,\alpha)$ will be a set of the form

$$\{z \in S(d,\alpha) \mid 0 < |z| < r(\arg z)\},$$

where r is some positive-valued continuous function on $(d-\alpha/2, d+\alpha/2)$. In particular we have $S(d,\alpha,\rho)$ being a neighbourhood of 0 in $S(d,\alpha)$.

1.2 Analytic functions in sectors

For $\mu > 0$ we say that a function, defined on (a neighbourhood of ∞[1] in) $S = S(d,\alpha)$, is of exponential growth of order $\leq \mu$ if to every closed subsector $\overline{S}_1 \subset S$ corresponds a positive constant c, depending upon $\overline{S}_1$, such that $f(z) = O(1)e^{c|z|^\mu}$ as $z \to \infty$ on $\overline{S}_1$, i.e. to every closed subsector $\overline{S}_1 \subset S$ there exist positive constants R, M and c, all depending upon $\overline{S}_1$, such that

$$|f(z)| \leq Me^{c|z|^\mu}, \quad z \in \overline{S}_1, \quad |z| \geq R.$$

Given a function f, analytic in some sector S, and a formal power series $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathbb{C}[[z]]$, one says that $f(z)$ asymptotically equals $\hat{f}(z)$ as $z \to 0$ on S, if to every non-negative integer N and every bounded closed subsector $\overline{S}_1$ of S there exists a positive constant $C = C(N, \overline{S}_1)$ such that

$$|R_f(z,N)| \leq C, \quad \forall z \in \overline{S}_1,$$

where $R_f(z,N) = z^{-N}\left(f(z) - \sum_{n=0}^{N-1} f_n z^n\right)$.
The notation for this is

$$f(z) \simeq \hat{f}(z) \quad \text{as} \quad z \to 0 \quad \text{on} \quad S.$$

From now on we use the following convention:

> *A function is called analytic on a closed set if it is continuous on this set and analytic in the interior.*

1.3 The Laplace and Borel transform

Most of the properties listed in this section are proven in the appendix. For details and omitted proofs the reader is referred to [1].

[1] The definition is analogous to the definition of a neighbourhood of 0.

Let $f : S(d,\alpha) \to \mathbb{C}$ be analytic and of exponential growth of order ≤ 1, whereas $f(u) = O(u^{\varepsilon-1})$ as $u \to 0$ on $S(d,\alpha)$ for some $\varepsilon > 0$. Then the Laplace transform in the direction $\theta \in \left(d - \frac{\alpha}{2}, d + \frac{\alpha}{2}\right)$ of f is defined by

$$(\mathcal{L}_\theta f)(z) = \int_0^{\infty e^{i\theta}} f(u) e^{-\frac{u}{z}} \mathrm{d}u,$$

with integration along $\arg u = \theta$.
By analytic continuation (i.e. changes of θ) one obtains a Laplace transform $(\mathcal{L}f)(z)$ which is analytic in a neighbourhood of 0 in $S(d, \alpha + \pi)$.
Let

$$\hat{f}(u) = \sum_{m=0}^{\infty} a_m u^m \in \mathbb{C}[[u]],$$

then the formal Laplace transform of $\hat{f}$, $\hat{\mathcal{L}}\hat{f}$, will be defined by

$$(\hat{\mathcal{L}}\hat{f})(z) := \sum_{m=0}^{\infty} a_m \Gamma(m+1) z^{m+1} \in z\mathbb{C}[[z]].$$

If $f(u) \simeq \hat{f}(u)$ as $u \to 0$ on $S(d,\alpha)$, and the assumptions above on f are satisfied, then $(\mathcal{L}f)(z) \simeq (\hat{\mathcal{L}}\hat{f})(z)$ as $z \to 0$ on a neighbourhood of 0 in $S(d, \alpha + \pi)$. A more precise formulation and a proof can be found in the appendix.

Let U be a neighbourhood of 0 in $S(d, \alpha + \pi)$, $\alpha > 0$. Let $g : U \to \mathbb{C}$ be analytic and $g(z) = O(z^\delta)$ as $z \to 0$ on U, where $\delta \in \mathbb{R}$. Moreover, let $d_\pm$ be directions in $S(d, \alpha + \pi)$ such that $d_+ > d + \pi/2$ and $d_- < d - \pi/2$. Then the Borel transform of g is defined by

$$(\mathcal{B}_\gamma g)(u) = \frac{1}{2\pi i} \int_\gamma g(z) e^{\frac{u}{z}} \mathrm{d}(z^{-1}),$$

where γ is a loop from 0 to 0 in U (i.e. $\gamma \setminus \{0\} \subset U$), with the first part in direction d_+ and the last part in direction d_-. Changing the loop γ gives an analytic continuation of $\mathcal{B}_\gamma g$ and in this way we find a Borel transform $(\mathcal{B}g)(u)$ which is analytic and of exponential growth of order ≤ 1 in $S(d,\alpha)$.
Let

$$\hat{g}(z) = \sum_{m=1}^{\infty} a_m z^m \in z\mathbb{C}[[z]],$$

then the formal Borel transform of $\hat{g}$, $\hat{\mathcal{B}}\hat{g}$, is defined by

$$(\hat{\mathcal{B}}\hat{g})(u) = \sum_{m=1}^{\infty} \frac{a_m}{\Gamma(m)} u^{m-1} \in \mathbb{C}[[u]].$$

If $g(z) \simeq \hat{g}(z)$ as $z \to 0$ on U and the assumptions above on g are satisfied, then $(\mathcal{B}g)(u) \simeq (\hat{\mathcal{B}}\hat{g})(u)$ as $u \to 0$ on $S(d,\alpha)$. This is also proven in the appendix.

We have $\mathcal{BL} = \text{id}$ and $\mathcal{LB} = \text{id}$ on the spaces of functions f and g which satisfy the assumptions above.
Furthermore, if f and g are analytic and of exponential growth of order ≤ 1 in a sector $S(d,\alpha)$ and $f(z)$, $g(z) = O(z^{\varepsilon-1})$ as $z \to 0$ on $S(d,\alpha)$ for some $\varepsilon > 0$, then the convolution of f and g,

$$(f * g)(z) := \int_0^z f(\tau)g(z-\tau)\mathrm{d}\tau,$$

is defined, analytic and of exponential growth of order ≤ 1 in the sector $S(d,\alpha)$. Moreover, $(f*g)(z)$ is of order $O(z^{2\varepsilon-1})$ as $z \to 0$ on $S(d,\alpha)$.
Hence, $\mathcal{L}(f*g)$ is defined and analytic in a neighbourhood of 0 in $S(d,\alpha+\pi)$ and the equation

$$\mathcal{L}(f*g) = \mathcal{L}f \cdot \mathcal{L}g$$

holds on a neighbourhood of 0 in $S(d,\alpha+\pi)$. From this and the identity formulas one easily obtains

$$\mathcal{B}(f \cdot g) = \mathcal{B}f * \mathcal{B}g,$$

on $S(d,\alpha)$, for functions f and g which satisfy the assumptions in the definition of $\mathcal{B}$.
A proof of the convolution property can be found in the appendix.

We will end this section with a definition and some properties of a class of Laplace integrable functions:

Definition 1.3.1 *Let $0 < \delta < \pi$ and define $S_c = S(\frac{\pi}{2}, \pi-\delta)$. Let $\overline{S_c} = \overline{S(\frac{\pi}{2}, \pi-\delta)}$ be the closure of S_c, so $\overline{S_c} = \{z \in \mathbb{C}^* \mid \frac{\delta}{2} \leq \arg z \leq \pi - \frac{\delta}{2}\} \cup \{0\}$. Then for arbitrary $\tau > 0$ the class $\mathcal{A}_\tau$ is defined to be the set of functions $\varphi : \overline{S_c} \to \mathbb{C}^n$ such that*

(i) φ is analytic in $\overline{S_c}$ (i.e. $\varphi \in C(\overline{S_c}, \mathbb{C}^n)$ and φ is analytic in the interior S_c of $\overline{S_c}$);

(ii) $\varphi(u) = O(1)e^{\tau_1|u|}$ as $u \to \infty$ on $\overline{S_c}$ for all $\tau_1 > \tau$;

(iii) There exist constants $\varphi_m \in \mathbb{C}^n$, $m = 0,1,2,\ldots$, such that:
For all $N \in \mathbb{N}$ a constant C_N, depending upon N, exists such that

$$\left|\varphi(u) - \sum_{m=0}^{N-1} \varphi_m u^m\right| \leq C_N |u|^N, \quad \forall\, u \in \overline{S_c}(1),$$

where $\overline{S_c}(1) := \overline{S_c} \cap \overline{\Delta_1(1)}$.

Remark 1.3.2

(1) $\mathcal{A}_\tau$ is a linear space and $\mathcal{A}_{\tau_1} \subset \mathcal{A}_{\tau_2}$ if $\tau_2 > \tau_1$.
If $n = 1$ every monomial u^m belongs to $\mathcal{A}_\tau$ (the third condition is satisfied by taking $\varphi_k = 1$ if $k = m$ and $\varphi_k = 0$ otherwise).
The class $\mathcal{A}_\tau$ depends on $\delta \in (0, \pi)$, but to keep notation simple we do not display this dependence.

(2) The third property in definition 1.3.1 obviously implies

$$\varphi(u) \simeq \sum_{m=0}^{\infty} \varphi_m u^m \quad \text{as } u \to 0 \text{ on } S_c.$$

(3) For a function $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in \mathcal{A}_\tau$ the Laplace transform $\mathcal{L}\varphi := (\mathcal{L}\varphi_1, \mathcal{L}\varphi_2, \cdots, \mathcal{L}\varphi_n)$ is defined and analytic in a neighbourhood U_δ of 0 in $S(\frac{\pi}{2}, 2\pi - \delta)$.
Then to every $\varepsilon \in (\delta, \pi)$ there exists a positive radius r_ε such that $\mathcal{L}\varphi$ is defined and analytic in the sector $S_d := S(\frac{\pi}{2}, 2\pi - \varepsilon, r_\varepsilon)$ (cf. [1]).
This sector S_d depends on $\varepsilon \in (\delta, \pi)$, but to keep notation simple we do not display this dependence.

(4) A similar definition will be used for matrix-valued functions.

Chapter 2

On a system of singular linear ordinary differential equations

Let the following system of singular linear ordinary differential equations

$$z^2 \frac{\mathrm{d}\Phi}{\mathrm{d}z} + A\Phi = z\tilde{A}\Phi + zB(z)\Phi + zc(z), \tag{2.1}$$

where $\Phi = \Phi(z) \in \mathbb{C}^n$, be given. We will study this system on a sector S with bisecting direction $\frac{\pi}{2}$, opening between π and 2π and finite radius.

2.1 A few hypotheses

We assume A and $\tilde{A}$ to be diagonal matrices and we write

$$A = \operatorname{diag}\{a_1, a_2, \dots, a_n\} \quad \text{and} \quad \tilde{A} = \operatorname{diag}\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\},$$

in which $a_i \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathcal{I}m\ z \geq 0\}$ for each $i \in \{1, 2, \dots, n\}$.
Suppose there exist $\delta \in (0, \pi)$ and U a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi - \delta)$ such that the matrix-valued function $B(z)$ and the vector-valued function $c(z)$ are analytic in U (i.e. all the entries of B and all the components of c are analytic in U). Moreover, we assume that B (resp. c) asymptotically equals the series $\sum_{m=1}^{\infty} B_m z^m$ (resp. $\sum_{m=1}^{\infty} c_m z^m$) as $z \to 0$ on U. Here $B_m \in \mathsf{M}_n(\mathbb{C})$ and $c_m \in \mathbb{C}^n$ for all $m \in \mathbb{N}_{\geq 1}$. This implies that both $\beta := \mathcal{B}B$ and $\gamma := \mathcal{B}c$ are defined and analytic in $S(\frac{\pi}{2}, \pi - \delta)$.
Finally, we assume that β belongs to $\mathcal{A}_{\tau'}$ for some $\tau' > 0$ and that γ belongs to $\mathcal{A}_\tau$ for some $\tau > \tau'$. As $\mathcal{A}_{\tau'} \subset \mathcal{A}_\tau$ we have $B = \mathcal{L}\beta$ and $c = \mathcal{L}\gamma$ on S_d, where S_d is defined as in remark 1.3.2(3).

Our aim is to find an analytic solution Φ of the system (2.1) which can be written as a Laplace integral $\Phi = \mathcal{L}w$, with w some vector-valued function in $\mathcal{A}_\tau$.

2.2 A formal solution

In this section we look for a formal solution of (2.1). In fact we will prove the following

Theorem 2.2.1 *Suppose that for all $i \in \{1, 2, \ldots, n\}$ with $a_i = 0$ we have $\tilde{a}_i \notin \mathbb{N}$, then the system (2.1) has a unique formal power series solution $\sum_{m=0}^{\infty} \Phi_m z^m$ with $\Phi_0 = 0$.*

Proof. The assumptions on B and c imply

$$B(z) \simeq \hat{B}(z) = \sum_{m=1}^{\infty} B_m z^m \quad \text{as } z \to 0 \text{ on } S_d,$$

and

$$c(z) \simeq \hat{c}(z) = \sum_{m=1}^{\infty} c_m z^m \quad \text{as } z \to 0 \text{ on } S_d.$$

When one formally substitutes the series $\hat{\Phi}(z) = \sum_{m=0}^{\infty} \Phi_m z^m$ in the equation

$$z^2 \frac{\mathrm{d}\Phi}{\mathrm{d}z} + A\Phi = z\tilde{A}\Phi + z\hat{B}(z)\Phi + z\hat{c}(z)$$

one obtains

$$\sum_{m=2}^{\infty} (m-1)\Phi_{m-1} z^m + \sum_{m=0}^{\infty} A\Phi_m z^m$$
$$=$$
$$\sum_{m=2}^{\infty} c_{m-1} z^m + \sum_{m=1}^{\infty} \tilde{A}\Phi_{m-1} z^m + \sum_{m=2}^{\infty} \left\{ \sum_{k=1}^{m-1} B_k \Phi_{m-1-k} \right\} z^m.$$

Comparing coefficients gives the following recurrence relation

$$\begin{cases} A\Phi_0 = 0, & m = 0 \\ A\Phi_1 = \tilde{A}\Phi_0, & m = 1 \\ (m-1)\Phi_{m-1} + A\Phi_m = c_{m-1} + \tilde{A}\Phi_{m-1} + \\ \qquad \sum_{k=1}^{m-1} B_k \Phi_{m-1-k}, & m \geq 2 \end{cases}.$$

If $a_i \neq 0$ for some $i \in \{1, 2, \ldots, n\}$, the first equation implies $\Phi_0^{(i)} = 0$. (Here $\Phi_0^{(i)}$ denotes the i^{th} coefficient of Φ_0.) In case $a_i = 0$ we have $\tilde{a}_i \notin \mathbb{N}$, so in particular $\tilde{a}_i \neq 0$ and thus in this case the equation $a_i \Phi_1^{(i)} = \tilde{a}_i \Phi_0^{(i)}$ implies $\Phi_0^{(i)} = 0$. Hence $\Phi_0 = 0$.
In a similar way we obtain $\Phi_1^{(i)} = 0$ for those $i \in \{1, 2, \ldots, n\}$ with $a_i \neq 0$ and if $a_i = 0$, we have $\Phi_1^{(i)} = (1 - \tilde{a}_i)^{-1} c_1^{(i)}$, which follows from the third equation with $m = 2$.

Using induction we see that for $N \geq 2$ the coefficients of Φ_N can be found from the third equation with $m = N$ in case $a_i \neq 0$ and with $m = N+1$ if $a_i = 0$.
Note that this formal solution is unique, for Φ_N only depends on the coefficients c_m, B_m and on the preceding Φ_k.
The solution is non-trivial if and only if there exists an integer n such that $c_n \neq 0$. ∎

2.3 Two useful lemmas

Lemma 2.3.1 *For two scalar functions φ_1, $\varphi_2 \in \mathcal{A}_\tau$ we have $\varphi_1 * \varphi_2 \in \mathcal{A}_\tau$.*

Proof. For $\xi \in S_c$ we have

$$\begin{aligned}(\varphi_1 * \varphi_2)(\xi) &= \int_0^\xi \varphi_1(\xi - s)\varphi_2(s)\mathrm{d}s \\ &= \int_0^{\frac{\xi}{2}} \varphi_1(\xi - s)\varphi_2(s)\mathrm{d}s + \int_0^{\frac{\xi}{2}} \varphi_1(s)\varphi_2(\xi - s)\mathrm{d}s.\end{aligned}$$

Moreover, if $\varphi \in \mathcal{A}_\tau$ we have for $\xi \in S_c$ and $h \in \mathbb{C}$ such that $\xi + h \in S_c$:

$$\varphi(\xi + h) = \varphi(\xi) + h\varphi'(\xi) + \int_0^h (h - t)\varphi''(\xi + t)\mathrm{d}t,$$

as is easily seen by computing the latter integral (using integration along the straight line $[0, h]$ and the fact that S_c is a convex set).
Hence

$$\begin{aligned}\frac{\varphi(\xi + h) - \varphi(\xi)}{h} - \varphi'(\xi) &= \frac{1}{h}\int_0^h (h - t)\varphi''(\xi + t)\mathrm{d}t \\ &= h\int_0^1 (1 - \sigma)\varphi''(\xi + h\sigma)\mathrm{d}\sigma\end{aligned}$$

and so

$$\begin{aligned}\left|\frac{\varphi(\xi + h) - \varphi(\xi)}{h} - \varphi'(\xi)\right| &\leq |h| \int_0^1 |(1 - \sigma)||\varphi''(\xi + h\sigma)||\mathrm{d}\sigma| \\ &\leq |h| \cdot \sup_{\zeta \in [\xi, \xi + h]} |\varphi''(\zeta)|.\end{aligned}$$

After these helpful remarks we can prove that $\varphi_1 * \varphi_2$ is analytic in S_c. To that end we choose $u \in S_c$ arbitrary and take $h \in \mathbb{C}$ such that $u + h \in S_c$.

We then have

$$\begin{aligned}
&\frac{(\varphi_1 * \varphi_2)(u+h) - (\varphi_1 * \varphi_2)(u)}{h} \\
= &\frac{1}{h}\Big[\int_0^{\frac{u+h}{2}} \varphi_1(u+h-s)\varphi_2(s)\mathrm{d}s - \int_0^{\frac{u}{2}} \varphi_1(u-s)\varphi_2(s)\mathrm{d}s\Big] \quad + \\
&\frac{1}{h}\Big[\int_0^{\frac{u+h}{2}} \varphi_1(s)\varphi_2(u+h-s)\mathrm{d}s - \int_0^{\frac{u}{2}} \varphi_1(s)\varphi_2(u-s)\mathrm{d}s\Big].
\end{aligned}$$

As φ_1 and φ_2 satisfy exact the same conditions, it is sufficient to consider one of these two terms in order to prove the existence of $(\varphi_1 * \varphi_2)'(u)$. We have

$$\begin{aligned}
&\frac{1}{h}\Big[\int_0^{\frac{u+h}{2}} \varphi_1(u+h-s)\varphi_2(s)\mathrm{d}s - \int_0^{\frac{u}{2}} \varphi_1(u-s)\varphi_2(s)\mathrm{d}s\Big] \\
= &\frac{1}{h}\Big[\int_0^{\frac{u}{2}} (\varphi_1(u+h-s) - \varphi_1(u-s))\varphi_2(s)\mathrm{d}s\Big] \quad + \\
&\frac{1}{2}\cdot\frac{2}{h}\cdot\Big[\int_{\frac{u}{2}}^{\frac{u+h}{2}} \varphi_1(u+h-s)\varphi_2(s)\mathrm{d}s\Big].
\end{aligned}$$

As S_c is an open set and both u and $\frac{u}{2}$ belong to S_c, there exists a positive δ such that for all $h \in \mathbb{C}$, $|h| < \delta$, we have $u+h$, $\frac{u}{2}+h \in S_c$.
By drawing a picture it is easily seen that there exists a compact set $K \subset S_c$ such that $[u-s, u-s+h] \subset K$ for all $h \in \mathbb{C}$, $|h| < \delta$, and for all $s \in [0, \frac{u}{2}]$.
Using the fact that we can integrate along the straight line $[0, \frac{u}{2}] \subset S_c$ we obtain for all $h \in \mathbb{C}$, $|h| < \delta$:

$$\begin{aligned}
&\left|\frac{1}{h}\int_0^{\frac{u}{2}} (\varphi_1(u+h-s) - \varphi_1(u-s))\varphi_2(s)\mathrm{d}s - \int_0^{\frac{u}{2}} \varphi_1{}'(u-s)\varphi_2(s)\mathrm{d}s\right| \\
\leq\ &\int_0^{\frac{u}{2}} \left|\frac{\varphi_1(u+h-s) - \varphi_1(u-s)}{h} - \varphi_1{}'(u-s)\right| |\varphi_2(s)||\mathrm{d}s| \\
\leq\ &\sup_{s\in[0,\frac{u}{2}]} |\varphi_2(s)| \cdot \int_0^{\frac{u}{2}} |h| \cdot \sup_{\zeta\in K}|\varphi_1{}''(\zeta)| \cdot |\mathrm{d}s| \\
\leq\ &\sup_{s\in[0,\frac{u}{2}]} |\varphi_2(s)| \cdot \sup_{\zeta\in K}|\varphi_1{}''(\zeta)| \cdot \left|\frac{u}{2}\right| \cdot |h|
\end{aligned}$$

and this tends to 0 as $|h| \to 0$.
The term $\frac{1}{2}\cdot\frac{2}{h}\cdot\big[\int_{\frac{u}{2}}^{\frac{u+h}{2}} \varphi_1(u+h-s)\varphi_2(s)\mathrm{d}s\big]$ tends to $\frac{1}{2}\varphi_1(\frac{u}{2})\varphi_2(\frac{u}{2})$ as $h \to 0$, for φ_1 and φ_2 both are continuous in $\frac{u}{2}$, the function φ_2 is bounded on K

and for all $h \in \mathbb{C}$, $|h| < \delta$, we have the estimate

$$\begin{aligned}
&\left| \frac{2}{h} \int_{\frac{u}{2}}^{\frac{u+h}{2}} \varphi_1(u+h-s)\varphi_2(s)\mathrm{d}s - \varphi_1\Big(\frac{u}{2}\Big)\varphi_2\Big(\frac{u}{2}\Big) \right| \\
\leq \; & \frac{2}{|h|} \int_{\frac{u}{2}}^{\frac{u+h}{2}} \left|\varphi_1(u+h-s) - \varphi_1\Big(\frac{u}{2}\Big)\right| |\varphi_2(s)|\, |\mathrm{d}s| + \\
& \frac{2}{|h|} \int_{\frac{u}{2}}^{\frac{u+h}{2}} \left|\varphi_1\Big(\frac{u}{2}\Big)\right| \left|\varphi_2(s) - \varphi_2\Big(\frac{u}{2}\Big)\right| |\mathrm{d}s|.
\end{aligned}$$

To prove the continuity of $\varphi_1 * \varphi_2$ on $\overline{S_c}$ we only have to consider this function on $\overline{S_c} \setminus S_c$. First take $R > 0$ arbitrary and define $X = \overline{S_c} \cap \overline{\Delta_1(R)}$. Next take $u \in \overline{S_c} \setminus S_c$, $|u| \leq R - 1$, and let $h \in \mathbb{C}$, $|h| \leq 1$, so that $u + h \in \overline{S_c}$.
As both 0 and $u + h$ belong to X, the convexity of X implies $\frac{1}{2}(u+h) \in X$. By halving h we can assume that both $u + h$ and $\frac{u}{2} + h$ belong to X. As before we split $(\varphi_1 * \varphi_2)(u+h) - (\varphi_1 * \varphi_2)(u)$ and we only have to consider the function

$$u \mapsto \begin{array}{l} \int_0^{\frac{u}{2}} (\varphi_1(u+h-s) - \varphi_1(u-s))\varphi_2(s)\mathrm{d}s \quad + \\ \int_{\frac{u}{2}}^{\frac{u+h}{2}} \varphi_1(u+h-s)\varphi_2(s)\mathrm{d}s, \end{array} \tag{2.2}$$

in order to prove that $\varphi_1 * \varphi_2$ is continuous in u. Before studying this function we first note that for all $s \in [0, \frac{u}{2}]$ the difference $u - s$ can be written as $u - s = t \cdot \frac{u}{2} + (1-t) \cdot u$ for some $t \in [0,1]$. Hence, both $u - s$ and $u + h - s = t \cdot (\frac{u}{2} + h) + (1-t) \cdot (u+h)$ belong to X. In a similar way we have $u + h - s \in X$ for all $s \in [\frac{u}{2}, \frac{u+h}{2}]$.
As φ_1 is continuous on $\overline{S_c}$, it is uniformly continuous on X. The function φ_2 is bounded on X. So the first integral in (2.2) tends to 0 if $|h| \to 0$.
Moreover, both φ_1 and φ_2 are bounded on X and thus the second integral in (2.2) also tends to 0 if $|h| \to 0$.
But $u \in \overline{S_c} \setminus S_c$, $|u| \leq R - 1$, and $R > 0$ were chosen arbitrary. This implies the continuity of $\varphi_1 * \varphi_2$ on $\overline{S_c}$.
Next choose $\tau_1 > \tau$ arbitrary, then for $i \in \{1,2\}$ there exist constants $M_i > 0$ such that $|\varphi_i(u)| \leq M_i \exp(\tau_1 |u|)$ for all $u \in \overline{S_c}$. This implies $|(\varphi_1 * \varphi_2)(u)| \leq M_1 M_2 |u| \exp(\tau_1 |u|)$. As $|u| \leq \frac{e^{\varepsilon|u|}}{\varepsilon}$, we have the estimate $|(\varphi_1 * \varphi_2)(u)| \leq \frac{M_1 M_2}{\varepsilon} \exp((\tau_1 + \varepsilon)|u|)$, $u \in \overline{S_c}$, for all $\varepsilon > 0$. Because $\tau_1 > \tau$ and $\varepsilon > 0$ were chosen arbitrary we conclude

$$|(\varphi_1 * \varphi_2)(u)| = O(1)e^{\tilde{\tau}|u|} \quad \text{as } u \to \infty \text{ on } \overline{S_c}$$

for all $\tilde{\tau} > \tau$.
Finally we will show the third property in definition 1.3.1. For $i \in \{1,2\}$ the functions φ_i belong to A_τ and thus for each $N \in \mathbb{N}$ constants $K_{N,i}$ can

be found such that

$$\left|\varphi_i(u) - \sum_{m=0}^{N-1} \varphi_{m,i} u^m\right| \le K_{N,i}|u|^N, \qquad \forall\, u \in \overline{S_c}(1),$$

where $\varphi_{m,i} \in \mathbb{C}$, $m = 0, 1, 2, \ldots$. We will show that

$$\left|(\varphi_1 * \varphi_2)(u) - \sum_{m=0}^{N-1} \left\{ \sum_{k=0}^{m} \varphi_{k,1}\varphi_{(m-k),2} B(k+1, m-k+1) \right\} u^{m+1}\right|$$

$$\le M_N |u|^{N+1}, \quad \forall\, u \in \overline{S_c}(1),$$

where B is the beta function and M_N is a positive constant depending upon N:

For $u \in \overline{S_c}(1)$ we have $\left(\text{with } \tilde{R}_{N,i}(u) = \varphi_i(u) - \sum_{m=0}^{N-1} \varphi_{m,i} u^m\right)$

$$\left|(\varphi_1 * \varphi_2)(u) - \sum_{m=0}^{N-1} \left\{ \sum_{k=0}^{m} \varphi_{k,1}\varphi_{(m-k),2} B(k+1, m-k+1) \right\} u^{m+1}\right|$$

$$= \left| \begin{array}{c} \left(\sum_{k=0}^{N-1} \varphi_{k,1} u^k + \tilde{R}_{N,1}\right) * \left(\sum_{l=0}^{N-1} \varphi_{l,2} u^l + \tilde{R}_{N,2}\right)(u) - \\ \sum_{m=0}^{N-1} \left\{ \sum_{k=0}^{m} \varphi_{k,1}\varphi_{(m-k),2} B(k+1, m-k+1) \right\} u^{m+1} \end{array} \right|$$

and this equals

$$\left| \int_0^u \left(\begin{array}{c} \sum_{k+l=N}^{2N-2} c(k,l)(u-s)^k s^l + \sum_{m=0}^{N-1} \varphi_{m,1}(u-s)^m \tilde{R}_{N,2}(s) + \\ \sum_{m=0}^{N-1} \varphi_{m,2} s^m \tilde{R}_{N,1}(u-s) + \tilde{R}_{N,1}(u-s)\tilde{R}_{N,2}(s) \end{array} \right) \mathrm{d}s \right|,$$

where $c(k,l) = \varphi_{k,1}\varphi_{l,2}$. Using the fact that we can integrate along the straight line $[0, u]$ it is easily seen that this last expression is $\le M_N |u|^{N+1}$, for some constant M_N. ∎

Remark 2.3.2 Since $u^k * u^l = u^{k+l+1} \int_0^1 (1-t)^k t^l \mathrm{d}t = B(k+1, l+1) u^{k+l+1}$, if $k, l > -1$, we see that in S_c the function $\varphi_1 * \varphi_2$ asymptotically equals the formal convolution of the series which are the asymptotic expansions of φ_1 and φ_2.

Definition 2.3.3 *For $N \in \mathbb{N}_{\ge 1}$ and $\tau_1 > 0$ we define $\mathcal{V}_{N,\tau_1}$ to be the space of functions $\vartheta : \overline{S_c} \to \mathbb{C}^n$ such that $u^{1-N}\vartheta(u)$ is analytic in $\overline{S_c}$ and*

$$\sup_{u \in \overline{S_c}} \left|u^{1-N}\vartheta(u)\right| e^{-\tau_1 |u|} < \infty.$$

Here $|\cdot|$ denotes a norm on $\mathbb{C}^n$.

Lemma 2.3.4 *The space* $(\mathcal{V}_{N,\tau_1}, \|\cdot\|_{N,\tau_1})$ *is a Banach space, with norm* $\|\cdot\|_{N,\tau_1}$ *defined by*

$$\|\vartheta\|_{N,\tau_1} := \sup_{u\in\overline{S_c}} \left|u^{1-N}\vartheta(u)\right| e^{-\tau_1|u|}.$$

Remark 2.3.5 Similarly we can formulate such a definition and lemma for matrix-valued functions. In that case $|\cdot|$ denotes a matrix norm. In the following we use the norm $|M| = \sup_{\substack{z\in\mathbb{C}^n\\ z\neq 0}} \frac{|Mz|}{|z|}$ for any $n\times n$ - matrix.

Proof of lemma 2.3.4. It is clear that $\mathcal{V}_{N,\tau_1}$ is a linear space, i.e. the function identically equal to 0 belongs to $\mathcal{V}_{N,\tau_1}$ and if $\vartheta_1,\ \vartheta_2 \in \mathcal{V}_{N,\tau_1}$ and $\alpha,\ \beta \in \mathbb{C}$ then $\alpha\vartheta_1 + \beta\vartheta_2 \in \mathcal{V}_{N,\tau_1}$. From this it follows that $\|\cdot\|_{N,\tau_1}$ is a seminorm on $\mathcal{V}_{N,\tau_1}$. Moreover, $\|\vartheta\|_{N,\tau_1} = 0$ implies $|u^{1-N}\vartheta(u)| = 0$ for every $u\in\overline{S_c}$ and thus $\vartheta\equiv 0$.

To prove the completeness of the linear space $\mathcal{V}_{N,\tau_1}$ we take a Cauchy-sequence $(\vartheta_k)_{k\in\mathbb{N}}$ in $\mathcal{V}_{N,\tau_1}$, i.e.

$$\forall\eta>0\ \ \exists M_\eta>0\ \text{ s.t. }\ \forall m_1,\ m_2\geq M_\eta \quad \|\vartheta_{m_1}-\vartheta_{m_2}\|_{N,\tau_1}\leq\eta$$

or

$$\begin{aligned}&\forall\eta>0\ \ \exists M_\eta>0\ \text{ s.t. }\ \forall m_1,\ m_2\geq M_\eta \quad \forall u\in\overline{S_c}\\ &\qquad\qquad\qquad \left|u^{1-N}\left(\vartheta_{m_1}(u)-\vartheta_{m_2}(u)\right)\right| e^{-\tau_1|u|}\leq\eta.\end{aligned} \tag{2.3}$$

This implies that for every $u\in\overline{S_c}$ we have:

$$\forall\eta>0\ \ \exists M_\eta>0\ \text{ s.t. }\ \forall m_1,\ m_2\geq M_\eta \quad |\vartheta_{m_1}(u)-\vartheta_{m_2}(u)|\leq\eta e^{\tau_1|u|}|u|^{N-1}.$$

Hence for every $u\in\overline{S_c}$ the sequence $(\vartheta_k(u))_{k\in\mathbb{N}}$ is a Cauchy-sequence in $\mathbb{C}^n$. Since this latter space is complete we conclude that to every $u\in\overline{S_c}$ there exists a vector $\vartheta_u\in\mathbb{C}^n$ such that

$$\lim_{k\to\infty}\vartheta_k(u)=\vartheta_u.$$

Now define the function $\vartheta:\overline{S_c}\to\mathbb{C}^n$ by

$$\vartheta(u) := \vartheta_u = \lim_{k\to\infty}\vartheta_k(u), \qquad u\in\overline{S_c}.$$

When we let $m_2\to\infty$, formula (2.3) transforms into

$$\begin{aligned}&\forall\eta>0\ \ \exists M_\eta>0\ \text{ s.t. }\ \forall m\geq M_\eta \ \ \forall u\in\overline{S_c}\\ &\qquad\qquad\qquad \left|u^{1-N}\left(\vartheta_m(u)-\vartheta(u)\right)\right| e^{-\tau_1|u|}\leq\eta.\end{aligned} \tag{2.4}$$

For $\eta>0$ and $m\geq M_\eta$ we have for all $u\in\overline{S_c}$

$$\left|u^{1-N}\vartheta(u)\right| e^{-\tau_1|u|}\leq\eta+\|\vartheta_m\|_{N,\tau_1}$$

and so

$$\sup_{u \in \overline{S_c}} \left| u^{1-N} \vartheta(u) \right| e^{-\tau_1 |u|} < \infty.$$

Now we will show that $u^{1-N}\vartheta(u)$ is analytic in $\overline{S_c}$. First take $R > 0$, then (2.4) gives

$$\begin{aligned} &\forall \eta > 0 \; \exists M_\eta > 0 \text{ s.t. } \forall m \geq M_\eta \quad \forall u \in \overline{S_c} \cap \overline{\Delta_1(R)} \\ &\qquad\qquad \left| u^{1-N} \left(\vartheta_m(u) - \vartheta(u) \right) \right| \leq \eta e^{\tau_1 R}, \end{aligned}$$

so $u^{1-N}\vartheta_m(u)$ converges to $u^{1-N}\vartheta(u)$ uniformly on $\overline{S_c} \cap \overline{\Delta_1(R)}$. This implies that $u^{1-N}\vartheta(u)$ is analytic in $\overline{S_c} \cap \overline{\Delta_1(R)}$. Since R was chosen arbitrary, we conclude that $u^{1-N}\vartheta(u)$ is analytic in $\overline{S_c}$. So $\vartheta \in \mathcal{V}_{N,\tau_1}$. Finally (2.4) tells us $\vartheta_m \xrightarrow[\|\cdot\|_{N,\tau_1}]{} \vartheta$ as $m \to \infty$ and this completes the proof. ■

2.4 From a formal solution to an analytic one

In section 2.2 we have found a formal solution $\hat{\Phi}(z) = \sum_{m=1}^{\infty} \Phi_m z^m$ of (2.1). Once we know there exists a formal solution we can prove the existence of an analytic solution which can be represented as a Laplace integral. This result is formulated more precisely in the following theorem.

Theorem 2.4.1 *Under the assumption of theorem 2.2.1 there exists an analytic solution $\Phi = \Phi(z)$ of (2.1) which can be written as $\Phi = \mathcal{L}w$ for some function $w \in \mathcal{A}_\tau$. This solution has $\hat{\Phi}$ as asymptotic expansion in S_d. Moreover, the solution Φ with these properties is unique.*

Proof. First define $u_N(z) = \sum_{m=1}^{N-1} \Phi_m z^m$, a partial sum of the formal solution, in which $N \geq 2$ will be specified later on. With $B_N(z) = \sum_{m=1}^{N-1} B_m z^m$ and $c_N(z) = \sum_{m=1}^{N-1} c_m z^m$ (see section 2.2) we have

$$\begin{aligned} & z^2 \frac{\mathrm{d}u_N}{\mathrm{d}z} + Au_N - z\tilde{A}u_N - zB(z)u_N - zc(z) \\ = \; & z^2 \frac{\mathrm{d}u_N}{\mathrm{d}z} + Au_N - z\tilde{A}u_N - zB_N(z)u_N - zc_N(z) + \\ & -z(B(z) - B_N(z))u_N - z(c(z) - c_N(z)). \end{aligned}$$

We substitute the series $\sum_{m=0}^{N-1} \Phi_m z^m$ for u_N and keep in mind that $\Phi_0 = 0$. We then obtain

$$\begin{aligned} & \sum_{m=2}^{N} (m-1)\Phi_{m-1} z^m + \sum_{m=0}^{N-1} A\Phi_m z^m - \sum_{m=1}^{N} \tilde{A}\Phi_{m-1} z^m + \\ & -\sum_{m=2}^{N} \left\{ \sum_{k=1}^{m-1} B_k \Phi_{m-1-k} \right\} z^m - P_{N+1}(z) - \sum_{m=2}^{N} c_{m-1} z^m + \\ & -z\left(B(z) - B_N(z)\right) u_N - z\left(c(z) - c_N(z)\right), \end{aligned}$$

where $P_{N+1}(z)$ is a (vector-valued) polynomial of order $O(z^{N+1})$ and degree $2N-1$. Because of the recurrence relation in the proof of theorem 2.2.1 we find

$$z^2\frac{\mathrm{d}u_N}{\mathrm{d}z}+Au_N=z\tilde{A}u_N+zB(z)u_N+zc(z)-R_N(z),$$

where

$$\begin{aligned}R_N(z)=A\Phi_N z^N+P_{N+1}(z)+z\left(B(z)-B_N(z)\right)u_N(z)+\\ z\left(c(z)-c_N(z)\right).\end{aligned}\tag{2.5}$$

With $\Psi=\Phi-u_N$ we get

$$z^2\frac{\mathrm{d}\Psi}{\mathrm{d}z}+A\Psi=z\tilde{A}\Psi+zB(z)\Psi+R_N(z)\tag{2.6}$$

as an equation equivalent with (2.1).

From $\big(\mathcal{B}(z^m)\big)(u)=\frac{1}{\Gamma(m)}u^{m-1}$, the convolution property of the Borel transform, remark 1.3.2(1) and lemma 2.3.1 we deduce that $\xi_N:=\mathcal{B}R_N$ belongs to $\mathcal{A}_\tau$. In fact, if $P_{N+1}(z)=\sum_{k=N+1}^{2N-1}a_kz^k$, then

$$\begin{aligned}\xi_N(u)\ =\ &\frac{1}{\Gamma(N)}A\Phi_Nu^{N-1}\ +\ \sum_{k=N}^{2N-2}\frac{1}{\Gamma(k+1)}a_{k+1}u^k\\ &+\\ &\left[1*\Big(\beta-\sum_{k=0}^{N-2}\beta_ku^k\Big)*\Big(\sum_{k=0}^{N-2}\frac{1}{\Gamma(k+1)}\Phi_{k+1}u^k\Big)\right](u)\\ &+\\ &\left[1*\Big(\gamma-\sum_{k=0}^{N-2}\gamma_ku^k\Big)\right](u),\end{aligned}\tag{2.7}$$

where $\beta_k=\frac{B_{k+1}}{\Gamma(k+1)}$ and $\gamma_k=\frac{c_{k+1}}{\Gamma(k+1)}$, for all $k\in\mathbb{N}$.
We first seek a solution Ψ of (2.6) which can be written as $\Psi=\mathcal{L}w$ for some function $w\in\mathcal{V}_{N,\tau_1}$, $\tau_1>\tau$ arbitrary. If $\Psi=\mathcal{L}w$ (or $w=\mathcal{B}\Psi$) then after Borel transformation the differential equation (2.6) becomes (compare theorem A.2.2)

$$uw(u)+Aw(u)=\tilde{A}(1*w)(u)+(1*\beta*w)(u)+\xi_N(u)$$

and we conclude:

The differential equation

$$z^2\frac{\mathrm{d}\Psi}{\mathrm{d}z}+A\Psi=z\tilde{A}\Psi+zB(z)\Psi+R_N(z)$$

has a solution Ψ which can be written as a Laplace integral of a function in $\mathcal{V}_{N,\tau_1}$ if the convolution equation

$$(u\mathrm{I}+A)w=\tilde{A}(1*w)+1*\beta*w+\xi_N \tag{2.8}$$

has a solution $w\in\mathcal{V}_{N,\tau_1}$.

The last equation can be written as

$$w=\mathcal{T}w, \tag{2.9}$$

where

$$(\mathcal{T}w)(u)=(u\mathrm{I}+A)^{-1}\tilde{A}(1*w)(u)+(u\mathrm{I}+A)^{-1}(1*\beta*w)(u)+ \\ (u\mathrm{I}+A)^{-1}\xi_N(u). \tag{2.10}$$

If we write $A=(a_{ij})_{i,j}$ then the definition of A implies $a_{ij}=0$ for all $i\neq j$. By rearranging the order of indices (if necessary) we can assume

$$\begin{cases} a_{ii}=a_i=0, & i\in\{0,1,\dots,n_1\} \\ a_{ii}=a_i\neq 0, & i\in\{n_1+1,n_1+2,\dots,n\}, \end{cases}$$

where $n_1\in\{0,1,\dots,n\}$.
In case $a_i\neq 0$ we have $\mathcal{R}e\ (a_i)\neq 0$ or $\mathcal{I}m\ (a_i)\neq 0$ and if $\mathcal{I}m\ (a_i)\neq 0$ then $\mathcal{I}m\ (a_i)>0$. So we can conclude that for all $i\in\{n_1+1,n_1+2,\dots,n\}$ and for all $u\in\overline{S_c}$ we have $u+a_i\neq 0$.
We can write

$$u\mathrm{I}+A=\begin{pmatrix} u\mathrm{I}_{n_1} & 0 \\ 0 & u\mathrm{I}_{n_2}+A^{22}\end{pmatrix},$$

where $n_2=n-n_1$ and $A^{22}=\mathrm{diag}\{a_{n_1+1},a_{n_1+2},\dots,a_n\}$. Our construction of n_1 implies that $u\mathrm{I}_{n_2}+A^{22}$ is non-singular on $\overline{S_c}$.
Moreover,

$$(u\mathrm{I}+A)^{-1}=\mathrm{diag}\left\{u^{-1}\mathrm{I}_{n_1},\left(u\mathrm{I}_{n_2}+A^{22}\right)^{-1}\right\}$$

and

$$\left(u\mathrm{I}_{n_2}+A^{22}\right)^{-1} \quad \text{and} \quad u(u\mathrm{I}+A)^{-1}$$

are uniformly bounded on $\overline{S_c}$:
The function $g:\overline{S_c}\to\mathsf{M}_n(\mathbb{C})$ defined by $g(u)=u\mathrm{I}_{n_2}+A^{22}$ is continuous and non-singular on $\overline{S_c}$, so the function $\{g(u)\}^{-1}$ also is continuous on $\overline{S_c}$. Furthermore we have $\lim_{u\to\infty}\{g(u)\}^{-1}=\lim_{u\to\infty}\frac{1}{u}(\mathrm{I}_{n_2}+A^{22}/u)^{-1}=0$. Hence $u\mapsto(u\mathrm{I}_{n_2}+A^{22})^{-1}$ is uniformly bounded on $\overline{S_c}$. Using the facts

$$\lim_{u\to 0}u(u\mathrm{I}+A)^{-1}=\begin{pmatrix}\mathrm{I}_{n_1} & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad \lim_{u\to\infty}u(u\mathrm{I}+A)^{-1}=\begin{pmatrix}\mathrm{I}_{n_1} & 0\\ 0 & \mathrm{I}_{n_2}\end{pmatrix},$$

we easily prove the same statement for the function $u\mapsto u(u\mathrm{I}+A)^{-1}$.

When we use a partitioning of vectors $f = \binom{f^1}{f^2}$ after the $n_1{}^{\text{th}}$ component, we deduce from (2.7), together with remark 1.3.2(1) and the last part of the proof of lemma 2.3.1, that $\xi_N^1(u) = O(u^N)$ and $\xi_N^2(u) = O(u^{N-1})$ as $u \to 0$ on $\overline{S_c}$.

We solve (2.9) in the Banach space $(\mathcal{V}_{N,\tau_1}, \|\cdot\|_{N,\tau_1})$, for arbitrary $\tau_1 > \tau$.

Claim 2.4.2

(*a*) *Both* $\tilde{A}$ *and* $1 * \beta$ *belong to* $\mathcal{V}_{1,\tau}$;

(*b*) *For arbitrary* $\tau_1 > \tau$ *both* $u^{-N}(\tilde{A} * \vartheta)$ *and* $u^{-N}(1 * \beta * \vartheta)$ *are analytic in* $\overline{S_c}$ *if* $\vartheta \in \mathcal{V}_{N,\tau_1}$;

(*c*) *The function* $(u\mathrm{I} + A)^{-1}\xi_N$ *belongs to* $\mathcal{V}_{N,\tau_1}$ *for arbitrary* $\tau_1 > \tau$.

Proof. Lemma 2.3.1 implies that $1 * \beta \in \mathcal{A}_{\tau'}$, so $1 * \beta$ is analytic in $\overline{S_c}$. Moreover, we have $(1 * \beta)(u) = O(1)\exp(\tau|u|)$ as $u \to \infty$ on $\overline{S_c}$ and this implies

$$\sup_{u \in \overline{S_c}} |(1 * \beta)(u)| e^{-\tau|u|} < \infty.$$

This proves the second statement in (*a*); the assertion $\tilde{A} \in \mathcal{V}_{1,\tau}$ is trivial. The functions $\tilde{A} * \vartheta$ and $1 * \beta * \vartheta$ are analytic in $\overline{S_c}$, which follows from lemma 2.3.1 and the fact that $\tilde{A}$, $1 * \beta$ and ϑ are analytic in $\overline{S_c}$. After multiplication with u^{-N} the results clearly are analytic in S_c and continuous on $\overline{S_c} \setminus \{0\}$. To prove the continuity in 0 of the function $u^{-N}(1 * \beta * \vartheta)$ for example, we note $(1 * \beta)(s) = (1 * \beta)(0) + o(1)$, $s \to 0$, and $\vartheta(s) = (\nu + o(1))s^{N-1}$, $s \to 0$, if $\lim_{\substack{s \to 0 \\ s \in S_c}} s^{1-N}\vartheta(s) = \nu$, so

$$\begin{aligned}
(1 * \beta * \vartheta)(u) &= u \int_0^1 (1 * \beta)(u(1-\sigma))\vartheta(u\sigma)\mathrm{d}\sigma \\
&= u^N \int_0^1 [(1 * \beta)(0) + o(1)][\nu + o(1)]\sigma^{N-1}\mathrm{d}\sigma \\
&= \frac{u^N}{N}[(1 * \beta)(0) + o(1)][\nu + o(1)] \quad \text{as } u \to 0 \text{ on } \overline{S_c}
\end{aligned}$$

and thus $\lim_{\substack{u \to 0 \\ u \in S_c}} u^{-N}(1 * \beta * \vartheta)(u)$ exists and equals $\frac{\nu}{N}(1 * \beta)(0) = 0$.

Now we still have to prove statement (*c*). It is easily seen that

$$u^{1-N}(u\mathrm{I} + A)^{-1}\xi_N(u) = \begin{pmatrix} u^{-N}\xi_N^1(u) \\ u^{1-N}\left(u\mathrm{I}_{n_2} + A^{22}\right)^{-1}\xi_N^2(u) \end{pmatrix}.$$

It is also clear that both components are analytic in S_c and continuous on $\overline{S_c} \setminus \{0\}$. As $\xi_N^i(u) = O(u^{N+1-i})$ as $u \to 0$ on $\overline{S_c}$ $(i = 1, 2)$, we also obtain continuity at the origin, for if $\xi_N^1(u) \simeq \sum_{m=N}^{\infty} \xi_{N,m}^1 u^m$ as $u \to 0$ on S_c, then for all integers $M \geq N$ a positive constant C_M can be found such that $\left|\xi^1_N(u) - \sum_{m=N}^{M-1} \xi^1_{N,m} u^m\right| \leq C_M |u|^M$ for all $u \in \overline{S_c}(1)$. Hence, with $M = N + 1$ we obtain $\left|\xi^1_N(u) - \xi^1_{N,N} u^N\right| \leq C_{N+1}|u|^{N+1}$ for all $u \in \overline{S_c}(1)$ and so

$$\left|u^{-N}\xi^1_N(u) - \xi^1_{N,N}\right| \leq C_{N+1}|u| \xrightarrow[u \in \overline{S_c}]{u \to 0} 0.$$

A similar reasoning proves the continuity at the origin of the function $u \mapsto u^{1-N}\left(u\mathrm{I}_{n_2} + A^{22}\right)^{-1}\xi^2_N(u)$. Moreover, for $u \in \overline{S_c} \setminus \{0\}$ we have

$$\begin{aligned} & \left|u^{1-N}\left(u\mathrm{I} + A\right)^{-1}\xi_N(u)\right| e^{-\tau_1|u|} \\ \leq\ & \left|u^{-N}\xi^1_N(u)\right| e^{-\tau_1|u|} \ +\ \left|u^{1-N}\left(u\mathrm{I}_{n_2} + A^{22}\right)^{-1}\xi^2_N(u)\right| e^{-\tau_1|u|}. \end{aligned}$$

The limit $u \to 0$ on $\overline{S_c}$ of this last expression exists, which proves the boundedness of $\left|u^{1-N}\left(u\mathrm{I} + A\right)^{-1}\xi_N(u)\right| e^{-\tau_1|u|}$ on $\overline{S_c} \cap \overline{\Delta_1(r)}$ for some $r > 0$. Moreover,

$$\left|u^{1-N}\left(u\mathrm{I} + A\right)^{-1}\xi_N(u)\right| e^{-\tau_1|u|} \leq |u|^{-N}\left|u\left(u\mathrm{I} + A\right)^{-1}\right| \cdot \frac{|\xi_N(u)|}{e^{\tau_1|u|}}$$

and the right hand side of this inequality tends to 0 as $u \to \infty$ on $\overline{S_c}$. This proves the boundedness of $\left|u^{1-N}\left(u\mathrm{I} + A\right)^{-1}\xi_N(u)\right| e^{-\tau_1|u|}$ in $\overline{S_c} \cap \complement\Delta_1(R)$ for some $R \gg 0$.
Hence, the continuity of $u^{1-N}(u\mathrm{I} + A)^{-1}\xi_N$ on $\overline{S_c}$ implies

$$\sup_{u \in \overline{S_c}} \left|u^{1-N}\left(u\mathrm{I} + A\right)^{-1}\xi_N(u)\right| e^{-\tau_1|u|} < \infty,$$

which proves claim 2.4.2. □

Next we will show that $\mathcal{T}$ maps $\mathcal{V}_{N,\tau_1}$ into itself for arbitrary $\tau_1 > \tau$: Choose $\vartheta \in \mathcal{V}_{N,\tau_1}$. From (2.10) and claim 2.4.2(c) we conclude that it is sufficient to show that $(u\mathrm{I} + A)^{-1}\tilde{A}(1 * \vartheta)$ and $(u\mathrm{I} + A)^{-1}(1 * \beta * \vartheta)$ belong to $\mathcal{V}_{N,\tau_1}$. First we note that after multiplication with u^{1-N} the results still are analytic in $\overline{S_c}$, which follows immediately from claim 2.4.2(b), and the fact that $u \mapsto u\left(u\mathrm{I}_{n_2} + A^{22}\right)^{-1}$ is analytic in $\overline{S_c}$.
For $u \in \overline{S_c}$ we have the estimate

$$\begin{aligned} & |\tilde{A}(1 * \vartheta)(u)| \\ \leq\ & |\tilde{A}| \int_0^u |\vartheta(s)||\mathrm{d}s| \\ \leq\ & \left(\max_{i \in \{1,2,\ldots,n\}} |\tilde{a}_i|\right) \cdot \|\vartheta\|_{N,\tau_1} \int_0^u |s|^{N-1} e^{\tau_1|s|}|\mathrm{d}s|. \end{aligned}$$

Using the fact that we integrate along a straight line, we find for $u \in \overline{S_c}$

$$|\tilde{A}(1 * \vartheta)(u)| \le \Big(\max_{i \in \{1,2,\dots,n\}} |\tilde{a}_i|\Big) \cdot e^{\tau_1 |u|} \cdot \|\vartheta\|_{N,\tau_1} \cdot \frac{|u|^N}{N}$$

and thus, with $M := \sup_{u \in \overline{S_c}} \left|u(u\mathrm{I} + A)^{-1}\right|$,

$$\begin{aligned}
&\left\|(u\mathrm{I} + A)^{-1}\,\tilde{A}\,(1 * \vartheta)\right\|_{N,\tau_1} \\
\le\ & \sup_{u \in \overline{S_c}} |u|^{1-N} \cdot \left|(u\mathrm{I} + A)^{-1}\right| \cdot |\tilde{A}(1 * \vartheta)(u)| \cdot e^{-\tau_1 |u|} \\
\le\ & \frac{1}{N} \cdot \Big(\max_{i \in \{1,2,\dots,n\}} |\tilde{a}_i|\Big) \cdot M \cdot \|\vartheta\|_{N,\tau_1}.
\end{aligned}$$

In a similar way we obtain for $u \in \overline{S_c}$

$$\begin{aligned}
&|(1 * \beta * \vartheta)(u)| \\
\le\ & \int_0^u |(1 * \beta)(u - s)\vartheta(s)||\mathrm{d}s| \\
\le\ & \|1 * \beta\|_{1,\tau_1} \int_0^u e^{\tau_1 |u-s|} |\vartheta(s)||\mathrm{d}s| \\
\le\ & \|1 * \beta\|_{1,\tau} \cdot \|\vartheta\|_{N,\tau_1} \int_0^u e^{\tau_1(|u-s|+|s|)} |s|^{N-1} |\mathrm{d}s| \\
\le\ & \|1 * \beta\|_{1,\tau} \cdot \|\vartheta\|_{N,\tau_1} \cdot e^{\tau_1 |u|} \cdot \frac{|u|^N}{N}
\end{aligned}$$

and so

$$\begin{aligned}
&\left\|(u\mathrm{I} + A)^{-1}\,(1 * \beta * \vartheta)\right\|_{N,\tau_1} \\
\le\ & \sup_{u \in \overline{S_c}} |u|^{1-N} \cdot \left|(u\mathrm{I} + A)^{-1}\right| \cdot |(1 * \beta * \vartheta)(u)| \cdot e^{-\tau_1 |u|} \\
\le\ & \frac{1}{N} \cdot \|1 * \beta\|_{1,\tau} \cdot M \cdot \|\vartheta\|_{N,\tau_1}.
\end{aligned}$$

We conclude that

$$\vartheta \in \mathcal{V}_{N,\tau_1} \quad \Longrightarrow \quad \mathcal{T}\vartheta \in \mathcal{V}_{N,\tau_1}.$$

For $\vartheta_1,\ \vartheta_2 \in \mathcal{V}_{N,\tau_1}$ we deduce

$$\begin{aligned}
&\|\mathcal{T}\vartheta_1 - \mathcal{T}\vartheta_2\|_{N,\tau_1} \\
=\ & \left\|(u\mathrm{I} + A)^{-1}\tilde{A}(1 * (\vartheta_1 - \vartheta_2)) + (u\mathrm{I} + A)^{-1}(1 * \beta * (\vartheta_1 - \vartheta_2))\right\|_{N,\tau_1} \\
\le\ & \frac{K}{N}\|\vartheta_1 - \vartheta_2\|_{N,\tau_1}
\end{aligned}$$

for some positive constant K, independent of N and τ_1.
Choosing $N_0 > K$ we see that $\mathcal{T}$ defines a contraction mapping on the Banach space $\mathcal{V}_{N,\tau_1}$ for all $N \geq N_0$. Consequently there exists a unique solution w of (2.8) in $\mathcal{V}_{N,\tau_1}$ if $N \geq N_0$ and a priori this solution depends on N and τ_1.

Claim 2.4.3 *Variation of $\tau_1 > \tau$ gives $w(u) = O(1)e^{\tilde{\tau}|u|}$ as $u \to \infty$ on $\overline{S_c}$ for all $\tilde{\tau} > \tau$.*

Proof. Choose two parameters $\tau_1,\ \tau_2 > \tau$, then for $i = 1, 2$ and $N \geq N_0$ we can find a solution

$$w_i \in \mathcal{V}_{N,\tau_i} = \left\{ \vartheta : \overline{S_c} \to \mathbb{C}^n \;\middle|\; \begin{array}{l} u^{1-N}\vartheta(u) \ \text{ analytic in } \overline{S_c} \ \text{ and} \\ \sup_{u\in\overline{S_c}} \left|u^{1-N}\vartheta(u)\right| e^{-\tau_i|u|} < \infty \end{array} \right\}$$

of (2.8). Without loss of generality we may assume $\tau_1 \leq \tau_2$ and from this one easily concludes

$$\mathcal{V}_{N,\tau_1} \subseteq \mathcal{V}_{N,\tau_2}.$$

Hence the uniqueness of the solution of (2.8) in $\mathcal{V}_{N,\tau_2}$ implies $w_1 = w_2$ and we conclude that the solution w is independent of the choice of $\tau_1 > \tau$. So for all $\tau_1 > \tau$ there exists a constant $M = M_{\tau_1}$ such that

$$\sup_{u\in\overline{S_c}} \left|u^{1-N}w(u)\right| e^{-\tau_1|u|} \leq M < \infty.$$

Using the fact

$$\frac{s^m}{m!} \leq e^s, \qquad \forall\, s \geq 0, \quad \forall\, m \in \mathbb{N}, \tag{2.11}$$

we obtain for all $u \in \overline{S_c}$:

$$|w(u)| \leq M|u|^{N-1}e^{\tau_1|u|} \leq M\frac{(N-1)!}{\eta^{N-1}}e^{(\tau_1+\eta)|u|}, \qquad \forall\, \tau_1 > \tau, \quad \forall\, \eta > 0.$$

So

$$w(u) = O(1)e^{\tilde{\tau}|u|} \quad \text{as } u \to \infty \ \text{ on } \overline{S_c}$$

for all $\tilde{\tau} > \tau$ and this was to be shown. □

We conclude that $\Psi = \mathcal{L}w$ satisfies (2.6). Note that Ψ in general depends on N. Now define

$$\tilde{w}(u) := \sum_{m=1}^{N-1} \frac{\Phi_m}{\Gamma(m)}u^{m-1} + w(u),$$

then $\Phi := \mathcal{L}\tilde{w}$ is a solution of the original equation (2.1) such that (*i*) $\tilde{w}$ is analytic in $\overline{S_c}$;

(*ii*) $\tilde{w}(u) = O(1)e^{\tilde{\tau}|u|}$ as $u \to \infty$ on $\overline{S_c}$ for all $\tilde{\tau} > \tau$;

(*iii*) $\tilde{w}(u) = \sum_{m=0}^{N-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + w(u)$ as $u \to 0$ on $\overline{S_c}$.

As $u \mapsto u^{1-N} w(u)$ is analytic in $\overline{S_c}$, it is bounded on $\overline{S_c}(1)$, so

$$\left| \tilde{w}(u) - \sum_{m=0}^{N-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m \right| = |w(u)| \le C_N |u|^{N-1}, \quad \forall\, u \in \overline{S_c}(1),$$

for some constant C_N, which may depend on N.

Next we will show that $\tilde{w}$ (and thus Φ) is independent of N. To that end we first prove the following

Claim 2.4.4 *If* $\tau_1 > \tau$ *and* $N_2 > N_1$ *then* $\mathcal{V}_{N_2,\tau_2} \subset \mathcal{V}_{N_1,\tau_1}$ *for all* $\tau_2 \in (\tau, \tau_1)$.

Proof. Take $\tau_2 \in (\tau, \tau_1)$ arbitrary and write $\eta = \tau_1 - \tau_2$. If $\vartheta \in \mathcal{V}_{N_2,\tau_2}$ the function $u \mapsto u^{1-N_1}\vartheta(u) = u^{N_2-N_1} \cdot u^{1-N_2}\vartheta(u)$ is analytic in $\overline{S_c}$. Moreover, for $u \in \overline{S_c} \setminus \{0\}$ we have (using (2.11)):

$$\begin{aligned} \left|u^{1-N_1}\vartheta(u)\right| e^{-\tau_1|u|} &= |u|^{N_2-N_1} \left|u^{1-N_2}\vartheta(u)\right| e^{-\tau_2|u|} e^{-(\tau_1-\tau_2)|u|} \\ &\le \|\vartheta\|_{N_2,\tau_2} |u|^{N_2-N_1} e^{-\eta|u|} \\ &\le \|\vartheta\|_{N_2,\tau_2} \cdot \frac{(N_2-N_1)!}{\eta^{N_2-N_1}}, \end{aligned}$$

which also holds for $u = 0$ by taking the limit $u \to 0$ on $\overline{S_c}$.
Hence

$$\sup_{u \in \overline{S_c}} \left|u^{1-N_1}\vartheta(u)\right| e^{-\tau_1|u|} < \infty.$$

This proves claim 2.4.4. □

Now take N_1, $N_2 \in \mathbb{N}$, $N_2 > N_1 \ge N_0$. Let

$$\tilde{w}_{N_1}(u) = \sum_{m=0}^{N_1-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + w_{N_1}(u),$$

with $w_{N_1} \in \mathcal{V}_{N_1,\tilde{\tau}}$ ($\forall\, \tilde{\tau} > \tau$, cf. the proof of claim 2.4.3) a solution of (2.8) with $N := N_1$ and let

$$\tilde{w}_{N_2}(u) = \sum_{m=0}^{N_2-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + w_{N_2}(u),$$

with $w_{N_2} \in \mathcal{V}_{N_2,\tilde{\tau}}$ ($\forall\, \tilde{\tau} > \tau$) a solution of (2.8) with $N := N_2$. Now $\tilde{w}_{N_2}$ can be written as

$$\tilde{w}_{N_2}(u) = \sum_{m=0}^{N_1-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + \sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + w_{N_2}(u).$$

Next, if $\tau_1 > \tau$ arbitrary, then take $\tau_2 \in (\tau, \tau_1)$. As $u \mapsto \sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m$ belongs to $\mathcal{V}_{N_1,\tau_1}$ and $\mathcal{V}_{N_2,\tau_2} \subset \mathcal{V}_{N_1,\tau_1}$ we conclude that both

$$w_{N_1}(u) \quad \text{and} \quad \sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + w_{N_2}(u)$$

are solutions of (2.8) with $N := N_1$, and these solutions belong to $\mathcal{V}_{N_1,\tau_1}$. Here we used that $\sum_{m=0}^{\infty} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m$ is a formal solution of the equation obtained from (2.1) by taking a formal Borel transform, and that both $\tilde{w}_{N_1}$ and $\tilde{w}_{N_2}$ are analytic solutions of this equation.
The uniqueness of the solution of (2.8) in $\mathcal{V}_{N_1,\tau_1}$ then implies

$$w_{N_1}(u) = \sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m + w_{N_2}(u).$$

As this can be done for every $N_2 > N_1 \geq N_0$, we conclude that $\tilde{w}$ is independent of N and

$$\left| \tilde{w}(u) - \sum_{m=0}^{N-2} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m \right| \leq C_N |u|^{N-1}, \quad \forall\, u \in \overline{S_c}(1), \quad \forall\, N \geq N_0.$$

An easy calculation shows this estimate for $N = N_0 - 1, N_0 - 2, \ldots, 1$, so in particular

$$\tilde{w}(u) \simeq \sum_{m=0}^{\infty} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m \quad \text{as } u \to 0 \text{ on } S_c.$$

Hence we have $\tilde{w} \in \mathcal{A}_\tau$, $\Phi = \mathcal{L}\tilde{w}$ being the unique solution of (2.1) (i.e. unique in the set of functions which can be written as Laplace integrals of functions in $\mathcal{A}_\tau$) and

$$\Phi(z) \simeq \hat{\mathcal{L}}\left(\sum_{m=0}^{\infty} \frac{\Phi_{m+1}}{\Gamma(m+1)} u^m \right)(z) = \hat{\Phi}(z) \quad \text{as } z \to 0 \text{ on } S_d,$$

which ends the proof of theorem 2.4.1. ∎

Remark 2.4.5 We did not specify the choice of $\varepsilon \in (\delta, \pi)$ in the definition of the sector S_d, so in fact we have proven theorem 2.4.1 with 'S_d' replaced by 'a neighbourhood U_δ of 0 in $S(\frac{\pi}{2}, 2\pi - \delta)$', where

$$\Phi(z) \simeq \hat{\Phi}(z) \quad \text{as } z \to 0 \text{ on } U_\delta$$

has to be interpreted as in theorem A.1.1.

Chapter 3

Some tools from complex analysis in several variables

3.1 Analytic functions in several variables

Before formulating some results which will help us in the near future, we first list a few properties from the theory of analytic functions in several complex variables. Omitted proofs and more details can be found in [6].

Definition 3.1.1 *Let $\varphi \in C^1(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is an open set. We say that φ is analytic in Ω if the Cauchy-Riemann equations are satisfied, i.e.*

$$\frac{\partial \varphi}{\partial \overline{y}_j} \equiv 0, \quad \forall\, j = 1, 2, \dots, n.$$

The set of all analytic functions in Ω, denoted by $A(\Omega)$, is a ring[1].

For a multi-index $Q = (q_1, q_2, \dots, q_n) \in \mathbb{N}^n$ we define the operator D^Q by $D^Q := \left(\frac{\partial}{\partial y_1}\right)^{q_1} \left(\frac{\partial}{\partial y_2}\right)^{q_2} \cdots \left(\frac{\partial}{\partial y_n}\right)^{q_n}$. Moreover, $Q! := q_1! \cdot q_2! \cdots q_n!$ and $|Q| := \sum_{i=1}^n q_i$. If $y = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n$, then $y^Q := \prod_{i=1}^n {y_i}^{q_i}$.

We shall now consider power series expansions of functions which are analytic in polydiscs. In doing so we say that a series $\sum_{Q\in\mathbb{N}^n} \Upsilon_Q(y)$ converges *normally* in an open set $\Omega \subset \mathbb{C}^n$ if

$$\sum_{Q\in\mathbb{N}^n} \sup_{y\in K} |\Upsilon_Q(y)| < \infty$$

on every compact set $K \subset \Omega$ (i.e. absolute uniform convergence on compact sets of Ω). This implies that $\sum_{Q\in\mathbb{N}^n} \Upsilon_Q(y)$ exists on Ω, is independent of the order of summation and is analytic in Ω if all Υ_Q are analytic in Ω.

[1] The collection $A(\Omega)$ is a linear space and $\varphi_1,\ \varphi_2 \in A(\Omega)$ implies $\varphi_1\varphi_2 \in A(\Omega)$.

Theorem 3.1.2 (Taylor) *For $j \in \{1, 2, \dots, n\}$ let $\rho_j > 0$. If φ is analytic in the polydisc*

$$D = \{y = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n \mid |y_j| < \rho_j, \quad j = 1, 2, \dots, n\},$$

we have

$$\varphi(y) = \sum_{Q \in \mathbb{N}^n} \frac{D^Q \varphi(0)}{Q!} y^Q, \qquad y \in D,$$

with normal convergence.

Theorem 3.1.3 (Cauchy's inequality) *If φ is analytic and $|\varphi| \leq M$ in the polydisc $\{y \in \mathbb{C}^n \mid |y_j| < \rho_j, \quad j = 1, 2, \dots, n\}$, it follows that*

$$|D^Q \varphi(0)| \leq \frac{MQ!}{\rho^Q},$$

where $\rho = (\rho_1, \rho_2, \dots, \rho_n)$.

Theorem 3.1.4 *Let φ be a complex-valued function, defined on the open set $\Omega \subset \mathbb{C}^n$. Then φ is analytic in Ω if and only if φ is analytic in each variable y_j when the other variables are given arbitrary fixed values.*

Remark 3.1.5 This theorem looks rather trivial (and the proof is even more trivial when we use definition 3.1.1), but a corresponding result would be false for functions of real variables: the function $f(x, y) = \frac{xy}{x^2+y^2}$, $f(0,0) = 0$, is a $\mathcal{C}^\infty$ function of x (resp. y) if y (resp. x) is kept fixed, but in spite of that f is not even continuous at the origin (consider the path $y = x$).

Lemma 3.1.6 *For $i = 1, 2$ let Ω_i be an open set in $\mathbb{C}^{n_i}$ and suppose that φ_i is analytic in Ω_i. Then the direct product of φ_1 and φ_2, denoted by $\varphi_1 \otimes \varphi_2$, belongs to $A(\Omega_1 \times \Omega_2)$.*

Proof of lemma 3.1.6. Let us introduce $y = (y_1, y_2, \dots, y_{n_1}) \in \mathbb{C}^{n_1}$ and $w = (w_1, w_2, \dots, w_{n_2}) \in \mathbb{C}^{n_2}$, then we have $(\varphi_1 \otimes \varphi_2)(y, w) = \varphi_1(y)\varphi_2(w)$. Take $j \in \{1, 2, \dots, n_1\}$ arbitrary, then the function φ_1 is analytic in y_j when the variables $y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{n_1}$ are given arbitrary fixed values (see theorem 3.1.4). This implies that the function $\varphi_1 \otimes \varphi_2$ is analytic in y_j when the other variables $y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{n_1}, w_1, \dots, w_{n_2}$ are given arbitrary fixed values and this holds for every $j \in \{1, 2, \dots, n_1\}$.
A similar reasoning, for arbitrary $j \in \{1, 2, \dots, n_2\}$, leads to $\varphi_1 \otimes \varphi_2$ being analytic in w_j when all the other variables are given arbitrary fixed values. Theorem 3.1.4 then completes the proof. ■

Now let Υ_Q be complex numbers, defined for all multi-indices $Q \in \mathbb{N}^n$. Let us consider the power series

$$\sum_{Q \in \mathbb{N}^n} \Upsilon_Q y^Q.$$

We define its *domain of convergence*, D_{conv}, as the set of all $y \in \mathbb{C}^n$ such that the series converges absolutely at every point in a neighbourhood of y, so $y \in D_{conv}$ is equivalent with

$$\exists\, \varepsilon > 0 \text{ such that } \forall w \in \mathbb{C}^n, \text{ with } \|y-w\| < \varepsilon, \text{ we have } \sum_{Q \in \mathbb{N}^n} |\Upsilon_Q w^Q| < \infty.$$

Here $\|\cdot\|$ denotes a norm on $\mathbb{C}^n$.

Theorem 3.1.7 *Suppose that the series $\sum_{Q\in\mathbb{N}^n} \Upsilon_Q y^Q$ has domain of convergence D_{conv}. Then this series converges normally in D_{conv} and thus the sum is analytic in D_{conv}.*

3.2 On interchanging $\mathcal{B}$ and summation

We are now ready to formulate and to prove some helpful results. The first proposition is about interchanging Borel transformation and summation in case of one variable:

Proposition 3.2.1 *Suppose the series $\sum_{m=1}^{\infty} \varphi_m z^m$ has radius of convergence $\varepsilon > 0$. Let, for $z \in \Delta_1(\varepsilon)$, $\varphi(z) = \sum_{m=1}^{\infty} \varphi_m z^m$ be the sum of the series, then $\mathcal{B}\varphi$ exists and is an entire function. Moreover, we have*

$$(\mathcal{B}\varphi)(u) = \sum_{m=0}^{\infty} \frac{\varphi_{m+1}}{\Gamma(m+1)} u^m, \qquad u \in \mathbb{C}.$$

Proof. We have φ being analytic in a full neighbourhood of 0 in $\mathbb{C}$ and thus $\varphi(z) = O(1)$, as $z \to 0$. This implies that the Borel transform of φ exists and is analytic in $\mathbb{C} \setminus \{0\}$. From $\varphi(z) \simeq \sum_{m=1}^{\infty} \varphi_m z^m$ as $z \to 0$ and theorem A.1.2 we conclude

$$(\mathcal{B}\varphi)(u) \simeq \sum_{m=0}^{\infty} \frac{\varphi_{m+1}}{\Gamma(m+1)} u^m \quad \text{as } u \to 0.$$

As $\lim_{u\to 0} u(\mathcal{B}\varphi)(u) = 0\cdot\varphi_1 = 0$, the function $\mathcal{B}\varphi$ has a removable singularity at $u = 0$. Riemann's theorem says that in this case $(\mathcal{B}\varphi)(u)$ can be assigned a value $(\mathcal{B}\varphi)(0)$ at $u = 0$ so that the resulting function is an entire function. Therefore it has a Taylor series expansion

$$(\mathcal{B}\varphi)(u) = \sum_{m=0}^{\infty} \psi_m u^m, \qquad u \in \mathbb{C}.$$

With the knowledge that a Taylor expansion also is an asymptotic expansion one can, using the uniqueness of the asymptotic expansion, easily complete the proof. ∎

Remark 3.2.2 If φ satisfies the conditions in proposition 3.2.1, it is easily seen that $\mathcal{B}\varphi$ is of exponential growth of order ≤ 1, for if $\eta < \varepsilon$ and $M = \max_{z\in\overline{\Delta_1(\eta)}} |\varphi(z)|$, we have $|\varphi_n| \leq \frac{M}{\eta^n}$ (Cauchy's inequality in one variable) and thus

$$|(\mathcal{B}\varphi)(u)| \leq \sum_{m=0}^{\infty} \left|\frac{\varphi_{m+1}}{\Gamma(m+1)}\right| |u|^m \leq \frac{M}{\eta} \sum_{m=0}^{\infty} \frac{\left(\frac{|u|}{\eta}\right)^m}{m!} = \frac{M}{\eta} e^{\frac{|u|}{\eta}}$$

for all $u \in \mathbb{C}$.

Let us now consider a function $\varphi = \varphi(z, y)$ which is analytic in the polydisc $\Delta_1(\varepsilon) \times \Delta_n(\rho)$. Theorem 3.1.2 then implies

$$\varphi(z, y) = \sum_{\substack{Q\in\mathbb{N}^n \\ m\in\mathbb{N}}} \varphi_{Q,m} z^m y^Q, \qquad (z, y) \in \Delta_1(\varepsilon) \times \Delta_n(\rho),$$

where $\varphi_{Q,m} = \frac{\left(\frac{d}{dz}\right)^m D^Q\varphi(0,0)}{m!\ Q!}$.
For each $Q \in \mathbb{N}^n$ the series $\sum_{m=0}^{\infty} \varphi_{Q,m} z^m$ converges normally in $\Delta_1(\varepsilon)$. To prove this we take $K \subset \Delta_1(\varepsilon)$ any compact subset, then $K \times \left\{\left(\frac{\rho}{2}, \dots, \frac{\rho}{2}\right)\right\}$ is a compact subset of $\Delta_1(\varepsilon) \times \Delta_n(\rho)$ and

$$\sum_{m=0}^{\infty} \sup_{z\in K} |\varphi_{Q,m} z^m| \leq \left(\frac{2}{\rho}\right)^{|Q|} \sum_{R\in\mathbb{N}^n} \sum_{m=0}^{\infty} \sup_{z\in K} |\varphi_{R,m} z^m| \left(\frac{\rho}{2}\right)^{|R|} =$$

$$\left(\frac{2}{\rho}\right)^{|Q|} \sum_{\substack{R\in\mathbb{N}^n \\ m\in\mathbb{N}}} \sup_{\substack{z\in K \\ y\in\{(\frac{\rho}{2},\dots,\frac{\rho}{2})\}}} \left|\varphi_{R,m} z^m y^R\right| < \infty.$$

Hence for each $Q \in \mathbb{N}^n$ the series $\sum_{m=0}^{\infty} \varphi_{Q,m} z^m$ represents an analytic function in $\Delta_1(\varepsilon)$. In a similar way we obtain normal convergence of the series $\sum_{Q\in\mathbb{N}^n} \varphi_{Q,m} y^Q$ in $\Delta_n(\rho)$ for each $m \in \mathbb{N}$ and therefore the series $\sum_{Q\in\mathbb{N}^n} \varphi_{Q,m} y^Q$ represents an analytic function in $\Delta_n(\rho)$, $\forall m \in \mathbb{N}$.

Theorem 3.2.3 *Let $\varphi = \varphi(z, y)$ be analytic in the polydisc $\Delta_1(\varepsilon) \times \Delta_n(\rho)$. We assume $\varphi(0, \cdot) \equiv 0$ and we write $\varphi(z, y) = \sum_{Q\in\mathbb{N}^n} \varphi_Q(z) y^Q$, where φ_Q is analytic in $\Delta_1(\varepsilon)$ for each $Q \in \mathbb{N}^n$.*
Then for all $Q \in \mathbb{N}^n$ $\psi_Q := \mathcal{B}\varphi_Q$ exists and is an entire function. Moreover, the series $\sum_{Q\in\mathbb{N}^n} \psi_Q(u) y^Q$ represents an analytic function $\psi = \psi(u, y)$ in $\mathbb{C} \times \Delta_n(\rho)$ and

$$(\mathcal{B}\varphi(\cdot, y))(u) = \psi(u, y), \qquad \forall\, y \in \Delta_n(\rho).$$

Proof. Each φ_Q can be written as a power series with radius of convergence at least $\varepsilon > 0$ (note that $\varphi(0, \cdot) \equiv 0$ implies $\varphi_Q(0) = 0$ for all $Q \in \mathbb{N}^n$):

$$\varphi_Q(z) = \sum_{m=1}^{\infty} \varphi_{Q,m} z^m, \qquad |z| < \varepsilon.$$

We have absolute convergence of the series

$$\sum_{Q\in\mathbb{N}^n}\sum_{m=1}^{\infty}\varphi_{Q,m}z^m y^Q \tag{3.1}$$

for all $(z,y)\in\Delta_1(\varepsilon)\times\Delta_n(\rho)$, for if $(z,y)\in\Delta_1(\varepsilon)\times\Delta_n(\rho)$ we can find a compact set $K\subset\Delta_1(\varepsilon)\times\Delta_n(\rho)$ with $(z,y)\in K$.

By proposition 3.2.1 the series

$$\sum_{m=0}^{\infty}\frac{\varphi_{Q,m+1}}{\Gamma(m+1)}u^m$$

converges in $\mathbb{C}$ for all $Q\in\mathbb{N}^n$ and therefore represents an entire function, which we will denote by ψ_Q. Moreover, we have $\psi_Q=\mathcal{B}\varphi_Q$.

Next take a compact set $K\subset\mathbb{C}\times\Delta_n(\rho)$, then there exist positive constants R,r, $0<r<\rho$, such that $K\subset\Delta_1(R)\times\Delta_n(r)$. So

$$\begin{aligned}\sum_{\substack{Q\in\mathbb{N}^n\\ m\in\mathbb{N}}}\sup_{(u,y)\in K}\left|\frac{\varphi_{Q,m+1}}{\Gamma(m+1)}u^m y^Q\right| &\le \sum_{m=0}^{\infty}\sum_{Q\in\mathbb{N}^n}\frac{|\varphi_{Q,m+1}|}{m!}R^m r^{|Q|}\\ &= \sum_{m=0}^{\infty}\left(\sum_{Q\in\mathbb{N}^n}|\varphi_{Q,m+1}|r^{|Q|}\right)\frac{R^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{a_{m+1}}{m!}R^m,\end{aligned}$$

where $a_n=a_n(r)=\sum_{Q\in\mathbb{N}^n}|\varphi_{Q,n}|r^{|Q|}<\infty$ for all $n\in\mathbb{N}_{\geq 1}$.

For all $z\in\Delta_1(\varepsilon)$ we have $\sum_{m=1}^{\infty}a_m|z|^m<\infty$, which follows immediately from the absolute convergence of (3.1) by taking $y=(r,r,\dots,r)\in\Delta_n(\rho)$. This implies that the radius of convergence of the series $\sum_{m=1}^{\infty}a_m z^m$ is at least ε. Proposition 3.2.1 then implies that the series $\sum_{m=0}^{\infty}\frac{a_{m+1}}{m!}u^m$ converges in $\mathbb{C}$ and therefore

$$\sum_{m=0}^{\infty}\frac{a_{m+1}}{m!}R^m<\infty.$$

So the series $\sum_{\substack{Q\in\mathbb{N}^n\\ m\in\mathbb{N}}}\frac{\varphi_{Q,m+1}}{\Gamma(m+1)}u^m y^Q$ converges absolutely on compact subsets of $\mathbb{C}\times\Delta_n(\rho)$. Hence it represents an analytic function in $\mathbb{C}\times\Delta_n(\rho)$.

Finally we take $y\in\Delta_n(\rho)$ fixed and we consider the function $\Phi:\Delta_1(\varepsilon)\to\mathbb{C}$ defined by $\Phi(z)=\varphi(z,y)$. Theorem 3.1.4 implies that Φ is analytic in $\Delta_1(\varepsilon)$ and

$$\Phi(z)=\sum_{m=1}^{\infty}\left(\sum_{Q\in\mathbb{N}^n}\varphi_{Q,m}y^Q\right)z^m,\qquad |z|<\varepsilon.$$

So Φ satisfies the conditions in proposition 3.2.1 and thus

$$(\mathcal{B}\Phi)(u) = \sum_{m=0}^{\infty} \frac{\sum_{Q\in\mathbb{N}^n} \varphi_{Q,m+1} y^Q}{m!} u^m = \sum_{Q\in\mathbb{N}^n} \sum_{m=0}^{\infty} \frac{\varphi_{Q,m+1}}{m!} u^m y^Q =$$

$$= \sum_{Q\in\mathbb{N}^n} \psi_Q(u) y^Q = \psi(u,y).$$

Because $y \in \Delta_n(\rho)$ was taken arbitrary we conclude

$$(\mathcal{B}\varphi(\cdot,y))(u) = (\mathcal{B}\Phi)(u) = \psi(u,y)$$

for all $y \in \Delta_n(\rho)$. ∎

Remark 3.2.4 In case we use the Taylor expansion

$$\varphi(z,y) = \sum_{m=1}^{\infty} \varphi_m(y) z^m,$$

with φ_m analytic in $\Delta_n(\rho)$, the series

$$\sum_{m=0}^{\infty} \frac{\varphi_{m+1}(y)}{m!} u^m$$

represents an analytic function in $\mathbb{C} \times \Delta_n(\rho)$ and

$$(\mathcal{B}\varphi(\cdot,y))(u) = \sum_{m=0}^{\infty} \frac{\varphi_{m+1}(y)}{m!} u^m, \quad \forall\, y \in \Delta_n(\rho).$$

This immediately follows from the proof of theorem 3.2.3.

Chapter 4

On a system of singular linear partial differential equations

In this chapter we only formulate a problem concerning a system of singular linear partial differential equations and we will prove the existence of a formal solution of this system. We shall often refer to chapter 2, which contains most of the proof of the existence of the above mentioned formal solution.

4.1 The problem

Let us consider the following system of linear partial differential equations with a singularity at the origin

$$z^2\frac{\partial\Phi}{\partial z} + \sum_{j=1}^{n}(\lambda_j+\alpha_j z)y_j\frac{\partial\Phi}{\partial y_j} + A(z,y)\Phi = za(z,y), \qquad (4.1)$$

where $z \in \mathbb{C}$, $y \in \mathbb{C}^n$, $\Phi = \Phi(z,y) \in \mathbb{C}^n$, $(\lambda_j,\alpha_j) \in \mathbb{C}\times\mathbb{C}$, $\mathcal{I}m\ \lambda_j \geq 0$, $A(z,y)$ an $n\times n$-matrix analytic in a neighbourhood of $(0,0)$, $A(z,y) = A_0 - A_1 z - zB(z,y)$ with A_0 and A_1 diagonal matrices and $a(z,y)$ an n-vector analytic in a neighbourhood of $(0,0)$. We write

$$A_0 = \mathrm{diag}\{c_1,c_2,\dots,c_n\} \quad \text{and} \quad A_1 = \mathrm{diag}\{d_1,d_2,\dots,d_n\}$$

and λ (resp. α) denotes the n-vector with components λ_j (resp. α_j).

Let $\langle\cdot,\cdot\rangle$ denote the bilinear form on $\mathbb{C}^n$ defined by

$$\langle z,w\rangle := \sum_{i=1}^{n} z_i w_i,$$

if both $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$ belong to $\mathbb{C}^n$.
We assume c_i to be chosen such that for all $i \in \{1, 2, \dots, n\}$ and for all $Q \in \mathbb{N}^n$ we have:

$$\mathcal{I}m\ (\langle Q, \lambda\rangle + c_i) \geq 0.$$

Moreover, we assume that the vector-valued function a and matrix-valued function B are analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$ for some $\tau' > 0$ and some $\rho' > 0$. Hence theorem 3.1.2 implies that both a and B can be expanded in a Taylor series

$$a(z,y) = \sum_{Q\in\mathbb{N}^n} a_Q(z)y^Q \quad \text{and} \quad B(z,y) = \sum_{Q\in\mathbb{N}^n} B_Q(z)y^Q,$$

if $(z, y) \in \Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$.
Finally, we assume

$$a(0,\cdot) \equiv 0 \quad \text{and} \quad B(0,\cdot) \equiv 0.$$

For each $Q \in \mathbb{N}^n$ the functions a_Q and B_Q are analytic in $\Delta_1(\frac{1}{\tau'})$, so we have $a_Q(z) \simeq \hat{a}_Q(z)$ and $B_Q(z) \simeq \hat{B}_Q(z)$ as $z \to 0$, where $\hat{a}_Q$ and $\hat{B}_Q$ are the Taylor expansions of a_Q and B_Q.
Proposition 3.2.1 and remark 3.2.2 imply that both $\mathcal{B}a_Q$ and $\mathcal{B}B_Q$ satisfy the first two conditions in the definition of $\mathcal{A}_{\tau'}$ (where $\delta \in (0, \pi)$ can be chosen arbitrary, cf. definition 1.3.1). That the third condition is satisfied follows from the fact that both $\mathcal{B}a_Q$ and $\mathcal{B}B_Q$ are entire functions.

For convenience we write (4.1) in the following form:

$$\begin{aligned} & z^2\frac{\partial\Phi}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^n \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}_{i=1}^n \cdot \Phi \\ = & \\ & z\cdot\operatorname{diag}\left\{d_i - \sum_{j=1}^n \alpha_j y_j \frac{\partial}{\partial y_j}\right\}_{i=1}^n \cdot \Phi + za(z,y) + zB(z,y)\Phi. \end{aligned} \tag{4.2}$$

The problem is to find a solution of (4.2), which is analytic in the product of a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi)$ and a polydisc $\Delta_n(\rho)$, for some $\rho \leq \rho'$. The following section deals with a formal power series solution $\sum_{Q\in\mathbb{N}^n} \Phi_Q(z)y^Q$ with both formal and analytic coefficients. The remaining chapters are concerned with the convergence of the formal series with analytic coefficients. This will be specified later.

In the remainder of this report we take $\tau > \tau'$ arbitrary but fixed.

4.2 A formal solution with analytic coefficients

Theorem 4.2.1 *Suppose that for all $i \in \{1,2,\dots,n\}$ and for all multi-indices $Q \in \mathbb{N}^n$ with $\langle Q,\lambda\rangle + c_i = 0$ we have $d_i - \langle Q,\alpha\rangle \notin \mathbb{N}$, then the equation (4.2) has a unique formal power series solution*

$$\hat{\Phi}(z,y) = \sum_{Q\in\mathbb{N}^n} \hat{\Phi}_Q(z)y^Q,$$

with coefficients $\hat{\Phi}_Q \in \mathbb{C}^n[[z]]$ satisfying $\hat{\Phi}_Q(0) = 0$.
Under the hypotheses listed in the preceding section, there also exists a unique formal power series solution

$$\Phi(z,y) = \sum_{Q\in\mathbb{N}^n} \Phi_Q(z)y^Q,$$

with coefficients Φ_Q being analytic in a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi)$.
Moreover, each Φ_Q can be written as a Laplace integral $\Phi_Q = \mathcal{L}w_Q$, where $w_Q \in \mathcal{A}_\tau$ (in the definition of $\mathcal{A}_\tau$ one can take $\delta \in (0,\pi)$ arbitrary).
Finally, for each multi-index Q the function Φ_Q asymptotically equals $\hat{\Phi}_Q$ as $z \to 0$ on a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi)$.

Proof. First of all we take $\delta \in (0,\pi)$ arbitrary.
Next, let us formally substitute the series $\Phi(z,y) = \sum_{Q\in\mathbb{N}^n} \Phi_Q(z)y^Q$ in (4.2) to obtain

$$\sum_{Q\in\mathbb{N}^n}\left(z^2\frac{\mathrm{d}\Phi_Q}{\mathrm{d}z}(z)\right)y^Q + \sum_{Q\in\mathbb{N}^n} \mathrm{diag}\,\{\langle Q,\lambda\rangle + c_i\}_{i=1}^n \cdot \Phi_Q(z)y^Q$$
$$=$$
$$\sum_{Q\in\mathbb{N}^n} z\cdot \mathrm{diag}\,\{d_i - \langle Q,\alpha\rangle\}_{i=1}^n \cdot \Phi_Q(z)y^Q + \sum_{Q\in\mathbb{N}^n} za_Q(z)y^Q +$$
$$z\cdot\left[\sum_{S\in\mathbb{N}^n} B_S(z)y^S\right]\cdot\left[\sum_{R\in\mathbb{N}^n}\Phi_R(z)y^R\right],$$

because

$$\left(\sum_{j=1}^n \lambda_j y_j\frac{\partial}{\partial y_j} + c_i\right)y^Q = \left(\sum_{j=1}^n \lambda_j q_j + c_i\right)y^Q = (\langle Q,\lambda\rangle + c_i)y^Q, \quad \forall\, Q\in\mathbb{N}^n.$$

The product of the two series mentioned above equals

$$\left[\sum_{S\in\mathbb{N}^n} B_S(z)y^S\right]\cdot\left[\sum_{R\in\mathbb{N}^n}\Phi_R(z)y^R\right]$$
$$= \sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\le R\le Q} B_{(Q-R)}(z)\Phi_R(z)\right\}y^Q,$$

in which $0 \leq R \leq Q$ has to be interpreted as $0 \leq r_i \leq q_i$ for each index $i \in \{1, 2, \dots, n\}$. Comparing coefficients y^Q for $Q \in \mathbb{N}^n$ gives the following system of singular linear ordinary differential equations

$$z^2 \frac{\mathrm{d}\Phi_Q}{\mathrm{d}z} + A_Q \Phi_Q = z\tilde{A}_Q \Phi_Q \; + \; z a_Q(z) \; + \sum_{0 \leq R \leq Q} z B_{(Q-R)}(z) \Phi_R,$$

where $\Phi_Q = \Phi_Q(z) \in \mathbb{C}^n$.
Here $A_Q = \mathrm{diag}\{\langle Q, \lambda \rangle + c_i\}_{i=1}^n$ and $\tilde{A}_Q = \mathrm{diag}\{d_i - \langle Q, \alpha \rangle\}_{i=1}^n$. Note that A_Q is a diagonal matrix with entries in the upper halfplane $\mathbb{H}$.

To solve this system we use an induction argument.
First we take the multi-index $Q = 0 \in \mathbb{N}^n$. We then have to solve the equation

$$z^2 \frac{\mathrm{d}\Phi_0}{\mathrm{d}z} + A_0 \Phi_0 = z\tilde{A}_0 \Phi_0 \; + \; z a_0(z) \; + \; z B_0(z) \Phi_0, \tag{4.3}$$

in which a_0 and B_0 should be replaced by $\hat{a}_0$ and $\hat{B}_0$ if we are looking for a formal solution. This equation has been studied in detail in chapter 2 and theorem 2.2.1 immediately gives the existence of a formal solution $\hat{\Phi}_0$, with $\hat{\Phi}_0(0) = 0$.
For the time being we assume (4.3) to have an analytic solution Φ_0, satisfying the conditions in the theorem (a proof of this assumption can be found at the end of this section), and we make the following induction hypothesis:
Let $k \in \mathbb{N}$ such that for all multi-indices Q' with $|Q'| \leq k$ there exist formal solutions $\hat{\Phi}_{Q'}$, with $\hat{\Phi}_{Q'}(0) = 0$, satisfying

$$z^2 \frac{\mathrm{d}\hat{\Phi}_{Q'}}{\mathrm{d}z} + A_{Q'} \hat{\Phi}_{Q'} = z\tilde{A}_{Q'} \hat{\Phi}_{Q'} \; + \; z\hat{a}_{Q'}(z) \; + \sum_{0 \leq R \leq Q'} z\hat{B}_{(Q'-R)}(z) \hat{\Phi}_R$$

and analytic functions $\Phi_{Q'}$, satisfying the conditions in the theorem, such that

$$z^2 \frac{\mathrm{d}\Phi_{Q'}}{\mathrm{d}z} + A_{Q'} \Phi_{Q'} = z\tilde{A}_{Q'} \Phi_{Q'} \; + \; z a_{Q'}(z) \; + \sum_{0 \leq R \leq Q'} z B_{(Q'-R)}(z) \Phi_R.$$

For a multi-index $Q \in \mathbb{N}^n$ with $|Q| = k + 1$ we then have the equation

$$\begin{aligned} & z^2 \frac{\mathrm{d}\hat{\Phi}_Q}{\mathrm{d}z} + A_Q \hat{\Phi}_Q \\ = \; & z\tilde{A}_Q \hat{\Phi}_Q \; + \; z\hat{a}_Q(z) \; + \sum_{0 \leq R \leq Q} z\hat{B}_{(Q-R)}(z) \hat{\Phi}_R \\ = \; & z\tilde{A}_Q \hat{\Phi}_Q \; + \; z\hat{a}_Q(z) \; + \sum_{\substack{0 \leq R \leq Q \\ R \neq Q}} z\hat{B}_{(Q-R)}(z) \hat{\Phi}_R \; + \; z\hat{B}_0(z) \hat{\Phi}_Q \end{aligned}$$

in the formal case. We have $\hat{a}_Q + \sum_{\substack{0 \leq R \leq Q \\ R \neq Q}} \hat{B}_{(Q-R)} \hat{\Phi}_R$ being a known formal series without constant term, which follows from the induction hypothesis. So again theorem 2.2.1 shows the existence of a formal solution $\hat{\Phi}_Q \in \mathbb{C}^n[[z]]$ with $\hat{\Phi}_Q(0) = 0$.
The equation in the analytic case equals

$$\begin{aligned} & z^2 \frac{\mathrm{d}\Phi_Q}{\mathrm{d}z} + A_Q \Phi_Q \\ = & \\ & z\tilde{A}_Q \Phi_Q \; + \; z a_Q(z) \; + \sum_{\substack{0 \leq R \leq Q \\ R \neq Q}} z B_{(Q-R)}(z) \Phi_R + z B_0(z) \Phi_Q. \end{aligned} \tag{4.4}$$

The induction hypothesis implies $a_Q \; + \; \sum_{\substack{0 \leq R \leq Q \\ R \neq Q}} B_{(Q-R)} \Phi_R$ to be a known analytic function and lemma 2.3.1, together with the fact $\mathcal{A}_{\tau'} \subset \mathcal{A}_\tau$, implies that the Borel transform of this function, $\mathcal{B}a_Q + \sum_{\substack{0 \leq R \leq Q \\ R \neq Q}} \mathcal{B}B_{(Q-R)} * w_R$, belongs to $\mathcal{A}_\tau$.
So both (4.3) and (4.4) are equations of the form studied in section 2.4. Theorem 2.4.1, together with remark 2.4.5, implies that the solution of (4.3) (resp. (4.4)) is analytic in a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi - \delta)$, can be written as a Laplace integral of a function in $\mathcal{A}_\tau$ and asymptotically equals the corresponding formal solution.
But $\delta \in (0, \pi)$ was chosen arbitrary. Taking δ smaller gives an analytic continuation of the solution of the corresponding convolution equation [i.e. the equation obtained from (4.3) (resp. (4.4)) by taking a Borel transform]. Hence the solution of (4.3) (resp. (4.4)) is analytic in a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi)$, asymptotically equals the corresponding formal series in this neighbourhood and can be written as a Laplace integral of a function in $\mathcal{A}_\tau$, for arbitrary $\delta \in (0, \pi)$. ■

Chapter 5

The non-resonance case

We will study the following system of singular linear partial differential equations

$$z^2\frac{\partial \Phi}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}_{i=1}^{n} \cdot \Phi$$
$$=$$
$$z \cdot \operatorname{diag}\left\{d_i - \sum_{j=1}^{n} \alpha_j y_j \frac{\partial}{\partial y_j}\right\}_{i=1}^{n} \cdot \Phi + za(z,y) + zB(z,y)\Phi.$$

Besides the assumptions in chapter 4, we assume that there are no resonance relations, i.e. we assume

$$\langle Q, \lambda\rangle + c_i \neq 0$$

for all $Q \in \mathbb{N}^n$ and all $i \in \{1, 2, \ldots, n\}$.
Furthermore, we assume the existence of two positive constants k and γ, both independent of Q and i, such that

$$|\langle Q, \lambda\rangle + c_i| \geq k|Q|^{-\gamma}, \quad \forall\, Q \in \mathbb{N}^n \setminus \{0\}, \quad \forall\, i \in \{1, 2, \ldots, n\}.$$

This latter estimate is called a Siegel condition.
In this chapter and in chapter 6 we assume α_i and d_i to be equal to zero for all $i \in \{1, 2, \ldots, n\}$ (this can be done without losing the validity of theorem 4.2.1). In chapter 7 we also consider a case where α_i might not be zero.
With the assumption $\alpha_i = d_i = 0$ for all $i \in \{1, 2, \ldots, n\}$ the differential equation above reduces to

$$z^2\frac{\partial \Phi}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}_{i=1}^{n} \cdot \Phi$$
$$= \tag{5.1}$$
$$za(z,y) + zB(z,y)\Phi.$$

5.1 A formal solution in terms of powers of z

We assume a and B to be analytic in a polydisc $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$, so we can write (remembering our assumption $a(0,\cdot) \equiv 0$ and $B(0,\cdot) \equiv 0$):

$$a(z,y) = \sum_{m=1}^{\infty} a_m(y) z^m \quad \text{and} \quad B(z,y) = \sum_{m=1}^{\infty} B_m(y) z^m,$$

if $(z,y) \in \Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$. (Note that section 3.2 tells us that both a_m and B_m are analytic in $\Delta_n(\rho')$ for each $m \in \mathbb{N}_{\geq 1}$.)
Substitution of the series $\sum_{m=0}^{\infty} \Phi_m(y) z^m$ in (5.1) gives

$$\begin{aligned}\sum_{m=2}^{\infty} (m-1)\Phi_{m-1}(y) z^m + \sum_{m=0}^{\infty} \operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} \Phi_m(y) z^m \\ = \sum_{m=2}^{\infty} a_{m-1}(y) z^m + \sum_{m=2}^{\infty} \left\{ \sum_{k=1}^{m-1} B_k(y) \Phi_{m-1-k}(y) \right\} z^m.\end{aligned}$$

Comparing coefficients results in the following recurrence relation

$$\operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} \Phi_0(y) = 0,$$

$$\operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} \Phi_1(y) = 0,$$

and for $m \geq 2$

$$\begin{aligned}(m-1)\Phi_{m-1}(y) + \operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} \Phi_m(y) \\ = \\ a_{m-1}(y) + \sum_{k=1}^{m-1} B_k(y) \Phi_{m-1-k}(y).\end{aligned}$$

Our aim in this section is to show that this recurrence relation has a unique solution in $A(\Delta_n(\rho'))$. The procedure is as follows:
We use induction after $m \in \mathbb{N}$ and for such a non-negative integer m we construct a formal series solution $\sum_{Q \in \mathbb{N}^n} \Phi_{Q,m} y^Q$. We will check the convergence of this formal solution afterwards.
Let us start with a series $\sum_{Q \in \mathbb{N}^n} \Phi_{Q,0} y^Q$. Formally we can take the operator $\operatorname{diag}\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\}$ through the sum to get

$$\sum_{Q \in \mathbb{N}^n} A_Q \Phi_{Q,0} y^Q = 0,$$

where $A_Q = \mathrm{diag}\{\langle Q, \lambda\rangle + c_i\}_{i=1}^n$.
Hence $A_Q \Phi_{Q,0} = 0$ for each $Q \in \mathbb{N}^n$. The assumption of no resonance implies that A_Q is invertible, and so $\Phi_{Q,0} = 0$, for each $Q \in \mathbb{N}^n$. We conclude $\Phi_0 \equiv 0$ and obviously $\Phi_0 \in A(\Delta_n(\rho'))$. (Exactly the same reasoning implies $\Phi_1 \equiv 0$.)
Next suppose that for all $m \in \{0, 1, \ldots, M\}$, $M \geq 1$, a (unique) solution Φ_m, analytic in $\Delta_n(\rho')$, exists, then for $m = M + 1$ we have the equation

$$\mathrm{diag}\left\{\sum_{j=1}^n \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}\Phi_{M+1}(y) = \zeta_M(y),$$

for some function $\zeta_M \in A(\Delta_n(\rho'))$. This function ζ_M can be expanded in a Taylor series $\zeta_M(y) = \sum_{Q\in\mathbb{N}^n} \zeta_{Q,M} y^Q$. Substitution of the series $\sum_{Q\in\mathbb{N}^n} \Phi_{Q,M+1} y^Q$ gives

$$\sum_{Q\in\mathbb{N}^n} A_Q \Phi_{Q,M+1} y^Q = \sum_{Q\in\mathbb{N}^n} \zeta_{Q,M} y^Q.$$

Hence for each $Q \in \mathbb{N}^n$ we have $\Phi_{Q,M+1} = A_Q{}^{-1}\zeta_{Q,M}$.
To prove the convergence of the series $\sum_{Q\in\mathbb{N}^n} \Phi_{Q,M+1} y^Q$ we need the Siegel condition, which implies for $Q \in \mathbb{N}^n \setminus \{0\}$:

$$|A_Q{}^{-1}| = \max_{i\in\{1,2,\ldots,n\}} |(\langle Q, \lambda\rangle + c_i)^{-1}| \leq \frac{1}{k}|Q|^\gamma.$$

Hence the coefficients $\Phi_{Q,M+1}$, $Q \in \mathbb{N}^n \setminus \{0\}$, can be estimated by

$$|\Phi_{Q,M+1}| \leq \frac{1}{k}|Q|^\gamma\, |\zeta_{Q,M}|.$$

Next take a compact subset K of $\Delta_n(\rho')$, then there exist positive numbers r and $\tilde{r}$, $0 < r < \tilde{r} < \rho'$, such that $K \subseteq \Delta_n(r) \subset \Delta_n(\tilde{r}) \subset \Delta_n(\rho')$ and we know

$$\sum_{Q\in\mathbb{N}^n} |\zeta_{Q,M}|\, \tilde{r}^{|Q|} < \infty.$$

Hence

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n} \sup_{y\in K} \left|\Phi_{Q,M+1} y^Q\right| &\leq \left|\Phi_{0,M+1}\right| + \sum_{Q\in\mathbb{N}^n\setminus\{0\}} |\Phi_{Q,M+1}|\, r^{|Q|} \\ &\leq \left|\Phi_{0,M+1}\right| + \frac{1}{k}\sum_{Q\in\mathbb{N}^n\setminus\{0\}} |Q|^\gamma\, |\zeta_{Q,M}|\, r^{|Q|}\end{aligned}$$

and we have to show that this latter sum converges, but this is an easy consequence of the following

Proposition 5.1.1 *Both* $\sum_{m=1}^{\infty} a_m x^m$ *and* $\sum_{m=1}^{\infty} m^\gamma a_m x^m$ *have the same radius of convergence.*

Proof. The proof is not very hard, for

$$\sqrt[m]{m^\gamma} = m^{\frac{\gamma}{m}} = e^{\frac{\gamma}{m}\log m} \xrightarrow[m\to\infty]{} 1,$$

hence $\lim_{m\to\infty} \sqrt[m]{m^\gamma}$ exists and we conclude

$$\limsup_{m\to\infty} \sqrt[m]{m^\gamma |a_m|} = \limsup_{m\to\infty} \sqrt[m]{|a_m|}.$$

This proves proposition 5.1.1. ∎

5.2 Formal reduction to a convolution equation

In the preceding section we have found a formal solution $\sum_{m=2}^{\infty} \Phi_m(y) z^m$ with coefficients $\Phi_m \in A(\Delta_n(\rho'))$. Let us consider a partial sum of this formal solution, $u_N = u_N(z,y) := \sum_{m=2}^{N-1} \Phi_m(y) z^m$, in which $N \geq 3$ will be specified later on. This function is analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$ (even in $\mathbb{C} \times \Delta_n(\rho')$, but this is inessential), for lemma 3.1.6 implies that for each $m \in \mathbb{N}$ the function $(z,y) \mapsto \Phi_m(y) z^m$ is analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$ and thus a finite sum of these terms also is analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$. To keep notation simple we substitute the series $\sum_{m=0}^{N-1} \Phi_m(y) z^m$ for u_N in (5.1) and keep in mind that $\Phi_0,\ \Phi_1 \equiv 0$. We obtain

$$\begin{aligned}
& z^2 \frac{\partial u_N}{\partial z} + \operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} u_N - z a(z,y) - z B(z,y) u_N \\
= \\
& \sum_{m=2}^{N} (m-1)\Phi_{m-1}(y) z^m + \sum_{m=0}^{N-1} \operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} \Phi_m(y) z^m + \\
& - \sum_{m=2}^{N} a_{m-1}(y) z^m - z \sum_{m=1}^{N-1} \left(\sum_{k=1}^{m} B_k(y) \Phi_{m-k}(y) \right) z^m + \\
& -z \sum_{m=N}^{2N-2} \left(\sum_{\substack{k,l \in \{1,\dots,N-1\} \\ k+l=m}} B_k(y) \Phi_l(y) \right) z^m - z \sum_{m=N}^{\infty} a_m(y) z^m + \\
& -z \left[\sum_{k=N}^{\infty} B_k(y) z^k \right] \left[\sum_{l=1}^{N-1} \Phi_l(y) z^l \right] \\
= \\
& -R(z,y).
\end{aligned}$$

Note that we only used the fact $\Phi_0 \equiv 0$. Using the recurrence relation derived in the preceding section we conclude

$$\begin{aligned} R(z,y) &= \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}\Phi_N(y)z^N + z\sum_{m=N}^{\infty} a_m(y)z^m + \\ & z\sum_{m=N}^{2N-2}\left(\sum_{\substack{k,l\in\{1,\dots,N-1\}\\ k+l=m}} B_k(y)\Phi_l(y)\right)z^m + \qquad (5.2) \\ & z\left[\sum_{k=N}^{\infty} B_k(y)z^k\right]\left[\sum_{l=1}^{N-1}\Phi_l(y)z^l\right] \end{aligned}$$

or

$$\begin{aligned} R(z,y) &= \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}\Phi_N(y)z^N + \sum_{m=N+1}^{\infty} a_{m-1}(y)z^m + \\ & \sum_{m=N+1}^{2N-1}\left(\sum_{\substack{k,l\in\{1,\dots,N-1\}\\ k+l=m-1}} B_k(y)\Phi_l(y)\right)z^m + \qquad (5.3) \\ & \sum_{m=N+2}^{\infty}\left(\sum_{\substack{k\in\{N,N+1,\dots\}\\ l\in\{1,\dots,N-1\}\\ k+l=m-1}} B_k(y)\Phi_l(y)\right)z^m. \end{aligned}$$

With $\Psi := \Phi - u_N$ we obtain

$$\begin{aligned} & z^2\frac{\partial\Psi}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^{n}\lambda_j y_j\frac{\partial}{\partial y_j} + c_i\right\}\Psi \\ = & \qquad (5.4) \\ & zB(z,y)\Psi + R(z,y). \end{aligned}$$

as an equation equivalent with (5.1).

We observe that the function R is analytic in $\Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$, which follows from (5.2), the fact that both $A(\Delta_1(\frac{1}{r'}) \times \Delta_n(\rho'))$ and $A(\Delta_n(\rho'))$ are rings and lemma 3.1.6.

[The function represented by the series $\sum_{m=N}^{\infty} a_m(y)z^m$ is analytic in the polydisc $\Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$ as a difference of two analytic functions, namely $a(z,y) - \sum_{m=1}^{N-1} a_m(y)z^m$, and similarly we have $\sum_{m=N}^{\infty} B_m(y)z^m$ representing an analytic function in $\Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$. Also observe that the function $(z,y) \mapsto z$ is analytic in $\Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$.]

In section 5.1 we have seen that each coefficient Φ_m of the formal solution of (5.1) can be written as a convergent power series

$$\Phi_m(y) = \sum_{Q\in\mathbb{N}^n} \Phi_{Q,m}y^Q, \quad y \in \Delta_n(\rho').$$

Moreover, for each $m \in \mathbb{N}_{\geq 1}$ the functions a_m and B_m are analytic in $\Delta_n(\rho')$, so they can be written as

$$a_m(y) = \sum_{Q \in \mathbb{N}^n} a_{Q,m} y^Q \quad \text{and} \quad B_m(y) = \sum_{Q \in \mathbb{N}^n} B_{Q,m} y^Q.$$

This implies for $(z, y) \in \Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$ (using the independence of the order of summation)

$$R(z, y) = \sum_{Q \in \mathbb{N}^n} R_Q(z) y^Q, \tag{5.5}$$

where for each $Q \in \mathbb{N}^n$:

$$\begin{aligned} R_Q(z) \quad = \quad & A_Q \Phi_{Q,N} z^N + \sum_{m=N+1}^{\infty} a_{Q,m-1} z^m + \\ & \sum_{0 \leq R \leq Q} \left[\sum_{m=N+1}^{2N-1} \left(\sum_{\substack{k,l \in \{1,\dots,N-1\} \\ k+l=m-1}} B_{(Q-R),k} \Phi_{R,l} \right) z^m \right] + \\ & \sum_{0 \leq R \leq Q} \left[\sum_{m=N+2}^{\infty} \left(\sum_{\substack{k \in \{N,N+1,\dots\} \\ l \in \{1,\dots,N-1\} \\ k+l=m-1}} B_{(Q-R),k} \Phi_{R,l} \right) z^m \right]. \end{aligned}$$

In section 4.1 we have seen that the functions a and B can be written as

$$a(z, y) = \sum_{Q \in \mathbb{N}^n} a_Q(z) y^Q \quad \text{and} \quad B(z, y) = \sum_{Q \in \mathbb{N}^n} B_Q(z) y^Q$$

if $(z, y) \in \Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$.

From now on we take $\delta \in (0, \frac{\pi}{2})$ arbitrary but fixed[1], and (as in chapter 2) we write $S_c = S(\frac{\pi}{2}, \pi - \delta)$, while $\overline{S_c}$ denotes the closure of S_c.

We first seek a solution Ψ of (5.4) which is analytic in the product of a neighbourhood U_δ of 0 in $S(\frac{\pi}{2}, 2\pi - \delta)$ and a polydisc $\Delta_n(\rho)$ for some $\rho \leq \rho'$. As the map $y \mapsto \Psi(z, y)$ is analytic in $\Delta_n(\rho)$, the function Ψ can be expanded in a Taylor series

$$\Psi(z, y) = \sum_{Q \in \mathbb{N}^n} \Psi_Q(z) y^Q, \quad (z, y) \in U_\delta \times \Delta_n(\rho),$$

where $\Psi_Q(z) = \frac{D^Q \Psi(z,0)}{Q!}$ is analytic in U_δ. Here $D^Q = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_n}\right)^Q$. Substitution of this series in (5.4) gives, using the Taylor expansions of B

[1] For later applications it is practical to take $\delta \in (0, \frac{\pi}{2})$ instead of $\delta \in (0, \pi)$.

and R,

$$\begin{aligned} & \sum_{Q\in\mathbb{N}^n}\Big(z^2\frac{\mathrm{d}\Psi_Q}{\mathrm{d}z}(z)+A_Q\Psi_Q(z)\Big)y^Q \\ = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.6) \\ & \sum_{Q\in\mathbb{N}^n}\Bigg\{\sum_{0\le R\le Q} zB_{(Q-R)}(z)\Psi_R(z)\Bigg\}y^Q + \sum_{Q\in\mathbb{N}^n}R_Q(z)y^Q. \end{aligned}$$

With theorem 3.2.3 we see that for each $y\in\Delta_n(\rho')$ the function $\xi(u,y):=(\mathcal{B}R(\cdot,y))(u)$ exists and represents an analytic function in $\mathbb{C}\times\Delta_n(\rho')$. Moreover, from (5.5) we deduce

$$\xi(u,y)=\sum_{Q\in\mathbb{N}^n}\xi_Q(u)y^Q, \tag{5.7}$$

where for each $Q\in\mathbb{N}^n$:

$$\begin{aligned} \xi_Q(u) \;=\; & \frac{A_Q\Phi_{Q,N}}{\Gamma(N)}u^{N-1}+\sum_{m=N}^{\infty}\frac{1}{\Gamma(m+1)}a_{Q,m}u^m + \\ & \sum_{0\le R\le Q}\left[\sum_{m=N}^{2N-2}\frac{1}{\Gamma(m+1)}\left(\sum_{\substack{k,l\in\{1,\dots,N-1\}\\ k+l=m}} B_{(Q-R),k}\Phi_{R,l}\right)u^m\right]+ \\ & \sum_{0\le R\le Q}\left[\sum_{m=N+1}^{\infty}\frac{1}{\Gamma(m+1)}\left(\sum_{\substack{k\in\{N,N+1,\dots\}\\ l\in\{1,\dots,N-1\}\\ k+l=m}} B_{(Q-R),k}\Phi_{R,l}\right)u^m\right]. \end{aligned}$$

An application of theorem 3.2.3 again, shows that for $y\in\Delta_n(\rho')$ fixed the functions $\alpha(u,y):=(\mathcal{B}a(\cdot,y))(u)$ and $\beta(u,y):=(\mathcal{B}B(\cdot,y))(u)$ exist. Moreover, α and β represent analytic functions in $\mathbb{C}\times\Delta_n(\rho')$ and they can be written as

$$\sum_{Q\in\mathbb{N}^n}\alpha_Q(u)y^Q \quad\text{resp.}\quad \sum_{Q\in\mathbb{N}^n}\beta_Q(u)y^Q,$$

where for each $Q\in\mathbb{N}^n$ we have $\alpha_Q=\mathcal{B}a_Q$ and $\beta_Q=\mathcal{B}B_Q$.

A formal application of $\mathcal{B}$ transforms (5.6) into the equation

$$\begin{aligned} & \sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)w_Q(u)y^Q \\ = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.8) \\ & \sum_{Q\in\mathbb{N}^n}\Bigg\{\sum_{0\le R\le Q}(1*\beta_{(Q-R)}*w_R)(u)\Bigg\}y^Q+\sum_{Q\in\mathbb{N}^n}\xi_Q(u)y^Q, \end{aligned}$$

where $w_Q = \mathcal{B}\Psi_Q$ for each $Q \in \mathbb{N}^n$.
Though this is a convolution equation, it is not the one we will study in the near future. In fact we will study the equation obtained from (5.8) by a termwise multiplication with $(u\mathrm{I} + A_Q)^{-1}$. Later we will prove that a solution of this last equation also is a solution of (5.8) in a certain sense.

5.3 Two auxiliary results

Definition 5.3.1 *For $N \in \mathbb{N}_{\geq 1}$, $\tau_1 > 0$ and $\rho > 0$ we define $\mathcal{W}_{N,\tau_1,\rho}$ to be the space of convergent series $\omega = \omega(u,y) = \sum_{Q\in\mathbb{N}^n} \omega_Q(u)y^Q$ such that $\omega_Q \in \mathcal{V}_{N,\tau_1}$ for all $Q \in \mathbb{N}^n$ (cf. definition 2.3.3) and*

$$\sum_{Q\in\mathbb{N}^n} \|\omega_Q\|_{N,\tau_1}\rho^{|Q|} < \infty.$$

Lemma 5.3.2 *The space $(\mathcal{W}_{N,\tau_1,\rho}, \|\cdot\|_{N,\tau_1,\rho})$ is a Banach space, with norm $\|\cdot\|_{N,\tau_1,\rho}$ defined by*

$$\|\omega\|_{N,\tau_1,\rho} = \sum_{Q\in\mathbb{N}^n} \|\omega_Q\|_{N,\tau_1}\rho^{|Q|}.$$

Proof. If both

$$\omega_1(u,y) = \sum_{Q\in\mathbb{N}^n} \omega_{Q,1}(u)y^Q \quad \text{and} \quad \omega_2(u,y) = \sum_{Q\in\mathbb{N}^n} \omega_{Q,2}(u)y^Q$$

are series in $\mathcal{W}_{N,\tau_1,\rho}$ and α, β are some complex numbers, then the linear combination $\alpha\omega_1 + \beta\omega_2$, defined by

$$(\alpha\omega_1 + \beta\omega_2)(u,y) := \sum_{Q\in\mathbb{N}^n} (\alpha\omega_{Q,1}(u) + \beta\omega_{Q,2}(u))\, y^Q,$$

also belongs to $\mathcal{W}_{N,\tau_1,\rho}$, for $\mathcal{V}_{N,\tau_1}$ is a linear space and $\|\cdot\|_{N,\tau_1}$ defines a norm on $\mathcal{V}_{N,\tau_1}$. From this one obtains that $\|\cdot\|_{N,\tau_1,\rho}$ is a seminorm on $\mathcal{W}_{N,\tau_1,\rho}$. Moreover, if $\omega = \omega(u,y) = \sum_{Q\in\mathbb{N}^n} \omega_Q(u)y^Q \in \mathcal{W}_{N,\tau_1,\rho}$, then $\|\omega\|_{N,\tau_1,\rho} = 0$ is equivalent with $\|\omega_Q\|_{N,\tau_1} = 0$ for all $Q \in \mathbb{N}^n$. The proof of lemma 2.3.4 implies the last statement to be equivalent with $\omega_Q \equiv 0$ for all $Q \in \mathbb{N}^n$. Hence $\omega \equiv 0$.

Next take a Cauchy sequence $(\omega_k)_{k\in\mathbb{N}}$ in $\mathcal{W}_{N,\tau_1,\rho}$. When we write

$$\omega_k(u,y) = \sum_{Q\in\mathbb{N}^n} \omega_{Q,k}(u)y^Q,$$

the definition tells us

$$\forall\eta > 0 \;\; \exists M_\eta > 0 \;\; \text{s.t.} \;\; \forall m_1,\, m_2 \geq M_\eta \qquad \sum_{Q\in\mathbb{N}^n} \|\omega_{Q,m_1} - \omega_{Q,m_2}\|_{N,\tau_1}\rho^{|Q|} \leq \eta. \tag{5.9}$$

This implies that for each multi-index $Q \in \mathbb{N}^n$ we have

$$\forall \eta > 0 \;\; \exists M_\eta > 0 \;\; \text{s.t.} \;\; \forall m_1,\; m_2 \geq M_\eta \quad \|\omega_{Q,m_1} - \omega_{Q,m_2}\|_{N,\tau_1} \leq \eta \rho^{-|Q|}.$$

Hence for every $Q \in \mathbb{N}^n$ the sequence $(\omega_{Q,k})_{k \in \mathbb{N}}$ is a Cauchy sequence in the complete space $\mathcal{V}_{N,\tau_1}$, so for each $Q \in \mathbb{N}^n$ there exists a function $\omega_Q \in \mathcal{V}_{N,\tau_1}$ such that

$$\lim_{m \to \infty} \|\omega_{Q,m} - \omega_Q\|_{N,\tau_1} = 0.$$

Next define the series $\omega = \omega(u,y) := \sum_{Q \in \mathbb{N}^n} \omega_Q(u) y^Q$. Formula (5.9) implies

$$\forall \eta > 0 \;\; \exists M_\eta > 0 \;\; \text{s.t.} \;\; \forall m_1,\; m_2 \geq M_\eta \;\; \forall N \geq 0$$
$$\sum_{\substack{Q \in \mathbb{N}^n \\ |Q| \leq N}} \|\omega_{Q,m_1} - \omega_{Q,m_2}\|_{N,\tau_1} \rho^{|Q|} \leq \eta.$$

Since $\|\cdot\|_{N,\tau_1}$ defines a norm on $\mathcal{V}_{N,\tau_1}$, the map $\|\cdot\|_{N,\tau_1} : \mathcal{V}_{N,\tau_1} \to [0,\infty)$ is a continuous one, so when we let $m_2 \to \infty$ the last mentioned formula transforms into

$$\forall \eta > 0 \;\; \exists M_\eta > 0 \;\; \text{s.t.} \;\; \forall m \geq M_\eta \;\; \forall N \geq 0$$
$$\sum_{\substack{Q \in \mathbb{N}^n \\ |Q| \leq N}} \|\omega_{Q,m} - \omega_Q\|_{N,\tau_1} \rho^{|Q|} \leq \eta.$$

Letting $N \to \infty$ we obtain

$$\forall \eta > 0 \;\; \exists M_\eta > 0 \;\; \text{s.t.} \;\; \forall m \geq M_\eta$$
$$\sum_{Q \in \mathbb{N}^n} \|\omega_{Q,m} - \omega_Q\|_{N,\tau_1} \rho^{|Q|} \leq \eta. \tag{5.10}$$

For $\eta > 0$ and $m \geq M_\eta$ we have

$$\sum_{Q \in \mathbb{N}^n} \|\omega_Q\|_{N,\tau_1} \rho^{|Q|} \leq \eta + \|\omega_m\|_{N,\tau_1,\rho} < \infty,$$

so the series ω belongs to $\mathcal{W}_{N,\tau_1,\rho}$.
Moreover, formula (5.10) tells us

$$\lim_{m \to \infty} \|\omega_m - \omega\|_{N,\tau_1,\rho} = 0.$$

This proves lemma 5.3.2. ■

Lemma 5.3.3 *Suppose $\varphi \in A\big(\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')\big)$ and assume that $\varphi(\cdot, y)$ has a zero of order $N \geq 1$ at $z = 0$ for all $y \in \Delta_n(\rho')$.*
Then φ can be written as

$$\varphi(z,y) = \sum_{Q \in \mathbb{N}^n} \varphi_Q(z) y^Q, \quad (z,y) \in \Delta_1(\tfrac{1}{\tau'}) \times \Delta_n(\rho'),$$

where for each $Q \in \mathbb{N}^n$ the function $\varphi_Q(z) = \frac{D^Q\varphi(z,0)}{Q!}$ is analytic in $\Delta_1(\frac{1}{\tau'})$ and has an N^{th} order zero at $z = 0$.
The Borel transform of φ, $\mathcal{B}\varphi(\cdot, y)$, exists, is analytic in $\mathbb{C} \times \Delta_n(\rho')$ and can be written as

$$\big(\mathcal{B}\varphi(\cdot, y)\big)(u) = \sum_{Q \in \mathbb{N}^n} \big(\mathcal{B}\varphi_Q\big)(u) y^Q.$$

Moreover, $\mathcal{B}\varphi(\cdot, y)$ belongs to the Banach space $\mathcal{W}_{N,\tilde{\tau},\rho}$ for all $\tilde{\tau} > \tau'$ and all $\rho < \frac{\rho'}{n}$.

Proof. The assumptions on φ imply that this function can be written as

$$\varphi(z,y) = z^N \tilde{\varphi}(z,y),$$

where $\tilde{\varphi}$ is analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$. Hence each coefficient φ_Q can be written as

$$\varphi_Q(z) = z^N \cdot \frac{D^Q \tilde{\varphi}(z,0)}{Q!},$$

and thus has a zero of order N at $z = 0$.
The following three statements in the lemma immediately follow from theorem 3.2.3.
Next choose $\tilde{\tau} > \tau'$ and $\tilde{\rho} < \rho'$ arbitrary. Let us define

$$M = M(\tilde{\tau}, \tilde{\rho}) = \max_{(z,y) \in K} \left| z^{-N+1} \varphi(z,y) \right|,$$

where K is the compact set $\overline{\Delta_1(\tilde{\tau}^{-1}) \times \Delta_n(\tilde{\rho})}$.
For arbitrary $Q \in \mathbb{N}^n$, the Cauchy inequality then implies

$$\left| z^{-N+1} \varphi_Q(z) \right| \leq M \tilde{\rho}^{-|Q|}, \quad \forall\, z \in \overline{\Delta_1(\tilde{\tau}^{-1})}.$$

As $z \mapsto z^{-N+1}\varphi_Q(z)$ is analytic in $\Delta_1(\frac{1}{\tau'})$ and has a zero at $z = 0$, we conclude from theorem 3.2.1 that $\mathcal{B}\big[z^{-N+1}\varphi_Q\big]$ is defined on $\mathbb{C}$ and represents an entire function. Moreover, remark 3.2.2 implies

$$\left| \big(\mathcal{B}\big[z^{-N+1}\varphi_Q\big]\big)(u) \right| \leq \frac{M\tilde{\tau}}{\tilde{\rho}^{|Q|}} e^{\tilde{\tau}|u|}. \tag{5.11}$$

In case $N \geq 2$ the convolution property of the Borel transform gives

$$\begin{aligned}
\left|(\mathcal{B}\varphi_Q)(u)\right| &\leq \left|(\mathcal{B}[z^{N-1}] * \mathcal{B}[z^{-N+1}\varphi_Q])(u)\right| \\
&\leq \int_0^u \frac{M\tilde{\tau}}{\Gamma(N-1)\tilde{\rho}^{|Q|}} e^{\tilde{\tau}|u-s|}|s|^{N-2}|\mathrm{d}s| \\
&= \frac{M\tilde{\tau}}{\Gamma(N-1)\tilde{\rho}^{|Q|}} \int_0^{|u|} e^{\tilde{\tau}(|u|-t)} t^{N-2} \mathrm{d}t.
\end{aligned}$$

Hence

$$\left|(\mathcal{B}\varphi_Q)(u)\right| \leq \frac{M\tilde{\tau}}{\Gamma(N)\tilde{\rho}^{|Q|}} |u|^{N-1} e^{\tilde{\tau}|u|}, \quad \forall\, u \in \mathbb{C} \tag{5.12}$$

and this estimate also holds for $N = 1$, by (5.11).
As $\mathcal{B}\varphi_Q$ is analytic in $\mathbb{C}$, the function $u \mapsto u^{1-N}(\mathcal{B}\varphi_Q)(u)$ is analytic in $\mathbb{C} \setminus \{0\}$. From (5.12) we deduce

$$\lim_{u \to 0} u \cdot u^{1-N}(\mathcal{B}\varphi_Q)(u) = 0,$$

so Riemann's theorem implies that $u^{1-N}\mathcal{B}\varphi_Q$ can be assigned a value at $u = 0$, so that the result is an entire function. This, together with (5.12), implies each coefficient $\mathcal{B}\varphi_Q$ to be an element of $\mathcal{V}_{N,\tilde{\tau}}$ and

$$\|\mathcal{B}\varphi_Q\|_{N,\tilde{\tau}} \leq \frac{C_N}{\tilde{\rho}^{|Q|}},$$

where $C_N = C_N(\tilde{\tau}, \tilde{\rho})$ is some positive constant, depending upon N, $\tilde{\tau}$ and $\tilde{\rho}$.
As $\tilde{\tau} > \tau'$ and $\tilde{\rho} < \rho'$ were chosen arbitrary, this holds for every $\tilde{\tau} > \tau'$ and every $\tilde{\rho} < \rho'$.
Next take $\tilde{\tau} > \tau'$ and $\rho < \frac{\rho'}{n}$ arbitrary, then $\tilde{\rho} < \rho'$ can be found such that $\rho < \frac{\tilde{\rho}}{n}$.
We have

$$\sum_{Q \in \mathbb{N}^n} \|\mathcal{B}\varphi_Q\|_{N,\tilde{\tau}} \rho^{|Q|} \leq C_N \sum_{Q \in \mathbb{N}^n} \left(\frac{\rho}{\tilde{\rho}}\right)^{|Q|}.$$

From the theory of combinatorics we have for $m \in \mathbb{N}$

$$\#\left\{(q_1, q_2, \ldots, q_n) \in \mathbb{N}^n \;\middle|\; \sum_{i=1}^n q_i = m\right\} = \binom{n+m-1}{m}.$$

A proof can be found in [3].
An easy induction argument (after m) shows that this last binomial coefficient can be estimated by

$$\binom{n+m-1}{m} \leq n^m,$$

so

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n}\|\mathcal{B}\varphi_Q\|_{N,\tilde{\tau}}\rho^{|Q|} &\leq C_N\sum_{m=0}^{\infty}\binom{n+m-1}{m}\left(\frac{\rho}{\tilde{\rho}}\right)^m\\ &\leq C_N\sum_{m=0}^{\infty}\left(n\frac{\rho}{\tilde{\rho}}\right)^m\end{aligned}$$

and this last sum converges, by the construction of $\tilde{\rho}$. ■.

From this it is also clear that both

$$\alpha(u,y)=\sum_{Q\in\mathbb{N}^n}\alpha_Q(u)y^Q \quad\text{and}\quad \beta(u,y)=\sum_{Q\in\mathbb{N}^n}\beta_Q(u)y^Q$$

belong to $\mathcal{W}_{1,\tilde{\tau},\rho}$ for arbitrary $\tilde{\tau}>\tau'$ and arbitrary $\rho<\frac{\varrho'}{n}$, because both a and B satisfy the conditions in lemma 5.3.3 with $N=1$.

5.4 A solution of the convolution equation

In this section we will study the convolution equation (5.8) after a termwise multiplication with the operator $(u\mathrm{I}+A_Q)^{-1}$. This gives the equation

$$\omega=\mathcal{T}\omega, \tag{5.13}$$

where, for $\omega=\omega(u,y)=\sum_{Q\in\mathbb{N}^n}\omega_Q(u)y^Q$:

$$\begin{aligned}(\mathcal{T}\omega)(u,y) &= \sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\leq R\leq Q}(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*\omega_R)(u)\right\}y^Q+\\ &\quad\sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\xi_Q(u)y^Q.\end{aligned}$$

We seek a solution of (5.13) in $\mathcal{W}_{N,\tau_1,\rho}$ for arbitrary $\tau_1>\tau$ and arbitrary $\rho<\frac{\varrho'}{n}$.
[Remember that $\tau>\tau'$ is chosen arbitrary but fixed.]

In this section we will prove

Lemma 5.4.1 *There exists an integer N_0 such that $\mathcal{T}$ defines a contraction mapping on the Banach space $\mathcal{W}_{N,\tau_1,\rho}$ for all $N\geq N_0$.*

Proof. We start the proof with formulating a lemma which will be proven in the following section.

Lemma 5.4.2 *The series $\sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\xi_Q(u)y^Q$ belongs to the Banach space $\mathcal{W}_{N,\tau_1,\rho}$.*

Assuming this lemma to be true, we can show that $\mathcal{T}$ maps $\mathcal{W}_{N,\tau_1,\rho}$ into itself:
If we take a series $\sum_{Q\in\mathbb{N}^n}\omega_Q(u)y^Q$ in $\mathcal{W}_{N,\tau_1,\rho}$ we thus have to show that

$$\sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\le R\le Q}(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*\omega_R)(u)\right\}y^Q$$

also belongs to $\mathcal{W}_{N,\tau_1,\rho}$.
In section 4.1 we have seen that for each $Q\in\mathbb{N}^n$ the function $\beta_Q=\mathcal{B}B_Q$ belongs to $\mathcal{A}_{\tau'}$, so lemma 2.3.1 implies $1*\beta_Q\in\mathcal{A}_{\tau'}$. Hence $1*\beta_Q\in\mathcal{V}_{1,\tau}$ for each $Q\in\mathbb{N}^n$. Claim 2.4.2(b) then implies that the function

$$u\mapsto u^{1-N}(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*\omega_R)(u)$$

is analytic in $\overline{S_c}$ for each $Q\in\mathbb{N}^n$ and each $R\in\mathbb{N}^n$, $0\le R\le Q$.
As in section 2.4 we see that for each $Q\in\mathbb{N}^n$ and each R, $0\le R\le Q$,

$$\left|(1*\beta_{(Q-R)}*\omega_R)(u)\right|\le\|1*\beta_{(Q-R)}\|_{1,\tau}\cdot\|\omega_R\|_{N,\tau_1}\cdot e^{\tau_1|u|}\cdot\frac{|u|^N}{N},\quad\forall\,u\in\overline{S_c}.$$

Proposition 5.4.3 *For each $Q\in\mathbb{N}^n$ we can estimate the inverse of $u\mathrm{I}+A_Q$ by*

$$\left|(u\mathrm{I}+A_Q)^{-1}\right|\le\frac{1}{|u|\sin\frac{\delta}{2}},\quad u\in\overline{S_c}\setminus\{0\}.$$

Proof. For each $i\in\{1,2,\dots,n\}$ we have

$$|u+\langle Q,\lambda\rangle+c_i|\ge\mathcal{I}m\,(u+\langle Q,\lambda\rangle+c_i)\ge\mathcal{I}m\,u\ge|u|\sin\frac{\delta}{2}$$

for all $u\in\overline{S_c}=\overline{S(\frac{\pi}{2},\pi-\delta)}$. Hence, for all $u\in\overline{S_c}\setminus\{0\}$:

$$\left|(u\mathrm{I}+A_Q)^{-1}\right|=\max_{i\in\{1,2,\dots,n\}}\left|(u+\langle Q,\lambda\rangle+c_i)^{-1}\right|\le\frac{1}{|u|\sin\frac{\delta}{2}}.$$

This proves the proposition. □

For $u\in\overline{S_c}\setminus\{0\}$ we thus have

$$\begin{aligned}&\left|u^{1-N}(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*\omega_R)(u)\right|\\ \le\;&\frac{1}{|u|^N\sin\frac{\delta}{2}}\cdot\|1*\beta_{(Q-R)}\|_{1,\tau}\cdot\|\omega_R\|_{N,\tau_1}\cdot e^{\tau_1|u|}\cdot\frac{|u|^N}{N}\\ \le\;&\frac{1}{N\sin\frac{\delta}{2}}\cdot\|1*\beta_{(Q-R)}\|_{1,\tau}\cdot\|\omega_R\|_{N,\tau_1}\cdot e^{\tau_1|u|},\end{aligned}$$

which also holds for $u = 0$ (by taking the limit $u \to 0$ on $\overline{S_c}$).
Hence, for each $Q \in \mathbb{N}^n$ the function

$$u \mapsto \sum_{0 \leq R \leq Q} (uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * \omega_R)(u)$$

belongs to $\mathcal{V}_{N,\tau_1}$ as a finite sum of functions belonging to $\mathcal{V}_{N,\tau_1}$. Moreover,

$$\begin{aligned}
&\sum_{Q\in\mathbb{N}^n}\left(\left\|\sum_{0\leq R\leq Q}(uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * \omega_R)\right\|_{N,\tau_1}\right)\rho^{|Q|} \\
\leq\ &\sum_{Q\in\mathbb{N}^n}\left(\sum_{0\leq R\leq Q}\left\|(uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * \omega_R)\right\|_{N,\tau_1}\right)\rho^{|Q|} \\
\leq\ &\sum_{Q\in\mathbb{N}^n}\left(\sum_{0\leq R\leq Q}\frac{1}{N\sin\frac{\delta}{2}}\cdot\|1 * \beta_{(Q-R)}\|_{1,\tau}\cdot\|\omega_R\|_{N,\tau_1}\right)\rho^{|Q|} \\
=\ &\frac{1}{N\sin\frac{\delta}{2}}\left[\sum_{Q\in\mathbb{N}^n}\|1 * \beta_Q\|_{1,\tau}\rho^{|Q|}\right]\cdot\left[\sum_{Q\in\mathbb{N}^n}\|\omega_Q\|_{N,\tau_1}\rho^{|Q|}\right].
\end{aligned}$$

Furthermore, for $u \in \overline{S_c}$ we have

$$|(1 * \beta_Q)(u)| \leq \int_0^u |\beta_Q(s)||\mathrm{d}s| \leq \|\beta_Q\|_{1,\tau}\int_0^u e^{\tau|s|}|\mathrm{d}s|.$$

Because we can integrate along a straight line we obtain

$$|(1 * \beta_Q)(u)| \leq \|\beta_Q\|_{1,\tau}\int_0^{|u|} e^{\tau r}\mathrm{d}r \leq \frac{\|\beta_Q\|_{1,\tau}}{\tau}e^{\tau|u|}.$$

Hence for each $Q \in \mathbb{N}^n$:

$$\|1 * \beta_Q\|_{1,\tau} \leq \frac{1}{\tau}\|\beta_Q\|_{1,\tau}$$

and we conclude

$$\begin{aligned}
&\sum_{Q\in\mathbb{N}^n}\left(\left\|\sum_{0\leq R\leq Q}(uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * \omega_R)\right\|_{N,\tau_1}\right)\rho^{|Q|} \\
\leq\ &\frac{1}{N}\frac{1}{\tau\sin\frac{\delta}{2}}\|\beta\|_{1,\tau,\rho}\cdot\|\omega\|_{N,\tau_1,\rho},
\end{aligned}$$

which is finite by lemma 5.3.3. This proves that $\mathcal{T}$ maps $\mathcal{W}_{N,\tau_1,\rho}$ into itself. Moreover, if $\omega_1,\ \omega_2 \in \mathcal{W}_{N,\tau_1,\rho}$ are two series, then we have

$$\begin{aligned}
&(\mathcal{T}\omega_1 - \mathcal{T}\omega_2)(u, y) \\
=\ &\sum_{Q\in\mathbb{N}^n}\left(\sum_{0\leq R\leq Q}(uI + A_Q)^{-1}\Big(1 * \beta_{(Q-R)} * (\omega_{R,1} - \omega_{R,2})\Big)(u)\right)y^Q
\end{aligned}$$

and thus

$$\|\mathcal{T}\omega_1 - \mathcal{T}\omega_2\|_{N,\tau_1,\rho} \le \frac{K}{N}\|\omega_1 - \omega_2\|_{N,\tau_1,\rho},$$

where $K = \frac{1}{\tau \sin\frac{\delta}{2}}\|\beta\|_{1,\tau,\rho}$, a positive constant independent of N and τ_1. Choosing $N_0 > K$ completes the proof of lemma 5.4.1. ■

So for all $N \ge N_0$ there exists a unique series $w(u,y) = \sum_{Q\in\mathbb{N}^n} w_Q(u)y^Q$, belonging to $\mathcal{W}_{N,\tau_1,\rho}$, such that $\mathcal{T}w = w$. A priori this solution depends on τ_1 and N.

It is easily seen that $\tau < \tau_1 \le \tau_2$ implies

$$\mathcal{W}_{N,\tau_1,\rho} \subseteq \mathcal{W}_{N,\tau_2,\rho}.$$

As in the proof of claim 2.4.3 we see that the solution of the contraction equation is independent of the choice of $\tau_1 > \tau$. Hence we have a solution $w = w(u,y) = \sum_{Q\in\mathbb{N}^n} w_Q(u)y^Q$ of (5.13), depending upon N and satisfying

- $\forall\ Q \in \mathbb{N}^n \ :\ w_Q \in \mathcal{V}_{N,\tilde{\tau}}$ for all $\tilde{\tau} > \tau$;
- $\sum_{Q\in\mathbb{N}^n} \|w_Q\|_{N,\tilde{\tau}}\rho^{|Q|} < \infty$ for all $\tilde{\tau} > \tau$.

By comparing coefficients we see that this solution also is a solution of (5.8) taking into account the following lemma:

Lemma 5.4.4 *If the series $\sum_{Q\in\mathbb{N}^n} \varphi_Q(u)y^Q$ belongs to $\mathcal{W}_{N,\tau_1,\rho}$, then the series $\sum_{Q\in\mathbb{N}^n}(u\mathrm{I} + A_Q)\varphi_Q(u)y^Q$ belongs to $\mathcal{W}_{N,\tau_2,\rho_1}$ for arbitrary $\tau_2 > \tau_1$ and arbitrary $\rho_1 < \rho$.*

Proof. First we note that for each $Q \in \mathbb{N}^n$ we have

$$|\langle Q,\lambda\rangle + c_i| \le \sum_{j=1}^{n} q_j|\lambda_j| + |c_i| \le C_i(|Q|+1),$$

for some constant C_i. Taking $C = \max_{1\le i\le n} C_i$ we obtain

$$|A_Q| \le C(|Q|+1).$$

Next choose $\rho_1 < \rho$ and $\tau_2 > \tau_1$ arbitrary and write $\tau_2 = \tau_1 + \varepsilon$, where $\varepsilon > 0$. The function $u \mapsto u^{1-N}(u\mathrm{I} + A_Q)\varphi_Q(u)$ is analytic in $\overline{S_c}$ and for $u \in \overline{S_c}\setminus\{0\}$ we have

$$\begin{aligned}
& \left|u^{1-N}(u\mathrm{I}+A_Q)\varphi_Q(u)\right|e^{-\tau_2|u|} \\
\le\ & |u|^{1-N}\big(|u| + C(|Q|+1)\big)|\varphi_Q(u)|e^{-\tau_2|u|} \\
\le\ & |u|^{1-N}|\varphi_Q(u)|e^{-\tau_1|u|}\cdot|u|e^{-\varepsilon|u|} \qquad + \\
& C\cdot(|Q|+1)\cdot|u|^{1-N}|\varphi_Q(u)|e^{-\tau_2|u|}\ .
\end{aligned}$$

Elementary calculus learns

$$\max_{x\in\mathbb{R}_{\geq 0}} \left(xe^{-\varepsilon x}\right) = \frac{1}{\varepsilon \cdot e},$$

so for all $u \in \overline{S_c} \setminus \{0\}$ we have

$$|u|^{1-N}|\varphi_Q(u)|e^{-\tau_1|u|} \cdot |u|e^{-\varepsilon|u|} \leq \|\varphi_Q\|_{N,\tau_1} \cdot \frac{1}{\varepsilon \cdot e}$$

and this estimate also holds for $u = 0$ (again by taking the limit $u \to 0$ on $\overline{S_c}$).

Moreover, for $u \in \overline{S_c} \setminus \{0\}$ we have

$$C \cdot (|Q| + 1) \cdot |u|^{1-N}|\varphi_Q(u)|e^{-\tau_2|u|} \leq C \cdot (|Q| + 1) \cdot \|\varphi_Q\|_{N,\tau_1},$$

because $e^{-\varepsilon|u|} \leq 1, \quad \forall\, u \in \overline{S_c}$. Hence

$$\|(u\mathrm{I} + A_Q)\varphi_Q\|_{N,\tau_2} \quad \leq \quad \|\varphi_Q\|_{N,\tau_1} \cdot \frac{1}{\varepsilon \cdot e} \quad + \quad C \cdot (|Q| + 1) \cdot \|\varphi_Q\|_{N,\tau_1},$$

which says that for each $Q \in \mathbb{N}^n$ the function

$$u \mapsto (u\mathrm{I} + A_Q)\varphi_Q(u)$$

belongs to $\mathcal{V}_{N,\tau_2}$.

Finally

$$\begin{aligned}
&\sum_{Q\in\mathbb{N}^n} \|(u\mathrm{I} + A_Q)\varphi_Q\|_{N,\tau_2}{\rho_1}^{|Q|} \\
\leq\ & \frac{1}{\varepsilon \cdot e} \sum_{Q\in\mathbb{N}^n} \|\varphi_Q\|_{N,\tau_1}{\rho_1}^{|Q|} \quad + \quad C \sum_{Q\in\mathbb{N}^n} (|Q| + 1) \cdot \|\varphi_Q\|_{N,\tau_1}{\rho_1}^{|Q|} \\
\leq\ & \left\{C + \frac{1}{\varepsilon \cdot e}\right\} \sum_{Q\in\mathbb{N}^n} \|\varphi_Q\|_{N,\tau_1}\rho^{|Q|} \quad + \quad C \sum_{Q\in\mathbb{N}^n} |Q| \cdot \|\varphi_Q\|_{N,\tau_1}{\rho_1}^{|Q|},
\end{aligned}$$

which is finite if we can show that $\sum_{Q\in\mathbb{N}^n} |Q| \cdot \|\varphi_Q\|_{N,\tau_1}{\rho_1}^{|Q|}$ converges, but this immediately follows from proposition 5.1.1 (with $\gamma = 1$). ■

5.5 Proof of lemma 5.4.2

To prove that the series $\sum_{Q\in\mathbb{N}^n} (u\mathrm{I} + A_Q)^{-1}\xi_Q(u)y^Q$ belongs to $\mathcal{W}_{N,\tilde{\tau},\rho}$ for arbitrary $\tilde{\tau} > \tau$ and arbitrary $\rho < \frac{\rho'}{n}$, we need the following two results.

Proposition 5.5.1 *For all $i \in \{1, 2, \ldots, n\}$ we have*

$$\sup_{u\in\overline{S_c}} \left|\frac{\langle Q, \lambda\rangle + c_i}{u + \langle Q, \lambda\rangle + c_i}\right| \leq \frac{1}{\sin\frac{\delta}{2}}.$$

Hence

$$\left|(u\mathrm{I} + A_Q)^{-1}A_Q\right| \leq \frac{1}{\sin\frac{\delta}{2}}, \quad \forall\, u \in \overline{S_c}.$$

Proof. First remember $S_c = S(\frac{\pi}{2}, \pi - \delta)$.
Choose $i \in \{1, 2, \ldots, n\}$ and write $\sigma_i = \langle Q, \lambda \rangle + c_i$. We consider the case $0 < \arg \sigma_i < \frac{\pi}{2}$, as drawn in figure 5.1, in detail.

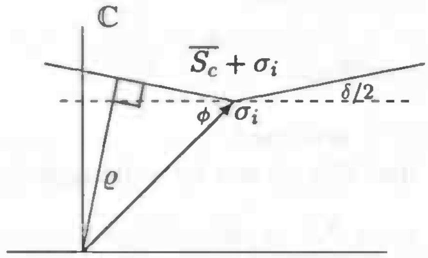

Figure 5.1: $\sup_{u \in \overline{S_c}} |\frac{\sigma_i}{u+\sigma_i}| = \frac{|\sigma_i|}{\varrho} \leq \frac{1}{\sin \frac{\delta}{2}}$

We have to estimate the fraction $|\frac{\sigma_i}{u+\sigma_i}|$. To that end we consider the sector $\overline{S_c} + \sigma_i$. It is easily seen that this translated sector is at distance ϱ from the origin. Hence for all $u \in \overline{S_c}$ we have $|u + \sigma_i| \geq \varrho$ and thus

$$\sup_{u \in \overline{S_c}} \left| \frac{\sigma_i}{u + \sigma_i} \right| = \frac{|\sigma_i|}{\varrho}.$$

By definition

$$\sin\Big(\phi + \frac{\delta}{2}\Big) = \frac{\varrho}{|\sigma_i|},$$

so

$$\frac{|\sigma_i|}{\varrho} = \frac{1}{\sin(\phi + \frac{\delta}{2})} \leq \frac{1}{\sin \frac{\delta}{2}}.$$

Note that this last inequality only holds for $\frac{\delta}{2}$ small (i.e. $\delta \in (0, \frac{\pi}{2})$ is small enough).
A similar reasoning proves proposition 5.5.1 in case $\arg \sigma_i \in (\frac{\pi}{2}, \pi)$. If $\arg \sigma_i = \frac{\pi}{2}$, then we easily see that $\sup_{u \in \overline{S_c}} |\frac{\sigma_i}{u+\sigma_i}| = 1 \leq \frac{1}{\sin \frac{\delta}{2}}$ (perhaps one should draw a picture again). In case the imaginary part of σ_i is zero, we have $\mathcal{R}e\ \sigma_i \neq 0$. Drawing a picture once more gives $\sup_{u \in \overline{S_c}} |\frac{\sigma_i}{u+\sigma_i}| = \frac{1}{\sin \frac{\delta}{2}}$.
This proves the proposition. □

Proposition 5.5.2 *If the series $\sum_{Q \in \mathbb{N}^n} \varphi_Q(u) y^Q$ belongs to $\mathcal{W}_{N+1,\tilde{\tau},\rho}$, then $\sum_{Q \in \mathbb{N}^n} (u\mathrm{I} + A_Q)^{-1} \varphi_Q(u) y^Q$ belongs to $\mathcal{W}_{N,\tilde{\tau},\rho}$.*

Proof. For each $Q \in \mathbb{N}^n$ the function $u \mapsto u^{-N} \varphi_Q(u)$ is analytic in $\overline{S_c}$. Hence $u \mapsto u^{1-N}(u\mathrm{I} + A_Q)^{-1} \varphi_Q(u)$ also is analytic in $\overline{S_c}$. Moreover, for all $u \in \overline{S_c}$, $u \neq 0$, we have

$$\begin{aligned} \left| u^{1-N}(u\mathrm{I} + A_Q)^{-1} \varphi_Q(u) \right| e^{-\tilde{\tau}|u|} &\leq \frac{1}{\sin \frac{\delta}{2}} \left| u^{-N} \varphi_Q(u) \right| e^{-\tilde{\tau}|u|} \\ &\leq \frac{\|\varphi_Q\|_{N+1,\tilde{\tau}}}{\sin \frac{\delta}{2}}. \end{aligned}$$

As this also holds for $u = 0$ (by taking the limit $u \to 0$ on $\overline{S_c}$) we obtain

$$\sum_{Q\in\mathbb{N}^n} \left\|(uI + A_Q)^{-1}\varphi_Q\right\|_{N,\tilde{\tau}}\rho^{|Q|} \leq \frac{1}{\sin\frac{\delta}{2}}\|\varphi\|_{N+1,\tilde{\tau},\rho} < \infty$$

and this proves the proposition. □

Now take $\tilde{\tau} > \tau$ and $\rho < \frac{\rho'}{n}$ arbitrary.
From (5.5) we deduce that $R(z,y)$ can be written as

$$R(z,y) = \sum_{Q\in\mathbb{N}^n} A_Q\Phi_{Q,N}z^N y^Q + \tilde{R}(z,y),$$

for some function $\tilde{R}$ analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$. Moreover, $\tilde{R}(\cdot,y)$ has a zero of order $N+1$ at $z=0$ for all $y \in \Delta_n(\rho')$, so lemma 5.3.3 implies that $\mathcal{B}\tilde{R}(\cdot,y)$ belongs to $\mathcal{W}_{N+1,\tilde{\tau},\rho}$. From proposition 5.5.2 we conclude that it is sufficient to show that

$$\sum_{Q\in\mathbb{N}^n} (uI + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}y^Q$$

belongs to $\mathcal{W}_{N,\tilde{\tau},\rho}$. To that end we take $\tilde{\rho} < \rho'$ such that $n\rho < \tilde{\rho} < \rho'$. From the Cauchy inequality we deduce

$$|\Phi_{Q,N}| \leq \frac{M}{\tilde{\tau}^{-N}\tilde{\rho}^{|Q|}}, \quad \forall\, Q \in \mathbb{N}^n,$$

where,

$$M = M(\tilde{\tau},\tilde{\rho}) := \sup_{(z,y)\in\overline{\Delta_1(\tilde{\tau}^{-1})\times\Delta_n(\tilde{\rho})}} \left|\Phi_N(y)z^N\right|.$$

$\Big($Remember $\Phi_{Q,N} = \frac{D^Q\Phi_N(y)}{Q!}\Big|_{y=0} = \frac{D^Q(\frac{d}{dz})^N\Phi_N(y)z^N}{Q!\,N!}\Big|_{(z,y)=(0,0)}$.$\Big)$

The assumption of no resonance implies the operator $uI + A_Q$ to be non-singular in $\overline{S_c}$, so

$$u \mapsto u^{1-N}(uI + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1} = (uI + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}$$

is analytic in $\overline{S_c}$. Moreover, for all $u \in \overline{S_c}\setminus\{0\}$ we have, by proposition 5.5.1,

$$\left|u^{1-N}(uI + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}\right| \leq \frac{M}{(N-1)!\,\tilde{\tau}^{-N}\tilde{\rho}^{|Q|}\sin\frac{\delta}{2}},$$

and this also holds for $u = 0$ (by taking the limit $u \to 0$ on $\overline{S_c}$). Hence for all $u \in \overline{S_c}$ we have

$$\left|u^{1-N}(uI + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}\right| \leq \frac{M}{(N-1)!\,\tilde{\tau}^{-N}\tilde{\rho}^{|Q|}\sin\frac{\delta}{2}}e^{\tilde{\tau}|u|}.$$

This implies that the function $u \mapsto (u\mathrm{I} + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}$ belongs to $\mathcal{V}_{N,\tilde{\tau}}$ and

$$\left\|(u\mathrm{I} + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}\right\|_{N,\tilde{\tau}} \leq \frac{C_1}{\tilde{\rho}^{|Q|}},$$

where $C_1 = \frac{M}{(N-1)!\tilde{\tau}^{-N}\sin\frac{\delta}{2}}$.
From this it is also clear that

$$\sum_{Q\in\mathbb{N}^n}\left\|(u\mathrm{I} + A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}\right\|_{N,\tilde{\tau}}\rho^{|Q|} < \infty.$$

This proves lemma 5.4.2. ■

5.6 Back to the differential equation

Lemma 5.6.1 *Suppose we have a series $\varphi = \varphi(u,y) = \sum_{Q\in\mathbb{N}^n}\varphi_Q(u)y^Q$ satisfying*

- $\forall\, Q \in \mathbb{N}^n \;:\; \varphi_Q \in \mathcal{V}_{N,\tilde{\tau}}$ *for all* $\tilde{\tau} > \tau$;
- $\sum_{Q\in\mathbb{N}^n}\|\varphi_Q\|_{N,\tilde{\tau}}\rho_1{}^{|Q|} < \infty$ *for all* $\tilde{\tau} > \tau$, *for all* $\rho_1 < \rho$,

then both $(\mathcal{L}\varphi(\cdot,y))(z)$ and $\sum_{Q\in\mathbb{N}^n}(\mathcal{L}\varphi_Q)(z)y^Q$ exist, are analytic in the set $U_\delta \times \Delta_n(\rho)$ and equal each other on this set. (Here U_δ is a neighbourhood of 0 *in $S(\frac{\pi}{2}, 2\pi - \delta)$.)*

Proof. For $y \in \Delta_n(\rho)$ fixed, a positive number $\rho_1 < \rho$ can be found such that $y \in \overline{\Delta_n(\rho_1)}$. If $\tilde{\tau} > \tau$ is chosen arbitrary, we have for $u \in \overline{S_c}$

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n}\left|\varphi_Q(u)y^Q\right| &\leq \sum_{Q\in\mathbb{N}^n}\|\varphi_Q\|_{N,\tilde{\tau}}|u|^{N-1}e^{\tilde{\tau}|u|}\rho_1{}^{|Q|}\\ &\leq \left(\sum_{Q\in\mathbb{N}^n}\|\varphi_Q\|_{N,\tilde{\tau}}\rho_1{}^{|Q|}\right)|u|^{N-1}e^{\tilde{\tau}|u|}.\end{aligned}$$

Hence, if $\theta \in \left(\frac{\delta}{2}, \pi - \frac{\delta}{2}\right)$ we have

$$\begin{aligned}&\int_0^{\infty e^{i\theta}}\sum_{Q\in\mathbb{N}^n}\left|\varphi_Q(u)y^Q\right|\cdot\left|e^{-\frac{u}{z}}\right||du|\\ &\leq \left(\sum_{Q\in\mathbb{N}^n}\|\varphi_Q\|_{N,\tilde{\tau}}\rho_1{}^{|Q|}\right)\int_0^{\infty e^{i\theta}}|u|^{N-1}e^{\tilde{\tau}|u|-\left|\frac{u}{z}\right|\cos(\theta-\arg z)}|du|,\end{aligned}$$

which is finite if and only if $\tilde{\tau} - \frac{1}{|z|}\cos(\theta - \arg z) < 0$.
On the set $\{z \in \mathbb{C}^* \mid \tilde{\tau}|z| - \cos(\theta - \arg z) < 0\}$ Lebesgue's dominated convergence theorem implies that

$$\int_0^{\infty e^{i\theta}} \sum_{Q\in\mathbb{N}^n} \varphi_Q(u) y^Q e^{-\frac{u}{z}} du$$

exists and equals

$$\sum_{Q\in\mathbb{N}^n} \left(\int_0^{\infty e^{i\theta}} \varphi_Q(u) e^{-\frac{u}{z}} du \right) y^Q.$$

As $\tilde{\tau} > \tau$ was chosen arbitrary, varying $\theta \in \left(\frac{\delta}{2}, \pi - \frac{\delta}{2}\right)$ gives

$$(\mathcal{L}\varphi(\cdot, y))(z) = \sum_{Q\in\mathbb{N}^n} (\mathcal{L}\varphi_Q)(z) y^Q, \quad \forall\, z \in U_\delta,$$

where U_δ is as in remark 1.3.2(3).
But $y \in \Delta_n(\rho)$ was chosen arbitrary, so

$$(\mathcal{L}\varphi(\cdot, y))(z) = \sum_{Q\in\mathbb{N}^n} (\mathcal{L}\varphi_Q)(z) y^Q, \quad \forall\, (z, y) \in U_\delta \times \Delta_n(\rho).$$

Finally, fix $(z, y) \in U_\delta \times \Delta_n(\rho)$. Then we can find $\theta \in \left(\frac{\delta}{2}, \pi - \frac{\delta}{2}\right)$, $\tilde{\tau} > \tau$ and $\rho_1 < \rho$ such that $\tilde{\tau}|z| - \cos(\theta - \arg z) < 0$ and $y \in \overline{\Delta_n(\rho_1)}$. With these choices we have

$$\begin{aligned}
&\sum_{Q\in\mathbb{N}^n} \left|(\mathcal{L}\varphi_Q)(z) y^Q\right| \\
\leq\ & \sum_{Q\in\mathbb{N}^n} \left(\int_0^{\infty e^{i\theta}} \left|\varphi_Q(u) e^{-\frac{u}{z}}\right| |du| \right) {\rho_1}^{|Q|} \\
\leq\ & \left(\sum_{Q\in\mathbb{N}^n} \|\varphi_Q\|_{N,\tilde{\tau}} {\rho_1}^{|Q|} \right) \int_0^{\infty e^{i\theta}} |u|^{N-1} e^{\tilde{\tau}|u| - \left|\frac{u}{z}\right| \cos(\theta - \arg z)} |du| \\
<\ & \infty.
\end{aligned}$$

Hence the series $\sum_{Q\in\mathbb{N}^n} (\mathcal{L}\varphi_Q)(z) y^Q$ converges absolutely for every pair $(z, y) \in U_\delta \times \Delta_n(\rho)$. Theorem 3.1.7 then implies that this series represents an analytic function in $U_\delta \times \Delta_n(\rho)$ (note that we have used the fact that $U_\delta \times \Delta_n(\rho)$ is open). This proves the lemma. ∎

From lemma 5.4.4 and lemma 5.6.1 we deduce that $\mathcal{L}$ is applicable to (5.8). This gives us a solution $\Psi(z, y) = (\mathcal{L}w(\cdot, y))(z)$ of the equation (5.4), which is analytic in $U_\delta \times \Delta_n(\rho)$ and which can be written as

$$\Psi(z, y) = \sum_{Q\in\mathbb{N}^n} \Psi_Q(z) y^Q,$$

where each Ψ_Q equals $\mathcal{L}w_Q$. It follows that $\Phi = \Psi + u_N$ is a solution of the original equation (5.1), which is analytic in $U_\delta \times \Delta_n(\rho)$. From the fact that u_N is analytic in $\Delta_1(\frac{1}{\tau'}) \times \Delta_n(\rho')$ we deduce that Φ can be expanded in a convergent series,

$$\Phi(z,y) = \sum_{Q\in\mathbb{N}^n} \Phi_Q(z)y^Q,$$

where each coefficient Φ_Q can be written as $\Phi_Q = \mathcal{L}\tilde{w}_Q$ for some function $\tilde{w}_Q$, analytic in $\overline{S_c}$ and satisfying $\tilde{w}_Q(u) = O(1)e^{\tau_1|u|}$ as $u \to \infty$ on $\overline{S_c}$ for all $\tau_1 > \tau$, for

$$\begin{aligned}\Phi(z,y) &= \Psi(z,y) + u_N(z,y) \\ &= \sum_{Q\in\mathbb{N}^n} \Psi_Q(z)y^Q + \sum_{Q\in\mathbb{N}^n}\sum_{m=2}^{N-1} \Phi_{Q,m}z^m y^Q \\ &= \sum_{Q\in\mathbb{N}^n} (\mathcal{L}\tilde{w}_Q)(z)y^Q,\end{aligned}$$

where for each $Q \in \mathbb{N}^n$ we have

$$\tilde{w}_Q(u) = w_Q(u) + \sum_{m=1}^{N-2} \frac{\Phi_{Q,m+1}}{\Gamma(m+1)}u^m.$$

Exactly the same reasoning as in chapter 2 gives that each $\tilde{w}_Q$ is independent of N:
If $N_2 > N_1 \geq N_0$, let

$$\tilde{w}_{N_1}(u,y) = \sum_{Q\in\mathbb{N}^n}\sum_{m=1}^{N_1-2} \frac{\Phi_{Q,m+1}}{\Gamma(m+1)}u^m y^Q \quad + \quad w_{N_1}(u,y)$$

and

$$\tilde{w}_{N_2}(u,y) = \sum_{Q\in\mathbb{N}^n}\sum_{m=1}^{N_2-2} \frac{\Phi_{Q,m+1}}{\Gamma(m+1)}u^m y^Q \quad + \quad w_{N_2}(u,y),$$

with w_{N_1}, resp. w_{N_2}, a solution of (5.8) belonging to $\mathcal{W}_{N_1,\tilde{\tau},\rho}$, resp. $\mathcal{W}_{N_2,\tilde{\tau},\rho}$, for all $\tilde{\tau} > \tau$.
Next, if $\tau_1 > \tau$ arbitrary, then take $\tau_2 \in (\tau, \tau_1)$. It is easily seen that

$$(u,y) \mapsto \sum_{Q\in\mathbb{N}^n}\sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{Q,m+1}}{\Gamma(m+1)}u^m y^Q$$

belongs to $\mathcal{W}_{N_1,\tau_1,\rho}$ (cf. lemma 5.3.3). Moreover, as in chapter 2 we have $\mathcal{W}_{N_2,\tau_2,\rho} \subset \mathcal{W}_{N_1,\tau_1,\rho}$, so both

$$w_{N_1}(u,y) \quad \text{and} \quad w_{N_2}(u,y) \; + \sum_{Q\in\mathbb{N}^n}\sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{Q,m+1}}{\Gamma(m+1)}u^m y^Q$$

are solutions of (5.8) belonging to $\mathcal{W}_{N_1,\tau_1,\rho}$. Hence

$$w_{N_1}(u,y) = w_{N_2}(u,y) + \sum_{Q\in\mathbb{N}^n} \sum_{m=N_1-1}^{N_2-2} \frac{\Phi_{Q,m+1}}{\Gamma(m+1)} u^m y^Q.$$

We conclude that Φ can be written as

$$\Phi(z,y) = \sum_{Q\in\mathbb{N}^n} (\mathcal{L}\tilde{w}_Q)(z) y^Q,$$

with $\tilde{w}_Q \in \mathcal{A}_\tau$ for each $Q \in \mathbb{N}^n$.

Now remember that $\delta \in (0, \frac{\pi}{2})$ was chosen arbitrary (but fixed). When we make δ smaller we obtain an analytic continuation of our solution. Hence we obtain a solution of (5.1) which is analytic in the product of a neighbourhood U of 0 in $S(\frac{\pi}{2}, 2\pi)$ and a polydisc $\Delta_n(\rho)$ with $\rho < \frac{\varrho'}{n}$ arbitrary. So the formal solution with analytic coefficients, found in section 4.2, converges and we immediately obtain (with the notation as in theorem 4.2.1)

$$\Phi_Q(z) \simeq \hat{\Phi}_Q(z) \quad \text{as } z \to 0 \text{ on } U$$

for each $Q \in \mathbb{N}^n$. This is often characterized as $\Phi(z,y)$ asymptotically equals $\sum_{Q\in\mathbb{N}^n} \hat{\Phi}_Q(z) y^Q$ in the sense of Gérard-Sibuya.

Hence we have proven the following theorem:

Theorem 5.6.2 *Under the assumptions listed in the introduction of this chapter there exists a solution $\Phi = \Phi(z,y)$ of (5.1), which is analytic in the product of a neighbourhood U of 0 in $S(\frac{\pi}{2}, 2\pi)$ and a polydisc $\Delta_n(\rho)$, for arbitrary $\rho < \frac{\varrho'}{n}$.*
This solution can be written as

$$\Phi(z,y) = \sum_{Q\in\mathbb{N}^n} \Phi_Q(z) y^Q,$$

where each Φ_Q equals $\Phi_Q(z) = (\mathcal{L}\tilde{w}_Q)(z)$ for some function $\tilde{w}_Q \in \mathcal{A}_\tau$, $\delta \in (0,\pi)$ arbitrary.
Moreover, $\Phi(z,y)$ asymptotically equals the series $\sum_{Q\in\mathbb{N}^n} \hat{\Phi}_Q(z) y^Q$ in the sense of Gérard-Sibuya.
Finally, this solution is unique in the set

$$\left\{ \Upsilon(z,y) = \sum_{Q\in\mathbb{N}^n} \Upsilon_Q(z) y^Q \;\middle|\; \begin{array}{c} \Upsilon \text{ is analytic in } U \times \Delta_n(\rho) \text{ and} \\ \forall\, Q \in \mathbb{N}^n \quad \exists\, \omega_Q \in \mathcal{A}_\tau \text{ such that} \\ \Upsilon_Q = \mathcal{L}\omega_Q. \end{array} \right\}$$

Chapter 6

Another proof in the non-resonance case

Again we will study the system

$$
\begin{aligned}
& z^2\frac{\partial\Phi}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^{n}\lambda_j y_j\frac{\partial}{\partial y_j} + c_i\right\}_{i=1}^{n}\cdot\Phi \\
= & \\
& za(z,y) + zB(z,y)\Phi,
\end{aligned}
\tag{6.1}
$$

where $\lambda_i \in \mathbb{H}$ for all $i \in \{1,2,\ldots,n\}$. As in chapter 4 we assume $c_i \in \mathbb{C}$ to be chosen such that for all $i \in \{1,2,\ldots,n\}$ and for all $Q \in \mathbb{N}^n$:

$$\mathcal{I}m\,\left(\langle Q,\lambda\rangle + c_i\right) \geq 0.$$

Moreover, we assume a and B to be analytic in $\Delta_1(\frac{1}{r'}) \times \Delta_n(\rho')$, a neighbourhood of $(0,0) \in \mathbb{C}\times\mathbb{C}^n$.
Finally we assume that there are no resonance relations between the eigenvalues λ_i, hence we assume that for each multi-index $Q \in \mathbb{N}^n$ and for each $i \in \{1,2,\ldots,n\}$ we have $\langle Q,\lambda\rangle + c_i \neq 0$.
Note that we have dropped the Siegel condition used in chapter 5.

6.1 Reduction to a convolution equation

To avoid the necessity of the Siegel condition we do not cut off a partial sum of the formal solution, found in section 5.1 (this was the only point where we used this Siegel condition). Remembering the Taylor series of both a and B,

$$a(z,y) = \sum_{Q\in\mathbb{N}^n} a_Q(z)y^Q \qquad \text{and} \qquad B(z,y) = \sum_{Q\in\mathbb{N}^n} B_Q(z)y^Q,$$

we formally substitute a series $\Phi = \Phi(z,y) = \sum_{Q\in\mathbb{N}^n} \Phi_Q(z)y^Q$ in (6.1) to obtain

$$\begin{aligned} &\sum_{Q\in\mathbb{N}^n}\Big(z^2\frac{\mathrm{d}\Phi_Q}{\mathrm{d}z}(z)+A_Q\Phi_Q(z)\Big)y^Q \\ = \\ &\sum_{Q\in\mathbb{N}^n} za_Q(z)y^Q \quad + \sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\le R\le Q} zB_{(Q-R)}(z)\Phi_R(z)\right\}y^Q. \end{aligned} \tag{6.2}$$

We use the same notation as in chapter 4 and 5, so we write $\alpha(\cdot,y) = \mathcal{B}a(\cdot,y)$ and $\beta(\cdot,y) = \mathcal{B}B(\cdot,y)$ which both are analytic in $\mathbb{C}\times\Delta_n(\rho')$.
With $w_Q = \mathcal{B}\Phi_Q$ a formal Borel transformation then gives the following convolution equation

$$\begin{aligned} &\sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)w_Q(u)y^Q \\ = \\ &\sum_{Q\in\mathbb{N}^n}(1*\alpha_Q)(u)y^Q + \sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\le R\le Q}(1*\beta_{(Q-R)}*w_R)(u)\right\}y^Q. \end{aligned} \tag{6.3}$$

After a termwise multiplication with $(u\mathrm{I}+A_Q)^{-1}$ we get

$$w = \mathcal{T}w, \tag{6.4}$$

where for a series $w = w(u,y) = \sum_{Q\in\mathbb{N}^n} w_Q(u)y^Q$ we have

$$\begin{aligned} (\mathcal{T}w)(u,y) \quad = \quad &\sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}(1*\alpha_Q)(u)y^Q \;+ \\ &\sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\le R\le Q}(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*w_R)(u)\right\}y^Q. \end{aligned}$$

We first take $\delta\in(0,\pi)$ arbitrary but fixed and we study (6.4) on the set $\overline{S_r}\times\Delta_n(\rho)$, where $\rho<\frac{\rho'}{n}$ arbitrary, and $\overline{S_r} := \{u\in\overline{S_c} \mid |u|\le r\}$, in which $r>0$ will be specified later on. The idea is to solve (6.4) on the set $\overline{S_r}\times\Delta_n(\rho)$ and after that translate the sector $\overline{S_r}$ over some vector in the complex plane. We shall see that the solution in this translated set is an analytic extension of the solution of (6.4) on $\overline{S_r}\times\Delta_n(\rho)$. Repeating this proces we find a solution $w(u,y)$, with $(u,y)\in\overline{S_c}\times\Delta_n(\rho)$. This solution can be written as $w(u,y)=\sum_{Q\in\mathbb{N}^n} w_Q(u)y^Q$, with coefficients analytic in $\overline{S_c}$. Finally we will obtain an exponential estimate for this solution, in order to be able to apply the Laplace transform to $w(\cdot,y)$.

6.2 Local solution of the integral equation

For $r \in \mathbb{R}_+$ we define the set $\overline{S_r}$ by $\overline{S_r} := \{u \in \overline{S_c} \mid |u| \leq r\}$, while S_r denotes the interior of $\overline{S_r}$. For a function $\vartheta : \overline{S_r} \to \mathbb{C}^n$ which is analytic in $\overline{S_r}$ (i.e. analytic in S_r and continuous on $\overline{S_r}$) we have

$$\sup_{u \in \overline{S_r}} |\vartheta(u)| < \infty,$$

where $|\cdot|$ denotes a norm on $\mathbb{C}^n$ (or a corresponding matrix norm, if we consider matrix-valued functions).
Let us introduce the abbreviation $\|\vartheta\|_r := \sup_{u \in \overline{S_r}} |\vartheta(u)|$ for functions ϑ described above. Then the space of these functions becomes a Banach space. The proof is similar to, but easier than, the proof of lemma 2.3.4.

Definition 6.2.1 *If ρ also is a positive number we define $\mathcal{H}_{r,\rho}$ to be the set of series $\omega = \omega(u,y) = \sum_{Q \in \mathbb{N}^n} \omega_Q(u) y^Q$ such that ω_Q is analytic in $\overline{S_r}$ for each multi-index $Q \in \mathbb{N}^n$, and such that*

$$\sum_{Q \in \mathbb{N}^n} \|\omega_Q\|_r \, \rho^{|Q|} < \infty.$$

When we use the abbreviation $\|\omega\|_{r,\rho} := \sum_{Q \in \mathbb{N}^n} \|\omega_Q\|_r \, \rho^{|Q|}$ for series described in the definition above, the space $(\mathcal{H}_{r,\rho}, \|\cdot\|_{r,\rho})$ becomes a Banach space and the proof is similar to the proof of lemma 5.3.2.

Now let us return to the convolution equation

$$\omega = \mathcal{T}\omega,$$

where $\mathcal{T}\omega$ is defined on $\mathcal{H}_{r,\rho}$ by

$$\begin{aligned}(\mathcal{T}\omega)(u,y) \quad = \quad & \sum_{Q \in \mathbb{N}^n} (u\mathrm{I} + A_Q)^{-1} (1 * \alpha_Q)(u) y^Q \quad + \\ & \sum_{Q \in \mathbb{N}^n} \left\{ \sum_{0 \leq R \leq Q} (u\mathrm{I} + A_Q)^{-1} (1 * \beta_{(Q-R)} * \omega_R)(u) \right\} y^Q.\end{aligned}$$

First we will see that $\mathcal{T}$ maps $\mathcal{H}_{r,\rho}$ into itself. To that end we choose a series $\omega = \omega(u,y) = \sum_{Q \in \mathbb{N}^n} \omega_Q(u) y^Q$ in the Banach space $\mathcal{H}_{r,\rho}$.

For every $Q \in \mathbb{N}^n$ the function $u \mapsto \alpha_Q(u)$ is an entire function. By definition $(1 * \alpha_Q)(u) = \int_0^u \alpha_Q(s) \mathrm{d}s$, so $1 * \alpha_Q$ is an entire function and if α_Q has the Taylor expansion $\alpha_Q(u) = \sum_{m=0}^{\infty} \alpha_{Q;m} u^m$, $u \in \mathbb{C}$, then, by uniform convergence, $(1 * \alpha_Q)(u) = \sum_{m=1}^{\infty} \frac{\alpha_{Q,m-1}}{m} u^m$. Hence $(u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)$ is analytic in $\overline{S_r}$.
For $u \in \overline{S_r}$ we have

$$|(1 * \alpha_Q)(u)| \;\leq\; \int_0^u |\alpha_Q(s)| \, |\mathrm{d}s| \;\leq\; |u| \cdot \sup_{|s| \leq r} |\alpha_Q(s)|,$$

so with proposition 5.4.3 we have for all $u \in \overline{S_r} \setminus \{0\}$

$$\left|(u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)(u)\right| \le \frac{1}{\sin\frac{\delta}{2}} \cdot \sup_{|s|\le r} |\alpha_Q(s)|.$$

But this also holds for $u = 0$ by taking the limit $u \to 0$ on $\overline{S_r}$, hence

$$\|(u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)\|_r \le \frac{1}{\sin\frac{\delta}{2}} \cdot \sup_{|s|\le r} |\alpha_Q(s)|.$$

We obtain

$$\left\|\sum_{Q\in\mathbb{N}^n} (u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)(u)y^Q\right\|_{r,\rho} \le \frac{1}{\sin\frac{\delta}{2}} \cdot \sum_{Q\in\mathbb{N}^n} \Big(\sup_{|s|\le r} |\alpha_Q(s)|\Big)\rho^{|Q|},$$

which is finite by the choice of ρ. This follows from the estimate (5.12) with $N = 1$ in the proof of lemma 5.3.3.

The same lemma also implies that for each $Q \in \mathbb{N}^n$ the function $u \mapsto \beta_Q(u)$ is analytic in $\overline{S_r}$, and that

$$\sum_{Q\in\mathbb{N}^n} \Big(\sup_{|s|\le r} |\beta_Q(s)|\Big)\rho^{|Q|} < \infty.$$

Next choose $Q \in \mathbb{N}^n$ and $R \in \mathbb{N}^n$ such that $0 \le R \le Q$, then the function $u \mapsto (1 * \beta_{(Q-R)} * \omega_R)(u)$ is analytic in $\overline{S_r}$ (cf. lemma 2.3.1). Moreover, for $u \in \overline{S_r}$ we have

$$\begin{aligned} \left|(1 * \beta_{(Q-R)} * \omega_R)(u)\right| &\le \int_0^u \left|(1 * \beta_{(Q-R)})(u-s)\right| \cdot |\omega_R(s)|\,|\mathrm{d}s| \\ &\le |u| \cdot \sup_{|s|\le|u|} \left|(1 * \beta_{(Q-R)})(s)\right| \cdot \|\omega_R\|_r \end{aligned}$$

and for all $s \in \overline{S_r}$, $|s| \le |u|$:

$$\left|(1 * \beta_{(Q-R)})(s)\right| \le |s| \cdot \sup_{|t|\le|u|} |\beta_{(Q-R)}(t)| \le |u| \cdot \sup_{|t|\le r} |\beta_{(Q-R)}(t)|.$$

Hence

$$\left|(1 * \beta_{(Q-R)} * \omega_R)(u)\right| \le |u|^2 \cdot \sup_{|t|\le r} |\beta_{(Q-R)}(t)| \cdot \|\omega_R\|_r. \tag{6.5}$$

We conclude that $u \mapsto (1 * \beta_{(Q-R)} * \omega_R)(u)$ is of order $O(u^2)$ as $u \to 0$ on $\overline{S_r}$. Hence the finite sum over all $R \in \mathbb{N}^n$, $0 \le R \le Q$, also is analytic in $\overline{S_r}$ and of order $O(u^2)$ as $u \to 0$ on $\overline{S_r}$. After multiplication with $(u\mathrm{I} + A_Q)^{-1}$,

the result still is analytic in $\overline{S_r}$.
From (6.5) we obtain for all $u \in \overline{S_r} \setminus \{0\}$:

$$\left|(uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * \omega_R)(u)\right| \le \frac{r}{\sin\frac{\delta}{2}} \cdot \left(\sup_{|s|\le r} |\beta_{(Q-R)}(s)|\right) \cdot \|\omega_R\|_r,$$

but this also holds for $u = 0$. Hence

$$\left\|(uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * \omega_R)\right\|_r \le \frac{r}{\sin\frac{\delta}{2}} \cdot \left(\sup_{|s|\le r} |\beta_{(Q-R)}(s)|\right) \cdot \|\omega_R\|_r.$$

From this we obtain

$$\begin{aligned}
&\left\| \sum_{Q\in\mathbb{N}^n} \left\{ \sum_{0\le R\le Q} (uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * w_R)(u) \right\} y^Q \right\|_{r,\rho} \\
= &\sum_{Q\in\mathbb{N}^n} \left\| \sum_{0\le R\le Q} (uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * w_R) \right\|_r \rho^{|Q|} \\
\le &\sum_{Q\in\mathbb{N}^n} \sum_{0\le R\le Q} \left\|(uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * w_R)\right\|_r \rho^{|Q|} \\
\le &\frac{r}{\sin\frac{\delta}{2}} \sum_{Q\in\mathbb{N}^n} \sum_{0\le R\le Q} \left(\sup_{|s|\le r} |\beta_{(Q-R)}(s)|\right) \cdot \|\omega_R\|_r \, \rho^{|Q-R|+|R|} \\
= &\frac{rM_r}{\sin\frac{\delta}{2}} \|\omega\|_{r,\rho},
\end{aligned}$$

where $M_r = \sum_{Q\in\mathbb{N}^n} \left(\sup_{|s|\le r} |\beta_Q(s)|\right) \rho^{|Q|} < \infty$.
We conclude that $\mathcal{T}$ maps $\mathcal{H}_{r,\rho}$ into itself. Moreover, $\mathcal{T}$ defines a contraction mapping on $(\mathcal{H}_{r,\rho}, \|\cdot\|_{r,\rho})$ if r is chosen such that $rM_r < \sin\frac{\delta}{2}$, for if ω_1 and ω_2 are two series in $\mathcal{H}_{r,\rho}$, we have

$$\begin{aligned}
&(\mathcal{T}\omega_1 - \mathcal{T}\omega_2)(u, y) \\
= &\sum_{Q\in\mathbb{N}^n} \left\{ \sum_{0\le R\le Q} (uI + A_Q)^{-1}[1 * \beta_{(Q-R)} * (\omega_{R,1} - \omega_{R,2})](u) \right\} y^Q.
\end{aligned}$$

Consequently, with this choice of r, there exists a unique solution $w(u, y)$ of (6.4), which belongs to $\mathcal{H}_{r,\rho}$.

6.3 Extension of the solution of the integral equation

Suppose we have an analytic solution w of (6.4) on $\overline{S_r} \times \Delta_n(\rho)$ for some $r > 0$ and some $\rho > 0$. Then w can be written as $w = w(u, y) = \sum_{Q\in\mathbb{N}^n} w_Q(u)y^Q$, with w_Q analytic in $\overline{S_r}$ for each $Q \in \mathbb{N}^n$. In the preceding section we have

seen that such a solution exists for $r > 0$ small enough and arbitrary $\rho < \frac{\varrho'}{n}$. Choose $u_0 \in \overline{S_r}$ with $\frac{1}{2}r < |u_0| < r$ and let $S_r^+ := \overline{S_r} + u_0 = \{u + u_0 \mid u \in \overline{S_r}\}$, the set $\overline{S_r}$ translated over u_0. Note that S_r^+ is a convex set.
We transform the integral equation (6.4) on S_r^+ using the following decomposition of $\varphi * \psi$ for scalar functions φ, ψ continuous on $\overline{S_r} \cup S_r^+$ and analytic in the interior:

$$(\varphi * \psi)(u + u_0) = [\varphi(u_0 + \cdot) * \psi](u) + [\psi(u_0 + \cdot) * \varphi](u) + \Xi(\varphi, \psi)(u), \tag{6.6}$$

where

$$\Xi(\varphi, \psi)(u) = \int_u^{u_0} \varphi(u + u_0 - s)\psi(s)\mathrm{d}s, \quad u \in \overline{S_r},$$

and the paths of integration $[0, u]$ and $[u, u_0]$ belong to $\overline{S_r}$. It is easily seen that $\Xi(\varphi, \psi)$ only depends on the values of φ and ψ on $[u_0, u]$.
For $(u, y) \in (\overline{S_r} - u_0) \times \Delta_n(\rho)$ we have

$$\begin{aligned}
&(\mathcal{T}w)(u + u_0, y) \\
= &\sum_{Q \in \mathbb{N}^n} \left((u + u_0)\mathrm{I} + A_Q\right)^{-1} (1 * \alpha_Q)(u + u_0) y^Q \quad + \\
&\sum_{Q \in \mathbb{N}^n} \left\{ \sum_{0 \le R \le Q} \left((u + u_0)\mathrm{I} + A_Q\right)^{-1} (1 * \beta_{(Q-R)} * w_R)(u + u_0) \right\} y^Q.
\end{aligned}$$

With (6.6) we obtain for $(u, y) \in \left(\overline{S_r} \cap (\overline{S_r} - u_0)\right) \times \Delta_n(\rho)$

$$(\mathcal{T}w)(u + u_0, y) = \left(\tilde{\mathcal{T}} w(u_0 + \cdot, \cdot)\right)(u, y),$$

where $\tilde{\mathcal{T}}$ is defined on series $\varphi = \varphi(u, y) = \sum_{Q \in \mathbb{N}^n} \varphi_Q(u) y^Q$ in $\mathcal{H}_{r,\rho}$ by

$$\begin{aligned}
&(\tilde{\mathcal{T}}\varphi)(u, y) \\
= &\sum_{Q \in \mathbb{N}^n} \left\{ \sum_{0 \le R \le Q} \left((u + u_0)\mathrm{I} + A_Q\right)^{-1} (1 * \beta_{(Q-R)} * \varphi_R)(u) \right\} y^Q \quad + \\
&\sum_{Q \in \mathbb{N}^n} \gamma_Q(u) y^Q, \qquad (u, y) \in \overline{S_r} \times \Delta_n(\rho),
\end{aligned}$$

where for each $Q \in \mathbb{N}^n$ we have

$$\begin{aligned}
&\gamma_Q(u) \\
= &\left((u + u_0)\mathrm{I} + A_Q\right)^{-1} (1 * \alpha_Q)(u + u_0) \\
+ &\sum_{0 \le R \le Q} \left((u + u_0)\mathrm{I} + A_Q\right)^{-1} \left((1 * \beta_{(Q-R)})(u_0 + \cdot) * w_R\right)(u) \qquad (6.7) \\
+ &\sum_{0 \le R \le Q} \left((u + u_0)\mathrm{I} + A_Q\right)^{-1} \Xi\left(1 * \beta_{(Q-R)}, w_R\right)(u).
\end{aligned}$$

Lemma 6.3.1 *The series $\sum_{Q\in\mathbb{N}^n}\gamma_Q(u)y^Q$ belongs to $\mathcal{H}_{r,\rho}$.*

We postpone the proof of this lemma till the following section.

For each $i \in \{1,2,\ldots,n\}$ we have

$$|u+u_0+\langle Q,\lambda\rangle+c_i| \geq \mathcal{I}m\ (u+u_0+\langle Q,\lambda\rangle+c_i) \geq |u|\sin\frac{\delta}{2},$$

if $u\in\overline{S_r}$, so $\left|\left((u+u_0)\mathrm{I}+A_Q\right)^{-1}\right| \leq \frac{1}{|u|\sin\frac{\delta}{2}}$, $\forall\ u\in\overline{S_r}\setminus\{0\}$.

Let $\varphi=\varphi(u,y)=\sum_{Q\in\mathbb{N}^n}\varphi_Q(u)y^Q$ be a series in $\mathcal{H}_{r,\rho}$, then we have (as before)

$$\left\|\left((u+u_0)\mathrm{I}+A_Q\right)^{-1}(1*\beta_{(Q-R)}*\varphi_R)\right\|_r \leq \frac{r}{\sin\frac{\delta}{2}}\cdot\left(\sup_{|s|\leq r}|\beta_{(Q-R)}(s)|\right)\cdot\|\varphi_R\|_r$$

and thus

$$\left\|\sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\leq R\leq Q}\left((u+u_0)\mathrm{I}+A_Q\right)^{-1}(1*\beta_{(Q-R)}*\varphi_R)(u)\right\}y^Q\right\|_{r,\rho}$$
$$\leq \frac{rM_r}{\sin\frac{\delta}{2}}\|\varphi\|_{r,\rho},$$

where $M_r=\sum_{Q\in\mathbb{N}^n}\left(\sup_{|s|\leq r}|\beta_Q(s)|\right)\rho^{|Q|}<\infty$. Assuming lemma 6.3.1 to be true we see that $\tilde{\mathcal{T}}$ maps $\mathcal{H}_{r,\rho}$ into itself. Moreover, $\tilde{\mathcal{T}}$ defines a contraction mapping on $(\mathcal{H}_{r,\rho},\|\cdot\|_{r,\rho})$ when we make the same assumption on r as in the preceding section ($rM_r<\sin\frac{\delta}{2}$), so there exists a unique solution $\varphi=\varphi(u,y)=\sum_{Q\in\mathbb{N}^n}\varphi_Q(u)y^Q$ of $\tilde{\mathcal{T}}\varphi=\varphi$, belonging to $\mathcal{H}_{r,\rho}$.

On $\left(\overline{S_r}\cap(\overline{S_r}-u_0)\right)\times\Delta_n(\rho)$ the function $w(u_0+\cdot,\cdot)$ is a solution of $\tilde{\mathcal{T}}\omega=\omega$, because of the relation of $\tilde{\mathcal{T}}$ and $\mathcal{T}$. Moreover, we have $(\tilde{\mathcal{T}}\varphi)(u,y)=\varphi(u,y)$ for all $(u,y)\in\left(\overline{S_r}\cap(\overline{S_r}-u_0)\right)\times\Delta_n(\rho)$.

Now there exists $0<r_1<r$ such that $\overline{S_{r_1}}\subset\left(\overline{S_r}\cap(\overline{S_r}-u_0)\right)$, so there exists a unique solution $\psi\in\mathcal{H}_{r_1,\rho}$ of $\tilde{\mathcal{T}}\omega=\omega$, for $r_1M_{r_1}<rM_r<\sin\frac{\delta}{2}$. As both $w(u_0+\cdot,\cdot)$ and φ belong to $\mathcal{H}_{r_1,\rho}$, the unicity of ψ implies

$$w(u_0+u,y)=\varphi(u,y)=\psi(u,y),\qquad\forall\ (u,y)\in\overline{S_{r_1}}\times\Delta_n(\rho).$$

Hence, for each $Q\in\mathbb{N}^n$ we have $\varphi_Q(u)=w_Q(u+u_0)$, $\forall u\in\overline{S_{r_1}}$. The identity and uniqueness theorem then implies $\varphi_Q(u)=w_Q(u+u_0)$ for all $u\in\overline{S_r}\cap(\overline{S_r}-u_0)$ and so the coefficient φ_Q is an analytic continuation of $w_Q(u_0+\cdot)$. Denoting this continuation also by $w_Q(u_0+\cdot)$, we see that $w=\mathcal{T}w$ on $(\overline{S_r}\cup S_r^+)\times\Delta_n(\rho)$, because of the relation of $\mathcal{T}$ and $\tilde{\mathcal{T}}$.

Varying u_0 we get a unique solution w of $\omega=\mathcal{T}\omega$, which can be written as

$$w(u,y)=\sum_{Q\in\mathbb{N}^n}w_Q(u)y^Q,$$

where each coefficient w_Q is analytic in $\{u \in \overline{S_c} \mid |u| \leq \frac{3}{2}r\}$.
By repeating this procedure we obtain a unique solution w of (6.4) in $\mathcal{H}_{r,\rho}$ for all $r > 0$. This solution can be written as $w = w(u, y) = \sum_{Q \in \mathbb{N}^n} w_Q(u) y^Q$ with coefficients continuous on $\overline{S_c}$ and analytic in the interior S_c.

6.4 Proof of lemma 6.3.1

We have to prove that the series $\sum_{Q \in \mathbb{N}^n} \gamma_Q(u) y^Q$, where each γ_Q is given by (6.7), belongs to the Banach space $\mathcal{H}_{r,\rho}$.
First of all we remark that the matrix-valued function $u \mapsto (u + u_0)\mathrm{I} + A_Q$ is non-singular in $\overline{S_r}$ (this immediately follows from the choice of u_0). As in the preceding section we have $1 * \alpha_Q$ being an entire function, hence the function $u \mapsto (1 * \alpha_Q)(u + u_0)$ is entire as a composition of two entire functions. In particular the function is analytic in $\overline{S_r}$. Clearly the same holds after multiplication with the inverse of $(u + u_0)\mathrm{I} + A_Q$. Moreover, for $u \in \overline{S_r}$ we have

$$\begin{aligned} |(1 * \alpha_Q)(u + u_0)| &\leq \int_0^{u+u_0} |\alpha_Q(s)|\,|ds| \\ &\leq |u + u_0| \cdot \sup_{|s| \leq 2r} |\alpha_Q(s)| \\ &\leq 2r \cdot \sup_{|s| \leq 2r} |\alpha_Q(s)|. \end{aligned}$$

For all $u \in \overline{S_r}$ and for all $i \in \{1, 2, \ldots, n\}$ we have

$$|u + u_0 + \langle Q, \lambda \rangle + c_i| \geq \mathcal{I}m\ u_0 \geq \frac{r}{2} \sin \frac{\delta}{2},$$

hence

$$\left| \left((u + u_0)\mathrm{I} + A_Q \right)^{-1} (1 * \alpha_Q)(u + u_0) \right| \leq \frac{4}{\sin \frac{\delta}{2}} \cdot \sup_{|s| \leq 2r} |\alpha_Q(s)|, \quad \forall\, u \in \overline{S_r}.$$

We conclude

$$\begin{aligned} &\left\| \sum_{Q \in \mathbb{N}^n} \left((u + u_0)\mathrm{I} + A_Q \right)^{-1} (1 * \alpha_Q)(u + u_0) y^Q \right\|_{r,\rho} \\ \leq\ & \frac{4}{\sin \frac{\delta}{2}} \sum_{Q \in \mathbb{N}^n} \Big(\sup_{|s| \leq 2r} |\alpha_Q(s)| \Big) \rho^{|Q|}, \end{aligned}$$

which is finite, because ρ is chosen such that $n\rho < \rho'$ (cf. the proof of lemma 5.3.3).

For $Q \in \mathbb{N}^n$ and $0 \leq R \leq Q$, the function $u \mapsto (1 * \beta_{(Q-R)})(u + u_0)$ is analytic in $\overline{S_r}$ (it even is an entire function). Hence the convolution with w_R, resp.

multiplication with $((u+u_0)\mathrm{I}+A_Q)^{-1}$, is analytic in $\overline{S_r}$, and this also holds for the finite sum over all $R \in \mathbb{N}^n$ with $0 \leq R \leq Q$.
Moreover, for all $u \in \overline{S_r}$ we have

$$\begin{aligned}
&\left|((u+u_0)\mathrm{I}+A_Q)^{-1}((1*\beta_{(Q-R)})(u_0+\cdot)*w_R)(u)\right| \\
\leq\ & \frac{2}{r\sin\frac{\delta}{2}}\cdot\Big(\sup_{|s|\leq 2r}|1*\beta_{(Q-R)}(s)|\Big)\cdot\|w_R\|_r\cdot|u| \\
\leq\ & \frac{2}{r\sin\frac{\delta}{2}}\cdot 2r^2\cdot\Big(\sup_{|s|\leq 2r}|\beta_{(Q-R)}(s)|\Big)\cdot\|w_R\|_r.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|\sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\leq R\leq Q}((u+u_0)\mathrm{I}+A_Q)^{-1}((1*\beta_{(Q-R)})(u_0+\cdot)*w_R)(u)\right\}y^Q\right\|_{r,\rho} \\
&\qquad\leq\quad \frac{4rM_{2r}}{\sin\frac{\delta}{2}}\|w\|_{r,\rho},
\end{aligned}$$

where $M_{2r}=\sum_{Q\in\mathbb{N}^n}\Big(\sup_{|s|\leq 2r}|\beta_Q(s)|\Big)\rho^{|Q|}<\infty$.

Next we take $Q \in \mathbb{N}^n$ and consider the function $u \mapsto \Xi(1*\beta_{(Q-R)}, w_R)(u)$, with $0 \leq R \leq Q$. This function can be written as

$$\Xi(1*\beta_{(Q-R)},w_R)(u)=\int_{u_0}^{u}K_{(Q-R)}(u,s)w_R(s)\mathrm{d}s,$$

where $K_{(Q-R)}(u,s)=-(1*\beta_{(Q-R)})(u+u_0-s)$. Hence $K_{(Q-R)}$ is an entire kernel as a composition of two entire functions, so if w_R is analytic in $\overline{S_r}$, the same holds for $\Xi(1*\beta_{(Q-R)},w_R)$. This doesn't change after multiplication with the inverse of $(u+u_0)\mathrm{I}+A_Q$, resp. summation over all $R \in \mathbb{N}^n$ with $0 \leq R \leq Q$. Moreover, for all $u \in \overline{S_r}$ we have

$$\begin{aligned}
&\left|((u+u_0)\mathrm{I}+A_Q)^{-1}\Xi(1*\beta_{(Q-R)},w_R)(u)\right| \\
\leq\ & \frac{2}{r\sin\frac{\delta}{2}}\int_u^{u_0}\left|(1*\beta_{(Q-R)})(u+u_0-s)\right|\cdot|w_R(s)|\cdot|\mathrm{d}s|.
\end{aligned}$$

But $\Xi(1*\beta_{(Q-R)},w_R)$ only depends on the values of $1*\beta_{(Q-R)}$ and w_R on $[u_0,u]\subset\overline{S_r}$. Hence

$$\begin{aligned}
&\left|((u+u_0)\mathrm{I}+A_Q)^{-1}\Xi(1*\beta_{(Q-R)},w_R)(u)\right| \\
\leq\ & \frac{2r}{r\sin\frac{\delta}{2}}\Big(\sup_{|s|\leq r}|\beta_{(Q-R)}(s)|\Big)\cdot\|w_R\|_r\cdot|u_0-u| \\
\leq\ & \frac{4r}{\sin\frac{\delta}{2}}\Big(\sup_{|s|\leq r}|\beta_{(Q-R)}(s)|\Big)\cdot\|w_R\|_r
\end{aligned}$$

and we conclude

$$\left\| \sum_{Q\in\mathbb{N}^n} \left\{ \sum_{0\le R\le Q} \left((u+u_0)\mathrm{I}+A_Q\right)^{-1}\Xi(1*\beta_{(Q-R)},w_R)(u) \right\} y^Q \right\|_{r,\rho}$$

$$\le \quad \frac{4rM_r}{\sin\frac{\delta}{2}}\|w\|_{r,\rho}.$$

This proves lemma 6.3.1. ∎

6.5 Exponential estimate for the solution

In this section we will obtain an exponential estimate for the solution $w = w(u,y) = \sum_{Q\in\mathbb{N}^n} w_Q(u)y^Q$ of (6.4) on $\overline{S_c} \times \Delta_n(\rho)$ in order to be able to apply the Laplace transform to $w(\cdot,y)$. For $r \ge 0$ let us define

$$g(r) = \sum_{Q\in\mathbb{N}^n} g_Q(r)\rho^{|Q|},$$

where each g_Q is defined by

$$g_Q(r) := \sup_{\substack{u\in\overline{S_c}\\|u|=r}} |w_Q(u)|.$$

Note that for each $r \ge 0$ we have

$$g(r) \quad \le \quad \sum_{Q\in\mathbb{N}^n} \|w_Q\|_r \rho^{|Q|} \quad < \infty,$$

so g is a well-defined, positive-valued function.

Lemma 6.5.1 *The function $g : [0,\infty) \to [0,\infty)$ is continuous.*

Proof. First we take $Q \in \mathbb{N}^n$ and consider the function $r \mapsto g_Q(r)$.
Take $\varepsilon > 0$ arbitrary.
As $u \mapsto w_Q(u)$ is continuous on $\overline{S_c}$, the same holds for $u \mapsto |w_Q(u)|$. So for $u_0 \in \overline{S_c}$ arbitrary there exists $\delta > 0$ such that for all $u \in \overline{S_c}$, $|u-u_0| < \delta$, we have $\big||w_Q(u)| - |w_Q(u_0)|\big| < \varepsilon$.
Next take $\tilde r \in [0,\infty)$ arbitrary and take $r \in [0,\infty)$ such that $|r-\tilde r| < \delta$. Weierstrass's theorem says that we can find $\tilde u$ and u in $\overline{S_c}$, $|\tilde u| = \tilde r$ and $|u| = r$, such that $g_Q(\tilde r) = |w_Q(\tilde u)|$ and $g_Q(r) = |w_Q(u)|$.
First suppose $g_Q(r) < g_Q(\tilde r)$.
As $|r-\tilde r| < \delta$ we can find $u' \in \overline{S_c}$, $|u'| = r$, so that $|\tilde u - u'| < \delta$. Because $|w_Q(u')| \le g_Q(r)$ we have

$$\begin{aligned} |g_Q(\tilde r) - g_Q(r)| &= g_Q(\tilde r) - g_Q(r) \\ &\le g_Q(\tilde r) - |w_Q(u')| \\ &= \big||w_Q(\tilde u)| - |w_Q(u')|\big| \\ &< \varepsilon. \end{aligned}$$

The case $g_Q(\tilde{r}) < g_Q(r)$ is proven similarly, by taking $u' \in \overline{S_c}$, $|u'| = \tilde{r}$, so that $|u - u'| < \delta$. Hence, g_Q is continuous on $[0, \infty)$.

If $R > 0$ we have for all $r \in [0, R]$

$$g(r) \;\le\; \sum_{Q \in \mathbb{N}^n} \|w_Q\|_R \, \rho^{|Q|} = \|w\|_{R,\rho} < \infty,$$

so the series $\sum_{Q \in \mathbb{N}^n} g_Q(r) \rho^{|Q|}$ converges uniformly on compact sets in $[0, \infty)$. Hence g is continuous on $[0, \infty)$. ∎

Remember the convolution equation which is satisfied by w

$$\begin{aligned} w(u, y) \;=\; & \sum_{Q \in \mathbb{N}^n} (u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)(u) y^Q \;+ \\ & \sum_{Q \in \mathbb{N}^n} \left\{ \sum_{0 \le R \le Q} (u\mathrm{I} + A_Q)^{-1}(1 * \beta_{(Q-R)} * w_R)(u) \right\} y^Q, \end{aligned}$$

so for each $Q \in \mathbb{N}^n$ we have

$$\begin{aligned} w_Q(u) \;=\; & (u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)(u) \;+ \\ & \sum_{0 \le R \le Q} (u\mathrm{I} + A_Q)^{-1}(1 * \beta_{(Q-R)} * w_R)(u), \end{aligned}$$

for all $u \in \overline{S_c}$.

Take $\tau_1 > \tau$ arbitrary (here τ is the same fixed number as in chapter 4).

Take $Q \in \mathbb{N}^n$ arbitrary, then for $u \in \overline{S_c} \setminus \{0\}$ we have

$$\begin{aligned} \left|(u\mathrm{I} + A_Q)^{-1}(1 * \alpha_Q)(u)\right| \;&\le\; \frac{1}{|u| \sin\frac{\delta}{2}} \left\|\alpha_Q\right\|_{1,\tau_1} \cdot |u| \cdot e^{\tau_1 |u|} \\ &\le\; \frac{\left\|\alpha_Q\right\|_{1,\tau_1}}{\sin\frac{\delta}{2}} e^{\tau_1 |u|} \\ &\le\; \frac{\left\|\alpha_Q\right\|_{1,\tau}}{\sin\frac{\delta}{2}} e^{\tau_1 |u|}, \end{aligned}$$

but, as before, this also holds for $u = 0$.
For $u \in \overline{S_c}$, $|u| = r$ and $r \ge 1$, we have (using $\|1 * \beta_Q\|_{1,\tau_1} \le \|1 * \beta_Q\|_{1,\tau} \le$

$\frac{1}{\tau}\|\beta_Q\|_{1,\tau}$ for every $Q \in \mathbb{N}^n$)

$$
\begin{aligned}
& \left|(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*w_R)(u)\right| \\
\leq\ & \frac{1}{|u|\sin\frac{\delta}{2}}\int_0^u \left|(1*\beta_{(Q-R)})(u-s)\right|\cdot\left|w_R(s)\right||\mathrm{d}s| \\
\underset{|u|=r\geq 1}{\leq}\ & \frac{1}{\sin\frac{\delta}{2}}\int_0^u \left|(1*\beta_{(Q-R)})(u-s)\right|\cdot\left|w_R(s)\right||\mathrm{d}s| \\
\leq\ & \frac{\|\beta_{(Q-R)}\|_{1,\tau}}{\tau\sin\frac{\delta}{2}}\int_0^u e^{\tau_1|u-s|}\left|w_R(s)\right||\mathrm{d}s| \\
\leq\ & \frac{\|\beta_{(Q-R)}\|_{1,\tau}}{\tau\sin\frac{\delta}{2}}\int_0^{|u|} e^{\tau_1(|u|-\sigma)}g_R(\sigma)\mathrm{d}\sigma \\
=\ & \frac{\|\beta_{(Q-R)}\|_{1,\tau}}{\tau\sin\frac{\delta}{2}}\left(e^{\tau_1 r}*g_R\right)(r).
\end{aligned}
$$

Hence, for each $u \in \overline{S_c}$, $|u| = r$ and $r \geq 1$, we have

$$
\begin{aligned}
\left|w_Q(u)\right| \leq\ & \frac{\left\|\alpha_Q\right\|_{1,\tau}}{\sin\frac{\delta}{2}}e^{\tau_1 r} \quad + \\
& \sum_{0\leq R\leq Q}\frac{\|\beta_{(Q-R)}\|_{1,\tau}}{\tau\sin\frac{\delta}{2}}\left(e^{\tau_1 r}*g_R\right)(r)
\end{aligned}
$$

and, as this holds for every $u \in \overline{S_c}$, $|u| = r$ and $r \geq 1$, $|w_Q(u)|$ can be replaced by $g_Q(r)$. Moreover, for all $(u, y) \in \overline{S_c} \times \Delta_n(\rho)$, $|u| = r$ and $r \geq 1$, we have

$$
|w(u,y)| \leq \sum_{Q\in\mathbb{N}^n}\left|w_Q(u)\right|\rho^{|Q|} \leq \sum_{Q\in\mathbb{N}^n} g_Q(r)\rho^{|Q|}
$$

and

$$
\begin{aligned}
\sum_{Q\in\mathbb{N}^n} g_Q(r)\rho^{|Q|} \leq\ & \sum_{Q\in\mathbb{N}^n}\frac{\left\|\alpha_Q\right\|_{1,\tau}}{\sin\frac{\delta}{2}}e^{\tau_1 r}\rho^{|Q|} \quad + \\
& \sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\leq R\leq Q}\frac{\|\beta_{(Q-R)}\|_{1,\tau}}{\tau\sin\frac{\delta}{2}}\left(e^{\tau_1 r}*g_R\right)(r)\right\}\rho^{|Q|}.
\end{aligned}
$$

The latter sum equals

$$
\begin{aligned}
& \frac{1}{\tau\sin\frac{\delta}{2}}\left(\sum_{Q\in\mathbb{N}^n}\|\beta_Q\|_{1,\tau}\rho^{|Q|}\right)\cdot\sum_{Q\in\mathbb{N}^n}\left(e^{\tau_1 r}*g_Q\right)(r)\rho^{|Q|} \\
=\ & c\sum_{Q\in\mathbb{N}^n}\left(e^{\tau_1 r}*g_Q\right)(r)\rho^{|Q|},
\end{aligned}
$$

where $c = c(\tau) = \frac{1}{\tau \sin\frac{\delta}{2}} \|\beta\|_{1,\tau,\rho}$ a positive constant.
But from the preceding two sections we obtain for all $r \leq 1$

$$g(r) = \sum_{Q \in \mathbb{N}^n} g_Q(r)\rho^{|Q|} \leq \sum_{Q \in \mathbb{N}^n} \|w_Q\|_1 \rho^{|Q|} = \|w\|_{1,\rho} < \infty.$$

Choosing $M > \max\{\|w\|_{1,\rho}\, , \ \frac{\|\alpha\|_{1,\tau,\rho}}{\sin\frac{\delta}{2}}\}$ we have for $r \geq 0$

$$g(r) < (\mathcal{T}_1 g)(r), \tag{6.8}$$

where

$$(\mathcal{T}_1 g)(r) \ = \ Me^{\tau_1 r} \ + \ c \sum_{Q \in \mathbb{N}^n} (e^{\tau_1 r} * g_Q)(r)\rho^{|Q|}.$$

Note that $(\mathcal{T}_1 g)(r)$ can be written as

$$(\mathcal{T}_1 g)(r) \ = \ Me^{\tau_1 r} + c(e^{\tau_1 r} * g)(r),$$

for interchanging summation and integration is allowed by Lebesgue's monotone convergence theorem.
Let us now study the equation

$$v(r) = (\mathcal{T}_1 v)(r), \tag{6.9}$$

and let v denote any continuous solution, then

$$v(r) = Me^{\tau_1 r} + c\int_0^r e^{\tau_1(r-\sigma)}v(\sigma)\mathrm{d}\sigma$$

is differentiable and

$$\begin{aligned}
v'(r) &= M\tau_1 e^{\tau_1 r} + cv(r) + c\int_0^r \tau_1 e^{\tau_1(r-\sigma)}v(\sigma)\mathrm{d}\sigma \\
&= M\tau_1 e^{\tau_1 r} + cv(r) + \tau_1\big(v(r) - Me^{\tau_1 r}\big) \\
&= (c+\tau_1)v(r).
\end{aligned}$$

Hence $v(r) = v(0)e^{(c+\tau_1)r} = Me^{(c+\tau_1)r}$. Moreover,

$$g(0) \leq \|w\|_{1,\rho} < M = v(0).$$

Next suppose there exists $r_0 > 0$ such that $0 \leq g(r) < v(r)$ if $0 \leq r < r_0$ and $g(r_0) = v(r_0)$. (Here we use the continuity of both g and v.) Then (6.8) implies

$$\begin{aligned}
g(r_0) &< (\mathcal{T}_1 g)(r_0) \\
&= Me^{\tau_1 r_0} + c\big(e^{\tau_1 r} * g\big)(r_0) \\
&\leq Me^{\tau_1 r_0} + c\big(e^{\tau_1 r} * v\big)(r_0) \\
&= (\mathcal{T}_1 v)(r_0) \\
&= v(r_0),
\end{aligned}$$

which gives a contradiction. Hence $g < v$ on $\mathbb{R}_{\geq 0}$.
We conclude that for $(u, y) \in \overline{S_c} \times \Delta_n(\rho)$:

$$|w(u,y)| \leq g(|u|) < v(|u|) = Me^{(c+\tau_1)|u|}.$$

As $\tau_1 > \tau$ was chosen arbitrary, this holds for every $\tau_1 > \tau$.

6.6 A solution of the differential equation

The norm of $u\mathrm{I} + A_Q$ can be estimated by

$$\left|u\mathrm{I} + A_Q\right| \leq |u| + C(|Q|+1), \quad \forall\, u \in \overline{S_c},$$

where C is some positive constant.
Next take $\tau_1 > \tau$ arbitrary and write $\tau_2 = \tau_1 + c$, where M and c are the same constants as in the preceding section. For arbitrary but fixed $y \in \Delta_n(\rho)$, a positive number $\rho_1 < \rho$ can be found such that $y \in \overline{\Delta_n(\rho_1)}$. Hence, for $u \in \overline{S_c}$ we have

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n} \left|(u\mathrm{I} + A_Q)w_Q(u)y^Q\right| &\leq |u| \sum_{Q\in\mathbb{N}^n} |w_Q(u)|\rho^{|Q|} \; + \\ &\qquad C\sum_{Q\in\mathbb{N}^n} (|Q|+1)|w_Q(u)|{\rho_1}^{|Q|}.\end{aligned}$$

But

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n} |Q||w_Q(u)|{\rho_1}^{|Q|} &= \sum_{Q\in\mathbb{N}^n} |Q| \left(\frac{\rho_1}{\rho}\right)^{|Q|} |w_Q(u)|\rho^{|Q|} \\ &\leq K \sum_{Q\in\mathbb{N}^n} |w_Q(u)|\rho^{|Q|},\end{aligned}$$

where $K = \sup_{Q\in\mathbb{N}^n} |Q| \left(\frac{\rho_1}{\rho}\right)^{|Q|}$, which is finite. [An easy calculation shows that the function $f_\alpha(x) = x\alpha^x$, $0 < \alpha < 1$, attains a maximum on $[0,\infty)$.]
Hence, as $\sum_{Q\in\mathbb{N}^n} |w_Q(u)|\rho^{|Q|} \leq g(|u|) \leq Me^{\tau_2|u|}$, we have by using (2.11)

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n} \left|(u\mathrm{I} + A_Q)w_Q(u)y^Q\right| &\leq \frac{M}{\varepsilon}e^{(\tau_2+\varepsilon)|u|} + MC(K+1)e^{\tau_2|u|} \\ &\leq \left(\frac{M}{\varepsilon} + MC(K+1)\right)e^{(\tau_2+\varepsilon)|u|}.\end{aligned}$$

Note that this holds for every $\tau_2 > \tau + c$ and every $\varepsilon > 0$, so to every $\tilde{\tau} > \tau + c$ corresponds a positive constant M_1, depending upon $\tilde{\tau}$, such that

$$\sum_{Q\in\mathbb{N}^n} \left|(u\mathrm{I} + A_Q)w_Q(u)y^Q\right| \leq M_1 e^{\tilde{\tau}|u|}, \quad \forall\, u \in \overline{S_c}.$$

Moreover, for arbitrary $\tau_2 > \tau + c > \tau$, we have for all $u \in \overline{S_c}$:

$$\begin{aligned}\sum_{Q\in\mathbb{N}^n} \left|(1 * \alpha_Q)(u)\right|\left|y^Q\right| &\leq \sum_{Q\in\mathbb{N}^n} \|1 * \alpha_Q\|_{1,\tau_2} e^{\tau_2|u|}\rho^{|Q|} \\ &\leq \sum_{Q\in\mathbb{N}^n} \frac{\|\alpha_Q\|_{1,\tau}}{\tau}\rho^{|Q|}e^{\tau_2|u|} \\ &\leq M_2 e^{\tau_2|u|}\end{aligned}$$

where $M_2 = \frac{1}{\tau}\|\alpha\|_{1,\tau,\rho}$.
Finally, for arbitrary $\tau_2 > \tau + c$, we have

$$\begin{aligned}&\sum_{Q\in\mathbb{N}^n}\left|\sum_{0\leq R\leq Q}(1 * \beta_{(Q-R)} * w_R)(u)\right|\left|y^Q\right| \\ \leq\; &\sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\leq R\leq Q}\frac{\|\beta_{(Q-R)}\|_{1,\tau}}{\tau}\left(e^{\tau_2|u|} * g_R\right)(|u|)\right\}\rho^{|Q|} \\ =\; &\frac{1}{\tau}\|\beta\|_{1,\tau,\rho}\cdot\sum_{Q\in\mathbb{N}^n}\left(e^{\tau_2|u|} * g_Q\right)(|u|)\rho^{|Q|} \\ =\; &\tilde{M}\left(e^{\tau_2|u|} * g\right)(|u|),\end{aligned}$$

where $\tilde{M} = \frac{1}{\tau}\|\beta\|_{1,\tau,\rho}$. From $v(|u|) = Me^{\tau_2|u|}$ we deduce

$$\left(e^{\tau_2|u|} * g\right)(|u|) \leq \int_0^{|u|} e^{\tau_2(|u|-s)}v(s)ds \leq M|u|e^{\tau_2|u|}$$

and we conclude (again using (2.11))

$$\begin{aligned}&\sum_{Q\in\mathbb{N}^n}\left|\sum_{0\leq R\leq Q}(1 * \beta_{(Q-R)} * w_R)(u)\right|\left|y^Q\right| \\ \leq\; &\tilde{M}M|u|e^{\tau_2|u|} \\ \leq\; &\frac{\tilde{M}M}{\varepsilon}e^{(\tau_2+\varepsilon)|u|}.\end{aligned}$$

But this holds for every $\tau_2 > \tau + c$ and every $\varepsilon > 0$, so to every $\tilde{\tau} > \tau + c$ one can find a constant M_3 such that

$$\sum_{Q\in\mathbb{N}^n}\left|\sum_{0\leq R\leq Q}(1 * \beta_{(Q-R)} * w_R)(u)\right|\left|y^Q\right| \leq M_3 e^{\tilde{\tau}|u|}, \quad \forall\, u \in \overline{S_c}.$$

Hence, each of the terms in (6.3) are less than or equal to *const.* $\cdot\, e^{\tau_2|u|}$ for every $\tau_2 > \tau + c$. As in the proof of lemma 5.6.1 we can apply $\mathcal{L}$ to the equation (6.3) to obtain a solution

$$\Phi(z, y) = \sum_{Q\in\mathbb{N}^n}\Phi_Q(z)y^Q$$

of (6.1). Here each Φ_Q equals $\mathcal{L}w_Q$.

Remark 6.6.1 For each $Q \in \mathbb{N}^n$ the coefficients $u \mapsto (uI + A_Q)w_Q(u)$, $u \mapsto (1 * \alpha_Q)(u)$ and $u \mapsto (1 * \beta_{(Q-R)} * w_R)(u)$ all are analytic in S_c and continuous on $\overline{S_c}$. Moreover, they all are less than or equal to $const. \cdot e^{\tau_2|u|}$, for every $\tau_2 > \tau + c$. Hence $\mathcal{L}$ is applicable to each of the coefficients.

As $\delta \in (0, \pi)$ was chosen arbitrary we can vary δ to obtain a solution $\Phi(z, y)$ which is analytic in the product of a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi)$ and a polydisc $\Delta_n(\rho)$, for arbitrary $\rho < \frac{\rho'}{n}$. Note that this neighbourhood is a subset of the neighbourhood found in the preceding chapter, for we can only estimate w by

$$|w(u, y)| \leq const. \cdot e^{(\tau_1 + c)|u|}, \quad \forall \tau_1 > \tau.$$

Next we will show that Φ asymptotically equals $\hat{\Phi}$ in the sense of Gérard-Sibuya. (Here $\hat{\Phi}$ is the formal solution with coefficients $\hat{\Phi}_Q \in \mathbb{C}^n[[z]]$, found in chapter 4.):
We have w being a solution of (6.3), hence each coefficient w_Q satisfies the equation

$$\begin{aligned} w_Q(u) &= (uI + A_Q)^{-1}(1 * \alpha_Q)(u) \quad + \\ & \sum_{0 \leq R \leq Q} (uI + A_Q)^{-1}(1 * \beta_{(Q-R)} * w_R)(u). \end{aligned} \tag{6.10}$$

Chapter 4 implies (6.10) to have the formal solution $\hat{\mathcal{B}}\hat{\Phi}_Q$. Moreover, we have (as in chapter 2)

$$w_Q(u) \simeq (\hat{\mathcal{B}}\hat{\Phi}_Q)(u) \quad \text{as } u \to 0 \text{ on } S(\tfrac{\pi}{2}, \pi - \delta).$$

So

$$\Phi_Q(z) = (\mathcal{L}w_Q)(z) \simeq (\hat{\mathcal{L}}\hat{\mathcal{B}}\hat{\Phi}_Q)(z) = \hat{\Phi}_Q(z)$$

as $z \to 0$ on a neighbourhood of 0 in $S(\frac{\pi}{2}, 2\pi)$. Hence we have proven the following theorem:

Theorem 6.6.2 *Under the assumptions listed in the introduction of this chapter there exists a solution $\Phi = \Phi(z, y)$ of (6.1), which is analytic in the product of a neighbourhood U of 0 in $S(\frac{\pi}{2}, 2\pi)$ and a polydisc $\Delta_n(\rho)$, for arbitrary $\rho < \frac{\rho'}{n}$.*
This solution can be written as

$$\Phi(z, y) \;\; = \sum_{Q \in \mathbb{N}^n} \Phi_Q(z) y^Q,$$

where each Φ_Q equals $\Phi_Q(z) = \big(\mathcal{L}\tilde{w}_Q\big)(z)$ for some function $\tilde{w}_Q \in \mathcal{A}_{\tau+c}$, $\delta \in (0, \pi)$ arbitrary.

Moreover, $\Phi(z,y)$ *asymptotically equals the series* $\sum_{Q\in\mathbb{N}^n}\hat{\Phi}_Q(z)y^Q$ *in the sense of Gérard-Sibuya.*
Finally, this solution is unique in the set

$$\left\{ \Upsilon(z,y)=\sum_{Q\in\mathbb{N}^n}\Upsilon_Q(z)y^Q \;\middle|\; \begin{array}{c} \Upsilon \text{ is analytic in } U\times\Delta_n(\rho) \text{ and} \\ \forall\, Q\in\mathbb{N}^n \quad \exists\, \omega_Q\in\mathcal{A}_{\tau+c} \text{ such that} \\ \Upsilon_Q=\mathcal{L}\omega_Q. \end{array} \right\}$$

Chapter 7

A special case with resonance

Contrary to both chapter 5 and chapter 6 we will now study a case with possible resonance relations between the eigenvalues λ_i of the well-known system

$$
\begin{aligned}
& z^2\frac{\partial\Phi}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^{n}\lambda_j y_j\frac{\partial}{\partial y_j} + c_i\right\}_{i=1}^{n}\cdot\Phi \\
= & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.1) \\
& z\cdot\operatorname{diag}\left\{d_i - \sum_{j=1}^{n}\alpha_j y_j\frac{\partial}{\partial y_j}\right\}_{i=1}^{n}\cdot\Phi \;+\; za(z,y) \;+\; zB(z,y)\Phi,
\end{aligned}
$$

i.e. the expression $\langle Q,\lambda\rangle + c_i$ might be zero for some $Q\in\mathbb{N}^n$ and some $i\in\{1,2,\dots,n\}$.

7.1 The hypotheses

Besides the assumptions in chapter 4 we assume each $c_i \neq 0$. Furthermore, we make the following two hypotheses for every $Q\in\mathbb{N}^n$:

H1 If $\langle Q,\lambda\rangle + c_i = 0$ for some $i\in\{1,2,\dots,n\}$ we assume $d_i - \langle Q,\alpha\rangle \notin \mathbb{N}$.

H2 If $|\langle Q,\lambda\rangle| > 1+\max_{i\in\{1,2,\dots,n\}}|c_i|$ we assume the existence of a positive constant K_1, independent of Q, such that

$$\left|u(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q\right| \le K_1, \quad \forall\, u\in\overline{S_c}.$$

If $|\langle Q,\lambda\rangle| \le 1+\max_{i\in\{1,2,\dots,n\}}|c_i|$ we assume to each $i\in\{1,2,\dots,n\}$ the existence of a positive constant $K_{2,i}$, independent of Q, such that

$$|d_i - \langle Q,\alpha\rangle| \le K_{2,i}.$$

Here A_Q and $\tilde{A}_Q$ are defined as in section 4.2.

Moreover, we assume the existence of four positive constants k_1, k_2, γ_1 and γ_2, all independent of Q and i, such that for all $Q \in \mathbb{N}^n \setminus \{0\}$ and all $i \in \{1, 2, \dots, n\}$ we have

$$\begin{cases} \left|\langle Q, \lambda\rangle + c_i\right| & \geq \; k_1 |Q|^{-\gamma_1}, \quad \text{if } \langle Q, \lambda\rangle + c_i \neq 0 \\ \inf_{m \in \mathbb{N}} \left| m - (d_i - \langle Q, \alpha\rangle)\right| & \geq \; k_2 |Q|^{-\gamma_2}, \quad \text{if } \langle Q, \lambda\rangle + c_i = 0 \end{cases}$$

We write

$$k = \min\{k_1,\ k_2\}, \qquad \gamma = \max\{\gamma_1,\ \gamma_2\} \qquad \text{and} \qquad K_2 = \max_{i \in \{1,2,\dots,n\}} K_{2,i}.$$

Remark 7.1.1 This remark is about the assumption $c_i \neq 0$.
Suppose there exists $i \in \{1, 2, \dots, n\}$ such that $c_i = 0$, then d_i might be zero as well. If there exists $Q_0 \in \mathbb{N}^n \setminus \{0\}$ such that $\langle Q_0, \lambda\rangle + c_i = \langle Q_0, \lambda\rangle = 0$, H1 implies $\langle Q_0, \alpha\rangle \neq 0$. Hence, the second assumption in H2 wouldn't make sense, for $\langle Q_0, \lambda\rangle = 0$ implies $\langle p\ Q_0, \lambda\rangle = 0$ for every $p \in \mathbb{N}$ and $\langle p\ Q_0, \alpha\rangle = p\ \langle Q_0, \alpha\rangle$ cannot be bounded by a constant independent of $Q = p\ Q_0$.

Using the same notations as in chapter 4 we will prove the following

Theorem 7.1.2 *Under the assumptions listed above there exists a solution $\Phi = \Phi(z, y)$ of (7.1), which is analytic in the product of a neighbourhood U of 0 in $S(\frac{\pi}{2}, 2\pi)$ and a polydisc $\Delta_n(\rho)$, for arbitrary $\rho < \frac{\rho'}{n}$.*
This solution can be written as

$$\Phi(z, y) = \sum_{Q \in \mathbb{N}^n} \Phi_Q(z) y^Q,$$

where each Φ_Q equals $\Phi_Q(z) = (\mathcal{L}\tilde{w}_Q)(z)$ for some function $\tilde{w}_Q \in \mathcal{A}_\tau$, $\delta \in (0, \pi)$ arbitrary.
Moreover, $\Phi(z, y)$ asymptotically equals the series $\sum_{Q \in \mathbb{N}^n} \hat{\Phi}_Q(z) y^Q$ in the sense of Gérard-Sibuya.
Finally, this solution is unique in the set

$$\left\{ \Upsilon(z, y) = \sum_{Q \in \mathbb{N}^n} \Upsilon_Q(z) y^Q \;\middle|\; \begin{array}{c} \Upsilon \textit{ is analytic in } U \times \Delta_n(\rho) \textit{ and} \\ \forall\, Q \in \mathbb{N}^n \;\; \exists\, \omega_Q \in \mathcal{A}_\tau \textit{ such that} \\ \Upsilon_Q = \mathcal{L}\omega_Q. \end{array} \right\}$$

To prove this theorem we use the same scheme as in chapter 5.

7.2 A formal solution in terms of powers of z

Substituting a series $\sum_{m=0}^{\infty} \Phi_m(y)z^m$ in (7.1) gives the following recurrence relation

$$\operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}\Phi_0(y) = 0 \tag{7.2}$$

$$\operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}\Phi_1(y) = \operatorname{diag}\left\{d_i - \sum_{j=1}^{n} \alpha_j y_j \frac{\partial}{\partial y_j}\right\}\Phi_0(y) \tag{7.3}$$

and for $m \geq 2$

$$\begin{aligned} &(m-1)\Phi_{m-1}(y) + \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\}\Phi_m(y) \\ = \quad & \\ &\operatorname{diag}\left\{d_i - \sum_{j=1}^{n} \alpha_j y_j \frac{\partial}{\partial y_j}\right\}\Phi_{m-1}(y) \quad + \quad a_{m-1}(y) \quad + \\ &\sum_{k=1}^{m-1} B_k(y)\Phi_{m-1-k}(y) \end{aligned} \tag{7.4}$$

To prove that this system has a unique solution in $A(\Delta_n(\rho'))$ we use induction after m.
For $m = 0$ we substitute a series $\sum_{Q\in\mathbb{N}^n} \Phi_{Q,0}y^Q$ to obtain from (7.2)

$$(\langle Q, \lambda\rangle + c_i)\Phi_{Q,0}^{(i)} = 0, \quad \forall\, Q \in \mathbb{N}^n, \quad \forall\, i \in \{1, 2, \ldots, n\},$$

where $\Phi_{Q,0}^{(i)}$ denotes the i^{th} coefficient of $\Phi_{Q,0}$. If $\langle Q, \lambda\rangle + c_i \neq 0$ we get $\Phi_{Q,0}^{(i)} = 0$ and if $\langle Q, \lambda\rangle + c_i = 0$ equation (7.3) gives $(d_i - \langle Q, \alpha\rangle)\Phi_{Q,0}^{(i)} = 0$. But the hypothesis H1 implies $d_i - \langle Q, \alpha\rangle \neq 0$, so $\Phi_{Q,0}^{(i)} = 0$. We conclude $\Phi_0 \equiv 0$.

For $m = 1$ we obtain in a similar way $\Phi_{Q,1}^{(i)} = 0$ for all $Q \in \mathbb{N}^n$ and all $i \in \{1, 2, \ldots, n\}$ with $\langle Q, \lambda\rangle + c_i \neq 0$. But if $\langle Q, \lambda\rangle + c_i = 0$ for certain $Q \in \mathbb{N}^n$ and certain $i \in \{1, 2, \ldots, n\}$, equation (7.4) (with $m = 2$) implies (using the expansion $a_1(y) = \sum_{Q\in\mathbb{N}^n} a_{Q,1}y^Q$, $y \in \Delta_n(\rho')$)

$$\left(1 - (d_i - \langle Q, \alpha\rangle)\right)\Phi_{Q,1}^{(i)} = a_{Q,1}^{(i)}.$$

Hence we obtain (using H1)

$$\Phi_1(y) = \sum_{i=1}^{n} \sum_{\substack{Q\in\mathbb{N}^n \\ \langle Q,\lambda\rangle + c_i = 0}} \left[\frac{a_{Q,1}^{(i)}}{1 - (d_i - \langle Q, \alpha\rangle)}\, e_i\right] y^Q,$$

where e_i is the i^{th} unit vector in $\mathbb{R}^n$.
Using the last assumption in the preceding section we deduce for all multi-indices $Q \in \mathbb{N}^n \setminus \{0\}$ and all $i \in \{1, 2, \dots, n\}$ with $\langle Q, \lambda \rangle + c_i = 0$:

$$\left|1 - (d_i - \langle Q, \alpha \rangle)\right| \geq k|Q|^{-\gamma}.$$

Next take a compact set $K \subset \Delta_n(\rho')$, then two positive numbers r and $\tilde{r}$ can be found such that $K \subset \Delta_n(r) \subset \Delta_n(\tilde{r}) \subset \Delta_n(\rho')$ and we know

$$\sum_{Q \in \mathbb{N}^n \setminus \{0\}} |a_{Q,1}| \tilde{r}^{|Q|} < \infty.$$

Hence

$$\begin{aligned} \sum_{Q \in \mathbb{N}^n \setminus \{0\}} \sup_{y \in K} \left|\Phi_{Q,1} y^Q\right| &\leq \sum_{Q \in \mathbb{N}^n \setminus \{0\}} |\Phi_{Q,1}| r^{|Q|} \\ &\leq \frac{1}{k} \cdot \sum_{Q \in \mathbb{N}^n \setminus \{0\}} |Q|^{\gamma} |a_{Q,1}| r^{|Q|}. \end{aligned}$$

Since this latter sum is finite (cf. proposition 5.1.1), the same holds for $\sum_{Q \in \mathbb{N}^n} \sup_{y \in K} \left|\Phi_{Q,1} y^Q\right|$ and we conclude that Φ_1 is analytic in $\Delta_n(\rho')$.

Now suppose that for all $m \in \{0, 1, \dots, M-1\}$, $M \geq 2$, a unique solution Φ_m, analytic in $\Delta_n(\rho')$, exists. To find Φ_M we have to consider (7.4) with $m = M$ and with $m = M + 1$. In case $m = M$ substitution of $\sum_{Q \in \mathbb{N}^n} \Phi_{Q,M} y^Q$ gives

$$\sum_{Q \in \mathbb{N}^n} A_Q \Phi_{Q,M} y^Q = \sum_{Q \in \mathbb{N}^n} \zeta_{Q,M} y^Q,$$

where the right hand side represents the analytic function

$$\begin{aligned} \zeta_M(y) &= \left(1 - M + \operatorname{diag}\{d_i - \textstyle\sum_{j=1}^n \alpha_j y_j \frac{\partial}{\partial y_j}\}\right) \Phi_{M-1}(y) + \\ & \quad a_{M-1}(y) + \sum_{k=1}^{M-1} B_k(y) \Phi_{M-1-k}(y). \end{aligned}$$

Hence

$$\Phi_{Q,M}^{(i)} = \left(\langle Q, \lambda \rangle + c_i\right)^{-1} \zeta_{Q,M}^{(i)},$$

for all $Q \in \mathbb{N}^n$ and all $i \in \{1, 2, \dots, n\}$ with $\langle Q, \lambda \rangle + c_i \neq 0$.
If $\langle Q, \lambda \rangle + c_i = 0$ then (7.4) with $m = M + 1$ gives

$$\Phi_{Q,M}^{(i)} = \left(M - (d_i - \langle Q, \alpha \rangle)\right)^{-1} \zeta_{Q,M+1}^{(i)},$$

where ζ_{M+1} equals

$$\zeta_{M+1}(y) = a_M(y) + \sum_{k=1}^{M} B_k(y) \Phi_{M-k}(y).$$

Both ζ_M and ζ_{M+1} belong to $A(\Delta_n(\rho'))$ and for multi-indices $Q \in \mathbb{N}^n \setminus \{0\}$ both $\left|\langle Q, \lambda\rangle + c_i\right|^{-1}$ and $\left|M - (d_i - \langle Q, \alpha\rangle)\right|^{-1}$ are less than or equal to $\frac{1}{k}|Q|^\gamma$. Moreover, our solution Φ_M can be written as

$$\begin{aligned}
\Phi_M(y) &= \sum_{i=1}^{n} \sum_{\substack{Q\in\mathbb{N}^n \\ \langle Q,\lambda\rangle + c_i \neq 0}} \left[\frac{\zeta^{(i)}_{Q,M}}{\langle Q, \lambda\rangle + c_i}\, e_i\right] y^Q \;+ \\
&\quad \sum_{i=1}^{n} \sum_{\substack{Q\in\mathbb{N}^n \\ \langle Q,\lambda\rangle + c_i = 0}} \left[\frac{\zeta^{(i)}_{Q,M+1}}{M - (d_i - \langle Q, \alpha\rangle)}\, e_i\right] y^Q
\end{aligned}$$

and twice an application of proposition 5.1.1 shows that both series converge absolutely on compact sets in $\Delta_n(\rho')$. Hence Φ_M is analytic in the polydisc $\Delta_n(\rho')$.

7.3 A solution of the differential equation

As in section 5.2 we substitute a partial sum $u_N = \sum_{m=0}^{N-1} \Phi_m(y) z^m$ of the formal solution with analytic coefficients in (7.1) and we keep in mind that $\Phi_0 \equiv 0$. A tedious computation gives

$$\begin{aligned}
& z^2 \frac{\partial u_N}{\partial z} + \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\} u_N \;+ \\
& -z \cdot \operatorname{diag}\left\{d_i - \sum_{j=1}^{n} \alpha_j y_j \frac{\partial}{\partial y_j}\right\} u_N - z a(z, y) - z B(z, y) u_N \\
= \;& \\
& -R(z, y),
\end{aligned}$$

where $R(z, y)$ is the same expression as in section 5.2:

$$\begin{aligned}
R(z, y) &= \operatorname{diag}\left\{\sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i\right\} \Phi_N(y) z^N + z \sum_{m=N}^{\infty} a_m(y) z^m \;+ \\
&\quad z \sum_{m=N}^{2N-2} \left(\sum_{\substack{k,l \in \{1,\dots,N-1\} \\ k+l=m}} B_k(y) \Phi_l(y) \right) z^m \;+ \\
&\quad z \left[\sum_{k=N}^{\infty} B_k(y) z^k\right] \left[\sum_{l=1}^{N-1} \Phi_l(y) z^l\right].
\end{aligned}$$

With $\Psi := \Phi - u_N$ we get the equation

$$\begin{aligned} & z^2 \frac{\partial \Psi}{\partial z} + \operatorname{diag}\left\{ \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j} + c_i \right\} \Psi \\ = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.5) \\ & z \cdot \operatorname{diag}\left\{ d_i - \sum_{j=1}^{n} \alpha_j y_j \frac{\partial}{\partial y_j} \right\} \Psi + zB(z,y)\Psi + R(z,y). \end{aligned}$$

We first seek a solution Ψ which is analytic in the product of a neighbourhood U_δ of 0 in $S(\frac{\pi}{2}, 2\pi - \delta)$ and a polydisc $\Delta_n(\rho)$, where $\delta \in (0, \frac{\pi}{2})$ and $\rho < \frac{\rho'}{n}$ arbitrary. Such a solution can be expanded in a Taylor series

$$\Psi(z,y) = \sum_{Q \in \mathbb{N}^n} \Psi_Q(z) y^Q, \quad (z,y) \in U_\delta \times \Delta_n(\rho)$$

and substitution in (7.5) gives

$$\begin{aligned} & \sum_{Q \in \mathbb{N}^n} \left(z^2 \frac{\mathrm{d}\Psi_Q}{\mathrm{d}z}(z) + A_Q \Psi_Q(z) \right) y^Q \\ = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.6) \\ & \sum_{Q \in \mathbb{N}^n} \left\{ \sum_{0 \le R \le Q} z B_{(Q-R)}(z) \Psi_R(z) \right\} y^Q + \\ & \sum_{Q \in \mathbb{N}^n} (z \tilde{A}_Q \Psi_Q(z)) y^Q + \sum_{Q \in \mathbb{N}^n} R_Q(z) y^Q. \end{aligned}$$

A formal Borel transformation, with $w_Q = \mathcal{B}\Psi_Q$, gives

$$\begin{aligned} & \sum_{Q \in \mathbb{N}^n} (u\mathrm{I} + A_Q) w_Q(u) y^Q \\ = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.7) \\ & \sum_{Q \in \mathbb{N}^n} \left\{ \sum_{0 \le R \le Q} (1 * \beta_{(Q-R)} * w_R)(u) \right\} y^Q + \\ & \sum_{Q \in \mathbb{N}^n} (\tilde{A}_Q (1 * w_Q)(u)) y^Q + \sum_{Q \in \mathbb{N}^n} \xi_Q(u) y^Q. \end{aligned}$$

As in section 5.4 we will study this convolution equation after a termwise multiplication with the inverse of $u\mathrm{I} + A_Q$. Hence we will study

$$\omega = \mathcal{T}\omega, \qquad (7.8)$$

where for a series $\omega = \omega(u,y) = \sum_{Q\in\mathbb{N}^n} \omega_Q(u)y^Q \in \mathcal{W}_{N,\tau_1,\rho}$ ($\tau_1 > \tau$ arbitrary):

$$\begin{aligned}(\mathcal{T}\omega)(u,y) \;=\; & \sum_{Q\in\mathbb{N}^n}\left\{\sum_{0\le R\le Q}(u\mathrm{I}+A_Q)^{-1}(1*\beta_{(Q-R)}*\omega_R)(u)\right\}y^Q \;+\\ & \sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\big(\tilde{A}_Q(1*\omega_Q)(u)\big)y^Q \;+\\ & \sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\xi_Q(u)y^Q\end{aligned}$$

For a series $\omega \in \mathcal{W}_{N,\tau_1,\rho}$, we have to prove that the series

$$\sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\big(\tilde{A}_Q(1*\omega_Q)(u)\big)y^Q \quad\text{and}\quad \sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\xi_Q(u)y^Q$$

also belong to this Banach space, for section 5.4 shows the same for the first series in the definition of $\mathcal{T}\omega$.
To prove that $\sum_{Q\in\mathbb{N}^n}(u\mathrm{I}+A_Q)^{-1}\xi_Q(u)y^Q$ belongs to $\mathcal{W}_{N,\tau_1,\rho}$ we only have to consider the series $\sum_{Q\in\mathbb{N}^n}\left[(u\mathrm{I}+A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}\right]y^Q$, for the proof of lemma 5.4.2 uses the non-singularity of A_Q only at this series.
The vector-valued function $u \mapsto (u\mathrm{I}+A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}$ has coefficients

$$\frac{\langle Q,\lambda\rangle + c_i}{u+\langle Q,\lambda\rangle + c_i}\frac{\Phi^{(i)}_{Q,N}}{\Gamma(N)},$$

which is zero if $\langle Q,\lambda\rangle + c_i = 0$ (also for $u = 0$, by definition). This proves that

$$u \mapsto u^{1-N}(u\mathrm{I}+A_Q)^{-1}\frac{1}{\Gamma(N)}A_Q\Phi_{Q,N}u^{N-1}$$

is analytic in $\overline{S_c}$. Now we can continue as in the proof of lemma 5.4.2.

Claim 2.4.2(b) implies that $u \mapsto u^{1-N}(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)(u)$ is analytic in $\overline{S_c}$ for each $Q \in \mathbb{N}^n$. Moreover, for $u \in \overline{S_c}$ we have

$$\big|(1*\omega_Q)(u)\big| \le \|\omega_Q\|_{N,\tau_1}\int_0^{|u|} r^{N-1}e^{\tau_1 r}\mathrm{d}r \le \|\omega_Q\|_{N,\tau_1}\cdot e^{\tau_1|u|}\cdot\frac{|u|^N}{N}.$$

Now we have to consider two cases:
If $|\langle Q,\lambda\rangle| > 1 + \max_{i\in\{1,2,\dots,n\}}|c_i|$, then for all $u \in \overline{S_c}\setminus\{0\}$ we have

$$\big|u^{1-N}(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)(u)\big| \;\le\; \frac{K_1}{N}\cdot\|\omega_Q\|_{N,\tau_1}\cdot e^{\tau_1|u|}.$$

As this also holds for $u = 0$ we obtain

$$\big\|(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)\big\|_{N,\tau_1} \le \frac{K_1}{N}\cdot\|\omega_Q\|_{N,\tau_1}.$$

If $|\langle Q,\lambda\rangle| \leq 1+\max_{i\in\{1,2,\dots,n\}}|c_i|$, then for all $u\in\overline{S_c}\setminus\{0\}$ we have (with proposition 5.4.3)

$$\left|u^{1-N}(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)(u)\right| \;\leq\; \frac{K_2}{N\sin\frac{\delta}{2}}\cdot\|\omega_Q\|_{N,\tau_1}\cdot e^{\tau_1|u|},$$

so

$$\left\|(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)\right\|_{N,\tau_1}\leq\frac{K_2}{N\sin\frac{\delta}{2}}\cdot\|\omega_Q\|_{N,\tau_1}.$$

With $K_3=\max\left\{K_1,\ \frac{K_2}{\sin\frac{\delta}{2}}\right\}$ we obtain

$$\left\|(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)\right\|_{N,\tau_1}\leq\frac{K_3}{N}\cdot\|\omega_Q\|_{N,\tau_1},$$

for every $Q\in\mathbb{N}^n$ and thus

$$\begin{aligned}&\sum_{Q\in\mathbb{N}^n}\left\|(u\mathrm{I}+A_Q)^{-1}\tilde{A}_Q(1*\omega_Q)\right\|_{N,\tau_1}\rho^{|Q|}\\ \leq\;&\frac{K_3}{N}\cdot\sum_{Q\in\mathbb{N}^n}\|\omega_Q\|_{N,\tau_1}\rho^{|Q|}\end{aligned}$$

Together with section 5.4 we see that $\mathcal{T}$ defines a contraction mapping on the Banach space $\mathcal{W}_{N,\tau_1,\rho}$ for all $N>\max\left\{\frac{1}{\tau\sin\frac{\delta}{2}}\|\beta\|_{1,\tau,\rho}\ ,\ K_3\right\}$. We denote this latter maximum by N_0 and so for all $N\geq N_0$ there exists a unique solution $w=w(u,y)=\sum_{Q\in\mathbb{N}^n}w_Q(u)y^Q\in\mathcal{W}_{N,\tau_1,\rho}$ such that $\mathcal{T}w=w$. We finish the proof of theorem 7.1.2 as in chapter 5.

Appendix A

Some important results

A.1 The relations between $\mathcal{L}$ and $\mathcal{B}$ and asymptotic expansions

We begin this section with a proof of Watsons lemma, which says that for an analytic function $f : S(d,\alpha) \to \mathbb{C}$ with asymptotic expansion $\hat{f}$ (and f such that $\mathcal{L}f$ exists), we have $\mathcal{L}f \simeq \hat{\mathcal{L}}\hat{f}$. Before formulating Watsons lemma we first remark that

$$(\mathcal{L}_\theta u^m)(z) = \Gamma(m+1)z^{m+1} \quad \text{if } |\theta - \arg z| < \frac{\pi}{2},$$

for all $m \geq 0$. The proof is not very hard, for

$$\int_0^{\infty e^{i\theta}} u^m e^{-\frac{u}{z}} \mathrm{d}u = z^{m+1} \int_0^{\infty e^{i(\theta - \arg z)}} s^m e^{-s} \mathrm{d}s.$$

The last integral equals $\Gamma(m+1)$ if we can show that it is allowed to take the positive real axis as path of integration. This follows quite easily by an application of Cauchy's theorem.

Theorem A.1.1 *Let $f : S(d,\alpha) \to \mathbb{C}$ be analytic and of exponential growth of order ≤ 1, whereas $f(u) = O(u^{\mu-1})$ as $u \to 0$ on $S(d,\alpha)$ for some $\mu > 0$. Moreover, we assume $f(u) \simeq \hat{f}(u) = \sum_{m=0}^{\infty} a_m u^m$ as $u \to 0$ on $S(d,\alpha)$. Then $(\mathcal{L}f)(z) \simeq (\hat{\mathcal{L}}\hat{f})(z)$ as $z \to 0$ on a neighbourhood of 0 in $S(d,\alpha+\pi)$, which has to be interpreted as follows:*
To every $\varepsilon > 0$ (and $\varepsilon < \alpha$) there exists a positive radius ρ_ε such that $\mathcal{L}f$ is analytic in the sector $S_\varepsilon := S(d, \alpha + \pi - \varepsilon, \rho_\varepsilon)$ and

$$(\mathcal{L}f)(z) \simeq (\hat{\mathcal{L}}\hat{f})(z) \quad \text{as } z \to 0 \text{ on } S_\varepsilon.$$

Proof. Choose $\varepsilon \in (0,\alpha)$ arbitrary, then clearly $\mathcal{L}f$ is analytic in S_ε for some $\rho_\varepsilon > 0$ (cf. [1]).
Next fix $\delta \in (0,\varepsilon)$ and let $\overline{S}_\delta := \left\{z \in S(d,\alpha) \mid d - \frac{\alpha}{2} + \frac{\delta}{2} \leq \arg z \leq d + \frac{\alpha}{2} - \frac{\delta}{2}\right\}$

be a closed subsector of $S(d,\alpha)$. Define $\overline{S}_{\delta,1} := \{z \in \overline{S}_\delta \mid |z| \le 1\}$. Then, by assumption, there exist positive constants c_1 and c_2 such that

$$|f(u)| \le c_1 e^{c_2|u|}, \quad \forall u \in \overline{S}_\delta \cap \complement\Delta_1(1).$$

For $N \in \mathbb{N}$ let

$$R_f(u,N) = u^{-N}\left(f(u) - \sum_{m=0}^{N-1} a_m u^m\right),$$

then to every $N \ge 0$ a constant $C(N, \overline{S}_{\delta,1})$ can be found such that for all $u \in \overline{S}_{\delta,1}$:

$$|R_f(u,N)| \le C(N, \overline{S}_{\delta,1}).$$

Using the fact that $a_n = \lim\limits_{\substack{u\to 0\\ u\in\overline{S}_{\delta,1}}} R_f(u,n)$ we deduce $|a_n| \le C(n, \overline{S}_{\delta,1})$ for all $n \ge 0$.

For all $u \in \overline{S}_\delta \cap \complement\Delta_1(1)$ and all $N \ge 0$ we have

$$\begin{aligned} e^{-c_2|u|}|R_f(u,N)| &\le e^{-c_2|u|}\left\{|u|^{-N}|f(u)| + |u|^{-N}\sum_{m=0}^{N-1}|a_m||u|^m\right\} \\ &\le \frac{1}{|u|^N}e^{-c_2|u|}|f(u)| + \frac{1}{|u|^N}e^{-c_2|u|}\sum_{m=0}^{N-1} C(m,\overline{S}_{\delta,1})|u|^m \\ &\le c_1 + \sum_{m=0}^{N-1} C(m, \overline{S}_{\delta,1}) \end{aligned}$$

and if $u \in \overline{S}_{\delta,1}$ we obtain

$$e^{-c_2|u|}|R_f(u,N)| \le |R_f(u,N)| \le C(N, \overline{S}_{\delta,1}).$$

Choosing $\tilde{C}(N,\overline{S}_{\delta,1}) = \max\{C(N,\overline{S}_{\delta,1})\ ,\ c_1 + \sum_{m=0}^{N-1} C(m,\overline{S}_{\delta,1})\}$ we find

$$|R_f(u,N)| \le \tilde{C}(N,\overline{S}_{\delta,1})e^{c_2|u|}, \quad \forall u \in \overline{S}_\delta.$$

Next we choose a bounded closed subsector $\overline{S} \subset S_\varepsilon$ and $\eta > 0$ is chosen such that $\overline{S} \subset S(d, \alpha + \pi - \varepsilon - 2\eta)$. By drawing a picture one can see that for $z \in \overline{S}$ fixed, a direction $\theta \in [d - (\alpha-\delta)/2\ ,\ d + (\alpha-\delta)/2]$ can be chosen such that

$$|\theta - \arg z| \le \frac{\pi}{2} - \eta.$$

It is easily seen that the Laplace transform of $u^N R_f(u,N)$ in direction θ equals $\xi^{N+1}R_{\mathcal{L}f}(\xi, N+1)$ if $|\theta - \arg\xi| < \frac{\pi}{2}$. Here $R_{\mathcal{L}f}$ is defined by

$$R_{\mathcal{L}f}(\xi, N+1) = \xi^{-N-1}\left((\mathcal{L}f)(\xi) - \sum_{m=1}^{N} a_{m-1}\Gamma(m)\xi^m\right).$$

Hence for all $N \geq 0$ we have (as $|\theta - \arg z| < \frac{\pi}{2}$)

$$\begin{aligned}
\left|z^{N+1} R_{\mathcal{L}f}(z, N+1)\right| &\leq \int_0^{\infty e^{i\theta}} |u|^N \, |R_f(u,N)| \, e^{-\mathcal{R}e\,(\frac{u}{z})} |\mathrm{d}u| \\
&\leq \tilde{C}(N, \overline{S}_{\delta,1}) \int_0^{\infty e^{i\theta}} |u|^N e^{c_2|u| - |\frac{u}{z}|\cos(\theta - \arg z)} |du| \\
&\leq \tilde{C}(N, \overline{S}_{\delta,1}) \int_0^{\infty e^{i\theta}} |u|^N e^{c_2|u| - |\frac{u}{z}|\cos(\frac{\pi}{2} - \eta)} |du|.
\end{aligned}$$

As we can integrate along straight lines the substitution $u = se^{i\theta}$ gives

$$\begin{aligned}
\left|z^{N+1} R_{\mathcal{L}f}(z, N+1)\right| &\leq \tilde{C}(N, \overline{S}_{\delta,1}) \int_0^{\infty} s^N e^{c_2 s - \frac{s}{|z|}\sin\eta} \mathrm{d}s \\
&= \tilde{C}(N, \overline{S}_{\delta,1}) |z|^{N+1} \int_0^{\infty} r^N e^{r(c_2|z| - \sin\eta)} \mathrm{d}r \\
&\leq \tilde{C}(N, \overline{S}_{\delta,1}) |z|^{N+1} \int_0^{\infty} r^N e^{r(c_2\rho_\epsilon - \sin\eta)} \mathrm{d}r.
\end{aligned}$$

By taking ρ_ε smaller (if necessary) we can achieve $c_2\rho_\varepsilon \leq \frac{1}{2}\sin\eta$ and thus

$$\begin{aligned}
\left|z^{N+1} R_{\mathcal{L}f}(z, N+1)\right| &\leq \tilde{C}(N, \overline{S}_{\delta,1}) |z|^{N+1} \int_0^{\infty} r^N e^{-\frac{1}{2} r \sin\eta} \mathrm{d}r \\
&= \tilde{C}(N, \overline{S}_{\delta,1}) \left(\frac{2}{\sin\eta}\right)^{N+1} |z|^{N+1} \int_0^{\infty} t^N e^{-t} \mathrm{d}t \\
&= \tilde{C}(N, \overline{S}_{\delta,1}) \left(\frac{2}{\sin\eta}\right)^{N+1} \Gamma(N+1) |z|^{N+1}.
\end{aligned}$$

The constant $\tilde{C}(N, \overline{S}_{\delta,1})(\frac{2}{\sin\eta})^{N+1}\Gamma(N+1)$ is independent of z and $z \in \overline{S}$ was chosen arbitrary, so for all $z \in \overline{S}$ we have

$$|R_{\mathcal{L}f}(z, N+1)| \leq \tilde{C}(N, \overline{S}_{\delta,1}) \left(\frac{2}{\sin\eta}\right)^{N+1} \Gamma(N+1).$$

The fact that this latter estimate holds for every non-negative integer N completes the proof. ∎

Our next theorem is about the relationship between Borel transformation and asymptotic expansion. As starting point we have a neighbourhood U of 0 in $S(d, \alpha + \pi)$, $\alpha > 0$, and an analytic function $g : U \to \mathbb{C}$ which is of order $g(z) = O(z^\delta)$ as $z \to 0$ on U, where $\delta \in \mathbb{R}$.
Moreover, we assume $g(z) \simeq \hat{g}(z) = \sum_{m=1}^{\infty} a_m z^m$ as $z \to 0$ on U. In section 1.3 we have defined a loop γ in U with which the Borel transform of g is defined. By deforming this loop (in U) we can assume

$$\gamma = \ell_1 \cup C \cup \ell_2,$$

with

$$\begin{cases} \ell_1 \text{ a straight line from } 0 \text{ to a point } re^{id_+} \in U, \text{ where } d_+ = d + \frac{\pi}{2} + \frac{\varepsilon}{2}; \\ \ell_2 \text{ a straight line from } re^{id_-} \text{ to } 0 \text{ where } d_- = d - \frac{\pi}{2} - \frac{\varepsilon}{2}; \\ \mathcal{C} \subset U \text{ a contour joining the points } re^{id_+} \text{ and } re^{id_-}. \end{cases}$$

Here $\varepsilon \in (0, \alpha)$. Note that $\mathcal{C}$ never contains the point 0. By choosing r small enough, we may assume $\mathcal{C}$ to be a part of the circle with centre 0 and radius r.

For $m \geq 1$ we then have

$$\begin{aligned} (\mathcal{B}_\gamma z^m)(u) &= \frac{1}{2\pi i} \int_\gamma z^m e^{\frac{u}{z}} \mathrm{d}\left(z^{-1}\right) \\ &= \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{u^m}{v^m} e^v u^{-1} \mathrm{d}v \\ &= u^{m-1} \cdot \frac{1}{2\pi i} \int_{\tilde{\gamma}} e^v v^{-m} \mathrm{d}v, \quad u \in S(d, \varepsilon). \end{aligned}$$

Using Hankels integral representation of the reciprocal Gamma function (cf. [4]) we see that the last integral equals $\Gamma(m)^{-1}$ and thus

$$(\mathcal{B}_\gamma z^m)(u) = \frac{u^{m-1}}{\Gamma(m)}.$$

We are now ready to prove the following

Theorem A.1.2 *Let U be a neighbourhood of 0 in $S(d, \alpha + \pi)$, $\alpha > 0$. Let $g : U \to \mathbb{C}$ be an analytic function of order $g(z) = O(z^\delta)$ as $z \to 0$ on U, where $\delta \in \mathbb{R}$.*
Moreover, we assume $g(z) \simeq \hat{g}(z) = \sum_{m=1}^\infty a_m z^m$ as $z \to 0$ on U.
Then we have

$$(\mathcal{B}g)(u) \simeq (\hat{\mathcal{B}}\hat{g})(u) \quad \text{as } u \to 0 \text{ on } S(d, \alpha).$$

Proof. Let us first consider the case $0 < \alpha < \pi$.
We choose a bounded closed subsector $\overline{S}_1 \subset S(d, \alpha)$. Let $\gamma = \ell_1 \cup \mathcal{C} \cup \ell_2$ be a loop in U as above. Following Balser in [1], chapter 2, we conclude that ε can be chosen arbitrary in $(0, \alpha)$. Let us choose $\varepsilon > 0$ such that $\overline{S}_1 \subsetneq S(d, \varepsilon) \subsetneq S(d, \alpha)$.
On the other hand there exists a bounded closed subsector $\overline{S}_2 \subset U \subset S(d, \alpha + \pi)$ such that $\gamma \subset \overline{S}_2 \cup \{0\}$. From $g(z) \simeq \hat{g}(z)$ as $z \to 0$ on U we deduce for all $N \geq 1$ the existence of a positive constant $C(N, \overline{S}_2)$ such that for all $z \in \overline{S}_2$

$$\left|z^N R_g(z, N)\right| = \left|g(z) - \sum_{m=1}^{N-1} a_m z^m\right| \leq C(N, \overline{S}_2)|z|^N. \tag{A.1}$$

But this estimate also holds for $z = 0$ (by taking the limit $z \to 0$ on $\overline{S}_2$, as $\lim_{\substack{z\to 0\\ z\in\overline{S}_2}} g(z) = a_0 = 0$), so we conclude (A.1) for all $z \in \gamma$.

From now on we write $\mathcal{B}$ instead of $\mathcal{B}_\gamma$.

It is easily seen that for all $N \geq 1$ and for all $u \in S(d,\varepsilon)$ we have

$$\left(\mathcal{B}z^N R_g(z,N)\right)(u) = (\mathcal{B}g)(u) - \sum_{m=1}^{N-1} \frac{a_m}{\Gamma(m)} u^{m-1} =: u^{N-1} R_{\mathcal{B}g}(u, N-1).$$

For all $N \geq 1$ and for all $u \in \overline{S}_1$ we thus have

$$\begin{aligned} \left|u^{N-1} R_{\mathcal{B}g}(u,N-1)\right| &= \left|\frac{1}{2\pi i}\int_\gamma z^N R_g(z,N) e^{\frac{u}{z}} \mathrm{d}\left(z^{-1}\right)\right| \\ &\leq \frac{1}{2\pi}\left\{|I_1| + |I_2| + |I_3|\right\}, \end{aligned}$$

where

$$\begin{cases} I_1 &= \int_{\ell_1} z^N R_g(z,N) e^{\frac{u}{z}} \mathrm{d}\left(z^{-1}\right); \\ I_2 &= \int_{C} z^N R_g(z,N) e^{\frac{u}{z}} \mathrm{d}\left(z^{-1}\right); \\ I_3 &= \int_{\ell_2} z^N R_g(z,N) e^{\frac{u}{z}} \mathrm{d}\left(z^{-1}\right). \end{cases}$$

There exists a positive η such that $\overline{S}_1 \subset \overline{S}(d, \varepsilon - 2\eta)$, where $\overline{S}(d, \varepsilon - 2\eta)$ is defined by $\overline{S}(d,\varepsilon - 2\eta) = \{z \in S(d,\varepsilon) \mid d - \varepsilon/2 + \eta \leq \arg z \leq d + \varepsilon/2 - \eta\}$. So $\forall z \in \ell_1,\ \ \forall u \in \overline{S}_1$ we have

$$\arg u - \arg z \in [-\pi/2 - \varepsilon + \eta, -\pi/2 - \eta] \subset [-(3\pi)/2 + \eta, -\pi/2 - \eta]$$

and $\forall z \in \ell_2,\ \ \forall u \in \overline{S}_1$

$$\arg u - \arg z \in [\pi/2 + \eta, \pi/2 + \varepsilon - \eta] \subset [\pi/2 + \eta, (3\pi)/2 - \eta].$$

We conclude the existence of a constant $c \in (0,1)$ such that

$$\cos(\arg u - \arg z) \leq -c, \quad \forall z \in \ell_1 \cup \ell_2, \quad \forall u \in \overline{S}_1.$$

From this we obtain for $i \in \{1,3\}$

$$\begin{aligned} |I_i| &\leq \int_{\ell_i} \left|z^N R_g(z,N) e^{\frac{u}{z}} z^{-2}\right| |\mathrm{d}z| \\ &\leq C(N,\overline{S}_2) \int_{\ell_i} e^{-c\left|\frac{u}{z}\right|} |z|^{N-2} |\mathrm{d}z| \\ &= C(N,\overline{S}_2) \int_0^r e^{-c\frac{|u|}{s}} s^{N-2} \mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} |I_2| &\leq \int_C \left| z^N R_g(z,N) e^{\frac{u}{z}} \right| \left| \mathrm{d}\left(z^{-1}\right) \right| \\ &= \int_C \left| z^{N-2} R_g(z,N) e^{\left|\frac{u}{z}\right| \cos(\arg u - \arg z)} \right| |\mathrm{d}z|. \end{aligned}$$

Using the fact $\cos x \leq 1$ for all $x \in \mathbb{R}$ we obtain, by using a parametrization of C,

$$\begin{aligned} |I_2| &\leq C(N, \overline{S}_2) \int_{d_-}^{d_+} r^{N-2} e^{\frac{|u|}{r}} r \mathrm{d}t \\ &= C(N, \overline{S}_2) \cdot (d_+ - d_-) \cdot r^{N-1} e^{\frac{|u|}{r}}. \end{aligned}$$

For $u \in \overline{S}_1$ fixed we have $\lim_{N\to\infty} \frac{|u|}{N} = 0$, so for large N, say $N \geq N_0$, we may choose γ in such a way that $r = \frac{|u|}{N}$. This gives

$$|I_2| \leq C(N, \overline{S}_2) \cdot (d_+ - d_-) \cdot |u|^{N-1} \frac{1}{N^{N-1}} e^N$$

and (after substituting $t = \frac{c|u|}{s}$)

$$\begin{aligned} |I_1|,\ |I_3| &\leq C(N, \overline{S}_2) \int_{cN}^{\infty} \left(\frac{c|u|}{t} \right)^{N-2} e^{-t} \cdot \frac{c|u|}{t^2} \mathrm{d}t \\ &= C(N, \overline{S}_2) c^{N-1} |u|^{N-1} \int_{cN}^{\infty} t^{-N} e^{-t} \mathrm{d}t \\ &\leq C(N, \overline{S}_2) c^{-1} N^{-N} |u|^{N-1} \int_0^{\infty} e^{-t} \mathrm{d}t \\ &= C(N, \overline{S}_2) c^{-1} N^{-N} |u|^{N-1}. \end{aligned}$$

So for all $N \geq N_0$ we have

$$|R_{\mathcal{B}g}(u, N-1)| \leq \frac{C(N, \overline{S}_2)}{2\pi} \left((d_+ - d_-) N^{1-N} e^N + 2c^{-1} N^{-N} \right).$$

Moreover,

$$R_{\mathcal{B}g}(u, N_0 - 2) \leq |u| R_{\mathcal{B}g}(u, N_0 - 1) + \frac{a_{N_0-1}}{\Gamma(N_0 - 1)}.$$

Using the fact that $a_n = \lim\limits_{\substack{z \to 0 \\ z \in \overline{S}_2}} R_g(z, n)$ $(n \geq 1)$ we see, by going backwards step by step, that to every $N \geq 0$ there exists a constant $\tilde{C}_N$ such that

$$|R_{\mathcal{B}g}(u, N)| \leq \tilde{C}_N.$$

From the fact that $\tilde{C}_N$ can be chosen independent of u, we obtain this latter estimate for every $u \in \overline{S}_1$. So $(\mathcal{B}g)(u) \simeq (\hat{\mathcal{B}}\hat{g})(u)$ as $u \to 0$ on $S(d, \alpha)$.

In the case $\alpha \geq \pi$ we can split $S(d,\alpha)$ in finitely many sectors of opening between 0 and π, say $S(d,\alpha) = S^{(1)} \cup S^{(2)} \cup \cdots \cup S^{(m)}$. An arbitrary bounded closed subsector $\overline{S} \subset S(d,\alpha)$ can be written as $\overline{S} = \overline{S}^{(1)} \cup \overline{S}^{(2)} \cup \cdots \cup \overline{S}^{(m)}$, where each $\overline{S}^{(j)}$ is a bounded closed subsector of $S^{(j)}$. To each $\overline{S}^{(j)}$ and each $N \geq 0$ there exists a constant $C(N, \overline{S}^{(j)})$ such that

$$|R_{\mathcal{B}g}(u,N)| \leq C(N, \overline{S}^{(j)}), \quad \forall u \in \overline{S}^{(j)}$$

and thus, with $C(N,\overline{S}) = \max_{j \in \{1,2,\ldots,m\}} C(N, \overline{S}^{(j)})$, we have

$$|R_{\mathcal{B}g}(u,N)| \leq C(N,\overline{S}), \quad \forall u \in \overline{S}.$$

This completes the theorem. ∎

A.2 Two properties of the Laplace transform

In this report we have introduced the Laplace transform in a somewhat different way than usual. Suppose f is a function for which the Laplace transform in direction θ is defined. Then with $F(t) = f(te^{i\theta})$ we have

$$\begin{aligned}(\mathcal{L}_\theta f)(z) &= \int_0^{\infty e^{i\theta}} f(u) e^{-\frac{u}{z}} \mathrm{d}u \\ &= \int_0^{\infty} f(te^{i\theta}) e^{-\frac{t}{z}e^{i\theta}} e^{i\theta} \mathrm{d}t \\ &= e^{i\theta} (\mathsf{L}F)(\frac{1}{z} e^{i\theta}).\end{aligned}$$

Note that for a function $g \in L^1_{loc}([0,\infty))$, which is of order $O(e^{pt})$ as $t \to \infty$ for some $p \in \mathbb{R}_{\geq 0}$, the (usual) Laplace transform

$$(\mathsf{L}g)(z) = \int_0^\infty e^{-zt} g(t) \mathrm{d}t$$

is defined and analytic in the halfplane $\mathcal{R}e\ z > p$.

Theorem A.2.1 *Let $\mathsf{L}f$ and $\mathsf{L}g$ be defined and analytic in the halfplane $\mathcal{R}e\ z > p$, then*

$$(f * g)(t) := \int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau$$

is defined a.e. (on $\mathbb{R}_{\geq 0}$), $\mathsf{L}(f * g)$ is defined and converges absolutely on $\mathcal{R}e\ z > p$. Moreover, we have

$$\mathsf{L}(f * g) = \mathsf{L}f \cdot \mathsf{L}g$$

on $\mathcal{R}e\ z > p$.

Proof. Take z_0 with $\mathcal{R}e\ z_0 > p$ arbitrary, then by assumption

$$\int_0^\infty \left|e^{-z_0 t} f(t)\right| \mathrm{d}t < \infty \quad \text{and} \quad \int_0^\infty \left|e^{-z_0 t} g(t)\right| \mathrm{d}t < \infty.$$

Without loss of generality f and g can be defined by $f(t) = g(t) = 0$ for all $t < 0$, so

$$\begin{aligned} & \int_0^\infty \left|\left((e^{-z_0 t} f) * (e^{-z_0 t} g)\right)(t)\right| \mathrm{d}t \\ \leq & \int_0^\infty \int_0^t \left|e^{-z_0 \tau} f(\tau) \cdot e^{-z_0 (t-\tau)} g(t-\tau)\right| \mathrm{d}\tau \mathrm{d}t \\ = & \int_0^\infty \int_0^\infty \left|e^{-z_0 \tau} f(\tau) \cdot e^{-z_0 (t-\tau)} g(t-\tau)\right| \mathrm{d}\tau \mathrm{d}t. \end{aligned}$$

Changing the order of integration is allowed for non-negative functions (Fubini's theorem), so

$$\begin{aligned} & \int_0^\infty \left|\left((e^{-z_0 t} f) * (e^{-z_0 t} g)\right)(t)\right| \mathrm{d}t \\ \leq & \int_0^\infty \left|e^{-z_0 \tau} f(\tau)\right| \mathrm{d}\tau \int_0^\infty \left|e^{-z_0 (t-\tau)} g(t-\tau)\right| \mathrm{d}t \\ = & \int_0^\infty \left|e^{-z_0 \tau} f(\tau)\right| \mathrm{d}\tau \int_0^\infty \left|e^{-z_0 s} g(s)\right| \mathrm{d}s \\ < & \ \infty. \end{aligned}$$

Hence $e^{-z_0 t} f * e^{-z_0 t} g$ is defined a.e. and belongs to $L^1([0, \infty))$.
From this one also concludes

$$\int_0^\infty \left|e^{-z_0 t} (f * g)(t)\right| \mathrm{d}t = \int_0^\infty \left|\left((e^{-z_0 t} f) * (e^{-z_0 t} g)\right)(t)\right| \mathrm{d}t \ < \ \infty$$

and thus we see that $\mathsf{L}(f * g)(z)$ converges absolutely for $z = z_0$. This implies that $\mathsf{L}(f * g)(z)$ converges absolutely for all $z \in \mathbb{C}$ with $\mathcal{R}e\ z \geq \mathcal{R}e\ z_0$ (cf. [4]). Because z_0 was chosen arbitrary in the halfplane $\mathcal{R}e\ z > p$ we conclude that $\mathsf{L}(f * g)$ converges absolutely on $\mathcal{R}e\ z > p$.

Furthermore

$$\begin{aligned} & \int_0^\infty e^{-z_0 t} (f * g)(t) \mathrm{d}t \\ = & \int_0^\infty e^{-z_0 t} \int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau \mathrm{d}t \\ = & \int_0^\infty e^{-z_0 t} \int_0^\infty f(\tau) g(t-\tau) \mathrm{d}\tau \mathrm{d}t. \end{aligned}$$

Because the function $(t,\tau) \mapsto e^{-z_0 t} f(\tau)g(t-\tau)$ is Lebesgue measurable and $\int_0^\infty \int_0^\infty \left|e^{-z_0 t} f(\tau)g(t-\tau)\right| \mathrm{d}\tau \mathrm{d}t < \infty$, we can apply Fubini's theorem to obtain

$$\begin{aligned}
\big(\mathsf{L}(f*g)\big)(z_0) &= \int_0^\infty e^{-z_0 t}(f*g)(t)\mathrm{d}t \\
&= \int_0^\infty e^{-z_0 t}\int_0^\infty f(\tau)g(t-\tau)\mathrm{d}\tau\mathrm{d}t \\
&= \int_0^\infty e^{-z_0 \tau} f(\tau)\mathrm{d}\tau \int_0^\infty e^{-z_0(t-\tau)}g(t-\tau)\mathrm{d}t \\
&= \int_0^\infty e^{-z_0 \tau} f(\tau)\mathrm{d}\tau \int_0^\infty e^{-z_0 s}g(s)\mathrm{d}s \\
&= (\mathsf{L}f)(z_0)(\mathsf{L}g)(z_0).
\end{aligned}$$

Because z_0 was chosen arbitrary, we have

$$\mathsf{L}(f*g) = \mathsf{L}f \cdot \mathsf{L}g$$

on the halfplane $\mathcal{R}e\ z > p$. ■

Now let us return to the Laplace transform $\mathcal{L}_\theta$ in direction θ. Let f and g be functions for which the Laplace transforms are defined, then

$$\begin{aligned}
\mathcal{L}_\theta(f*g)(z) &= e^{i\theta}\int_0^\infty e^{-\frac{1}{z}e^{i\theta}t}(f*g)(te^{i\theta})\mathrm{d}t \\
&= e^{i\theta}\int_0^\infty e^{-\frac{1}{z}e^{i\theta}t}\int_0^{te^{i\theta}} f(\tau)g(te^{i\theta}-\tau)\mathrm{d}\tau\mathrm{d}t \\
&= e^{2i\theta}\int_0^\infty e^{-\frac{1}{z}e^{i\theta}t}\int_0^{t} f(se^{i\theta})g(e^{i\theta}(t-s))\mathrm{d}s\mathrm{d}t \\
&= e^{2i\theta}\int_0^\infty e^{-\frac{1}{z}e^{i\theta}t}(F*G)(t)\mathrm{d}t,
\end{aligned}$$

where F and G are defined as before.
We conclude

$$\mathcal{L}_\theta(f*g)(z) = e^{2i\theta}\mathsf{L}(F*G)(\frac{1}{z}e^{i\theta}).$$

From this one easily obtains

$$\mathcal{L}_\theta(f*g) = \mathcal{L}_\theta f \cdot \mathcal{L}_\theta g$$

and the identity and uniqueness theorem then implies

$$\mathcal{L}(f*g) = \mathcal{L}f \cdot \mathcal{L}g$$

on a neighbourhood of 0 in $S(d, \alpha+\pi)$.

Theorem A.2.2 *Let $f : S(d,\alpha) \to \mathbb{C}$ be analytic and of exponential growth of order ≤ 1, whereas $f(u) = O(u^{\varepsilon-1})$ as $u \to 0$ on $S(d,\alpha)$ for some $\varepsilon > 0$. Then we have*

$$z^2 \frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}f)(z) = (\mathcal{L}[uf(u)])(z)$$

on a neighbourhood of 0 *in* $S(d,\alpha+\pi)$.

Proof. It follows straight from the definition that $\mathcal{L}[uf(u)]$ is well defined on a neighbourhood of 0 in $S(d,\alpha+\pi)$. Moreover, if $\theta \in (d-\frac{\alpha}{2}, d+\frac{\alpha}{2})$ then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}_\theta f)(z) &= \int_0^{\infty e^{i\theta}} f(u)e^{-\frac{u}{z}}\frac{u}{z^2}\mathrm{d}u \\ &= \frac{1}{z^2}\int_0^{\infty e^{i\theta}} uf(u)e^{-\frac{u}{z}}\mathrm{d}u.\end{aligned}$$

Hence

$$z^2\frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}_\theta f)(z) = (\mathcal{L}_\theta[uf(u)])(z).$$

Using the identity and uniqueness theorem one obtains

$$z^2\frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}f)(z) = \mathcal{L}[uf(u)])(z)$$

on a neighbourhood of 0 in $S(d,\alpha+\pi)$. ■

Now let U is a neighbourhood of 0 in $S(d,\alpha+\pi)$, $\alpha > 0$. Let $g : U \to \mathbb{C}$ be analytic and of order $O(z^\delta)$ as $z \to 0$ on U for some $\delta \in \mathbb{R}$, then with $f(u) := (\mathcal{B}g)(u)$ we have

$$u(\mathcal{B}g)(u) = uf(u) = (\mathcal{B}[\mathcal{L}[uf]])(u) = \mathcal{B}\Big(z^2\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{L}f\Big)(u) = \mathcal{B}\Big(z^2\frac{\mathrm{d}}{\mathrm{d}z}g\Big)(u).$$

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