The Hopf Saddle-Node Bifurcation

Khairul Saleh
Supervisor: H.W. Broer

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Supervisor: H.W. Broer

Rijksuniversiteit Groningen
Department of Mathematics
Postbus 800
9700 AV Groningen

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Abstract

This paper concerns the normal form of the three-dimensional vector fields with one zero and a purely imaginary pair of eigenvalues at an equilibrium. It is studied by methods from pertubation theory. Application of the Implicit Function Theorem is disscused to investigate the persistence of equilibrium and periodic orbit. We use KAM theory to study persistence of parallel dynamics on 2-torus.

Contents

1 Introduction

This paper deals with a local analysis of three-dimensional vector fields in neighbourhood of an equilibrium. We assume that the linearized vector field at the equilibrium has one zero and a purely imaginary pair of eigenvalues. To study the dynamical properties of this vector field, we bring it into normal form, using a standard normal form procedure. The resulting system can be studied by methods from perturbation theory. The truncation of the normal form at second order constitutes the 'unperturbed' part and the higher order terms the perturbation. Chow (et al.) [7] and Guckenheimer and Holmes [8] distinguish four cases in the unfolding of unperturbed system. We here restrict to two cases, since the other two are similar. As in perturbation theory, we shall discuss the persistence of certain dynamical properties, that are known for the unperturbed case. The Implicit function theorem and a KAM theorem will be used to investigate the persistence of certain dynamical properties of unperturbed system.

1.1 Setting of the Problem

We consider the vector field

$$\begin{array}{l} \dot{x} = f(x), \quad x \in \mathbb{R}^3, \quad f \in C^\infty(\mathbb{R}^3, \mathbb{R}^3), \\ f(0) = 0, \end{array} \tag{1}$$

with linear part

$$Df(0) = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega > 0. \tag{2}$$

Note that this linear part has one pure imaginary pair and one simple zero eigenvalues.

1.2 Outline

The main aim of this paper is to study the dynamical properties of system (1) under generic conditions. We apply a standard normal form procedure to the system. First, the attention is focussed on the normal form truncation at second order. After introducing two unfolding parameters , we rescale the variables and the time to simplify the normal form. Then the family of vector fields both at the central codimension two singularity, and outside this singularity will be discussed. Next, we consider the higher order terms as a perturbation. The persistence of dynamical properties under such a perturbation will be investigated.

2 Normal Form

Consider a vector field

$$\dot{x} = Ax + f(x), \quad x \in \mathbb{R}^3, \tag{3}$$

where A is linear, $f(0) = 0$, and $Df(0) = 0$. The matrix A induces a map $ad_m A : H^m(\mathbb{R}^3) \to H^m(\mathbb{R}^3)$, where $H^m(\mathbb{R}^3)$ is the linear space of vector fields whose coefficients are homogeneous polynomials of degree m. Indeed, for $Y \in H^m$, the map $ad_m A$ is defined by

$$ad_m A(Y) = [Y, L] = DLY - DYL,$$

where L is the linear vector field $L : x \to Ax$. Let $B^m := im(ad_m A)$, the image of the map ad_m in $H_m(\mathbb{R}^3)$. Then for any complement G^m, in the sense that $B^m \bigoplus G^m = H^m(\mathbb{R}^3)$, we define the corresponding notion of 'simpleness' by requiring the homogeneous part of degree m to be in G^m. We now quote a well-known theorem, compare Guckenheimer and Holmes [8], section 3.3 or Broer [2], section 1.3.

Theorem 2.1 *Let X be a C^∞ vector field, defined in the neighbourhood of $0 \in \mathbb{R}^3$, with $X(0) = 0$ and $D_0 X = A$. Also let $N \in \mathbb{N}$ be given. Then there exists, near $0 \in \mathbb{R}^3$, an analytic change of coordinates $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, with $\Phi(0) = 0$ such that*

$$\Phi_* X(y) = Ay + g_2(y) + \cdots + g_N(y) + O(|y|^{N+1}), \tag{4}$$

with $g_m \in G_m$, for all $m = 2, 3, \cdots, N$.

We apply theorem 2.1 to system (1) for $N = 3$, so that the normal form in cylindrical coordinates reads

$$\begin{aligned} \dot{r} &= a_1 rz + a_2 r^3 + a_3 rz^2 + O(|r,z|^4), \\ \dot{z} &= b_1 r^2 + b_2 z^2 + b_3 r^2 z + b_4 z^3 + O(|r,z|^4), \\ \dot{\theta} &= \omega + O(|r,z|^2). \end{aligned} \tag{5}$$

A versal deformation of system (5) (see Arnol'd [1] and Chow, Li and Wang [6], section 2.9) is given by

$$\begin{aligned} \dot{r} &= \mu_1 r + a_1 rz + a_2 r^3 + a_3 rz^2 + O(|r,z|^4), \\ \dot{z} &= \mu_2 + b_1 r^2 + b_2 z^2 + b_3 r^2 z + b_4 z^3 + O(|r,z|^4), \\ \dot{\theta} &= \omega + O(|r,z|^2). \end{aligned} \tag{6}$$

where μ_1 and μ_2 are parameters.

3 Unperturbed System

In this section, we consider as the unperturbed system the normal form truncation at second order. So, we truncate (6) at $O(|r,z|^2)$ and remove the $\dot{\theta}$-term (since it decoupled from the radial component r) to obtain the planar system

$$\begin{aligned}\dot{r} &= \mu_1 r + a_1 r z,\\ \dot{z} &= \mu_2 + b_1 r^2 + b_2 z^2,\end{aligned} \tag{7}$$

which was shown by Takens [12] that $a_1, b_1, b_2 \neq 0$ and $b_2 - a_1 \neq 0$. In this case we can rescale to remove of the two coefficients. Letting $\bar{r} = \alpha r$, $\bar{z} = \beta z$, with $\beta = -b_2$ and $\alpha = -\sqrt{|b_1 b_2|}$ and dropping the bars, (7) then yields

$$\begin{aligned}\dot{r} &= \mu_1 r + arz,\\ \dot{z} &= \mu_2 + br^2 - z^2; \qquad b = \pm 1,\end{aligned} \tag{8}$$

where $a = -a_1/b_2$.
Now, we consider system (8) for $(\mu_1, \mu_2) = (0,0)$:

$$\begin{aligned}\dot{r} &= arz,\\ \dot{z} &= br^2 - z^2; \qquad b = \pm 1,\end{aligned} \tag{9}$$

The coefficient a can be either positive or negative (assuming that $a \neq 0$). we get the topological classification of (9) by using this information. For background information regarding the topological classification see, e.g., Arnol'd [1], Chapter 3, and Palis-de Melo [11], Chapter 2. In order to determine the invariant lines $z = kr$ for the vector field, we substitute $z = kr$ into (9). The slopes k then satisty

$$\frac{dz}{dr} = k = \frac{br^2 - (kr)^2}{ar(kr)} = \frac{b - k^2}{ak}, \tag{10}$$

or

$$k = \sqrt{b/(a+1)}. \tag{11}$$

Note that the z-axis($r = 0$) is always invariant, and that other invariant lines $z = kr$ exist if $b/(a+1) > 0$. There are six distinct topological types for this normal form (see Guckenheimer-Holmes [8]) as given in fig. 1.

- Case I: $b = 1, a > 0$.

- Case IIa: $b = 1, a \in (-1, 0)$,

- Case IIb: $b = 1, a \leq -1$,
- Case III: $b = -1, a > 0$,
- Case IVa: $b = -1, a \in (-1, 0)$,
- Case IVb: $b = -1, a < -1$.

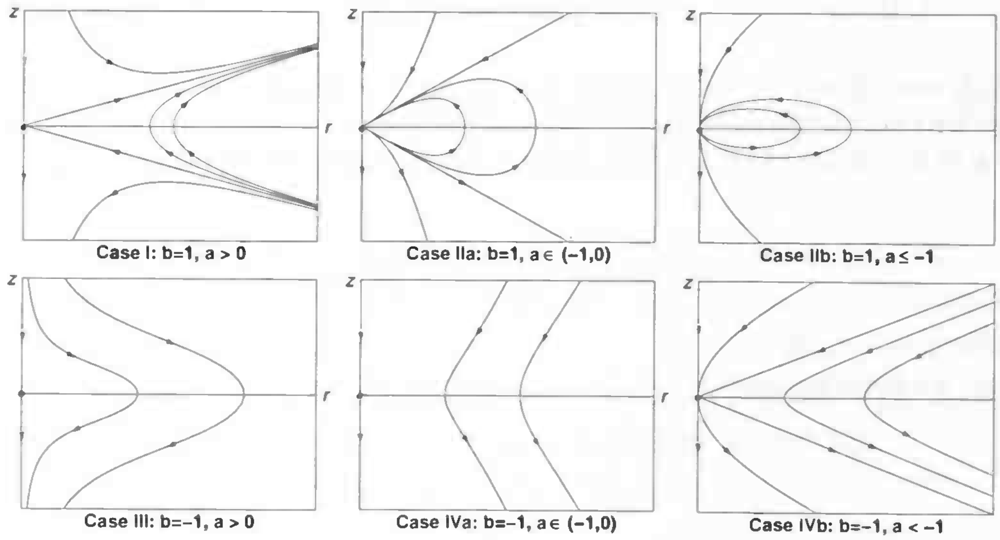

Figure 1: Phase portraits in the (r, z) half plane $(r \geq 0)$ for the 2^{nd}-order normal form truncation (9).

3.1 Inside The Center Manifold

In this subsection we discuss system (8) with $(\mu_1, \mu_2) \neq (0, 0)$. We find that the system has equilibria at

$$(r, z) = (0, \pm\sqrt{\mu_2}), \text{ for } \mu_2 \geq 0,$$

and

$$(r, z) = \left(\sqrt{\frac{\mu_1^2}{a^2} - \mu_2}, -\frac{\mu_1}{a}\right), \text{ for } \mu_1^2 \geq a^2\mu_2, \; b = 1;$$

$$(r, z) = \left(\sqrt{\mu_2 - \frac{\mu_1^2}{a^2}}, -\frac{\mu_1}{a}\right), \text{ for } \mu_1^2 \leq a^2\mu_2, \; b = -1.$$

3.1.1 Case I : $b = 1$, $a > 0$.

The linearized part of (8) at equilibria $(r, z) = (0, \pm\sqrt{\mu_2})$ is diagonal with eigenvalues $\mu_1 \pm a\sqrt{\mu_2}$ and $\mp 2\sqrt{\mu_2}$, respectively. At the equilibrium $(r, z) = (\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a)$, this linearized part reads

$$\begin{bmatrix} 0 & \sqrt{\mu_1^2 - a^2\mu_2} \\ \frac{2}{a}\sqrt{\mu_1^2 - a^2\mu_2} & \frac{2\mu_1}{a} \end{bmatrix}. \tag{12}$$

In case $\mu_1 \neq 0$, the invariant line $r = 0$ is a center eigenspace. The reduced system, which determines stability on this line, is given by

$$\dot{z} = \mu_2 - z^2,$$

showing that a saddle-node bifurcation occurs as μ_2 passes through zero. Next, we want to verify that, crossing $\mu_2 = \mu_1^2/a^2$ with decreasing μ_2, and for $\mu_1 \neq 0$, symmetric pitchfork bifurcations occur at the equilibrium $(r, z) = (0, \sqrt{\mu_2})$ $(\mu_1 < 0)$ and $(r, z) = (0, -\sqrt{\mu_2})$ $(\mu_1 > 0)$. To see this, we translate the equilibrium to the origin. Set $\mu_1 \neq 0$ and let ε be a parameter defined by $\sqrt{\mu_2} = |\mu_1|/a - \varepsilon$. We consider $\mu_1 > 0$ and the corresponding bifurcation at $(r, z) = (0, -\sqrt{\mu_2})$. Let $z = -\sqrt{\mu_2} + \xi$, then (7) becomes

$$\begin{aligned} \dot{r} &= \varepsilon a r + a r \xi, \\ \dot{\xi} &= 2\left(\tfrac{|\mu_1|}{a} - \varepsilon\right)\xi + r^2 - \xi^2, \end{aligned} \tag{13}$$

We look for a center manifold

$$\xi = h(r, \varepsilon) = \alpha r^2 + \beta r\varepsilon + \gamma\varepsilon^2 + O(3) \tag{14}$$

where $O(3)$ means terms of orders r^3, $r^2\varepsilon$, $r\varepsilon^2$, and ε^3. For a detailed discussion of center manifolds, see Chow (et al.) [7], section 1.3, Guckenheimer and Holmes [8], section 2.3, or Takens [12]. Substituting (14) in (13), we obtain

$$2\left(\frac{|\mu_1|}{a}\right)(\alpha r^2 + \beta r\varepsilon + \gamma\varepsilon^2) + r^2 = O(3).$$

So we find that

$$\alpha = \frac{-a}{2|\mu_1|} < 0, \qquad \beta = \gamma = 0,$$

for small ε, and thus $\xi = -\delta r^2 + h.o.t.$, where $\delta = -\alpha$. The reduced system, which determines stability, is given by

$$\dot{r} = \varepsilon a r - a\delta r^3 + h.o.t., \tag{15}$$

showing that a supercritical pitchfork bifurcation occurs as ε increases through zero. Since $|\mu_1|/a > 0$, in this case the center manifold repels nearby solutions. Now we obtain the stability of equilibria when μ_1 and μ_2 change as follows:

- For $\mu_2 > 0$ and $\mu_2 \geq \mu_1^2/a^2$, two equilibria exist, namely $(0, \pm\sqrt{\mu_2})$. The two equilibria are saddles (see fig. 2a).

- For $\mu_2 < 0$, we have one equilibrium $((\sqrt{\mu_1^2/a^2} - \mu_2, -\mu_1/a))$, which is a saddle (see fig. 2e).

- For $\mu_1 < 0$, and $0 < \mu_2 < \mu_1^2/a^2$, the equilibrium $(r, z) = (0, +\sqrt{\mu_2})$ is a sink. The other equilibria $(0, -\sqrt{\mu_2})$ and $(\sqrt{\mu_1^2/a^2} - \mu_2, -\mu_1/a)$ are saddles (see fig. 2c).

- For $\mu_1 > 0$, and $0 < \mu_2 < \mu_1^2/a^2$, we have one source $((0, -\sqrt{\mu_2}))$ and two saddle points, namely, $(0, \sqrt{\mu_2})$ and $(\sqrt{\mu_1^2/a^2} - \mu_2, -\mu_1/a)$ (see fig. 2b).

- For $\mu_2 = 0$, there are two equilibria $(0, 0)$ and $(\sqrt{\mu_1^2/a^2} - \mu_2, -\mu_1/a)$. Both of these are saddles when $\mu_1 \neq 0$ (see fig. 3d and 3f).

3.1.2 Case IIa-IIb : $b = 1$, $a < 0$.

Computations similar to those above show that saddle-node and pitchfork bifurcations occur at the lines $\mu_2 = 0$ and $\mu_2 = \mu_1^2/a^2$ in the parameter plane. The behavior of the third equilibrium $(r, z) = (\sqrt{\mu_1^2/a^2} - \mu_2, -\mu_1/a)$ is rather different in this case. Here the linearized part has eigenvalues

$$\lambda_{1,2} = \frac{\mu_1}{a} \pm \sqrt{\frac{\mu_1^2}{a^2} + \frac{2}{a}(\mu_1^2 - a^2\mu_2)},$$

and therefore the equilibrium is a sink for $\mu_1 > 0$, since $\mu_1/a < 0$, and a source for $\mu_1 < 0$ (the eigenvalues are complex conjugate for $\mu_2 < \mu_1^2(2 + 1/a)/2a^2$). These eigenvalues will pass the imaginary axis when $\mu_1 = 0$ and $\mu_2 < 0$. So, passing transversely through $\mu_1 = 0$ for $\mu_2 < 0$, a Hopf bifurcation at this equilibrium. At least cubic order must be included in the normal form to determine the dynamics of this secondary bifurcation occurs (see Guckenheimer and Holmes [8], or Kuznetsov [10]). We will discuss it in section 3.2.3. In fact, for $\mu_1 = 0$ the system

$$\begin{aligned} \dot{r} &= arz \\ \dot{z} &= \mu_2 + r^2 - z^2, \end{aligned} \tag{16}$$

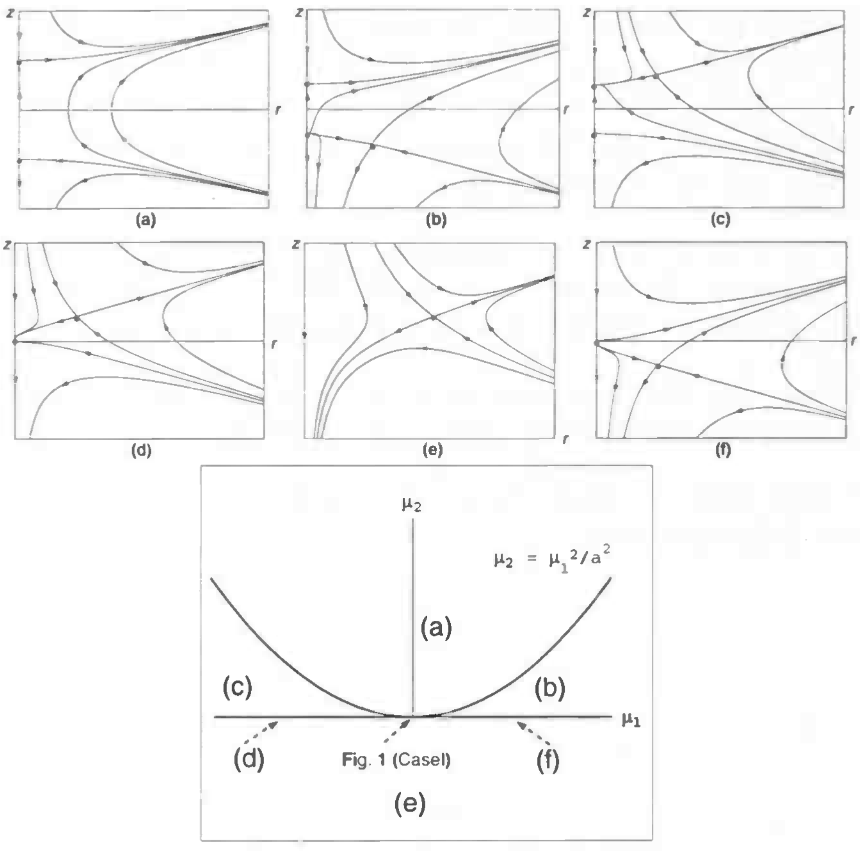

Figure 2: Bifurcation set and phase portraits of system (8) for case I; $b = 1$, $a > 0$.

is integrable, since the function

$$F(r, z) = \frac{a}{2} r^{2/a} \left[\mu_2 + \frac{r^2}{1+a} - z^2 \right], \ a \neq -1$$

is constant on solution curves. Thus, this function is a first integral of system (16), and orbits correspond with level curves $F(r, z) = \text{constant}$ (see fig. 3h). For more details concerning integrability of vector fields we refer to Verhulst [13].

Now we can determine the stability of equilibria, when μ_1 and μ_2 change, as follows:

- For $\mu_2 > \mu_1^2/a^2$ and $\mu_1 > 0$, two equilibria occur at $(r, z) = (0, \pm\sqrt{\mu_2})$. From the eigenvalues of the linearized part at these equilibria, we verify that the equilibrium $(0, +\sqrt{\mu_2})$ is a sink, and the equilibrium $(0, -\sqrt{\mu_2})$ is a source. See fig. 3a.

- For $0 < \mu_2 < \mu_1^2/a^2$ there are three equilibria. If $\mu_1 > 0$, the equilibrium $(r, z) = (\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a)$ is a sink, $(0, \sqrt{\mu_2}$ is a saddle, and $(0, \sqrt{\mu_2}$ is a source. If $\mu_1 < 0$, the equilibrium $(r, z) = (\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a)$ is a source, $(0, \sqrt{\mu_2})$ is a sink, and $(0, \sqrt{\mu_2})$ is a saddle. See fig. 3c and fig. 3b.

- For $\mu_2 = 0$, we have two equilibria. The equilibrium $(0, 0)$ is a saddle in this case. The point $(r, z) = (\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a)$ is a sink when $\mu_1 > 0$, and is a source when $\mu_1 < 0$ (see fig. 3d and 3e).

- For $\mu_2 < 0$, the fixed point $(r, z) = (\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a)$ is a sink when $\mu_1 > 0$ and a source when $\mu_1 < 0$, (see fig. 3f and 3g).

For the other cases, see Chow (et al.) [7], Guckenheimer and Holmes [8], or Kuznetsov [10], section 8.5.

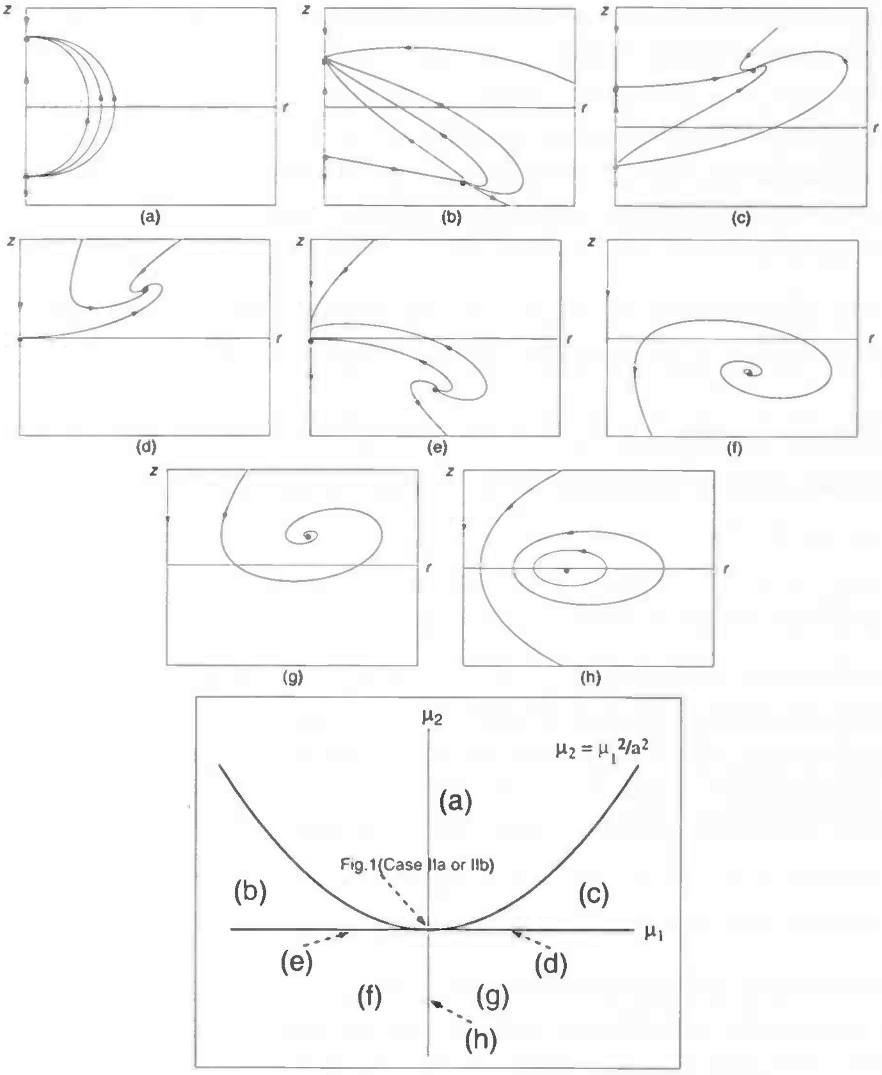

Figure 3: Bifurcation set and phase portraits for case IIa-IIb; $b = 1$, $a < 0$.

3.2 The Three-dimensional System

In this subsection, the interpretation of the above results for the full three-dimensional vector field is discussed. We start by restoring the θ-term to (8):

$$\begin{aligned} \dot{r} &= \mu_1 r + arz, \\ \dot{z} &= \mu_2 + br^2 - z^2, \\ \dot{\theta} &= \omega. \end{aligned} \tag{17}$$

This system is considered as unperturbed system in $3D$ and the higher order terms as perturbation. Note that the equilibria of the planar system (8) on the z-axis ($r = 0$) are equilibria of this system. The other equilibria of system (8) correspond to periodic orbits in the full three-dimensional system. To see this, let (r_0, z_0) ($r_0 > 0$) be an equilibrium of system (8). Then the corresponding orbit in the three-dimensional system is $(r_0, \omega t, z_0)$ (modulo 2π), which is periodic with period $2\pi/\omega$. So we conclude that:

- The fixed points $(0, \pm\sqrt{\mu_2})$ of the planar system correspond to fixed points $(0, 0, \pm\sqrt{\mu_2})$ in the three-dimensional case.
- The fixed point $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a\right)$ corresponds to a periodic orbit in the three-dimensional case, namely $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$, with period $2\pi/\omega$.

3.2.1 Case I: $b = 1, a > 0$

The saddle-node bifurcations occurring on $\mu_2 = 0$ remain saddle-nodes, since the orbit on z-axis does not depend on θ . The symmetric pitchfork bifurcations on $\mu_2 = \mu_1^2/a^2$ of system (8) correspond to Hopf bifurcations in the three-dimensional vector field. To see this, we consider again system (15), which for three-dimensional vector field becomes

$$\begin{aligned}\dot{r} &= \varepsilon a r - a\delta r^3 + O(|r|^5),\\ \dot{\theta} &= \omega.\end{aligned}$$

At $\varepsilon = 0$ we find a Hopf bifurcation (see Broer [2]).
Using the results of subsection 3.1.1., we obtain the stability types of the equilibria and the periodic orbit in the three-dimensional case, depending on μ_1 and μ_2, as follows. Observe that since $\omega > 0$, the fixed points and periodic orbits in the three-dimensional case have the same stability type with the corresponding fixed points in the planar system.

- For $\mu_2 > 0$ and $\mu_2 \geq \mu_1^2/a^2$, the fixed points $(0, 0, \pm\sqrt{\mu_2})$ are saddles, since the corresponding fixed points in the planar system are saddles.
- For $\mu_2 < 0$, the periodic orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ (the second coordinate is modulo 2π) is hyperbolic.
- For $\mu_1 < 0$ and $0 < \mu_2 < \mu_1^2/a^2$, the fixed points $(0, 0, \sqrt{\mu_2})$ is a sink, the fixed point $(0, 0, -\sqrt{\mu_2})$ is a saddle, and the periodic orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ is hyperbolic.

- For $\mu_1 > 0$ and $0 < \mu_2 < \mu_1^2/a^2$, the fixed point $\left(0, 0, -\sqrt{\mu_2}\right)$ is a source, the fixed point $\left(0, 0, +\sqrt{\mu_2}\right)$ is a saddle, and the periodic orbit $\left(\sqrt{\mu_1^2/a^2} - \mu_2, \omega t, -\mu_1/a\right)$ is hyperbolic.

- For $\mu_2 = 0$ and $\mu_1 \neq 0$, the fixed point $(0, 0, 0)$ is a saddle, and the periodic orbit $\left(\sqrt{\mu_1^2/a^2} - \mu_2, \omega t, -\mu_1/a\right)$ is hyperbolic.

3.2.2 Case IIa-IIb: $b = 1, a < 0$

Similar to case I, we have saddle-node and Hopf bifurcations in the three-dimensional vector field.
Using the results of subsection 3.1.2., we obtain the stability types of the equilibria and the periodic orbit in the three-dimensional case, depending on μ_1 and μ_2 , as follows.

- For $\mu_2 > \mu_1^2/a^2$ and $\mu_1 > 0$, two fixed point occur at $\left(0, 0, \pm\sqrt{\mu_2}\right)$. The fixed point $\left(0, 0, +\sqrt{\mu_2}\right)$ is a sink, and the fixed point $\left(0, 0, -\sqrt{\mu_2}\right)$ is a source.

- For $0 < \mu_2 < \mu_1^2/a^2$ there are two fixed points and one periodic orbit. If $\mu_1 > 0$, the fixed point $\left(0, 0, \sqrt{\mu_2}\right)$ is a saddle, $\left(0, 0, \sqrt{\mu_2}\right)$ is a source, and the periodic orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ is an attractor. If $\mu_1 < 0$, the periodic orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ is a repellor, and the fixed points $\left(0, 0, \pm\sqrt{\mu_2}\right)$ become a sink and a saddle respectively.

- For $\mu_2 = 0$, we have one fixed point and one periodic orbit. The fixed point $(0, 0, 0)$ is a saddle, and the orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ is an attractor when $\mu_1 > 0$, and is a repellor when $\mu_1 < 0$.

- For $\mu_2 < 0$, the periodic orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ is an attractor when $\mu_1 > 0$ and a repellor when $\mu_1 < 0$.

- For $\mu_1 = 0$ and $\mu_2 < 0$, the periodic orbit $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, \omega t, -\mu_1/a\right)$ is hyperbolic.

Next, we want to verify the presence of Hopf bifurcations (which are occuring in the planar system) in the three-dimensional vector field. In fact, if the planar system has a closed orbit, then the corresponding three-dimensional vector field has an invariant 2-torus, which is encircling the z-axis. From this information we verify that the peiodic orbit of a Hopf bifurcation, creating an invariant 2-torus in the three-dimensional case.

3.2.3 The Limit Cycle

Considering the third order terms in the planar system, there exists limit cycle for case IIa-IIb and case III (see Guckenheimer and Holmes [8], or Kuznetsov [7], section 8.5). The area of occuring limit cycle for these cases is sketched by fig. 4c-4d.

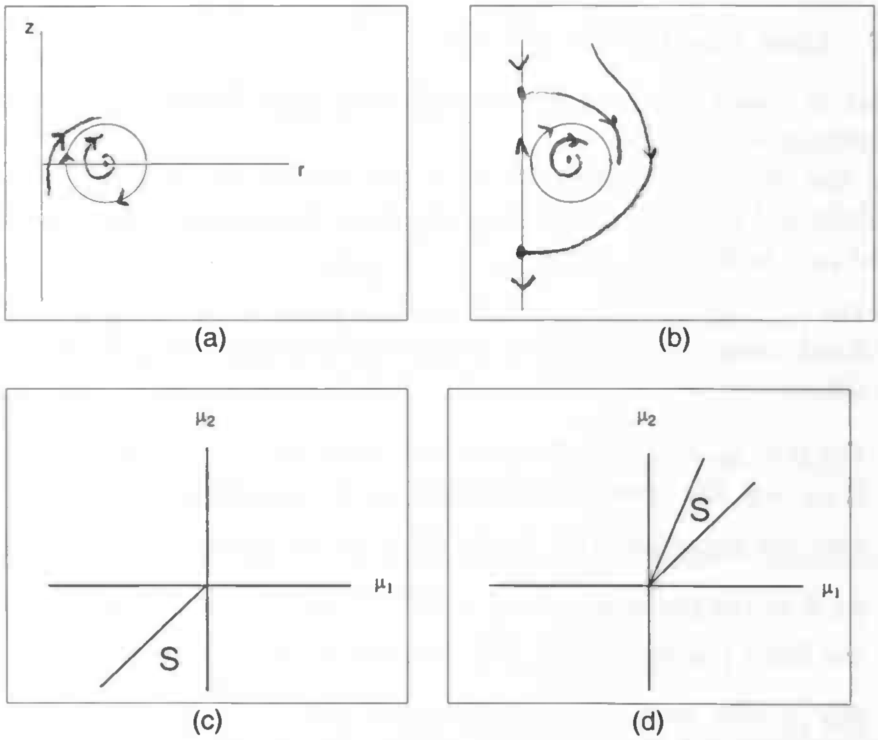

Figure 4: *Above*: The attracting Limit cycle: (a) Case IIa-IIb; (b) Case III. *Below*: The area of occuring limit cycle: (c) Case IIa-IIb; (d) Case III.

4 Perturbed System

In this section, we investigate the persistence of dynamical properties in the previous section under the perturbation given by the higher order terms of the normal form for small values of $|r, z|$.

4.1 Persistence of equilibria and periodic orbits

We rewrite our perturbed system as

$$\begin{aligned} \dot{r} &= \mu_1 r + arz + O(|r,z|^3), \\ \dot{z} &= \mu_2 + br^2 - z^2 + O(|r,z|^3, \\ \dot{\theta} &= \omega + O(|r,z|^2), \end{aligned} \tag{18}$$

Now, we study the persistence of equilibria and the periodic orbits under such a perturbation, using the Implicit Function Theorem.

4.1.1 Persistence of equilibria

In the (x, y, z) coordinates, system (18) reads

$$\begin{aligned} \dot{x} &= \mu_1 x - \omega y + axz + O(|x,y,z|^3), \\ \dot{y} &= \mu_1 y + \omega x + ayz + O(|x,y,z|^3), \\ \dot{z} &= \mu_2 + b(x^2+y^2) - z^2 + O(|x,y,z|^3). \end{aligned} \tag{19}$$

The linearized part of the unperturbed system at the equilibria $(0, 0, \pm\sqrt{\mu_2})$ is

$$\begin{pmatrix} \mu_1 + a \pm \sqrt{\mu_2} & -\omega & 0 \\ \omega & a \pm \sqrt{\mu_2} & 0 \\ 0 & 0 & \mp\sqrt{\mu_2} \end{pmatrix}, \tag{20}$$

which has determinant $(\mu_1 + a\sqrt{\mu_2})^2(\mp 2\sqrt{\mu_2}) \mp 2\omega^2\sqrt{\mu_2} \neq 0$ for $\mu_2 \neq 0$. So, according to the Implicit Function Theorem, for $|x, y, z|$ small enough, the fixed points $(x, y, z) = (0, 0, \pm\sqrt{\mu_2})$ are persistent for $\mu_2 \neq 0$.
Remark: The point (0,0,0) as an equilibrium is not persistent (so certainly we cannot apply the Implicit Function Theorem, indeed, the conditions do not hold at this equilibrium). We can verify that for $\mu_2 < 0$, there is no equilibrium in the neighbourhood of (0,0,0).

4.1.2 Persistence of periodic orbit

Consider a two-dimensional Poincaré cross section

$$\Sigma = \{(r, \theta, z)| \ \ \theta = 0; \ r > 0; \ r \text{ and } |z| \text{ sufficiently small}\},$$

for the full three-dimensional vector field (18). Let $P : \Sigma \to \Sigma$ be a corresponding, locally defined, Poincaré-mapping. So the point $\left(\sqrt{\mu_1^2/a^2 - \mu_2}, -\mu_1/a\right)$, which corresponds to a periodic orbit in the three-dimensional case, is a fixed point of P. We obtain

$$\begin{aligned} D_{(r,z)}P &= D_{(r,z)}\Phi^{2\pi}(r,z,0) \\ &= \exp(2\pi M), \end{aligned} \tag{21}$$

at this point, where Φ is the flow of the unperturbed system and

$$M = \begin{pmatrix} 0 & \sqrt{\mu_1^2 - a^2\mu_2} \\ \frac{2}{a}\sqrt{\mu_1^2 - a^2\mu_2} & \frac{2\mu_1}{a} \end{pmatrix}, \tag{22}$$

with eigenvalues

$$\lambda_{1,2} = \frac{\mu_1}{a} \pm \sqrt{\frac{\mu_1^2}{a^2} + \frac{2}{a}(\mu_1^2 - a^2\mu_2)}.$$

Observe that $D_{(r,z)}P_0$ has no eigenvalue 1 if M has no eigenvalue on the imaginary axis. So, by the Implicit Function Theorem, for $|r, z|$ sufficiently small, we have

- For $\mu_1 \neq 0$, the periodic orbit persists in all cases.
- For $\mu_1 = 0$, the periodic orbit persists if $\mu_2 \leq 0$ for case I, and if $\mu_2 \geq 0$ for case II.

4.2 KAM Theory

In this section, the unperturbed system is the normal form truncation at third order, and the higher order terms of the normal form as perturbation. We find that the resulting doubly periodic flow on the 2-torus has one 'fast' frequency (ω) associated with the angular variable θ, and a slow frequency ($\omega_1(\mu_1, \mu_2)$) associated with the limit cycle of the planar system. The hyperbolicity of the limit cycle now leads to normal hyperbolicity of the 2-torus. According to the center manifold theorem, see Hirch (et al.) [9], it follows that the 2-torus as invariant manifold is persistent. This means that the system, for $|r, z|$ small enough, still has a smooth invariant 2-torus. Next, using KAM theory, we investigate how far the parallel dynamics is persistent under such a perturbation.
An expected perturbation problem inside the center manifold is

$$\begin{aligned} \dot{\theta}_1 &= \omega + l.o.t(|\mu_1, \mu_2|) + h.o.t(|\mu_1, \mu_2|) \\ \dot{\theta}_2 &= \omega_1(\mu_1, \mu_2) + l.o.t(|\mu_1, \mu_2|) + h.o.t(|\mu_1, \mu_2|), \end{aligned} \tag{23}$$

where $h.o.t(|\mu_1, \mu_2|)$ is the small perturbation. It is not easy to compute $\omega_1(\mu_1, \mu_2)$, since an eliptic integral is involved in computing. The $l.o.t(|\mu_1, \mu_2|)$-terms can be determined by averaging method. Let $\Omega_1(\mu_1, \mu_2) = \omega + l.o.t(|\mu_1, \mu_2|)$ and $\Omega_2(\mu_1, \mu_2) = \omega_1(\mu_1, \mu_2) + l.o.t(|\mu_1, \mu_2|)$. The frequency map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ now is given by $(\mu_1, \mu_2) \longmapsto (\Omega_1(\mu_1, \mu_2), \Omega_2(\mu_1, \mu_2))$. The invariant 2-torus with parallel dynamics of the unperturbed system is *Diophantine* if for some constants $\tau > 1$ and $\gamma > 0$ the corresponding frequency vector (Ω_1, Ω_2) satisfies the following infinite system of innequalities:

$$|k_1\Omega_1 + k_2\Omega_2| \geq \gamma|k|^{-\tau}$$

for all $k = (k_1, k_2) \in \mathbb{Z}^2 - \{0\}$, where $|k| = |k_1| + |k_2|$. The frequency map Ω can be determined If we know the $l.o.t(|\mu_1, \mu_2|)$-terms. In this case we guess that the frequency map has maximal rank at a given point (μ_{10}, μ_{20}). We conclude quasi-periodic stability under preservation of the frequency ratios. The set of all frequency vectors (Ω_1, Ω_2) that are Diophantine in the above sense (denoted by $\mathbb{R}^2{}_{\tau,\gamma}$) creates a Cantor set of lines in the (Ω_1, Ω_2)-plane and has positive Lebesque measure (see Broer (et al.) [5]).

Conjecture 4.1 *The set $S_{\tau,\gamma} = \{(\mu_1, \mu_2) \in S \mid \Omega(\mu_1, \mu_2) \in \mathbb{R}^2{}_{\tau,\gamma}\}$ creates a Cantor set of lines inside the sector S with quasi-periodic 2-tori. This Cantor set has positive lebesque measure and the origin is a lebesque density point.*

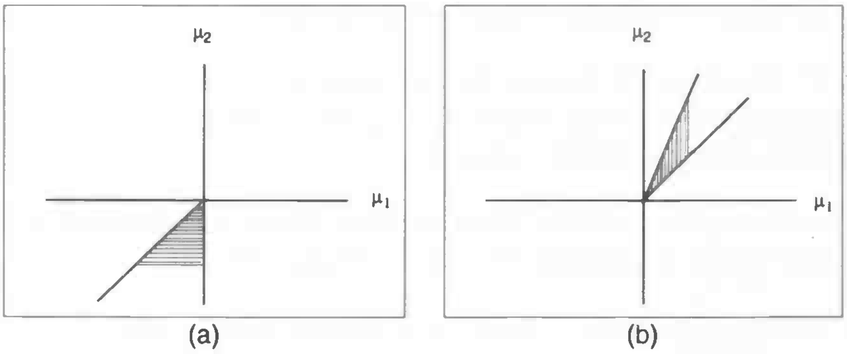

Figure 5: Cantor set of lines inside the sections of fig. 4c-4d: (a) Cantor set of horizontal lines for case IIa-IIb; (b) Cantor set of vertical lines for case III.

Remark: The parallel dynamics of the unperturbed vector field on the $\mathbb{T}^2$ can be translated to parallel dynamics of the Poincaré map inside the 2-torus, which in turn can be expressed by conjugacies to rigit rotations. These conjugacies between the maps translate back to equivalences between the vector fields. So, we can study our problem by means of the Poincaré map $P : \mathbb{T}^1 \to \mathbb{T}^1$.

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