Regular polytopes: symmetry groups and 3D-sections by Alicia Boole Stott.

Irene Polo Blanco

Supervisors: Prof.Dr. M. van der Put
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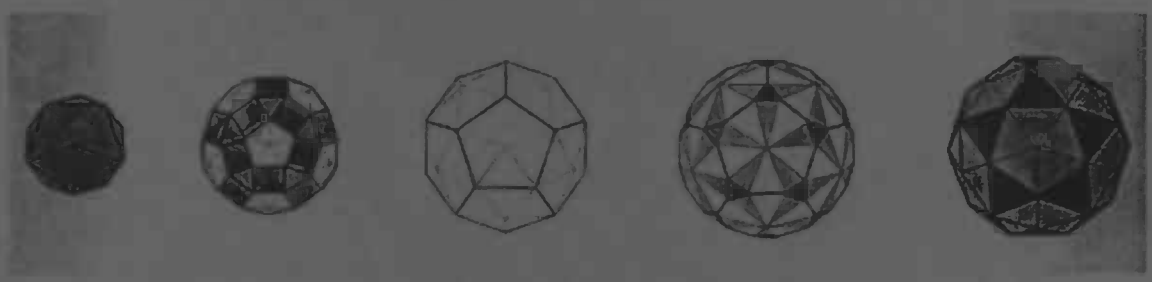

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Mathematics

Masters thesis

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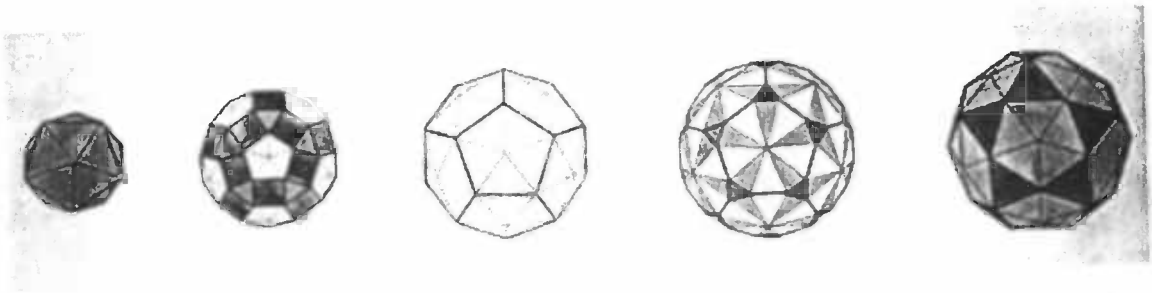

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Contents

1 Regular Polygons, Polyhedrons and Polytopes

1.1 In the plane

Definition. By a *polygon* we mean one figure in the plane which is bounded by a number of segments connected to each other in their end points. For any polygon, we define a *flag* (P, L) to be the figure consisting of a vertex P, and an edge L containing P. Then, the polygon is said to be *regular* if the following property holds: If (P, L) and $(P'L')$ are two flags, then there exists a unique symmetry σ of $\mathbf{R}^2$ such that,

$$(\sigma(P), \sigma(L)) = (P', L').$$

Using the previous property, it is easy to see that every edge of a regular polygon must have the same length, since it can be mapped to another by a symmetry which preserves the distances. We will now introduce the following definition:

Definition. A regular *k-gon* is one regular polygon with k vertices and k edges, where $k \geq 3$.

Let's consider the angles α_k, from the centre of a k-gon to two consecutive vertices, and β_k, the angle in the common vertex of two consecutive edges. By the symmetry condition, it is possible to map an edge and its respective angle α_k to the one next to it, and that must be done k times until the edge and angle are back to their initial place, that is:

$$k\alpha_k = 2\pi.$$

Since the triangle with vertices the center and two consecutive vertices is an isosceles triangle, the angle β_k is α_k's supplement. Then, we have just proved the following Proposition,

Proposition 1. *Let R be a k-gon, then,*

$$\alpha_k = \frac{2\pi}{k} \quad \beta_k = \pi - \frac{2\pi}{k}.$$

1.2 In the 3-dimensional space

Definition. By a *polyhedron* in the 3-dimensional space we mean one subset of $\mathbf{R}^3$ bounded by polygons and such that every edge of a polygon is also an edge of a unique other polygon. For any polyhedron, we define a *flag* (P, L, V) to be the figure consisting of a vertex P, an edge L containing P, and a face V containing L. Then, we call the polyhedron *regular* if the following symmetries exist:
If (P, L, V) and (P', L', V') are two flags of the regular polyhedron, then there exists a unique symmetry σ of $\mathbf{R}^3$ such that,

$$(\sigma(P), \sigma(L), \sigma(V)) = (P', L', V').$$

It immediately follows from this property, that all the faces of a regular polyhedron are regular polygons, and that there are the same number of edges meeting at every vertex. Then, if P is a vertex of the polygon, there are r edges which have P as an end point, where r is a fixed number. Let's call $L_1, \ldots, L_r$ the edges and m_i the mid points of L_i $\forall i = 1, \ldots, r$.

Proposition 2. *Let R be a regular polyhedron and let P be any vertex of R. If $L_1, \ldots L_r$ are the edges which have P as an end point, and $m_1, \ldots m_r$ their mid points respectively, then m_i are all in the same circle, $\forall i = 1, \ldots, r$.*

Proof. Let's call O the centre of R and let's consider the rotation with axe $\overline{P0}$. This symmetry clearly fixes the mid points m_i (since it fixes every vertex) and also the centre O. Then, $\overline{m_k O} = c$ and $\overline{m_k P} = c'$ are both constant $\forall k = 1, \ldots, r$. By intersecting the sphere with centre O and radius c with the sphere with centre P and radius c' we obtain a circle that contains $m_1, \ldots, m_r$. □

The figure with vertices $m_1, \ldots m_r$ is a regular polygon called the *link* of P. Let's now consider a regular polyhedron R with p-gonal faces and q-gonal links. We will denote R by $\{p, q\}$. Since in every vertex end q angles β_p and the sum of these angles must be less than 2π, then

$$q\beta_p < 2\pi \Leftrightarrow \beta_p < \frac{2\pi}{q} = \alpha_q.$$

We have just proven the following proposition:

Proposition 3. *If R is a regular polyhedron represented by $\{p, q\}$, then*

$$\beta_p < \alpha_q.$$

Let's see how many convex regular polyhedron there are. By the last property,

$$\beta_p < \alpha_q \Leftrightarrow \pi - \frac{2\pi}{p} < \frac{2\pi}{q} \Leftrightarrow \pi\left(1 - \frac{2}{p}\right) < \pi\frac{2}{q} \Leftrightarrow \frac{1}{2} < \frac{1}{p} + \frac{1}{q}.$$

Then, the only possible values for p and q are:

$$\{3,3\}\ ,\ \{3,4\}\ ,\ \{4,3\}\ ,\ \{3,5\}\ ,\ \{5,3\}.$$

We have just proven that there are at most five regular convex polyhedrons. It is easy to check that they all exist (see Coxeter, 1948). The five regular polyhedrons are called the *Platonic Solids*.
Let's now consider R' the figure with vertices the mid points of the edges of R. R' is called the *dual* of R. By the symmetry condition, R' is also a regular polyhedron (for a precise proof, see Coxeter, ref 1948). Let us suppose that v, e and f are the number of vertices, edges and faces of R respectively, and v', e' and f' the number of vertices, edges and faces of R'. By the definition of R', we also know that $v' = f$, and $f' = v$. By Euler's formula,

$$\left\{\begin{array}{c} v - e + f = 2 \\ v' - e' + f' = 2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} v - e + f = 2 \\ f - e' + v = 2 \end{array}\right. \Leftrightarrow e = e'$$

That is, if R is represented by $\{p,q\}$, then R' is represented by $\{q,p\}$.
From a regular polyhedron $\{p,q\}$, it is possible to calculate v, e and f from the values p and q. We will do it in the following way: There are p edges in every face, and each edge belongs to two faces, that is,

$$e = \frac{pf}{2} \Leftrightarrow f = \frac{2e}{p}. \tag{1}$$

Similarly, there are q edges meeting in every vertex and 2 vertices in every edge, that is,

$$e = \frac{qv}{2} \Leftrightarrow v = \frac{2e}{q}. \tag{2}$$

Substituting these values in Euler's formula, we obtain:

$$\frac{2e}{q} - e + \frac{2e}{p} = 2 \Leftrightarrow e\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{p}\right) = 1$$

Then, by (1) and (2) we have:

$$v = \frac{2}{q\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{p}\right)}, \quad e = \frac{1}{\frac{1}{q} - \frac{1}{2} + \frac{1}{p}}, \quad f = \frac{2}{p\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{p}\right)}.$$

Thus, we obtain the following table for the five Platonic Solids:

Polyhedron	Symbol	v	e	f
Tetrahedron	$\{3,3\}$	4	6	4
Cube	$\{4,3\}$	8	12	6
Octahedron	$\{3,4\}$	6	12	8
Dodecahedron	$\{5,3\}$	20	30	12
Icosahedron	$\{3,5\}$	12	30	20

Let's now consider the angle in R from the centre of the polyhedron to two consecutive vertices, and let's call it $\alpha_{\{p,q\}}$. Then, the following formula holds:

$$\cos\left(\frac{\alpha_{\{p,q\}}}{2}\right) = \frac{\cos \pi/p}{\sin \pi/q}$$

1.3 In the 4-dimensional space

Definition. By a *polytope* in 4-dimensional space, we mean one subset of $\mathbf{R}^4$ bounded by polyhedrons such that every face of a polyhedron is also a face of a unique other polyhedron. For any polytope, we define a *flag* (P, L, V, F) to be the figure consisting of a vertex P, an edge L containing P, a face V containing L, and a 3D-face F containing V. The polytope is said to be *regular* if the following symmetries exist:

If (P, L, V, R) and (P', L', V', R') are two flags of R regular polytope, then there exists a unique symmetry σ of $\mathbf{R}^4$ such that,

$$(\sigma(P), \sigma(L), \sigma(V), \sigma(R)) = (P', L', V', R')$$

It follows from this property that all the '3D-faces' of a regular polytope are regular polyhedrons having the same symbol.

Let R be a regular polytope. We will represent R by $\{p, q, r\}$ if the 3D-faces of R have symbol $\{p, q\}$ and if r 3D-faces meet in each edge. We can also define the dual of a polytope in n dimensions as follows:

Definition. The *dual* of a regular polytope in n dimensions is another polytope having one vertex in the centre of each $n-1$-face of the polytope we started with.

For example, polygons in 2 dimensions are all *self-dual*, that is, the dual of a polygon is again the same polygon. In 3 dimensions, the tetrahedron is self dual, the cube and octahedron are duals of each other, and the dodecahedron and icosahedron are also duals of each other (as we saw in the previous section).

By connecting the corresponding links of each $\{p,q\}$ meeting in an edge L, we obtain a regular polyhedron called the *link* of L. Since there are r faces of type $\{q\}$ meeting in each vertex, the link of R can be represented by $\{q,r\}$.

Since the dual of $\{p,q,r\}$ is $\{r,p,q\}$, $\{q,r\}$ is also a regular polyhedron. By applying the 3-dimensional theory to them, we conclude that the only possible values for the polytope $\{p,q,r\}$ are:

$\{3,3,3\}$, $\{3,3,4\}$, $\{3,3,5\}$, $\{3,4,3\}$, $\{3,5,3\}$, $\{4,3,3\}$, $\{4,3,4\}$, $\{4,3,5\}$, $\{5,3,3\}$, $\{5,3,4\}$, $\{5,3,5\}$.

These are 11 polytopes, 5 of which don't exist. To see that, we will first prove the following Proposition:

Proposition 4. *Let R be a regular polytope represented by $\{p,q,r\}$, then*

$$\beta_p < \alpha_{\{q,r\}}$$

where β_p is the angle in two consecutive vertex in a regular p-gon, and $\alpha_{\{q,r\}}$ is the angle in a regular polyhedron $\{q,r\}$ from its centre to two consecutive vertices.

The prove of this Proposition is given in (Coxeter, 1948). Doing the corresponding calculations, we conclude that the only regular polytopes are:

$$\{3,3,3\},\{4,3,3\},\{3,3,4\},\{3,4,3\},\{5,3,3\},\{3,3,5\}$$

called the hypertetrahedron, hypercube, hyperoctahedron, 24-cell, 120-cell, and 600-cell respectively.

It is easy to see by applying Euler's formula, that the dual of the regular polytope represented by $\{p,q,r\}$ is $\{r,q,p\}$. That is, the hypertetrahedron

and the 24-cell are self-duals, the hypercube and the hyperoctahedron are duals of each other, and the 120-cell and the 600-cell are also duals of each other. In the following table, we show the six regular polytopes:

Polytope	Symbol	v	e	f	c
Hypertetrahedron	$\{3,3,3\}$	5	10	10	5
Hypercube	$\{4,3,3\}$	16	32	24	8
Hyperoctahedron	$\{3,3,4\}$	8	24	32	16
24-cell	$\{3,4,3\}$	24	96	96	24
120-cell	$\{5,3,3\}$	600	1200	720	120
600-cell	$\{3,3,5\}$	120	720	1200	600

In higher dimensions there are only 3 regular polytopes, which are the equivalent of the tetrahedron, cube and octahedron in 3 dimensions. They are usually called the n-simplex, n-cube, and n-crosspolytope respectively, where n denotes the dimension. The corresponding symbols are:

$$\{3,\ldots,3\},\{3,\ldots,3,4\},\{4,3,\ldots,3\}$$

2 Symmetry group of the Platonic Solids

In the following, we will try to find the symmetry group of a given regular polyhedron, more particularly, the group of symmetries that preserve the orientation: the rotation group. Since every rotation induces a rotation in the *inscribed* polyhedron, the rotation group of a regular polyhedron and its dual are equal. Then, we need only to study the rotation group of the tetrahedron, the cube and the dodecahedron. In each case, we will apply the following Proposition:

Proposition 5. *Let R be a regular polytope in n dimensions and G its symmetry group. If $(P, L, F, V, \ldots)$ is any flag of R, then*

$$|G| = \#P \cdot \#L \cdot \#F \cdot \#V\# \cdots$$

where $\#P$ is the number of vertices of R, $\#L$ the number of edges meeting in a vertex, $\#F$ the number of faces meeting in an edge, $\#V$ the number of 3-dimensional faces meeting in a face, and so on.

Proof. This follows immediately from the definition of regular polytope. □

2.1 Tetrahedron {3,3}

Let $O_{\{3,3\}}$ be the group of symmetries of the tetrahedron. If (P, L, F) is a flag in $\{3, 3\}$, by proposition 5 we know that:

$$|O_{\{3,3\}}| = \#P \cdot \#L \cdot \#F = 4 \cdot 3 \cdot 2 = 24.$$

Let $X = \{v_1, v_2, v_3, v_4\}$ be the set of vertices. Clearly, $O_{\{3,3\}}$ acts on X. In this action, for every v_i, v_j in X, there exists a symmetry s in $O_{\{3,3\}}$ such that $s(v_i) = v_j$, $s(v_j) = v_i$ and $s(v_k) = v_k$ $\forall k \neq i, j$ (namely, the symmetry with respect to the plane generated by $0, v_{k_1}, v_{k_2}$ with $k_1, k_2 \neq i, j$. Since $|O_{\{3,3\}}| = |\Sigma_4| = 24$, we have that $O_{\{3,3\}} \simeq \Sigma_4$.

If $SO_{\{3,3\}}$ is the rotation group, $SO_{\{3,3\}} \leq O_{\{3,3\}}$ with $|O_{\{3,3\}} : SO_{\{3,3\}}| = 2$, that is, $|SO_{\{3,3\}}| = 12$. For all $r \in SO_{\{3,3\}}$, r preserves the orientation $\Leftrightarrow$ r is an even permutation $\Leftrightarrow$ $r \in A_4$.

2.2 Cube {4,3} and Octahedron {3,4}

Let $O_{\{4,3\}}$ be now the group of symmetries of the cube, and (P, L, F) a flag in it. Again by Proposition 5,

$$|O_{\{4,3\}}| = \#P \cdot \#L \cdot \#F = 8 \cdot 3 \cdot 2 = 48.$$

Let $X = \{d_1, d_2, d_3, d_4\}$ be the set of diagonals of the cube. We will consider now the rotation group $SO_{\{4,3\}}$. By the symmetry property, $SO_{\{4,3\}}$ acts on X in the following way: For any two diagonals d_i, d_j there exists a symmetry $s \in SO_{\{4,3\}}$ such that s interchanges d_i and d_j and fixes the rest of the diagonals (namely, the rotation of 180 degrees). Since $|SO_{\{4,3\}}| = |\Sigma_4| = 24$, we conclude that $SO_{\{4,3\}} \simeq \Sigma_4$. The antipodal map is a symmetry which inverts the orientation, and it is central, so $O_{\{4,3\}} \simeq \Sigma_4 \times C_2$.

Since the octahedron $\{3, 4\}$ is the dual of the cube, it has the same symmetry group.

2.3 Dodecahedron {5,3} and Icosahedron {3,5}

We consider $O_{\{5,3\}}$ to be the group of symmetries of the dodecahedron, and we calculate its order as before: $|O_{\{5,3\}}| = 20 \cdot 3 \cdot 2 = 120$. We can consider five cubes inscribed in the dodecahedron, the edges of the cube being diagonals of the dodecahedron's faces. Let $X = \{c_1, c_2, c_3, c_4, c_5\}$ be the set of these cubes. For any $c_i, c_j \in X$, there exists a symmetry $s \in O_{5,3}$ such that $s(c_i) = c_j$, $s(c_j) = c_i$ and $s(c_k) = c_k$ $\forall k \neq i, j$. Since $|O_{\{5,3\}}| = |\Sigma_5|$, we have that $O_{\{5,3\}} \simeq \Sigma_5$. It's known that the only normal subgroup of Σ_5 of index 2 is A_5, so we can conclude that $SO_{\{5,3\}} \simeq A_5$.

3 Root systems and Coxeter groups

To be able to understand the symmetry group of the six regular polytopes, we will first need several concepts. The symmetry groups of the hypertetrahedron $\{3,3,3\}$ and the hypercube $\{4,3,3\}$ can be understood by using root systems. We will give the following definitions:

Definition. 1. Let V be a real euclidean space and $\alpha \in V - \{0\}$. A *symmetry with vector* α is an automorphism s_α of V which sends α to $-\alpha$ and fixes the hyperplane H_α orthogonal to α.

2. A finite subset R of a vector space $V - \{0\}$ is called a *root system* if the following conditions hold:

   (i) R is finite and $\text{span}(R) = V$.

   (ii) For all $\alpha \in R$ there exists a symmetry with vector α, s_α, such that $s_\alpha(R) = R$.

   (iii) For all α, $\beta \in R$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

3. Let S be a subset of R. S is called a *base for* R if the following properties are satisfied:

   (i) S is a basis for V

   (ii) For every $\alpha \in R$, α can be written as

$$\alpha = \sum_{s \in S} n_s \cdot s$$

   where n_s are integers with the same sign $\forall s \in S$

It can be shown that for every root system R, there exists a base for it. It is also known that every root system is uniquely represented by its Dynkin diagram (See Serre, 1987)

Definition. The group $W \leq GL(V)$ generated by the symmetries s_α, with $\alpha \in R$, is called *the Weyl group of* R. By choosing a bilinear form $(\ ,\ )$ on V, we can describe every symmetry s_α as
follows:

$$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha \quad \forall x \in V.$$

It can be proved that $W = \{s_\alpha \mid \alpha \in S\}$ where S denotes a base of R. We can now give a more general definition:

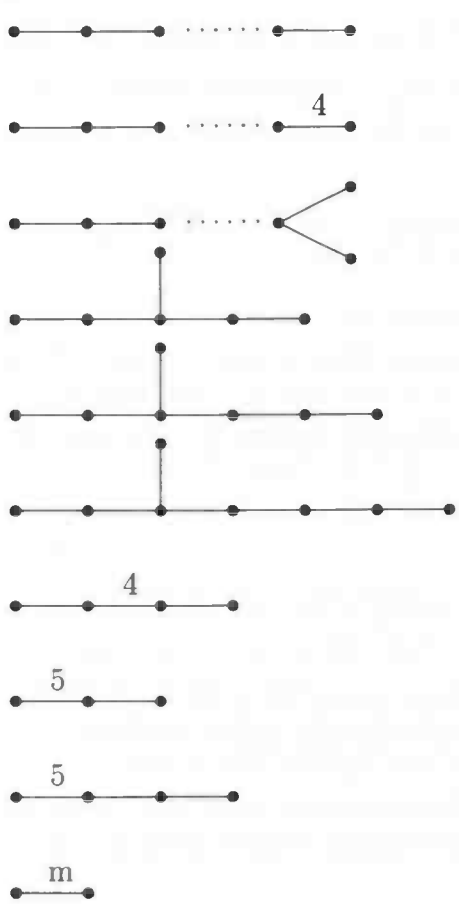
Definition. A group W generated by reflections is called a *Coxeter group* if the following relations are hold:

$$W =< s_1, \ldots, s_n | (s_i s_j)^{m_{ij}} = 1 \ \forall i \neq j > .$$

We assume that $m_{ij} = m_{ji} \geq 1$ and that $m_{ij} = 1 \Leftrightarrow i = j$.

Every Coxeter group W can be represented by its *Coxeter graph.* If W is finite, the corresponding diagram will be an edge-labelled graph with vertex the set of generators $s_1, \ldots, s_n$ and an edge joining two vertices s_i, s_j whenever $m_{ij} \geq 3$. The edge will be labelled with m_{ij} if $m_{ij} > 3$. If the resulting diagram is connected, W is said to be *irreducible.*

All finite irreducible Coxeter groups have been classified, and have the following Coxeter graphs:



$A_n \ n \geq 1$

$B_n \ n \geq 2$

$D_n \ n \geq 4$

E_6

E_7

E_7

F_4

H_3

H_4

$I_2(m)$

In the following, we will consider some of the last Coxeter graphs. Then, we will construct the root systems whose Weyl groups are represented by those graphs (by identifying the roots of the Dynking diagram with the generators of the Coxeter graph). Finally, we will try to relate these Weyl groups with the symmetry groups of our 4-dimensional polytopes. With that purpose, we will first give the following definition:

Definition. Let $\{x_1, \ldots, x_n\}$ denote a basis in $\mathbf{R}^n$. A *lattice* in R^n is a set

$$\mathbf{Z}x_1 + \cdots + \mathbf{Z}x_n$$

consisting of all integer linear combinations of the basis $\{x_1, \ldots, x_n\}$

Let us consider (α, β) to be the standard inner product $\forall \alpha, \beta \in \mathbf{R}^n$. We will begin with the Coxeter graph of type A_n:

3.1 A_n for $n \geq 1$

Let $L_{n+1} = \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_{n+1}$ denote the lattice in $\mathbf{R}^{n+1}$ with standard basis $\{e_1, \ldots, e_{n+1}\}$. Let V be the hyperplane

$$V = \{(y_1, \ldots, y_{n+1}) \in \mathbf{R}^{n+1} \mid \sum_{i=1}^{n+1} y_i = 0\}.$$

Let us consider the vectors $R = \{\alpha \in V \cap L_{n+1} \text{ with } (\alpha, \alpha) = 2\}$. It is easy to check that this vectors form a root system. R consists of the vectors: $e_i - e_j, 1 \leq i \neq j \leq n+1$. We will now consider the base $S = \{e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}\}$ for the root system R. The corresponding Coxeter graph looks as follows:

$A_n, \ n \geq 1$

To see what the Weyl group W is, we will look at the symmetries with vector $e_i - e_j$. For all $e_i - e_j \in S$, and $1 \leq k \leq n+1$ we have,

$$s_{e_i - e_j}(e_k) = e_k - (e_k, e_i - e_j) \cdot (e_i - e_j) = \begin{cases} e_i & \text{if } k = j \\ e_j & \text{if } k = i \\ e_k & \text{if } k \neq i, j. \end{cases}$$

That is, $W \simeq \Sigma_{n+1}$. Let's consider $\{e_1, \ldots, e_{n+1}\}$ to be the set of vertices of the n-simplex. We have just seen that W is the symmetry group of the n-simplex $\{3, \ldots, 3\}$.

3.2 B_n for $n \geq 2$

Let R now be the set of vectors $\alpha \in \mathbf{R}^n$ with $(\alpha, \alpha) = 1$ or 2 in the standard lattice L_n. These are the roots $\pm e_i$ and $\pm e_i \pm e_j$, with $i \neq j$. Let's take $S = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}$ as a base for R. Its Coxeter graph has the following form:

$B_n \; n \geq 2$

Then, if W is the Weyl group of R and $s_\alpha \in W$, we have the following two cases:

(i) If $s_{e_i - e_{i+1}} \in W$ then

$$s_{e_i - e_{i+1}}(e_k) = e_k - (e_k, e_i - e_{i+1}) \cdot (e_i - e_{i+1}) = \begin{cases} e_i & \text{if } k = i+1 \\ e_{i+1} & \text{if } k = i \\ e_k & \text{if } k \neq i, i+1. \end{cases}$$

(ii) If $s_{e_i} \in W$ then

$$s_{e_i}(e_k) = e_k - 2(e_k, e_i) \cdot e_i = \begin{cases} e_k & \text{if } k \neq i \\ -e_i & \text{if } k = i. \end{cases}$$

where $1 \leq k \leq n$.

In the first case, the symmetries generate a group isomorphic to Σ_n, and in the second place, a group isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$. The whole group W will be the semi-direct product $(\mathbf{Z}/2\mathbf{Z})^n \rtimes \Sigma_n$.

Let's now consider the set of roots $\{\pm e_1 \pm e_2, \pm e_2 \pm e_3, \pm e_3 \pm e_4, \pm e_1 \pm e_4\} \subseteq R$. This 8 vectors can be seen as the vertices of a hypercube.

To construct the hyperoctahedron, we can consider the set of roots $\{\pm e_i\}_{i=1}^4 \subseteq R$. These roots can be seen as the 8 vertices of the hyperoctahedron $\{3, 3, 4\}$.

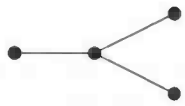
Let $O_{\{4,3,3\}}$ now be the symmetry group of the hypercube. By considering a flag (P, L, F, C) in $\{4, 3, 3\}$, we calculate $|O_{\{4,3,3\}}| = \#P \cdot \#L \cdot \#F \cdot \#C = 16 \cdot 4 \cdot 3 \cdot 2 = 2^4 \cdot 4! = |W|$. Since we have seen the vertices of the hypercube as a subset of R, we can conclude that its symmetry group is the Weyl group of the root system represented by B_4. By duality this group is also the group of symmetries of the hyperoctahedron.

3.3 D_4

For the construction of the root system D_4, let's start with the standard lattice L_4. We define the root system R to be the set of vectors $\alpha \in L_4$ with $(\alpha, \alpha) = 2$. Then, R will consist of 24 element, namely:

$$\pm e_i \pm e_j \ (i < j).$$

As a base for R we will take $S = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\}$. The root system D_4 has the following Coxeter graph:



D_4

We would now like to find out what the Weyl group of D_4 is, as we did for the other cases. This time we will do it in a different manner. Before that, some new concepts are required:

Definition. 1. We define the *Hamilton's quaternion field* $\mathbb{H}(\mathbb{Q})$ to be the vector space over $\mathbb{Q}$ with basis $\{1, i, j, k\}$. A multiplication is introduced in $\mathbb{H}$, and it's defined by the following relations:

$$ij = -ji = k \quad \text{and} \quad i^2 = j^2 = k^2 = -1$$

$\mathbb{H}$ with the multiplication defined above is a skew field.

2. Suppose $q = q_1 + q_2 i + q_3 j + q_4 k$ is a quaternion. We define de *conjugate of* q to be the element $\overline{q} = q_1 - q_2 i - q_3 j - q_4 k$. In this way, we shall define the *norm of* q as:

$$N(q)^2 := q\overline{q}$$

3. An *order* $\mathcal{O}$ in $\mathbb{H}$ is a subring generated over $\mathbb{Z}$ by some basis of $\mathbb{H}(\mathbb{Q})$ over $\mathbb{Q}$. Let's consider the order

$$\mathcal{O} = \{\sum_{i=1}^{4} a_i e_i \mid a_i \in \frac{1}{2}\mathbb{Z} \text{ and } \sum_{i=1}^{4} a_i \in \mathbb{Z}\}.$$

This order is normally called *integer quaternions* and denoted by $\mathbb{H}(\mathbb{Z})$.

We will now identify vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with elements $x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$. We would like to identify our root system D_4 with a subgroup of $\mathcal{O}$, in that manner that we are able to calculate easily its symmetry group.

Let G be the collection of invertible elements of $\mathcal{O}$. The group G will consist on the elements $G = \{a \in \mathcal{O} \text{ such that } a^{-1} \in \mathcal{O}\}$. Since $N(a \cdot a^{-1}) = 1 = N(a) \cdot N(a^{-1})$, and $a \in \mathcal{O}$, it follows that a is invertible in $\mathcal{O} \Leftrightarrow N(a) = 1$, that is

$$G = \{a \in \mathcal{O} \mid N(a) = 1\}$$

Let's now consider the 24 elements:

$$\pm 1, \pm i, \pm j, \pm k, 1/2(\pm 1 \pm i \pm j \pm k)$$

These elements are the only elements of $\mathbb{H}(\mathbb{R})$ which belong to $\mathcal{O}$ and have norm equal to 1, that is, they are the 24 elements of G. We would like to relate the group G with with our original root system D_4. For that, we will use the following theorem in Humphreys' book *Reflection Groups and Coxeter Groups* (see Humphreys, 1990, pg 47):

Theorem 1. *Any finite subgroup G of even order in $\mathbb{H}(\mathbb{R}) - \{0\}$ is a root system.*

In our case, since the elements of G are closed under multiplication, the theorem implies that G is a root system: in fact, G is the root system D_4. G is also the set of vertices of the 24-cell.

We will finally like to find out how the symmetries of the 24-cell look like. For elements $a, b \in \mathbb{H}(\mathbb{R})$ with $N(a) = N(b) = 1$, the map

$$\phi : v \to avb$$

is an orthonormal transformation of $\mathbb{H}(\mathbb{R})$. If $a\mathcal{O}b = \mathcal{O}$ holds, then $\phi(G) = G$ and ϕ is a symmetry of the 24-cell. We shall prove that all the symmetries of

the 24-cell are, in fact, of that form: Suppose that $\phi : G \longrightarrow G$ is a symmetry of the 24-cell, and that $\phi(1,0,0,0) = p$, for $p \in G$. Let's consider the transformation $\mathcal{L} : \mathbb{H}(\mathbb{R}) \longrightarrow \mathbb{H}(\mathbb{R})$ defined as: $\mathcal{L}(q) = pq$. This transformation verifies that $\mathcal{L}(1) = p$. $\mathcal{L}$ is linear, and since $N(p) = 1$, it leaves invariant the norm of any quaternion q, that is, $\mathcal{L}$ is an orthonormal transformation.

Let ψ be equal to $\mathcal{L}^{-1} \circ \phi$. Then ψ fixes the x-axis, since

$$\psi(x) = \mathcal{L}^{-1} \circ \phi(x) = \mathcal{L}^{-1}(p(x)) = (x).$$

ψ also preserves the norm of any quaternion. Then, it is a orthonormal transformation in the vector space over $\mathbb{R}$ with basis: $\{i, j, k\}$. It's known that ψ can be written as: $\psi : q \longrightarrow rqr^{-1}$ where $N(r) = 1$, that is,

$$\psi = \mathcal{L}^{-1} \circ \phi \Rightarrow \phi = \mathcal{L} \circ \psi : q \longrightarrow prqr^{-1} = arb$$

with $a = pr$, $b = r^{-1}$ and $N(a) = N(b) = 1$.

We have seen that ϕ is a symmetry of the 24-cell if and only if ϕ has the form $\phi : v \to avb$, for $a, b \in \mathbb{H}(\mathbb{R})$ with $\| a \| = \| b \| = 1$, such that $a\mathcal{O}b = \mathcal{O}$.

3.4 H_4

Finally, we would like to find the symmetries of the 600-cell (and also of its dual, the 120-cell), as we did for the 24-cell. In this case, we will prove that the group H_4 with graph:

H_4

is the symmetry group of the 600-cell. The Coxeter group H_4 can be described as follows:

$$H_4 = \{a_1, a_2, a_3, a_4 \mid a_i^2 = 1, (a_1 \circ a_2)^5 = (a_2 \circ a_3)^3 = (a_3 \circ a_4)^3 = 1\}$$

First, we would like to construct a root system whose symmetry group is H_4. With that purpose, we will again make use of the quaternions: Let's first consider the element $\tau := \frac{1+\sqrt{5}}{2}$ known as the *golden ratio*. It's known that $\mathbb{Z}[\tau]$ is the ring of integers of the field $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\tau)$. Let $\mathbb{H}(\mathbb{Q}(\sqrt{5}))$ be the quaternion field over $\mathbb{Q}(\sqrt{5})$, that is,

$$\mathbb{H}(\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\sqrt{5}) + \mathbb{Q}(\sqrt{5})i + \mathbb{Q}(\sqrt{5})j + \mathbb{Q}(\sqrt{5})k.$$

This quaternion field has also an order, namely $\mathcal{O}[\tau] = \mathcal{O} + \mathcal{O}\tau$. This order is known as the *icosians* and is denoted by $\mathbb{H}(\mathbb{Z}[\tau])$. Let's now consider G to be the group of invertible elements of $\mathbb{H}(\mathbb{Z}[\tau])$. We can prove, as we did for the case before, that G has the form:

$$G := \{a + b\tau \in \mathcal{O}[\tau] \mid \| a + b\tau \| = 1\}.$$

To find the elements of the group, let's suppose that $\| a + b\tau \| = 1$, where $a, b \in \mathbb{H}(\mathbb{R})$. Then:

$$\| a + b\tau \| = (a + b\tau)(\overline{a} + \overline{b}\tau) = (a\overline{a} + b\overline{b}) + (a\overline{b} + b\overline{a} + b\overline{b})\tau = 1.$$

The last holds if and only if $a\overline{a} + b\overline{b} = 1$ and $a\overline{b} + b\overline{a} + b\overline{b} = 0$. This gives us the following 120 possibilities:

$$G = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

$$\text{and } \frac{1}{2}(\pm i \pm \sigma j \pm \tau k) \text{ with all even permutations of coordinates}\}$$

where $\sigma = \frac{1-\sqrt{5}}{2}$. Since these elements are closed under multiplication, they form a group of 120 elements. By applying Humphreys' theorem, we obtain that G is a root system (in fact, it is the root system of H_4). This root system has a base, namely:

$$\{1/2(\tau - i - \sigma j), 1/2(\tau - i + \sigma j, 1/2(1 - \sigma i - \tau j), 1/2(-1 - \tau i - \sigma k)\}$$

The group G can also be seen as the set of vertices of the 600-cell.
In the same manner as we did for the symmetries of the 24-cell, it can be proved that ϕ is a symmetry of the 600-cell if and only if ϕ has the form: $v \to avb$ for $a, b \in \mathbb{H}(\mathbb{R}), \| a \| = \| b \| = 1$ and $a\mathcal{O}[\tau]b = \mathcal{O}[\tau]$. By duality, these will also be the symmetries of the 120-cell.

4 Alicia Boole Stott (1860-1940) and her four-dimensional polytopes

In this chapter I will speak about the mathematician Alicia Boole Stott. In the beginning of the 20th century, Alicia came to the Netherlands to work together with Proffesor Pieter Schoute at the University of Groningen. Some results of her mathematical work are still in Groningen. The interest for Alicia's life and work started when several big rolled papers were found at the basement of the WSN-building, at the University of Groningen. These papers contain colour-designs for the three-dimensional sections of some of the regular polytopes drawn by Alicia. Also the Groningen University Museum still keeps a big showcase, which happens to contain Alicia's models of sections of some polytopes. These two facts motivated the present research, and together with some other information obtained at the Groningen Provincial Archive, have reconstructed part of Alicia's stay in the Netherlands. However, there are still several questions that have not yet found an answer. All this will be discussed in the following sections.

The sources from which I have obtained the information are mainly: (Batchelor 1996), (MacHale 1985), (Coxeter 1978), (Coxeter 1987). The first of them is a biography of the nephew of Alicia, G. I. Taylor, and it has been my main source of information for the first section, for considering Taylor very close to Alicia's life. The information in the second section about the Boole's family is described in detail in both, MacHale's and Batchelor's books. I have combined all this information with more that I have obtained in the State Archive of Groningen.

4.1 Alicia's life

Alicia Boole Stott is still remembered for her contributions to four-dimensional geometry. She was born on the 8th of June in 1860 in Cork (Ireland). She was the third daughter of the famous mathematician George Boole, and Mary Everest. George Boole was a professor of mathematics in Cork. He died from fever at the age of 49, on the 8th of December in 1864, and his wife and five daughters were left with very little money. Alicia's mother had to move to London with four daughters while Alicia stayed with Mrs Boole's mother and uncle in Cork. She lived with them until she was eleven. Then she moved to London, where she lived for seven years with her invalid mother and her

four sisters in a very poor and dirty place. They could not have any kind of education.

Alicia's mother had a friend called James Hinton, whose son Charles Howard was a school teacher. Howard Hinton started stimulating the five girls by teaching them some geometry and Latin. He used to bring a set of wooden cubes and put them together trying to visualize the four dimensional hypercube. This strongly inspired Alicia in her later work. Alicia soon started surprising Howard with her ability for visualizing the fourth dimension.

In 1900 Alicia, who was working near Liverpool as a secretary, married Walter Stott and they had two children: Mary and Leonard. Alicia was much interested in geometry, in particular, in four-dimensional figures. She found out that there are six regular polytopes, and she studied their three dimensional sections, which she built in wood. She found these sections by using only Euclidean constructions. The sections Alicia studied are parallel to the 3-dimensional space that contains one of the 3-dimensional cells of the polytope. At the same time, professor Pieter Schoute (1846-1913) of the University of Groningen was studying, with analytical methods, the central sections of these four dimensional polytopes. In 1893, Schoute published a paper on this subject in the *Proceedings of the Amsterdam Academy.* According to (Batchelor 1996), one of Alicia's friend noticed Schoute's paper, and Alicia sent to Schoute some photographs of her work, showing that her middle sections agreed with his results [1]. Schoute answered very enthusiastically, asking whether he could meet her and could collaborate with her. They became very good friends, and they worked together for nearly twenty years. They used to visit Ethel Boole, the youngest sister of Alicia, where they worked together. Schoute insisted that Alicia would publish her results, what she did in two papers in 1900 and 1910. The first one is about the parallel sections of the four dimensional polytopes. The second one deals with expansion and contraction of some regular solids. Alicia published other results together with Schoute. I shall speak about all publications in section 4.3, discussing with more detail the first paper in 1900.

Alicia introduced the notion of *polytope* and the *Cartesian product* of any two polytopes. After Schoute's death in 1913, the University of Groningen

[1]Since the photograph with her models that Batchelor refers to (and the only one that has been found) was taken later in the Netherlands, we tent to believe that Alicia did in fact not sent to Schoute such a photograph.

invited Alicia to its 300th anniversary celebration, where they gave her an honorary doctorate. As I discuss in section (4.4), Alicia did not assist to the 1914 festivities.

According to (MacHale 1985), in 1930 Alicia's nephew G. I. Taylor introduced her to the Canadian geometer Coxeter, who was also interested in polytopes. She kept in contact with him, and worked in the building of four-dimensional kaleidoscopes. Coxeter referred to her in several publications. At the age of 80, on the 17th of December of 1940, Alicia died of thrombosis in a Catholic Nursing Home in England.

4.2 Alicia's family

Alicia came from a very interesting and talented family. Her father George Boole was born in Lincoln, England, on the 2nd of November in 1815. He was professor of mathematics at Queen's College in Cork. George married Mary Everest in 1855, and they had five daughters: Mary, Margaret, Alicia, Lucy, and Ethel. The next picture shows a photograph of Boole's family:
Mary Everest, her five daughters and five of her grandchildren

George died from pneumonia on December 8 in 1864. Today, Boole's work on symbolic logic is used in information theory, graph theory, computer science, and artificial intelligence research.

Alicia's mother, Mary Everest, was born in England in 1832. She was the niece of Sir George Everest, the Surveyor-General of India, after whom Mount Everest was named. She met George Boole in Cork, and they became good friends. George introduced her to mathematics. In 1865, Mary started working as a librarian at Queen's College. She made interesting contributions to the psychology of education. She published books and articles frequently, until her death in 1916.

The five daughters of George Boole and Mary Everest were all very special women: Alicia's eldest sister, Mary Ellen, was married to Howard Hinton. Hinton became famous for writing a book in geometry called *The fourth dimension*, where he worked with semi-regular four-dimensional polytopes and honeycombs. He also invented the word "tesseract" to describe the hypercube. The second daughter of George Boole and Mary Everest was called Margaret. She is best remembered for her son, the physicist Geoffrey Ingram Taylor, who was a student at Trinity. His Smith's Prize paper was the first convincing demonstration of the phenomenon of shock waves. The third daughter, Lucy, was a chemist, and also the first female professor of chemistry at the Royal Free Hospital in London. She died at the age of 43. Finally, the youngest daughter, Ethel Lilian Voynich, also became famous. She became involved in a group of Russian revolutionary workers. She was the author of the novel *The Gadfly* which she wrote after an affair she had during the summer of 1895, with the secret agent SydneyReilly. She died in 1960.

4.3 Alicia's work

Alicia made a very important contribution to four-dimensional geometry. She found elegant methods to describe the sections of the four-dimensional polytopes. She also worked on other geometrical concepts that I shall discuss during this section.

Alicia's first paper was published in 1900 (see Stott, 1900). In this article, thanks to her amazing ability to visualize the four dimension, Alicia was able to describe in detail the six regular polytopes, and their 3-dimensional sections. Before starting describing Alicia's work, we will give the following definitions:

Definition. 1. A *parallel section* of a regular 4-dimensional polytope is a section obtained by intersecting the polytope with a 3-dimensional space parallel to one of the 3-dimensional faces (called *cell*) of the polytope.

2. A *cross section* of a regular 4-dimensional polytope is a section obtained by intersecting the polytope with a 3-dimensional space at equal distance from

one of the vertices of the polytope.

In both cases, the figure obtained is a regular polyhedron. In the first part of the article, she begins with the proof of the existence of the six regular polytopes. To prove this, she argues as follows: Suppose that a regular polytope is bounded by cubes. If we consider a 3-dimensional section that cuts the edges which meet in one vertex at equal distances from that vertex, the intersection with each cube will give an equilateral triangle. Then, the corresponding 3-dimensional section will be a regular polyhedron bounded by equilateral triangles. Since there are only three regular polyhedrons bounded by triangles: the tetrahedron, the octahedron and the icosahedron, the polytope can only have 4, 8, or 20 cubes meeting at each vertex. Considering the possible angles, Alicia shows that the only possibility is in fact having 4 cubes at a vertex, which give us the 24-cell.

Proceeding in the same manner with the other cases, she shows that there are at most six regular polytopes: the hypercube (or 8-cell), the hypertetrahedron, the 16-cell, the 24-cell, the 120-cell and the 600-cell.

We will begin by explaining the sections of the hypercube:

The 8-cell or hypercube: The hypercube is the analogue of the cube in four dimensions. It is bounded by 8 cubes, and as we just saw, it is characterized by having 4 cubes meeting at every vertex. The figure in next page shows the four cubes meeting at the vertex A, where the three vertices K must be identified, so must the two vertices L, and the vertices M (this, of course, can only be understood in the fourth dimension).

We would like to calculate the sections of the hypercube that are parallel to the cube $ABCDEFGH$. To calculate the first section, let's consider the 3-dimensional space S_1 which contains the cube $ABCDEFGH$. If we intersect with the whole polytope, we will of course get the cube itself (in the figure, we can see the faces: $ABCD$, $ADEF$, $ABFG$). For the second section, let's consider the 3-dimensional space S_2 parallel to the cube $ABCDEFGH$, and passing through the point a (the mid point of $\bar{AK}$), as it's shown in figure 2. Making S_2 intersect with the polytope, we get again a cube of the same size (in the figure, we can see the faces $abcd$, $adef$, $abfg$ that will be "glued" by identifying the three points a, the b's, the d's and the f's). Finally, the third section will be obtained by intersecting the whole polytope with a 3-dimensional space parallel to $ABCDEFG$ and passing through the vertex K. Again, we will get a cube (faces $KLMN$, $KLOP$ and $KMSO$ are shown in the figure).

This is a trivial example. In the following case (the 16-cell), the corresponding sections are more interesting:

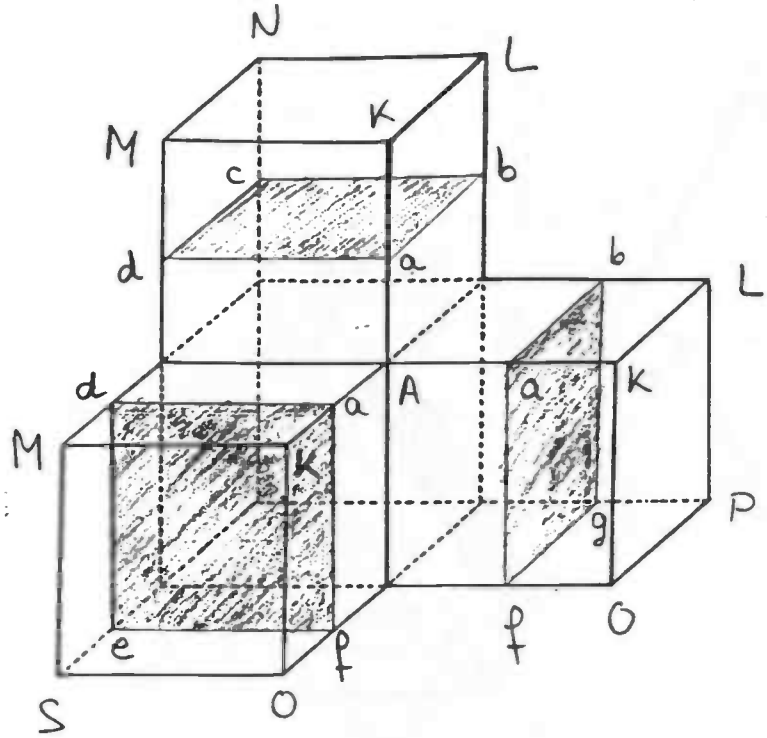


Figure 1: Hypercube: four cubes meeting at a vertex A

The 16-cell or hyperoctahedron: The 16-cell is the analogue of the octahedron in four dimensions. As its name suggests, it is bounded by 16 polyhedrons, in this case 16 tetrahedrons. As Alicia proved in the first part of her paper, the 16-cell has 8 tetrahedra meeting at a vertex A. In the figure below, five of them are drawn, and the four vertices D' are identified. Let's now consider, as we did for the hypercube, the space S_1 that contains the tetrahedron $ABCD$, and let's intersect it with the 16-cell: we will get the tetrahedron itself. Let's now take the point a, which is at distance $1/4$ from A of $\overline{AD'}$, and it is lying on the edge $\overline{AD'}$. Let's consider the space S_2 parallel to S_1 and passing through the point a: If we intersect it with the polytope, we obtain the following sections, each of them parallel to a face of the initial tetrahedron (see Figure 2):

1. In the tetrahedron $ABCD'$: An equilateral triangle abc parallel to the face ABC.
2. In the tetrahedron $AB'CD'$: A rectangle $adce$ parallel to the face ABD.
3. In the tetrahedron $ABC'D'$: A rectangle $abgf$ parallel to the face ACD.
4. In the tetrahedron $AB'C'D'$: An equilateral triangle adf parallel to the face BCD.

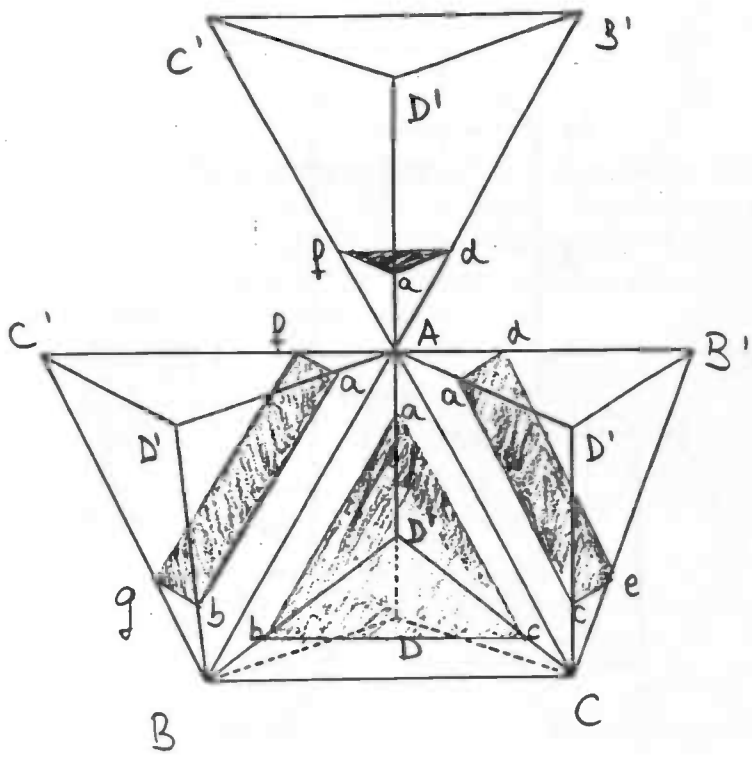


Figure 2: 16-cell being cut by a space S_2

In the same way, we proceed with spaces S_3, S_4 and S_5 passing through the points a_1, a_2 and D' respectively. The corresponding sections will be similar to those drawn before, but of different sizes. By identifying the corresponding points in every section, we get part of some polyhedrons (see Figure 4 at the end of the section).

The 24-cell: The 24-cell is the only polytope with no analogue in dimension three. It is bounded by 24 octahedrons, and it has 6 of them meeting at every vertex. To find out what its sections are, we will begin with a representation of the octahedrons meeting at a vertex A (see figure 3).

The first section occurs by intersecting the 24-cell with the space S_1 that contains the octahedron $ABCDEF$: this gives us the octahedron itself. For the second one, consider a to be the mid point of $\overline{A(AC)}$. By intersecting S_2 (parallel to S_1 and passing through a) with the polytope, we obtain rectangular sections on the octahedrons $ABCDEF$ and $A(AF)(AE)A'(AB)(AC)$, and hexagonal ones on the others (see figure above). The third sections will be obtained by intersecting the 24-cell with the space S_3 parallel to S_1 and passing through the vertex AC. In that manner, we get rectangular faces of maximal size on the octahedrons $ABCDEF$ and $A(AF)(AE)A'(AB)(AC)$,

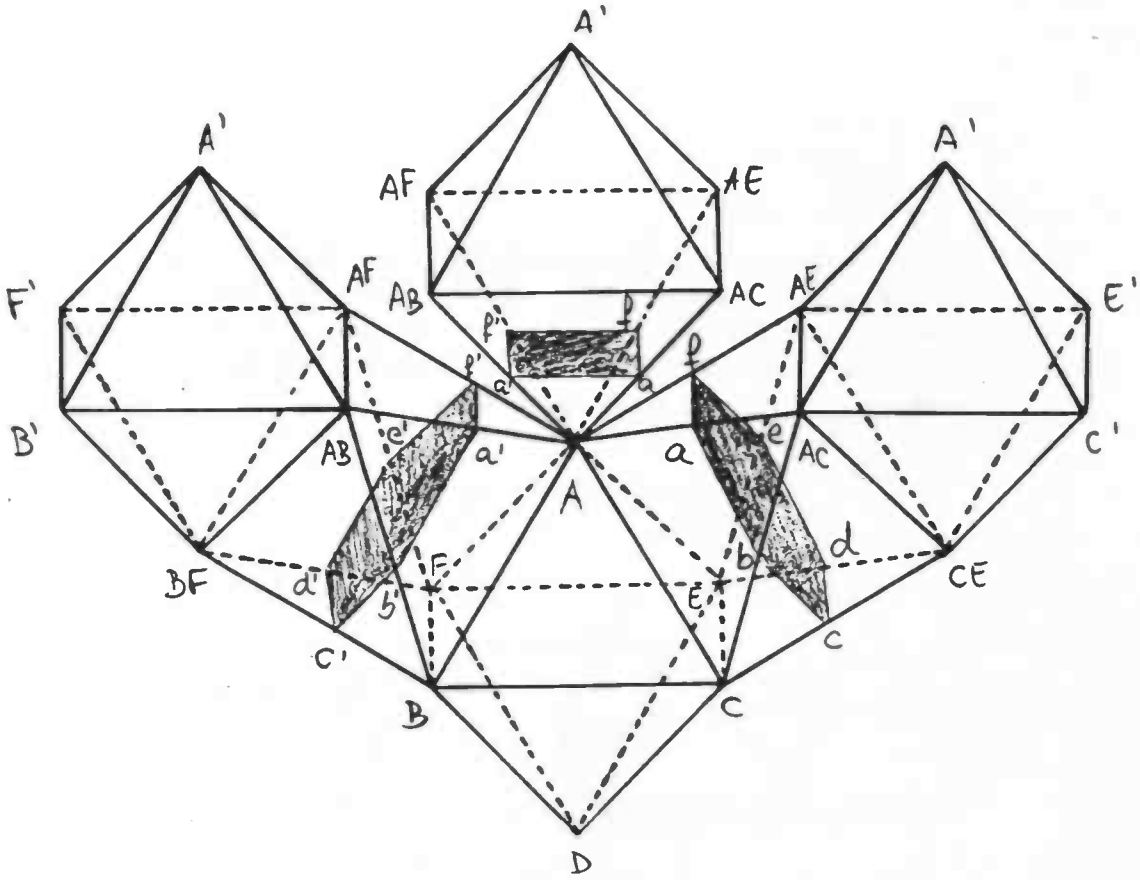


Figure 3: 24-cell being cut by a space S_2

and triangular ones on the others (which coincide with the faces of the octahedrons). The fourth and fifth sections will be similar to the 2nd and 1st ones respectively. By identifying the corresponding vertices, we will obtain the 3-dimensional sections in Figure (5).

Alicia also describes the sections of the 120-cell and the 600-cell, and gives plans to construct them.

Coxeter found Alicia's contribution to geometry very important. In his article (Coxeter, 1978), he points out the relevance of the contents of Alicia's second paper (Alicia, 1910). In this paper, she gives a new construction for the Archimedean solids and their n-dimensional analogue. She makes the following definitions:

Definition. 1. A regular p-gon with sides AB, BC, CD, ... is *expanded* by translating its sides AB, BC, CD, ... away from the center, all the same distance, to positions A_1B_1, B_2C_2, C_3D_3, ... such that the resulting figure is a regular $2p$-gon.

2. A regular $2p$-gon is *contracted* by translating its sides B_1B_2, C_2C_3, D_3D_4, ... towards the centre, all the same distance, to positions P_1P_2, P_2P_3, P_3P_4, ... such that the resulting figure is a p-gon, reciprocal to the original one.

She gives the analogue to these definitions in n dimensions, and introduces the concept of *Cartesian product* of a p-gon and a line segment, generalizing the idea to the Cartesian product of any two polytopes. An example of this is the product of two triangles: the new figure is a four dimensional polytope with nine vertices and six prisms as cells.

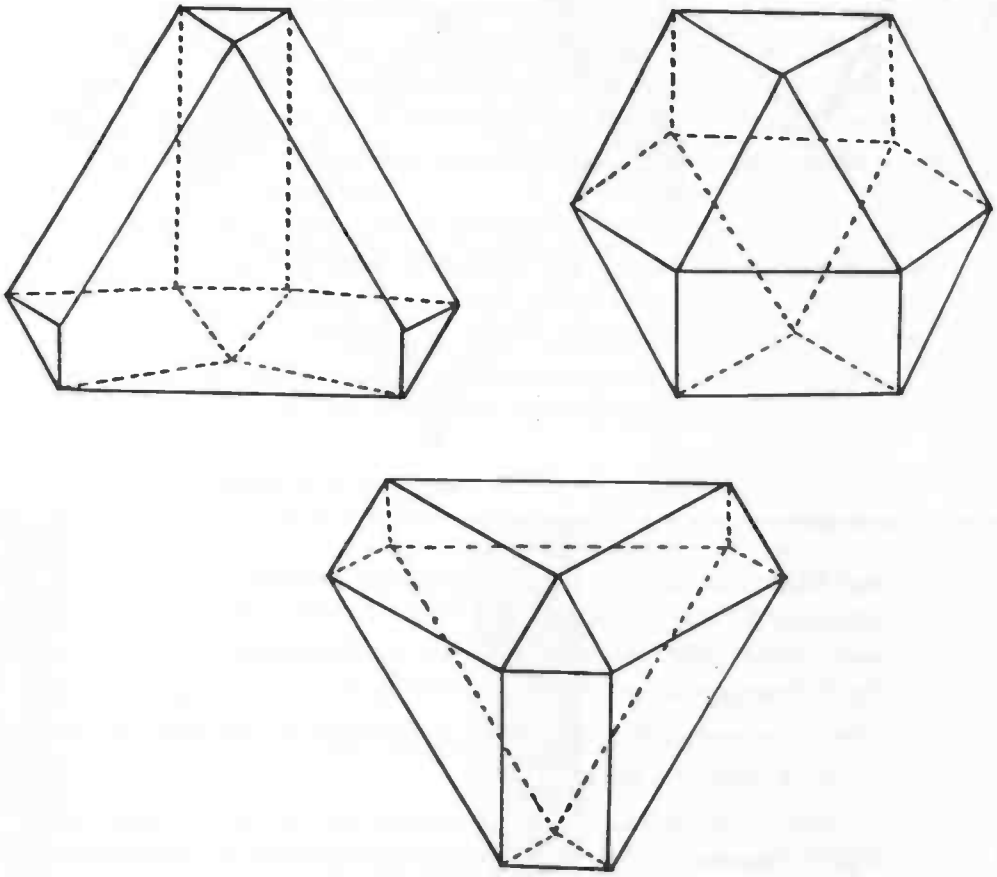


Figure 4: 2nd, 3rd and 4th sections of the 16-cell

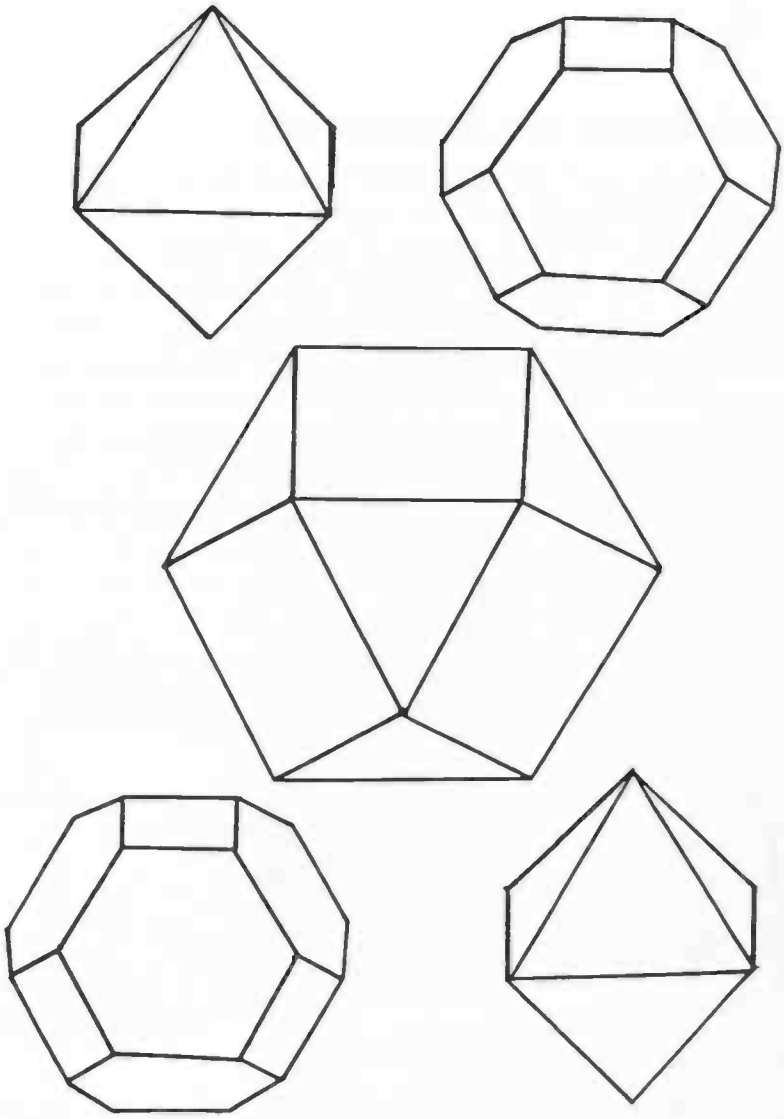
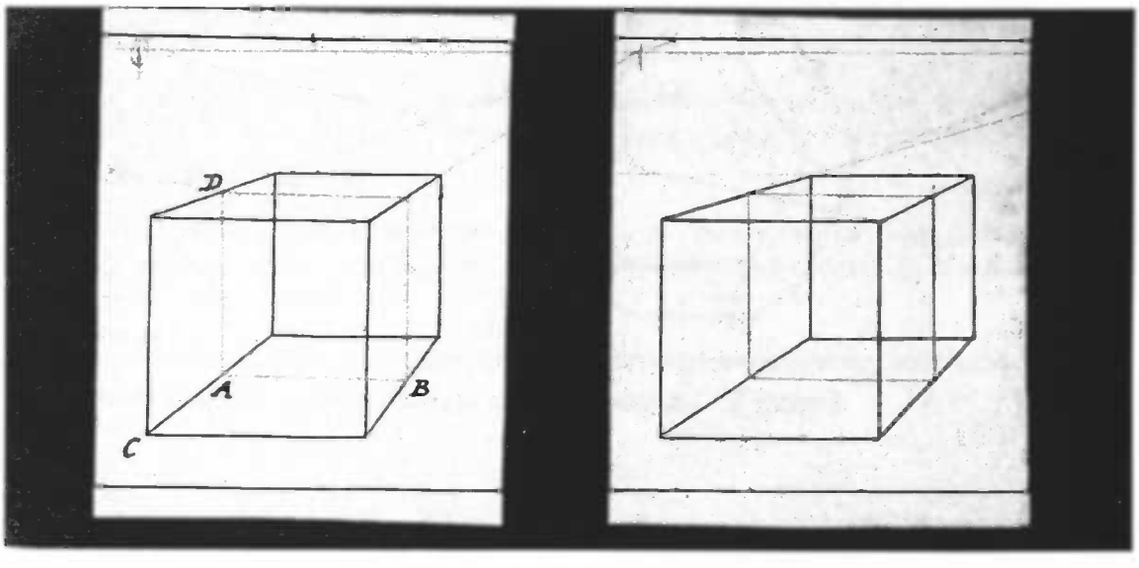


Figure 5: 1st, 2nd, 3rd, 4th and 5th sections of the 24-cell

4.4 Alicia and the Netherlands

At the end of the 19th century, Pieter Schoute went to England to meet Alicia Boole Stott, and to speak about her sections of polytopes. Some time later, she came to Groningen to collaborate with him, and they worked together for more than ten years. At that time, Schoute was a professor of mathematics at the University of Groningen. He had graduated as a civil engineer in 1867, and received his doctorate in Leiden in 1870 for a dissertation *Homography applied to the theory of quadric surfaces.* He taught mathematics in secondary schools in several places in the Netherlands until 1881, when he went to Groningen and became a professor of mathematics at the University.

Jan van Maanen visited Schoute's granddaughter Mrs. Stuiveling–van Vierssen Trip, in October 2000 and interviewed her (see section 4.5). She still kept two stereograms belonging to her grandfather. These are figures in three dimensions, that can be viewed by a stereoscope, also in Mrs. Stuiveling's possession. They are representing two polyhedrons in 3 dimensions: a cube and a section of a 4-dimensional polytope, the last belonging to (Schoute,1911). A copy of the first stereogram is shown below:

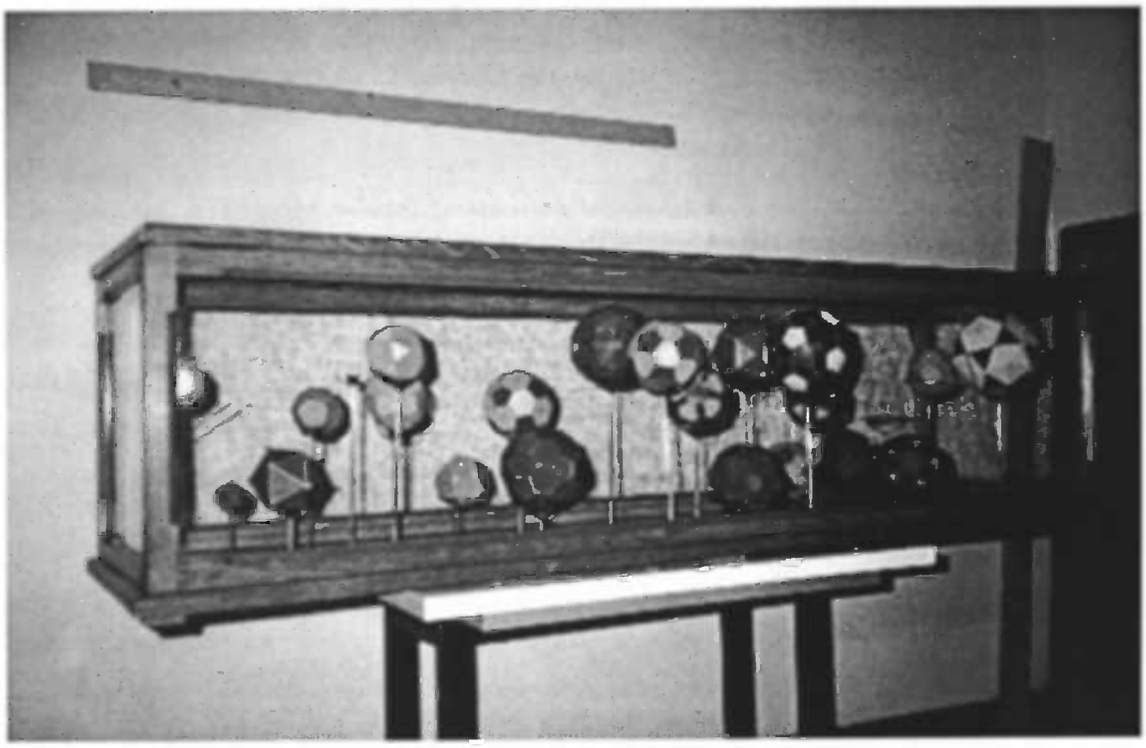


Sterereogram representing a cube

When Schoute met Alicia, he was working on the central sections of the four-dimensional polytopes. Although their methods to calculate these sections where completely different (while Schoute's were analytical, Alicia's were completely geo-

metrical), they worked together for more than ten years, combining Alicia's ability to visualize geometry with Schoute's analytical method.

Most of Alicia's papers can be found in the *Verhandelingen der Koninklijke Akademie van Wetenschappen.* The Groningen University Museum still keeps a big showcase which contains models of sections of some polytopes made by Alicia. These models represent the parallel sections of the 600-cell and 120-cell and they complete the contents of her article (Stott, 1900). The picture below was taken in the University Museum, and it shows the models mentioned above:



Models of parallel sections of the 600*-cell and* 120*-cell*

The legend, pinned down in a corner of the showcase, can be translated as follows:

N.1: Models prepared by Ms A. Boole-Stott, Liscard, Cheshire, England. Of intersections of the cell C_{120} *(perpendicular to* R_o*) and of the cell* C_{600} *(perpendicular to* OE_0*) by placement in parallel position, showing the decomposition of the* 120 *centers of the boundary dodecagons of the* C_{120} *and the* 120 *vertices of the* C_{600} *in the vertices of* 5 *cells* C_{24}.

(Compare about the horizontal scales "Regelmässige Schnitte usw.", Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, eerste sectie, deel IX, N.4, 1907).

Alicia also made the designs of these sections, which she drawn in colours (see Alicia, 'Designs for the three-dimensional models'). In the next page, some of them are shown.

Thanks to Alicia's important contributions to mathematics during her stay in the Netherlands, she was proposed to receive an honorary doctorate. Professor Barrau, the successor of Pieter Schoute, wrote the list of Alicia's publications (Barrau, 1914), as a proposal for the University to give her the doctorate. The list is shown at the end of this section. This paper is dated 7 February 1914, and contains the following articles:

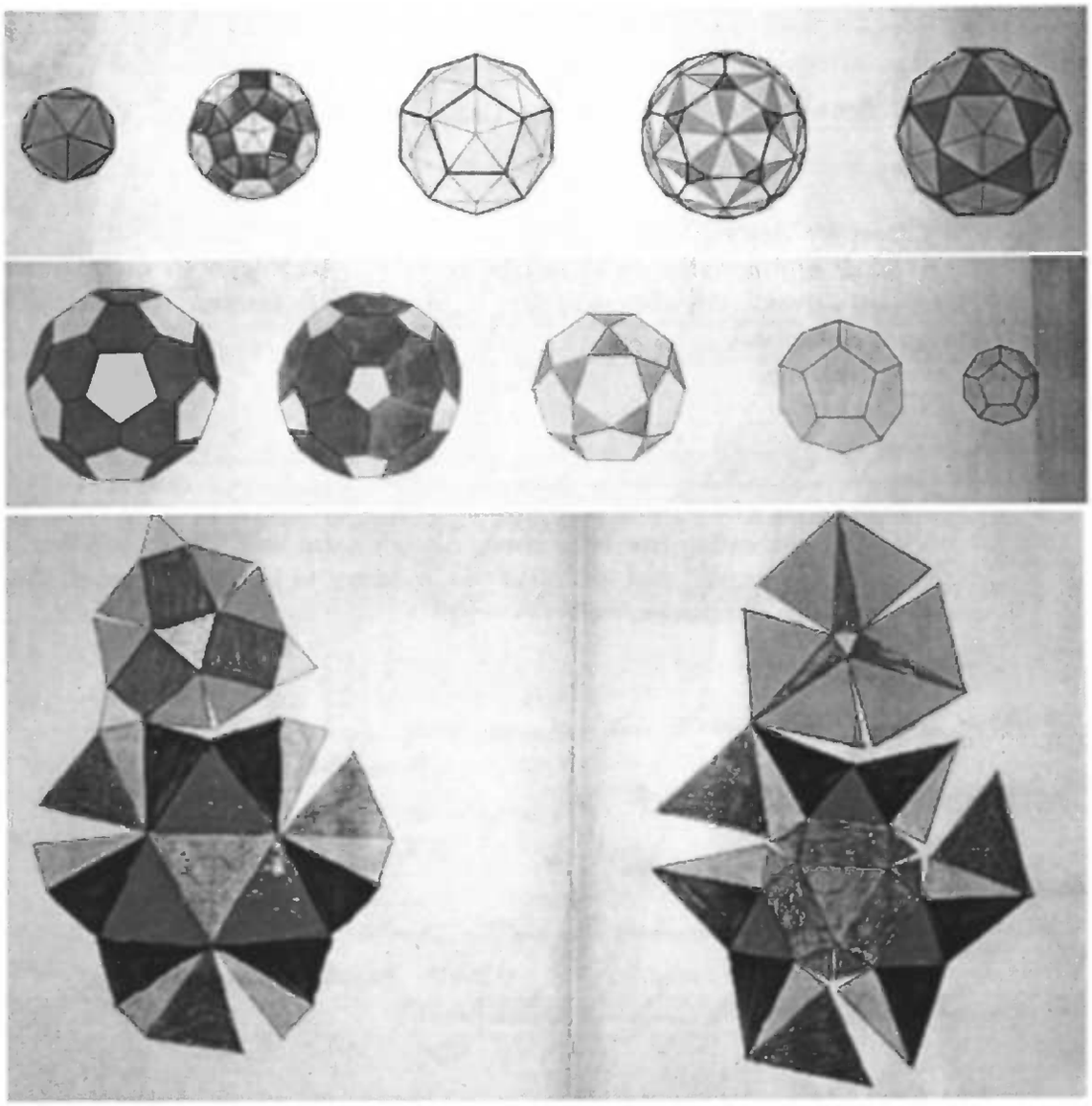
1. *On Models of* 3*-dimensional sections of regular hypersolids in space of* 4 *dimensions:* Report of the British Association for the Advancement of Science. Leicester 1907, p.406. (With P. H. Schoute).

2. *On the sections of a block of eightcells by a space rotating about a plane:* Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam IX, N.7., 1908. (With P. H. Schoute).

3. *Over vijf paren uit een zelfde bron afgeleide vierdimensionale cellen:* Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam XVI, p.482.(date unknown).

4. *Geometrical deduction of semiregular from regular polytopes and space fillings:* Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam XI, N.1, 1910.

5. *Over wederkeerigheid in verband met halfregelmatige polytopen en netten.(date unknown)*



Alicia's designs of parallel sections of the 600-*cell and* 120-*cell*

Barrau's proposal was accepted, and Alicia was invited to attend the tercentenary celebrations of the University of Groningen, where she was conferred with an honorary doctorate in mathematics and physics. In the arrangements for the celebrations, the University had to contact the people that were receiving a doctorate. The postcard (Schoute, J.C., 1914) is about Alicia's address in England. It was written by Dr J.C. Schoute (Pieter Schoute's son) to Professor J.A. Barrau before the tercentenary celebrations, and can be translated as follows:

Dear Mr Barrau!
On behalf of my mother, who resides here for a while with an indisposition, I give you herewith the address of Mrs A. Boole-Stott: Liscard, Withenslane 155, Cheshire, England.
Yours faithfully,
J.C. Schoute

However, in a later list of the places where the honorary doctorates were staying during the celebration (see references), Alicia's name was written together with the following remark: *does not come.* So, contrary to the general belief, she did not attend the ceremonies.

Lijst der Geschriften van Mrs. A. Boole-Stott.

On Models of 3. dimensional sections of regular hyper solids in Space of 4 dimensions.
(Report of the British Assoc. for the Advancem. of Science. Leicester 1907, p. 460).

met P. H. Schoute.
On the sections of a block of eight cells by a space rotating about a plane.
(Verhandelingen der Kon. Akad. van Wetenschappen IX No 7.)

met P. H. Schoute
Over vijf paren uit een zelfde bron afgeleide vierdimensionale cellen.
(Verslagen der Kon. Akad. van Wetensch. XVI p. 482).

Geometrical deduction of semiregular from regular polytopes and space fillings.
(Verhandelingen der Kon. Akad. v. Wetens. XI No 1.)

met P. H. Schoute.
Over wederkeerigheid in verband met halfregelmatige polytopen en netten.

Barrau's list with Alicia's publications

4.5 Interview with the granddaughter of P.H. Schoute

In October 2000 Jan van Maanen visited Mrs. Stuiveling–van Vierssen Trip. She is a granddaughter to Schoute, born 1907 and at the time of the interview living in Hilversum. Mrs. Stuiveling had a vivid memory of her grandfather, whom she much liked, and could also give information about the family relations. Schoute and his wife, Mathilde Pekelharing, had four children, three sons (Dirk, Kees and Johan) and one daughter, born 1879, who was the mother of Mrs. Stuiveling. Later there was a fifth child, which was idiot and died at the age of five. When the Queen-Mopossessionther Emma visited Groningen, Mrs. Stuiveling's mother offered flowers to Princess Wilhelmina. This happened when Schoute was Rector of Groningen University. Mrs. Stuiveling's mother also often told about the work she had to do for her father, in putting strings through all kinds of mathematical models that Schoute produced. Most of these were burnt 1906, when the Academy building went on fire. Mrs. Stuiveling told that she received three lovely presents from her grandfather, which were a kind of puzzles that made her wonder, and that stimulated the imagination and raised questions about physics. One was an object that balanced on a wire, on of the others a puppet which could walk on its own from a slanted plane. Still in her possession were the two stereograms, discussed in section 4.4. She also still owned the stereoscope by which these can be viewed. In both the family of her grandfather and the family of her grandmother there were topics one would avoid to speak about. In her grandmothers family, Mrs. Stuiveling told, this was the fact that an ancestor, a diplomate working in Africa, had a child with a black woman. In her grandfather's family one was not expected to speak about this English lady with whom her grandfather worked together. Pieter Hendrik Schoute was born at Wormerveer, on 21 January 1848. He was a civil engineer and had a PhD in Mathematics and Physics of Leiden University (1870). He was appointed professor of geometry at Groningen university in 1881, and stayed there until his death, on 18 April 1913. In 1872 he married to Mathilde Pekelharing, who was born in Zaandam on 11 March 1846 and died at Groningen on 10 October 1930. These data come from *Nederlands Patriciaat* 83 (2000–2001), 234.

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* 'Notices relating to the honorary doctorates: draft list where the honorary doctors would be lodged', Groningen Provincial Archive, Inv. Nr. 554., 1914.

* 'Nederlands Patriciaat' 83, Groningen Provincial Archive, (2000-2001).