Discrete-time LQG-balancing in infinite dimensions

Mark Opmeer

RuG

Mathematics

Masters thesis

Discrete-time LQG-balancing in infinite dimensions

Mark Opmeer

Rijksuniversiteit Groningen
Department of Mathematics
Postbus 800
9700 AV Groningen

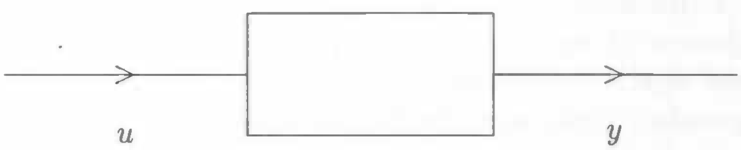
July 2002

Contents

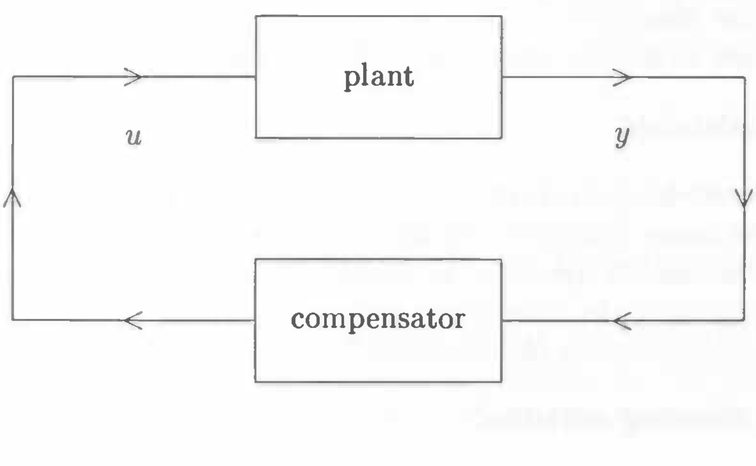
Chapter 1

Introduction

In control engineering we study systems that can be graphically portrayed as follows



We have a system (portrayed as a box) called the *plant* with input u and output y. The goal of control engineering is to choose an input such that the output has some desired property. The usual way to do this is to design another system called the *compensator* that takes the output of the plant as its input and has as its output the input for the plant



The first step in designing a compensator for a plant is to describe the plant mathematically. Many plants can be mathematically described as

$$\dot{x} = Ax + Bu \quad x(0) = x_0 \tag{1.1}$$

$$y = Cx + Du$$

where A, B, C, D are operators on Hilbert spaces. For mechanical and electrical systems these Hilbert spaces are finite-dimensional. For acoustical, thermodynamical and hydrodynamical systems these Hilbert spaces are infinite-dimensional. Both from a mathematical and an engineering perspective finite-dimensional systems are easier. The main aim in infinite-dimensional control engineering is to design finite-dimensional compensators for infinite-dimensional plants (that is: 'easy' compensators for 'hard' plants).

In this thesis we will not consider systems of the form (1.1) but of the form

$$x_{n+1} = Ax_n + Bu_n \quad x(0) = x_0 \tag{1.2}$$

$$y_n = Cx_n + Du_n$$

where $A \in \mathcal{L}(X), B \in \mathcal{L}(U, X), C \in \mathcal{L}(X, Y), D \in \mathcal{L}(U, Y)$ and X, U, Y are Hilbert spaces. Systems of the form (1.1) are called *continuous-time systems* and systems of the form (1.2) are called *discrete-time systems.* Discrete-time systems are important in their own right and are also a powerfull tool in understanding continuous-time systems.

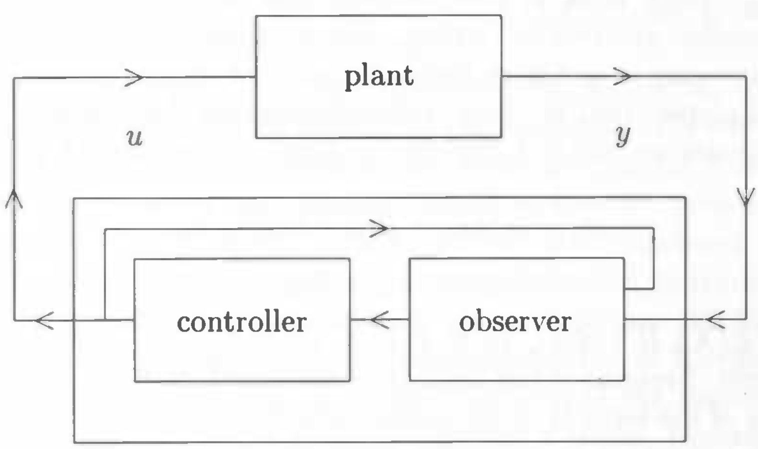
In this thesis we make a start with a new method of designing a finite dimensional compensator for an infinite-dimensional plant. That is: we assume that we are given a plant of the form (1.2) with X infinite-dimensional and U, Y (for the most part of this thesis) also infinite-dimensional and we want to design a compensator of the form

$$\tilde{x}_{n+1} = K\tilde{x}_n + Ly_n \quad \tilde{x}(0) = \tilde{x}_0 \tag{1.3}$$

$$u_n = M\tilde{x}_n + Ny_n$$

with $K \in \mathcal{L}(\tilde{X}), L \in \mathcal{L}(U, \tilde{X}), M \in \mathcal{L}(\tilde{X}, Y), N \in \mathcal{L}(U, Y)$ and $\tilde{X}$ a finite-dimensional Hilbert space. There are many design methods that give a compensator for which $\tilde{X} = X$. Our approach is to approximate the plant (1.2) by a system with a finite-dimensional state space $\tilde{X}$ and use one of these design methods to design a compensator with this $\tilde{X}$ as its state space and prove that the system that we get by interconnecting this compensator to the plant satisfies the specifications. In this thesis we concentrate on the first step: the approximation of the plant. Our approximation procedure takes the context of the approximation (designing a compensator for the plant based on this approximation) into account.

The idea behind this approximation is as follows. A compensator actually consists of two parts: an observer and a controller. The observer takes the output of the plant (and the output of the controller) as its input and gives an estimate of the state of the plant as its output. The controller takes this estimate of the state of the plant as its input and gives the input of the plant as its output.



The idea of the approximation is to omit the states that are unimportant in the design of the compensator (that is: unimportant in the design of the observer *and* in the design of the controller) and retain the states that are important in the design of the compensator (that is: important in the design of the observer *and* in the design of the controller). For this idea to work we need that all states are as important for the design of the observer as they are for the design of the controller. One possible measure of how important a state x is for the design of an observer is

$$f(x) := \frac{< Px, x >}{< x, x >}$$

where P is the smallest nonnegative solution of

$$APA^*-P+BB^* = (APC^*+BD^*)(R+CPC^*)^{-1}(CPA^*+DB^*) \quad R := I+DD^*.$$

This equation is called the Filter Algebraic Riccati Equation (FARE) of the system. A state x for which $f(x)$ is large is important for the design of the observer and a state x for which $f(x)$ is small is unimportant for the design of the observer. A possible measure of how important a state x is for the design of a controller is

$$c(x) := \frac{< Qx, x >}{< x, x >}$$

where Q is the smallest nonnegative solution of

$$A^*QA - Q + C^*C = (C^*D + A^*QB)(S + B^*QB)^{-1}(B^*QA + D^*C) \quad S := I + D^*D.$$

This equation is called the Control Algebraic Riccati Equation (CARE) of the system. A state x is important in the design of a controller if $c(x)$ is large and a state x is unimportant for the design of a controller if $c(x)$ is small. See Jonckheere and Silverman [5] for more details. Thus if we want all states to be as important in the design of an observer as they are in the design of a controller we actually want P and Q to be equal.

We now take a closer loop at equation (1.2). It can be shown that different quadruples (A, B, C, D) of operators give the same output y for the same input u; from an input-output point of view these quadruples of operators are equivalent. What we will prove in this thesis is that for a given quadruple of operators (with certain properties) there exists an equivalent quadruple of operators such that for this quadruple of operators the solutions P and Q of the FARE and CARE are equal. Such a quadruple of operators is called a LQG-balanced realization of the system.

Outline

In Chapter 2 the general theory of discrete-time systems is reviewed. In Definition 1 the concept of a discrete-time well-posed linear system is introduced. A discrete-time well-posed linear system consists of a quadruple of operators $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ that satisfy certain relations. A system can be described by given this quadruple of operators or the quadruple of operators (A, B, C, D) from (1.2). The quadruple of operators $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is obtained from the quadruple of operators (A, B, C, D) by 'solving' the difference equation (1.2) (see Definitions 7 and 8 for the precise relationships between these two quadruples). The description as a well-posed linear system has some advantages: the important system theoretic notions of stability (Definitions 1 and 2), controllability and observability (Definition 5) and Hankel operator (Definition 4) are easily defined; the concept of equivalent quadruples that was mentioned in the introduction can be rigorously defined as quadruples that have the same '$\mathfrak{D}$' operator (this leads to the concept of realization introduced in Definition 3) and some of the proofs become more transparent.

In section 2.2 we study the Z-transform (Definition 14) that we will use in some calculations in Chapters 5 and 7.

In Chapter 3 we study the Riccati equations mentioned in the introduction. In section 3.3 we will give some equivalent versions of the Riccati equations. The versions (3.1) and (3.2) will be especially important. In section 3.2 we discuss the relation between the operators A_P and A_Q that appear in this version, we will need this relation in Chapter 7. In section 3.4 we will see how the solutions of the Riccati equations transform under a similarity transformation in the state space X, we will need this in Chapter 6 as the last step to prove the existence of a LQG-balanced realization. In section 3.5 conditions on the four operators (A, B, C, D) are given to ensure that the Riccati equations have solutions.

To prove the existence of a LQG-balanced realization we translate the problem to a problem about an associated stable system. We will associate a stable system to our original system in two ways. In Chapter 4 we will factor $\mathfrak{D}$ as $\mathfrak{M}^{-1}\mathfrak{N}$ and consider the operator $[\mathfrak{M}, \mathfrak{N}]$ as the '$\mathfrak{D}$' operator of a new (stable) system. Special kinds of factorizations are considered in Chapter 4: normalized factorizations, coprime factorizations and factorizations with a compact Hankel. In Chapter 5 a stable system that has $[\mathfrak{M}, \mathfrak{N}]$ as its '$\mathfrak{D}$' operator is studied. In particular, the relation between the Riccati equations of the original system and the Lyapunov equations of the associated stable system is studied. All of this leads to Chapter 6 where Theorem 69 gives the existence of a LQG-balanced realization of the original system provided that $\mathfrak{D}$ has a factorization with certain properties. The other way that we associate a stable system with a given system is presented in Chapter 7. There we assume that we are given a quadruple $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ such that the corresponding Riccati equations have a solution. These are then used to construct the so called FARE and CARE closed loop systems (see Definitions 70 and 71). In section 7.2 the relation between the Riccati equations of the original system and the Lyapunov equations of the FARE and CARE closed loop systems is studied. In section 7.3 it is proven that two of the four operators defining the FARE and CARE closed loop systems are stable and in section 7.4 the connection with factorization is studied. In Chapter 8 we return to the problem of the existence of a LQG-balanced realization and prove that such a realization exists if the given quadruple $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ satisfies certain assumptions (Theorem 89).

Theorems, Propositions, Lemmas

The most important results in this thesis are labeled as *Theorem* (there are two of them: Theorem 69 and Theorem 89), the important results are labeled as *Proposition* and the 'technical' results are labeled as *Lemma.* This distinction can be of help when reading. Often the statements of the Lemmas are chosen to facilitate the proof and the statements of the Propositions are chosen in the form that they are used in the rest of this thesis.

If no proof is given this indicates that the result follows easily from the previous results in the section.

How to read this thesis

The reader is warned that the rest of this thesis is more or less one big proof. To avoid getting lost in the technical details and keep some idea of where this thesis is heading it is probably a good idea to return to the introduction and especially the outline given above several times and maybe take a look at the two most important results (Theorem 69 and Theorem 89) before you reach them linearly.

Chapter 2

Discrete-time systems

In this Chapter we review some theory about discrete-time systems.

2.1 Time domain

For a Hilbert space H we define the following maps on the space of functions from $\mathbb{Z}$ to H

$$(\tau h)_k := h_{k+1} \quad (\pi^- h)_k := \begin{cases} h_k & k \in \mathbb{Z}^- \\ 0 & k \in \mathbb{Z}^+ \end{cases} \quad (\pi^+ h)_k := \begin{cases} 0 & k \in \mathbb{Z}^- \\ h_k & k \in \mathbb{Z}^+ \end{cases}$$

Here $\mathbb{Z}^- := \{-1, -2, \ldots\}$ and $\mathbb{Z}^+ := \{0, 1, \ldots\}$. An operator $\mathfrak{D}$ that satisfies $\tau\mathfrak{D} = \mathfrak{D}\tau$ is called *time-invariant.* An operator $\mathfrak{D}$ that satisfies $\pi^-\mathfrak{D}\pi^+ = 0$ is called *causal.* Define $l^2_r(\mathbb{Z}; H)$ to be the space of sequences $(h_k)_{k\in\mathbb{Z}}$ such that $r^{-k}h_k \in l^2(\mathbb{Z}; H)$ with the norm $||h_k||_r := ||r^{-k}h_k||$ (note that $l^2_1 = l^2$). We define $ti_r(U, Y)$ to be the space of bounded time-invariant operators from $l^2_r(\mathbb{Z}; U)$ to $l^2_r(\mathbb{Z}; Y)$ and $tic_r(U, Y)$ to be the subset of $ti_r(U, Y)$ consisting of the operators that are also causal. In the case $r = 1$ we omit the r and just write ti and tic. We further define ti_∞ to be the union over all $r > 0$ of ti_r and we define tic_∞ similarly. It is obvious that tic_r is a subset of $tic_{r'}$ if $r' > r$. We now introduce the concept of a well-posed linear system.

Definition 1 (wpls) *Let $r > 0$ and X, U, Y be Hilbert spaces. An r-stable* discrete-time well-posed linear system *is a quadruple of operators $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ for which*

- $A \in \mathcal{L}(X)$ *satisfies* $\sup_{k\in\mathbb{Z}^+} ||r^{-k}A^k|| < \infty$
- $\mathfrak{B} \in \mathcal{L}(l^2_r(\mathbb{Z}^-, U), X)$ *satisfies* $\mathfrak{B}\tau\pi^- = A\mathfrak{B}$
- $\mathfrak{C} \in \mathcal{L}(X, l^2_r(\mathbb{Z}^+, Y)$ *satisfies* $\mathfrak{C}A = \pi^+\tau\mathfrak{C}$

- $\mathfrak{D} \in tic_r(U, Y)$ *satisfies* $\pi^+ \mathfrak{D} \pi^- = \mathfrak{C}\mathfrak{B}$.

We define $wpls_r$ *to be the set of all r-stable well-posed linear systems and wlps to be the union of these sets over all* $r > 0$. *It is clear that* $wpls_r$ *is a subset of* $wpls_{r'}$ *if* $r' > r$. *An element of* $wpls_1$ *is called* stable. *A wpls that is r-stable for a* $r < 1$ *is called* exponentially *or* power stable.

The components of a well-posed linear system are named as follows: A is called the state operator, $\mathfrak{B}$ is called the controllability map, $\mathfrak{C}$ the observability map and $\mathfrak{D}$ the input-output map of the well-posed linear system.

Definition 2 (State, input, output and input-output stability)
$(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls$ *is called*

- state stable *if* $\sup_{k \in \mathbb{Z}^+} ||A^k|| < \infty$,
- input stable *if* $\mathfrak{B} \in \mathcal{L}(l^2(\mathbb{Z}^-, U), X)$,
- output stable *if* $\mathfrak{C} \in \mathcal{L}(X, l^2(\mathbb{Z}^+, Y)$,
- input-output stable *if* $\mathfrak{D} \in tic(U, Y)$.

We shall also refer to components of a *wpls* as r-stable; for example we call A r-stable if $\sup_{k \in \mathbb{Z}^+} ||r^{-k} A^k|| < \infty$ and $\mathfrak{B}$ r-stable if $\mathfrak{B} \in \mathcal{L}(l_r^2(\mathbb{Z}^-, U), X)$.

Remark In this thesis we will actually never use state stabililty. The notion of 'stability' in the outline should be read as input-output stability.

Remark The reader should note that $wpls_\infty$ would be a more consistent notation when we compare '*wpls*' and '*ti*'. But this notational inconsistency seems to be standard.

Definition 3 (Realization) *A quadruple* $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls$ *is called a* realization *of* $\mathfrak{D}$.

It can be shown that every time-invariant causal operator has a realization; in fact it has infinitely many.

Definition 4 (Hankel operator) *The* Hankel operator Γ *of a* tic_∞ *operator* $\mathfrak{D}$ *is defined to be* $\Gamma := \pi^+ \mathfrak{D} \pi^-$. *The Hankel operator of* $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls$ *is defined to be the Hankel operator of* $\mathfrak{D}$.

It should be noted that the Hankel operator of $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls$ is also equal to $\mathfrak{C}\mathfrak{B}$.

Definition 5 (Controllability and observability) *An element of wpls is called* approximately controllable *if* $\ker \mathfrak{B}^* = \{0\}$ *and* approximately observable *if* $\ker \mathfrak{C} = \{0\}$.

Definition 6 (Gramians) *The controllability gramian L_B of an element of wpls is defined as $L_B = \mathfrak{B}\mathfrak{B}^*$. The observability gramian L_C of an element of wpls is defined as $L_C = \mathfrak{C}^*\mathfrak{C}$.*

Definition 7 (Generating operators) *Let $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls$. Define*

$$(\mathfrak{e}_j)_k := \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \qquad Bu := \mathfrak{B}(ue_{-1}) \quad Cx := (\mathfrak{C}x)_0 \quad Du := (\mathfrak{D}(ue_0))_0$$

The operators $(A, B, C, D) \in \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y)$ are called the generating operators of the well-posed linear system. The operator B is called the control operator, C is called the observation operator and D the feedthrough operator.

Definition 8 *Let $(A, B, C, D) \in \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y)$. Let $r > r(A)$ (here $r(A)$ is the spectral radius of A). Define for $x \in X$ and $u \in l_r^2(\mathbb{Z}, U)$*

$$\mathfrak{B}u := \sum_{k=0}^{\infty} A^k B u_{-k-1}$$

$$(\mathfrak{C}x)_k := \begin{cases} 0 & k \in \mathbb{Z}^- \\ CA^k x & k \in \mathbb{Z}^+ \end{cases}$$

$$(\mathfrak{D}u)_k := \sum_{i=0}^{\infty} CA^i B u_{k-i-1} + Du_k \quad k \in \mathbb{Z},$$

where the convergence of the sums is in the strong toplogy. Then $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls_r$ and it is called the well-posed linear system generated by the quadruple (A, B, C, D).

We thus have a one-to-one correspondence between *wpls* and $\mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y)$ if we identify $wpls_r$ with a subset of $wpls_{r'}$ if $r' > r$. We shall refer to both a well-posed linear system $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ and its quadruple of generating operators (A, B, C, D) as a *system*. For a stable well-posed linear system we can express the gramians in terms of the generating operators.

Lemma 9 *Let the system* (A, B, C, D) *be input stable. Then its controllability gramian is given by the formula:*

$$L_B = \sum_{k=0}^{\infty} A^k BB^* A^{*k}$$

and it is the smallest nonnegative solution of the control Lyapunov equation*:*

$$ALA^* - L + BB^* = 0 \tag{2.1}$$

in the unknown $L \in \mathcal{L}(X)$. *Let the system* (A, B, C, D) *be output stable. Then its observability gramian is given by the formula:*

$$L_C = \sum_{k=0}^{\infty} A^{*k} C^* C A^k$$

and it is the smallest nonnegative solution of the observer Lyapunov equation*:*

$$A^* LA - L + C^* C = 0 \tag{2.2}$$

in the unknown $L \in \mathcal{L}(X)$ *(the convergence of the sums is in the strong topology).*

Proof The stability assumption implies that the sums converge and it is easily checked that L_B and L_C as given above satisfy their respective equations. We will prove that L_C is the smallest nonnegative solution of the observer Lyapunov equation, the proof for L_B and the control Lyapunov equation is similar. Suppose that L is another nonnegative solution of the observer Lyapunov equation. Then for all $x \in X$:

$$< LAx, Ax > - < Lx, x > + < Cx, Cx >= 0.$$

If we subsitute $x = A^k z$ we get:

$$< CA^k z, CA^k z >=< LA^k z, A^k z > - < LA^{k+1} z, A^{k+1} z > .$$

We now sum from $k = 0$ to N and note that the right-hand side telescopes:

$$\sum_{k=0}^{N} < CA^k z, CA^k z >=< Lz, z > - < LA^{N+1} z, A^{N+1} z > .$$

Since L is nonnegative we thus have for every N:

$$\sum_{k=0}^{N} < z, A^{*k} C^* CA^k z > \leq < Lz, z >$$

and letting $N \to \infty$ we have $L_C \leq L$ as desired. □

More on well-posed linear system can be found in Mikkola [7] and Staffans [12].

2.2 Frequency domain

In this section we introduce some spaces that we need in our study of well-posed linear systems.

Definition 10 *Let H be a Hilbert space. We denote $\mathbf{L}^2(H)$ for the space of (equivalence classes of) square integrable functions from the unit circle to H and $\mathbf{H}^2(H)$ for the space of functions holomorphic on the open unit disc with the property that*

$$\sup_{r<1} \int_0^{2\pi} ||f(re^{i\theta})|| d\theta < \infty.$$

We collect some properties of these spaces:

- The space $\mathbf{L}^2$ can be shown to be a Hilbert space and each function in $\mathbf{L}^2$ has a representation

  $$f(z) = \sum_{n=-\infty}^{\infty} < f, e_n > e_n,$$

  where $e_n(\theta) = e^{in\theta}$.

- It can be shown that for every $f \in \mathbf{H}^2$ $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all $\theta \in [0, 2\pi)$ and that the resulting function on the unit disc is in $\mathbf{L}^2$. This map from $\mathbf{H}^2$ to $\mathbf{L}^2$ is injective and $\mathbf{H}^2$ can thus be seen as a subspace of $\mathbf{L}^2$. This subspace is closed and consists exactly of those functions for which $< f, e_n >= 0$ for all $n < 0$.

- The above representation gives us an isometric isomorphism between $\mathbf{L}^2(H)$ and $l^2(\mathbb{Z}; H)$ that maps $\mathbf{H}^2(H)$ onto $l^2(\mathbb{Z}^+; H)$.

Definition 11 *For $r > 0$ we define $\mathbf{L}^2_r$ to be the space of functions such that $f(r\cdot) \in \mathbf{L}^2$ and $\mathbf{H}^2_r$ to be the space of functions such that $f(r\cdot) \in \mathbf{H}^2$.*

Definition 12 *Let W be a Banach space. We denote $\mathbf{L}^\infty(W)$ for the space of (equivalence classes of) essentially bounded functions from the unit circle to W and $\mathbf{H}^\infty(W)$ for the space of functions holomorpic on the open unit disc with the property that*

$$\sup_{|z|<1} ||f(z)|| < \infty.$$

The space $\mathbf{L}^\infty$ can be shown to be a Banach space. It can also be shown that for every $f \in \mathbf{H}^\infty$ $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all $\theta \in [0, 2\pi)$ and the resulting function is in $\mathbf{L}^\infty$. This map from $\mathbf{H}^\infty$ to $\mathbf{L}^\infty$ is injective and $\mathbf{H}^\infty$ can thus be seen as a subspace of $\mathbf{L}^\infty$. This subspace is closed.

Definition 13 *For $r > 0$ we define $\mathbf{L}^\infty_r$ to be the space of functions such that $f(r\cdot) \in \mathbf{L}^\infty$ and $\mathbf{H}^\infty_r$ to be the space of functions such that $f(r\cdot) \in \mathbf{H}^\infty$. Further we define $\mathbf{H}^\infty_\infty$ to be the union over all $r \in (0,1]$ of $\mathbf{H}^\infty_r$.*

The above representation of $\mathbf{L}^2$ functions leads to the following concept:

Definition 14 (Z-transform) *The Z-transform $\hat{h}$ or $\mathcal{Z}h$ of $h : \mathbb{Z} \to H$ is defined as*

$$\hat{h}(z) := \sum_{j \in \mathbb{Z}} h_j z^j$$

for those z for which the sum converges absolutely.

We now study how the maps $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ 'transform' under the Z-transform.

Definition 15 *Let $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in wpls_r$ with generating operators (A, B, C, D). We define for $z < 1/r(A)$ (here $r(A)$ is the spectral radius of A)*

$$\hat{\mathfrak{B}}(z) := z(I - zA)^{-1}B = (\frac{1}{z}I - A)^{-1}B = \sum_{k=0}^{\infty} A^k B z^{k+1}$$

$$\hat{\mathfrak{C}}(z) := C(I - zA)^{-1} = \frac{1}{z}C(\frac{1}{z}I - A)^{-1} = \sum_{k=0}^{\infty} C A^k z^k$$

$$\hat{\mathfrak{D}}(z) := D + Cz(I - zA)^{-1}B = D + C(\frac{1}{z}I - A)^{-1}B =$$

$$D + \sum_{k=0}^{\infty} C A^k B z^{k+1} = D + \hat{\mathfrak{C}}(z)Bz = D + C\hat{\mathfrak{B}}(z).$$

With these definitions we have $\widehat{\mathfrak{B}\tau u} = \hat{\mathfrak{B}}\hat{u}$, $\widehat{\mathfrak{C}x} = \hat{\mathfrak{C}}x$ and $\widehat{\mathfrak{D}u} = \hat{\mathfrak{D}}\hat{u}$ on $\mathbb{D}_{1/r}$ (the disc of radius $1/r$ centered at the origin) for all $x \in X$ and $u \in l^2_r(\mathbb{Z}^+, U)$.

The map $\hat{\mathfrak{D}}$ is an element of $\mathbf{H}^\infty_{1/r}(\mathcal{L}(U, Y))$ and is called the *transfer function* of the system. We shall often write $\mathbf{H}^\infty_{1/r}(U, Y)$ instead of $\mathbf{H}^\infty_{1/r}(\mathcal{L}(U, Y))$. The map $\mathfrak{D} \mapsto \hat{\mathfrak{D}}$ from $tic_r(U, Y)$ to $\mathbf{H}^\infty_{1/r}(U, Y)$ is an isometric isomorphism.

The stability properties of the system translate to the frequency domain as follows. The system is input stable iff $B^*(I - zA^*)^{-1}x \in \mathbf{H}^2(U)$ for all $x \in X$. The system is output stable iff $C(I - zA)^{-1}x \in \mathbf{H}^2(Y)$ for all $x \in X$. The system is input-output stable iff $\hat{\mathfrak{D}} \in \mathbf{H}^\infty(U, Y)$. These stability properties lead to the following definition.

Definition 16 *Let X, H be Hilbert spaces. With $\mathbf{H}^2_{strong}(X, H)$ we denote the space of $\mathcal{L}(X, H)$ valued functions T such that $T(\cdot)x \in \mathbf{H}^2(H)$ for all $x \in X$.*

We will later need the following lemmas.

Lemma 17 *For all* $0 < r < 1$ $\mathbf{H}^2_{strong}(X,H)$ *is a subset of* $\mathbf{H}^\infty_r(X,H)$.

Proof Let $T \in \mathbf{H}^2_{strong}$. A function in $\mathbf{H}^2_{strong}$ is holomorphic on the open unit disc, we thus only have to prove that T is bounded on a disc of radius r for every $r \in (0,1)$. Since a holomorphic function is continuous and the closed disc of radius r is compact T is bounded on this closed disc and thus on the open disc of radius r. $\square$

Lemma 18 *A function that is an element of both* $\mathbf{H}^2_{strong}$ *and* $\mathbf{L}^\infty$ *is an element of* $\mathbf{H}^\infty$.

Proof Let $T \in \mathbf{H}^2_{strong} \cap \mathbf{L}^\infty$. Then T is analytic on the open unit disc and bounded on the unit circle. This implies that T is analytic on the open unit disc and bounded on the closed unit disc. $\square$

More on the spaces $\mathbf{L}^2, \mathbf{H}^2, \mathbf{L}^\infty, \mathbf{H}^\infty$ can be found in Rosenblum and Rovnyak [11] and Hoffman [2]. More on the use of these spaces in the theory of well-posed linear systems can be found in [7] and [12].

Chapter 3

Riccati equations

To a well-posed linear system we can associate the following equations

$$A_P P(I + C^* R^{-1} C P) A_P^* - P + B S^{-1} B^* = 0, \tag{3.1}$$

where

$$A_P := A - (BD^* + APC^*)(R + CPC^*)^{-1} C \quad R := I + DD^* \quad S := I + D^* D.$$

This equation is called the Filter Algebraic Riccati Equation (FARE) of the system. The following equation is called the Control Algebraic Riccati Equation (CARE) of the system

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - Q + C^* R^{-1} C = 0, \tag{3.2}$$

where

$$A_Q := A - B(S + B^* Q B)^{-1}(D^* C + B^* Q A).$$

The topic of this Chapter is the properties of the nonnegative solutions P and Q of these equations.

Throughout this Chapter (A, B, C, D) will be the generating operators of a well-posed linear system and $R := I + DD^*$ and $S := I + D^* D$.

3.1 Existence of inverses

In the rest of this thesis we take inverses of operators without explicitly verifying that these exist, but the reader can easily check that they do using the following Lemma.

Lemma 19 *If E and F are nonnegative operators on a Hilbert space $I + EF$ is boundedly invertible.*

Proof We use the well-known fact that for bounded linear operators S and T the operators ST and TS have the same spectrum. Thus EF and $\sqrt{E}F\sqrt{E}$ have the same spectrum. Since $\sqrt{E}F\sqrt{E}$ is nonnegative -1 is not in its spectrum and thus not in the spectrum of EF. That is: $I + EF$ is boundedly invertible. □

3.2 Some algebraic relations

Lemma 20 *Let P and Q be nonnegative operators. We recall that*

$$A_P := A - (BD^* + APC^*)(R + CPC^*)^{-1}C \tag{3.3}$$

$$A_Q := A - B(S + B^*QB)^{-1}(D^*C + B^*QA) \tag{3.4}$$

and define

$$\bar{A} := A - BS^{-1}D^*C. \tag{3.5}$$

Then

$$A_P(I + PC^*R^{-1}C) = \bar{A} = (I + BS^{-1}B^*Q)A_Q \tag{3.6}$$

$$A_Q = (I + BS^{-1}B^*Q)^{-1}A_P(I + PC^*R^{-1}C) \tag{3.7}$$

$$A_P = (I + BS^{-1}B^*Q)A_Q(I + PC^*R^{-1}C)^{-1}. \tag{3.8}$$

Proof We prove that $A_P(I + PC^*R^{-1}C) = \bar{A}$. The equality $\bar{A} = (I + BS^{-1}B^*Q)A_Q$ is proven similarly. By writing out A_P in full we write $A_P(I + PC^*R^{-1}C)$ as

$$A(I + PC^*R^{-1}C) - (BD^* + APC^*)(R + CPC^*)^{-1}C(I + PC^*R^{-1}C),$$

which is equal to

$$A(I + PC^*R^{-1}C) - (BD^* + APC^*)(R + CPC^*)^{-1}(R + CPC^*)R^{-1}C,$$

which is equal to

$$A + APC^*R^{-1}C - (BD^* + APC^*)R^{-1}C = A - BD^*R^{-1}C = A - BS^{-1}D^*C,$$

since $D^*R^{-1} = S^{-1}D^*$. This completes the proof of (3.6). Equations (3.7) and (3.8) easily follow from (3.6) and Lemma 19 that guaranties the existence of the given inverses. □

Note that in the above Lemma we have not assumed that P and Q are solutions of the Riccati equations.

Proposition 21 *Let P and Q be nonnegative solutions of the Riccati equations of (A, B, C, D) and define A_P and A_Q by (3.3) and (3.4). Then*

$$(I + PQ)A_Q = A_P(I + PQ).$$

Proof We use the FARE (3.1) to write

$$P = A_P P(I + C^* R^{-1} C P)A_P^* + BS^{-1}B^*,$$

which leads to

$$I + PQ = I + A_P P(I + C^* R^{-1} C P)A_P^* Q + BS^{-1}B^* Q$$

and so

$$(I + PQ)A_Q = (I + BS^{-1}B^* Q)A_Q + A_P P(I + C^* R^{-1} C P)A_P^* Q A_Q.$$

We use (3.7) to write the right-hand side as

$$A_P(I + PC^* R^{-1} C) + A_P P(I + C^* R^{-1} C P)A_P^* Q A_Q.$$

Rearranging gives

$$A_P + A_P P[C^* R^{-1} C + (I + C^* R^{-1} C P)A_P^* Q A_Q]$$

and using (3.7) again we obtain

$$A_P + A_P P[C^* R^{-1} C + A_Q^*(I + QBS^{-1}B^*)QA_Q].$$

According to the CARE (3.2) the term in square brackets equals Q. So the above is equal to $A_P(I + PQ)$. □

3.3 Equivalent versions of the Riccati equations

Lemma 22 *1. P is a nonnegative solution of (3.1) iff it is a nonnegative solution of*

$$\bar{A}P(I + C^* R^{-1} C P)^{-1}\bar{A}^* - P + BS^{-1}B^* = 0, \qquad (3.9)$$

where $\bar{A}$ is defined by (3.5).

2. P is a nonnegative solution of (3.1) iff it is a nonnegative solution of

$$APA^* - P + BB^* = (APC^* + BD^*)(R + CPC^*)^{-1}(CPA^* + DB^*). \tag{3.10}$$

3. Q is a nonnegative solution of (3.2) iff it is a nonnegative solution of

$$\bar{A}^*Q(I + BS^{-1}B^*Q)^{-1}\bar{A} - Q + C^*R^{-1}C = 0, \tag{3.11}$$

where $\bar{A}$ is defined by (3.5).

4. Q is a nonnegative solution of (3.2) iff it is a nonnegative solution of

$$A^*QA - Q + C^*C = (C^*D + A^*QB)(S + B^*QB)^{-1}(B^*QA + D^*C). \tag{3.12}$$

Proof We shall prove the equivalence of the filter equations; the equivalence of the control equations is similar.
1. The equations (3.1) and (3.9) are equivalent iff the following holds

$$\bar{A}P(I + C^*R^{-1}CP)^{-1}\bar{A}^* = A_P P(I + C^*R^{-1}CP)A_P^*. \tag{3.13}$$

We use Lemma 20 (which tells us that $\bar{A} = A_P(I + PC^*R^{-1}C)$) to write the left-hand side of equation (3.13) as

$$A_P(I + PC^*R^{-1}C)P(I + C^*R^{-1}CP)^{-1}(I + C^*R^{-1}CP)A_P^*,$$

which is indeed equal to the right-hand side of equation (3.13).
2. To prove the equivalence of (3.1) and (3.10) we substitute in (3.1) for A_P from (3.3) and for $(I + C^*R^{-1}CP)A_P^*$ we substitute $\bar{A}^*$ (using (3.6)) and then substitute (3.5) for $\bar{A}$. We then get

$$(A-(BD^*+APC^*)(R+CPC^*)^{-1}C)P(A^*-C^*DS^{-1}B^*)-P+BS^{-1}B^* = 0.$$

Rewriting this gives

$$APA^* - P + BB^* = (BD^* + APC^*)(R + CPC^*)^{-1}CPA^*$$
$$-(BD^* + APC^*)(R + CPC^*)^{-1}CPC^*DS^{-1}B^* + APC^*DS^{-1}B^*$$
$$-BS^{-1}B^* + BB^*.$$

We now focus on the last two lines of this last equation. We note that $I - S^{-1} = D^*DS^{-1}$ and we can thus rewrite these last two lines as

$$-(BD^*+APC^*)(R+CPC^*)^{-1}CPC^*DS^{-1}B^*+APC^*DS^{-1}B^*+BD^*DS^{-1}B^*$$

and this can be rewritten as

$$(BD^* + APC^*)(R + CPC^*)^{-1}[-CPC^* + R + CPC^*]DS^{-1}B^*.$$

Noting that $RDS^{-1} = D$ we see that this is equal to

$$(BD^* + APC^*)(R + CPC^*)^{-1}DB^*.$$

This completes the proof of the equivalence of (3.1) and (3.10) □

3.4 Similarity transformations

Proposition 23 *Let (A, B, C, D) be a system with nonnegative solutions of its Riccati equations P and Q and let V be an invertible operator on the state space. Then $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) := (VAV^{-1}, VB, CV^{-1}, D)$ has the same transfer function and $\bar{P} := VPV^*$ and $\bar{Q} := V^{-*}QV^{-1}$ are solutions of the Riccati equations corresponding to this new realization.*

Proof The transfer function of $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is

$$\bar{C}(\frac{1}{z}I - \bar{A})^{-1}\bar{B} + \bar{D} = CV^{-1}(\frac{1}{z}I - VAV^{-1})^{-1}VB + D$$

$$= CV^{-1}[V(\frac{1}{z}I - A)V^{-1}]^{-1}VB + D = CV^{-1}[V(\frac{1}{z}I - A)^{-1}V^{-1}]VB + D$$

$$= C(\frac{1}{z}I - A)^{-1}B + D,$$

which is the transfer function of (A, B, C, D). That proves the first part of the Proposition. For the second part we use the version (3.10) of the FARE. We want to show that

$$\bar{A}\bar{P}\bar{A}^* - \bar{P} + \bar{B}\bar{B}^* = (\bar{A}\bar{P}\bar{C}^* + \bar{B}\bar{D}^*)(R + \bar{C}\bar{P}\bar{C}^*)^{-1}(\bar{C}\bar{P}\bar{A}^* + \bar{D}\bar{B}^*),$$

which is equivalent to

$$VAV^{-1}VPV^*V^{-*}A^*V^* - VPV^* + VBB^*V^* = (VAV^{-1}VPV^*V^{-*}C^* + VBD^*)$$
$$(R + CV^{-1}VPV^*V^{-*}C^*)^{-1}(CV^{-1}VPV^*V^{-*}A^*V^* + DB^*V^*)$$

and this is is equivalent to

$$V[APA^* - P + BB^*]V^* = V(APC^* + BD^*)(R + CPC^*)^{-1}(CPA^* + DB^*)V^*.$$

Finally, the last expression is true because P is a solution of the FARE of (A, B, C, D). The proof for the CARE is similar. □

3.5 Sufficient conditions for existence of solutions

In the next definitions the '-' stands for an unimportant entry, $[\alpha, \beta]$ is a row vector of operators and $[\alpha; \beta]$ is a column vector of operators.

Definition 24 (output stabilizable) *A system* (A, B, C, D) *is called* output stabilizable *if there exists an operator* $F \in \mathcal{L}(X, U)$ *such that* $(A + BF, -, [F; C + DF], -)$ *is output stable.*

Definition 25 (input stabilizable) *A system* (A, B, C, D) *is called* input stabilizable *if there exists an operator* $L \in \mathcal{L}(Y, X)$ *such that* $(A + LC, [L, B + LD], -, -)$ *is input stable.*

Proposition 26 *If a system is output stabilizable then there exists a nonnegative solution of its CARE.*

Proof The proof is outlined in exercise 6.34 of Curtain and Zwart [1]. □

The dual result is.

Proposition 27 *If a system is input stabilizable then there exists a nonnegative solution of its FARE.*

Chapter 4

Factorizations

In this Chapter we study the concept of a factor of a tic_∞ operator.

Definition 28 (Factor) *An operator* $[\mathfrak{M};\mathfrak{N}] \in tic(U, U \times Y)$ *is called a* right factor *of an operator* $\mathfrak{D} \in tic_\infty(U,Y)$ *if* $\mathfrak{M}$ *has an inverse in* $tic_\infty(U)$ *and* $\mathfrak{D} = \mathfrak{N}\mathfrak{M}^{-1}$. *An operator* $[\tilde{\mathfrak{M}},\tilde{\mathfrak{N}}] \in tic(Y \times U, Y)$ *is called a* left factor *of an operator* $\mathfrak{D} \in tic_\infty(U,Y)$ *if* $\tilde{\mathfrak{M}}$ *has an inverse in* $tic_\infty(Y)$ *and* $\mathfrak{D} = \tilde{\mathfrak{M}}^{-1}\tilde{\mathfrak{N}}$.

We can translate the above concepts to the frequency domain:

Definition 29 (Factor) *A function* $[M;N] \in \mathbf{H}^\infty(U, U \times Y)$ *is called a* right factor *of a function* $G \in \mathbf{H}^\infty_\infty(U,Y)$ *if* M *has an inverse in* $\mathbf{H}^\infty_\infty(U)$ *and* $G = NM^{-1}$. *A function* $[\tilde{M},\tilde{N}] \in \mathbf{H}^\infty(Y \times U, Y)$ *is called a* left factor *of a function* G *in* $\mathbf{H}^\infty_\infty(U,Y)$ *if* $\tilde{M}$ *has an inverse in* $\mathbf{H}^\infty_\infty(Y)$ *and* $G = \tilde{M}^{-1}\tilde{N}$.

4.1 Normalized factorizations

Definition 30 (Inner and co-inner) *An operator* $\mathfrak{D} \in tic(U,Y)$ *is called* inner *if* $\mathfrak{D}^*\mathfrak{D} = I$ *and* co-inner *if* $\mathfrak{D}\mathfrak{D}^* = I$.

The frequency domain equivalent is as follows.

Definition 31 (Inner and co-inner) *A function* $G \in \mathbf{H}^\infty(U,Y)$ *is called* inner *if* $G(z)^*G(z) = I$ *for almost all* z *on the unit circle and* co-inner *if* $G(z)G(z)^* = I$ *for almost all* z *on the unit circle.*

We now derive some useful properties of realizations of inner *tic* operators.

Lemma 32 *Suppose that* (A,B,C,D) *is an approximately controllable and output stable realization of the inner tic operator* $\mathfrak{D}$. *Let* L_C *be its observability gramian. Then*

$$B^*L_CA + D^*C = 0. \tag{4.1}$$

Proof We have

$$\begin{aligned}
&\pi^+\mathfrak{D}^*\mathfrak{C}\mathfrak{B} & \\
&= \pi^+\mathfrak{D}^*\pi^+\mathfrak{D}\pi^- && \text{since } \mathfrak{C}\mathfrak{B} = \pi^+\mathfrak{D}\pi^- \\
&= \pi^+\mathfrak{D}^*(\pi^+ + \pi^-)\mathfrak{D}\pi^- && \text{since } \pi^+\mathfrak{D}^*\pi^- = 0 \\
&= \pi^+\mathfrak{D}^*\mathfrak{D}\pi^- && \text{since } \pi^+ + \pi^- = I \\
&= \pi^+\pi^- && \text{since } \mathfrak{D} \text{ is inner} \\
&= 0 &
\end{aligned}$$

This last equality comes from the fact that π^+ and π^- are projections on spaces that are each others orthogonal complements. So we have that $\pi^+\mathfrak{D}^*\mathfrak{C}\mathfrak{B} = 0$ and hence $\mathfrak{B}^*\mathfrak{C}^*\mathfrak{D}\pi^+ = 0$. Since the system is approximately controllable (see Definition 5) this implies that $\mathfrak{C}^*\mathfrak{D}\pi^+ = 0$, which in turn implies that $\pi^+\mathfrak{D}^*\mathfrak{C} = 0$. In particular, $(\mathfrak{D}^*\mathfrak{C})_0 = (\pi^+\mathfrak{D}^*\mathfrak{C})_0 = 0$. Translated in terms of the generating operators this is

$$\sum_{i=0}^{\infty} B^*A^{*i}C^*CA^{i+1} + D^*C = 0.$$

From Lemma 9 the left-hand side of this equation equals $B^*L_CA + D^*C$ and so we have proven (4.1). □

Lemma 33 *Suppose that (A, B, C, D) is an output stable realization of the inner tic operator $\mathfrak{D}$. Let L_C be its observability gramian. Then*

$$B^*L_CB + D^*D = I. \tag{4.2}$$

Proof Let $u \in l^2(\mathbb{Z}; U)$ be arbitrary. Then using Definition 8 we have

$$u_n = (\mathfrak{D}^*\mathfrak{D}u)_n = \sum_{j=0}^{\infty} B^*A^{*j}C^*(\mathfrak{D}u)_{n+j+1} + D^*(\mathfrak{D}u)_n$$

$$= \sum_{j=0}^{\infty} B^*A^{*j}C^*(\sum_{i=0}^{\infty} CA^iBu_{n+j-i} + Du_{n+j+1}) + D^*(\sum_{i=0}^{\infty} CA^iBu_{n-i-1} + Du_n)$$

$$= \sum_{j=0}^{\infty} B^*A^{*j}C^* \sum_{i=0, i\neq j}^{\infty} CA^iBu_{n+j-i} + \sum_{j=0}^{\infty} B^*A^{*j}C^*Du_{n+j+1} + \sum_{i=0}^{\infty} D^*CA^iBu_{n-i-1}$$

$$+\sum_{j=0}^{\infty} B^* A^{*j} C^* C A^j B u_n + D^* D u_n.$$

The coefficient of u_n in this last expression is:

$$\sum_{j=0}^{\infty} B^* A^{*j} C^* C A^j B + D^* D = B^* L_C B + D^* D,$$

where the last equality follows from the formula for the observability gramian from Lemma 9. Since u was arbitrary, this coefficient must be the identity and we have proven (4.2). □

Similar Lemmas hold for co-inner operators. Since the proofs are similar we shall omit them.

Lemma 34 *Suppose that (A, B, C, D) is an approximately observable and input stable realization of the co-inner tic operator $\mathfrak{D}$. Let L_B be its controllability gramian. Then*

$$CL_B A^* + DB^* = 0. \tag{4.3}$$

Lemma 35 *Suppose that (A, B, C, D) is an input stable realization of the co-inner tic operator $\mathfrak{D}$. Let L_B be its controllability gramian. Then*

$$CL_B C^* + DD^* = I. \tag{4.4}$$

Definition 36 (Normalized factor) *A left factor is called normalized if it is co-inner and a right factor is called normalized if it is inner.*

Proposition 37 *Suppose that (A, B, C, D) is an approximately controllable and output stable realization of a normalized right factor. Let L_C be its observability gramian. Then the following identities hold*

$$B^* L_C A + D^* C = 0 \tag{4.5}$$

$$B^* L_C B + D^* D = I. \tag{4.6}$$

Proposition 38 *Suppose that (A, B, C, D) is an approximately observable and input stable realization of a normalized left factor. Let L_B be its controllability gramian. Then the following identities hold*

$$CL_B A^* + DB^* = 0 \tag{4.7}$$

$$CL_B C^* + DD^* = I. \tag{4.8}$$

4.2 Coprime factorizations

Lemma 39 *A tic operator $\mathfrak{D}$ that is inner and has a left inverse in tic has a Hankel operator with norm strictly smaller than one.*

Proof We first prove that an inner operator $\mathfrak{D}$ has norm one

$$||\mathfrak{D}|| = \sup_{x:||x||=1} < \mathfrak{D}x, \mathfrak{D}x >= \sup_{x:||x||=1} < \mathfrak{D}^*\mathfrak{D}x, x >= \sup_{x:||x||=1} < x, x >= 1.$$

This implies that the Hankel operator of $\mathfrak{D}$ has norm smaller than or equal to one

$$||\pi^+\mathfrak{D}\pi^-|| \leq ||\pi^+||||\mathfrak{D}||||\pi^-|| = 1.$$

We shall now show that the norm of the Hankel operator cannot be equal to one. Suppose it is. Then there exists a sequence u_n with norm one such that $||\pi^+\mathfrak{D}\pi^-u_n|| \to 1$. This implies that π^-u_n cannot be zero for large n and we can thus assume that $u_n = \pi^-u_n$ for all n (omit the first terms if necessary and replace u_n by $\pi^-u_n/||\pi^-u_n||$). Define $g_n := \mathfrak{D}\pi^-u_n$, $g_n^+ := \pi^+g_n$, $g_n^- := \pi^-g_n$. Then

$$||g_n^+||^2 + ||g_n^-||^2 = ||g_n||^2 = ||\mathfrak{D}\pi^-u_n||^2 \leq ||\mathfrak{D}||^2||\pi^-||^2||u_n||^2 = 1$$

and since $||g_n^+|| = ||\pi^+\mathfrak{D}\pi^-u_n|| \to 1$ we have $||g_n^-|| \to 0$. We now use that $\mathfrak{D}$ has a left inverse in *tic*, call this left inverse $\mathfrak{X}$. We have

$$\pi^-u_n = \mathfrak{X}\mathfrak{D}\pi^-u_n = \mathfrak{X}g_n = \mathfrak{X}g_n^+ + \mathfrak{X}g_n^-.$$

Using the fact that $\mathfrak{X}$ is causal we obtain

$$< \pi^-u_n, \mathfrak{X}g_n^+ >=< u_n, \pi^-\mathfrak{X}\pi^+g_n >= 0.$$

So

$$0 =< \pi^-u_n, \mathfrak{X}g_n^+ >=< \pi^-u_n, \pi^-u_n - \mathfrak{X}g_n^- >=$$
$$< \pi^-u_n, \pi^-u_n > - < \pi^-u_n, \mathfrak{X}g_n^- >\to 1,$$

since $\pi^-u_n = u_n$, u_n has norm one, $g_n^- \to 0$ and $\mathfrak{X}$ is continuous. This is a contradiction. So the norm of the Hankel operator must be strictly smaller than one. □

We have the following dual result. Since the proof is similar, we shall omit it.

Lemma 40 *A tic operator $\mathfrak{D}$ that is co-inner and has a right inverse in tic has a Hankel operator with norm strictly smaller than one.*

Definition 41 (Coprime) *Two operators* $\mathfrak{M} \in tic(U, Z)$ *and* $\mathfrak{N} \in tic(U, Y)$ *are called* right coprime *if* $[\mathfrak{M}; \mathfrak{N}]$ *has a left inverse in* $tic(Z \times Y, U)$. *Two operators* $\tilde{\mathfrak{M}} \in tic(Z, Y)$ *and* $\tilde{\mathfrak{N}} \in tic(U, Y)$ *are called* left coprime *if* $[\tilde{\mathfrak{M}}, \tilde{\mathfrak{N}}]$ *has a right inverse in* $tic(Y, Z \times U)$.

These concepts translate to frequency domain as follows.

Definition 42 (Coprime) *Two functions* $M \in \mathbf{H}^\infty(U, Z)$ *and* $N \in \mathbf{H}^\infty(U, Y)$ *are called* right coprime *if* $[M; N]$ *has a left inverse in* $\mathbf{H}^\infty(Z \times Y, U)$. *Two functions* $\tilde{M} \in \mathbf{H}^\infty(Z, Y)$ *and* $\tilde{N} \in \mathbf{H}^\infty(U, Y)$ *are called* left coprime *if* $[\tilde{M}, \tilde{N}]$ *has a right inverse in* $\mathbf{H}^\infty(Y, Z \times U)$.

Definition 43 (Coprime factors) *A right factor* $[\mathfrak{M}; \mathfrak{N}]$ *is called coprime if* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are right coprime. A left factor*$[\tilde{\mathfrak{M}}, \tilde{\mathfrak{N}}]$ *is called coprime if* $\tilde{\mathfrak{M}}$ *and* $\tilde{\mathfrak{N}}$ *are left coprime.*

The next two Lemmas are special cases of the previous two Lemmas.

Lemma 44 *The Hankel operator of a normalized right coprime factor has norm stricly smaller than one.*

Lemma 45 *The Hankel operator of a normalized left coprime factor has norm stricly smaller than one.*

Lemma 46 *Consider a system with gramians* L_B *and* L_C *and Hankel operator* $\Gamma = \mathfrak{C}\mathfrak{B}$. *Then the spectral radius* $r(L_B L_C)$ *of* $L_B L_C$ *satisfies* $r(L_B L_C) = ||\Gamma||^2$.

Proof For an operator T we have $||T||^2 = ||T^*T|| = r(T^*T)$ and hence we have $||\Gamma||^2 = r(\mathfrak{B}^*\mathfrak{C}^*\mathfrak{C}\mathfrak{B})$. For operators S and T we have $r(ST) = r(TS)$ and thus we have: $||\Gamma||^2 = r(\mathfrak{B}\mathfrak{B}^*\mathfrak{C}^*\mathfrak{C}) = r(L_B L_C)$. □

A direct consequence of Lemmas 44-46 are the following results.

Proposition 47 *A realization of a normalized right coprime factor with gramians* L_B *and* L_C *has the property that* $I - L_B L_C$ *is boundedly invertible.*

Proposition 48 *A realization of a normalized left coprime factor with gramians* L_B *and* L_C *has the property that* $I - L_B L_C$ *is boundedly invertible.*

4.3 Factorizations with compact Hankel

Lemma 49 *Let* $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in$ *wpls with Hankel operator* Γ *and gramians* L_B *and* L_C. *If* $I - \Gamma\Gamma^*$ *is injective and the system is approximately observable, then* $I - L_B L_C$ *is injective.*

Proof We have

$$\mathfrak{C}(I - L_B L_C) = \mathfrak{C}(I - \mathfrak{B}\mathfrak{B}^*\mathfrak{C}^*\mathfrak{C}) = (I - \mathfrak{C}\mathfrak{B}\mathfrak{B}^*\mathfrak{C}^*)\mathfrak{C} = (I - \Gamma\Gamma^*)\mathfrak{C}.$$

Suppose $(I - L_B L_C)x = 0$, then $(I - \Gamma\Gamma^*)\mathfrak{C}x = 0$ according to the above. Since both $(I - \Gamma\Gamma^*)$ and $\mathfrak{C}$ are injective by assumption we have $x = 0$. □

Lemma 50 *Let* $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in$ *wpls with Hankel operator* Γ *and gramians* L_B *and* L_C. *Assume that* Γ *is compact,* $I - \Gamma\Gamma^*$ *is boundedly invertible and the system is approximately observable. Then* $I - L_B L_C$ *is boundedly invertible.*

Proof According to the previous Lemma $I - L_B L_C$ is injective. That is, one is not in the point spectrum of $L_B L_C$. Since $L_B L_C = \mathfrak{B}\mathfrak{B}^*\mathfrak{C}^*\mathfrak{C} = \mathfrak{B}\Gamma^*\mathfrak{C}$, we have that $L_B L_C$ is compact. Since for compact operators the spectrum is contained in the union of the point spectrum and zero, one is not in the spectrum of $L_B L_C$. Thus $I - L_B L_C$ is boundedly invertible. □

We have the following dual Lemmas.

Lemma 51 *Let* $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in$ *wpls with Hankel operator* Γ *and gramians* L_B *and* L_C. *If* $I - \Gamma^*\Gamma$ *is injective and the system is approximately controllable, then* $I - L_C L_B$ *is injective.*

Lemma 52 *Let* $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) \in$ *wpls with Hankel operator* Γ *and gramians* L_B *and* L_C. *If* Γ *is compact,* $I - \Gamma^*\Gamma$ *is boundedly invertible and the system is approximately controllable, then* $I - L_B L_C$ *is boundedly invertible.*

Proposition 53 *Suppose* $\mathfrak{D}$ *has a normalized left factor with compact Hankel operator* Γ *and such that* $I - \Gamma\Gamma^*$ *is boundedly invertible. Then for every approximately observable realization of this factor* $I - L_B L_C$ *is boundedly invertible, where* L_B *and* L_C *are the gramians of this realization.*

Proposition 54 *Suppose* $\mathfrak{D}$ *has a normalized right factor with compact Hankel operator* Γ *and such that* $I - \Gamma^*\Gamma$ *is boundedly invertible. Then for every approximately controllable realization of this factor* $I - L_B L_C$ *is boundedly invertible, where* L_B *and* L_C *are the gramians of this realization.*

Chapter 5

Factor systems

In this Chapter we prove that we can construct a realization of a transfer function if we are given a realization of a left or right factor. We will also study the relation between the Lyapunov equations of the factor system and the Riccati equations of this realization.

5.1 Realizations

If we are given a realization of a left factor $[\tilde{M}, \tilde{N}]$ we can construct a realization of $G := \tilde{M}^{-1}\tilde{N}$. To prove this we first need three Lemmas.

Lemma 55 *If G is invertible in $\mathbf{H}^{\infty}_{\infty}$ its feedthrough operator is invertible.*

Proof By the definition of transfer function we have $G(z) = D + Cz(I - zA)^{-1}B$. From this it is clear that $D = G(0)$. Since G is invertible we have $G^{-1}(z)G(z) = I = G(z)G^{-1}(z)$ for all z in some disc centered at the origin and thus in particular $G^{-1}(0)G(0) = I = G(0)G^{-1}(0)$. So we have that the feedthrough operator of G is invertible (and its inverse is the feedthrough operator of G^{-1}). □

Lemma 56 *If G is a transfer function with an invertible feedthrough operator then G is invertible. If G has the realization (A, B, C, D) then a realization of G^{-1} is given by $\bar{A} := A - BD^{-1}C, \bar{B} := BD^{-1}, \bar{C} := -D^{-1}C, \bar{D} := D^{-1}$.*

Proof We prove that $(\bar{C}(\frac{1}{z}I - \bar{A})^{-1}\bar{B} + \bar{D})(C(\frac{1}{z}I - A)^{-1}B + D) = I$. The reverse identity is similar. We start with

$$(\bar{C}(\frac{1}{z}I - \bar{A})^{-1}\bar{B} + \bar{D})(C(\frac{1}{z}I - A)^{-1}B + D)$$

and multiply out to obtain

$$\bar{C}(\frac{1}{z}I-\bar{A})^{-1}\bar{B}C(\frac{1}{z}I-A)^{-1}B+\bar{C}(\frac{1}{z}I-\bar{A})^{-1}\bar{B}D+\bar{D}C(\frac{1}{z}I-A)^{-1}B+\bar{D}D.$$

Substituting $\bar{B}=BD^{-1}$ in the second term, $\bar{D}C=D^{-1}C=-\bar{C}$ in the third term and $\bar{D}=D^{-1}$ in the fourth term yields

$$\bar{C}(\frac{1}{z}I-\bar{A})^{-1}\bar{B}C(\frac{1}{z}I-A)^{-1}B+\bar{C}(\frac{1}{z}I-\bar{A})^{-1}B-\bar{C}(\frac{1}{z}I-A)^{-1}B+I.$$

Next we factor $\bar{C}(\frac{1}{z}I-\bar{A})^{-1}$ from the left and $(\frac{1}{z}I-A)^{-1}B$ from the right to obtain

$$\bar{C}(\frac{1}{z}I-\bar{A})^{-1}[\bar{B}C+(\frac{1}{z}I-A)-(\frac{1}{z}I-\bar{A})](\frac{1}{z}I-A)^{-1}B+I.$$

Since the term in square brackets is zero, this completes the proof. □

Lemma 57 *Let $(\tilde{A},[\tilde{B}_1,\tilde{B}_2],\tilde{C},[\tilde{D}_1,\tilde{D}_2])$ be a system with output space Y and input space $\tilde{U}:=Y\times U$ and let $[\tilde{M},\tilde{N}]$ be its transfer function. If $\tilde{M}$ is invertible over tic_∞ then $\tilde{D}_1$ is invertible and*

$$A:=\tilde{A}-\tilde{B}_1\tilde{D}_1^{-1}\tilde{C}\quad B:=\tilde{B}_2-\tilde{B}_1\tilde{D}_1^{-1}\tilde{D}_2\quad C:=\tilde{D}_1^{-1}\tilde{C}\quad D:=\tilde{D}_1^{-1}\tilde{D}_2 \tag{5.1}$$

is a realization of $G:=\tilde{M}^{-1}\tilde{N}$.

Proof We have to prove that $\tilde{M}^{-1}(z)\tilde{N}(z)=C(\frac{1}{z}I-A)^{-1}B+D$. We start by writing out the left-hand side

$$\tilde{M}^{-1}(z)\tilde{N}(z)=[\tilde{D}_1^{-1}-\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-A)^{-1}\tilde{B}_1\tilde{D}_1^{-1}][\tilde{D}_2+\tilde{C}(\frac{1}{z}I-\tilde{A})^{-1}\tilde{B}_2]$$

and multiply out to obtain

$$\tilde{D}_1^{-1}\tilde{D}_2-\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-A)^{-1}\tilde{B}_1\tilde{D}_1^{-1}\tilde{D}_2+\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-\tilde{A})^{-1}\tilde{B}_2$$
$$-\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-A)^{-1}\tilde{B}_1\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-\tilde{A})^{-1}\tilde{B}_2.$$

Now factoring $\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-A)^{-1}$ from the left and $(\frac{1}{z}I-\tilde{A})^{-1}\tilde{B}_2$ from the right in the last two terms yields

$$\tilde{D}_1^{-1}\tilde{D}_2-\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I-A)^{-1}\tilde{B}_1\tilde{D}_1^{-1}\tilde{D}_2$$

$$+\tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I - A)^{-1}[(\frac{1}{z}I - A) - \tilde{B}_1\tilde{D}_1^{-1}\tilde{C}](\frac{1}{z}I - \tilde{A})^{-1}\tilde{B}_2.$$

We note that the term in square brackets equals $(\frac{1}{z}I - \tilde{A})$ and we simplify to obtain

$$\tilde{D}_1^{-1}\tilde{D}_2 + \tilde{D}_1^{-1}\tilde{C}(\frac{1}{z}I - A)^{-1}(\tilde{B}_2 - \tilde{B}_1\tilde{D}_1^{-1}\tilde{D}_2),$$

which is equal to $D + C(\frac{1}{z}I - A)^{-1}B$. □

The previous lemma gives us the following result.

Proposition 58 *Let $[\tilde{M}, \tilde{N}]$ be a left factor of G with realization $(\tilde{A}, [\tilde{B}_1, \tilde{B}_2], \tilde{C}, [\tilde{D}_1, \tilde{D}_2])$. Then the system given by (5.1) is a realization of G.*

A corresponding result hold for right factors.

Proposition 59 *Let $[M; N]$ be a right factor of G with realization $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$. Then the system*

$$A := \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 \quad B := \check{B}\check{D}_1^{-1} \quad C := \check{C}_2 - \check{D}_2\check{D}_1^{-1} \quad D := \check{D}_2\check{D}_1^{-1}$$

is a realization of G.

5.2 Some equalities

In this section we prove some equalities that we shall need in the next section to study the relation between the Lyapunov equations of a factor and the Riccati equations of the realization of the transfer function constructed in the previous section.

Lemma 60 *Let $(\tilde{A}, [\tilde{B}_1, \tilde{B}_2], \tilde{C}, [\tilde{D}_1, \tilde{D}_2])$ be a system with output space Y and input space $\tilde{U} := Y \times U$. Assume that $\tilde{D}_1$ is invertible and define the system (A, B, C, D) by (5.1). Further assume that there exists a nonnegative operator X that satisfies*

$$\tilde{B}\tilde{D}^* + \tilde{A}X\tilde{C}^* = 0.$$

We then have the following equalities

1. $\tilde{B}_1 R = -(BD^* + \tilde{A}XC^*)$
2. $\tilde{A} = A - (BD^* + AXC^*)(R + CXC^*)^{-1}C$
3. $\tilde{B}\tilde{B}^* = BS^{-1}B^* + \tilde{A}XC^*R^{-1}CX\tilde{A}^*$

4. $\tilde{A}X\tilde{A}^* - X + \tilde{B}\tilde{B}^* = \tilde{A}X(I + C^*R^{-1}CX)\tilde{A} - X + BS^{-1}B^*$,

where we denote $R := I + DD^*$ *and* $S := I + D^*D$.

Proof 1. We prove the first equality. With the equation for X and 5.1 we obtain

$$\tilde{B}_1\tilde{D}_1^* + \tilde{B}_2\tilde{D}_2^* + \tilde{A}XC^*\tilde{D}_1^* = 0.$$

So

$$\tilde{B}_1 = -\tilde{B}_2D^* - \tilde{A}XC^* = -(B + \tilde{B}_1D)D^* - \tilde{A}XC^*$$

and this yields

$$\tilde{B}_1R = \tilde{B}_1(I + DD^*) = -BD^* - \tilde{A}XC^*.$$

2. We now turn to the second equality. We take the first equality and substitute $\tilde{A} = A + \tilde{B}_1C$ to obtain

$$\tilde{B}_1R = -(BD^* + (A + \tilde{B}_1C)XC^*).$$

So

$$\tilde{B}_1(R + CXC^*) = -BD^* - AXC^*$$

or equivalently

$$\tilde{B}_1 = -(BD^* + AXC^*)(R + CXC^*)^{-1}.$$

So now we obtain

$$\tilde{A} = A + \tilde{B}_1C = A - (BD^* + AXC^*)(R + CXC^*)^{-1}C.$$

3. We prove the third equality. Now

$$\tilde{B}\tilde{B}^* = \tilde{B}_1\tilde{B}_1^* + \tilde{B}_2\tilde{B}_2^*$$

and substituting for $\tilde{B}_2$ from (5.1) gives

$$\tilde{B}_1\tilde{B}_1^* + (B + \tilde{B}_1D)(B + \tilde{B}_1D)^* = \tilde{B}_1R\tilde{B}_1^* + BB^* + BD^*\tilde{B}_1^* + \tilde{B}_1DB^*.$$

Next substituting for $\tilde{B}_1$ from the first equality yields

$$(BD^* + \tilde{A}XC^*)R^{-1}(DB^* + C^*X\tilde{A}^*) + BB^*$$

$$-BD^*R^{-1}(DB^* + C^*X\tilde{A}^*) - (BD^* + \tilde{A}XC^*)R^{-1}DB^*$$

and cancelling and noting that $I - D^*R^{-1}D = S^{-1}$ gives the result.

4. And finally the fourth equality. We use the third equality to write the left-hand side as

$$\tilde{A}X\tilde{A}^* - X + BS^{-1}B^* + \tilde{A}XC^*R^{-1}CX\tilde{A}^*$$

which is equal to the right-hand side. □

Lemma 61 *Let* $(\tilde{A},[\tilde{B}_1,\tilde{B}_2],\tilde{C},[\tilde{D}_1,\tilde{D}_2])$ *be a system with output space* Y *and input space* $\tilde{U}:=Y\times U$. *Assume that* $\tilde{D}_1$ *is invertible and define the system* (A,B,C,D) *by (5.1). Further assume that there exists a nonnegative operator* V *that satisfies*

$$\tilde{D}\tilde{D}^*+\tilde{C}V\tilde{C}^*=I.$$

Then we have

1. $R+CVC^*=\tilde{D}_1^{-1}\tilde{D}_1^{-*}$
2. $(I+C^*R^{-1}CV)\tilde{C}^*\tilde{C}=C^*R^{-1}C.$

Proof 1. We first prove the first equality. The given equation for V translates to

$$\tilde{D}_1\tilde{D}_1^*+\tilde{D}_1DD^*\tilde{D}_1^*+\tilde{D}_1CVC^*\tilde{D}_1^*=I$$

and multiplying from the left with $\tilde{D}_1^{-1}$ and from the right with $\tilde{D}_1^{-*}$ gives the result.
2. The first equality implies that $(R+CVC^*)^{-1}=\tilde{D}_1^*\tilde{D}_1$ and so $C^*(R+CVC^*)^{-1}C=\tilde{C}^*\tilde{C}$. Hence

$$\begin{aligned}(I+C^*R^{-1}CV)\tilde{C}^*\tilde{C}&=(I+C^*R^{-1}CV)C^*(R+CVC^*)^{-1}C\\&=C^*(R+CVC^*)^{-1}C+C^*R^{-1}CVC^*(R+CVC^*)^{-1}C\\&=C^*R^{-1}[R+CVC^*](R+CVC^*)^{-1}C=C^*R^{-1}C,\end{aligned}$$

which is the second equality. □

5.3 From Lyapunov equations to Riccati equations

Lemma 62 *Suppose that* $(\tilde{A},[\tilde{B}_1,\tilde{B}_2],\tilde{C},[\tilde{D}_1,\tilde{D}_2])$ *is an approximately observable and input stable realization of the normalized left factor* $\tilde{G}=[\tilde{M},\tilde{N}]$. *Let* L_B *be the controllability gramian of this realization. Define the system* (A,B,C,D) *by (5.1). Then* L_B *is a solution of the FARE of* (A,B,C,D). *Moreover,*

$$\tilde{A}L_B(I+C^*R^{-1}CL_B)\tilde{A}^*=L_B-BS^{-1}B^* \tag{5.2}$$

and

$$\tilde{A}=A-(BD^*+AL_BC^*)(R+CL_BC^*)^{-1}C. \tag{5.3}$$

Proof Proposition 38 tells us that

$$\tilde{D}\tilde{B}^* + \tilde{C}L_B\tilde{A}^* = 0.$$

Lemma 60 with $X := L_B$ now tells us that

$$\tilde{A}L_B\tilde{A}^* - L_B + \tilde{B}\tilde{B}^* = \tilde{A}L_B(I + C^*R^{-1}CL_B)\tilde{A}^* - L_B + BS^{-1}B^* \qquad (5.4)$$

and

$$\tilde{A} = A - (BD^* + AL_BC^*)(R + CL_BC^*)^{-1}C.$$

The left-hand side of equation (5.4) is zero because L_B is a solution of the control Lyapunov of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and so the right-hand side is zero. That is: L_B is a solution of the FARE (3.10) of (A, B, C, D). □

Lemma 63 *Suppose that $(\tilde{A}, [\tilde{B}_1, \tilde{B}_2], \tilde{C}, [\tilde{D}_1, \tilde{D}_2])$ is an approximately observable and input stable realization of the normalized left factor $\tilde{G} = [\tilde{M}, \tilde{N}]$. Let L_B and L_C be the gramians of this realization. Assume that $I - L_BL_C$ is boundedly invertible. Define the system (A, B, C, D) by (5.1) and $Q := L_C(I - L_BL_C)^{-1}$. Then*

$$(I + BS^{-1}B^*Q)(I - L_BL_C)\tilde{A} = \tilde{A}(I + L_BC^*R^{-1}C)(I - L_BL_C). \qquad (5.5)$$

Proof The left-hand side of equation (5.5) is

$$\tilde{A} - L_BL_C\tilde{A} + BS^{-1}B^*L_C\tilde{A},$$

which according to (5.2) equals

$$\tilde{A} - \tilde{A}L_B(I + C^*R^{-1}CL_B)\tilde{A}^*L_C\tilde{A}.$$

The observer Lyapunov equation for $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ tells us that $\tilde{A}^*L_C\tilde{A} = L_C - \tilde{C}^*\tilde{C}$ and so the left-hand side of equation (5.5) is equal to

$$\tilde{A} - \tilde{A}L_B(I + C^*R^{-1}CL_B)(L_C - \tilde{C}^*\tilde{C})$$

$$= \tilde{A} - \tilde{A}L_B(I + C^*R^{-1}CL_B)L_C + \tilde{A}L_B(I + C^*R^{-1}CL_B)\tilde{C}^*\tilde{C}.$$

From Proposition 38 we know that $\tilde{D}\tilde{D}^* + \tilde{C}L_B\tilde{C}^* = I$. So Lemma 61 part 2 with $V = L_B$ is applicable and hence the left-hand side of equation (5.5) is equal to

$$\tilde{A} - \tilde{A}L_B(I + C^*R^{-1}CL_B)L_C + \tilde{A}L_BC^*R^{-1}C.$$

And this is equal to the right-hand side of equation (5.5). □

Lemma 64 *Suppose that $(\tilde{A}, [\tilde{B}_1, \tilde{B}_2], \tilde{C}, [\tilde{D}_1, \tilde{D}_2])$ is an approximately observable and input stable realization of the normalized left factor $\tilde{G} = [\tilde{M}, \tilde{N}]$. Let L_B and L_C be the gramians of this realization. Assume that $I - L_B L_C$ is boundedly invertible. Define the system (A, B, C, D) by (5.1) and $Q := L_C(I - L_B L_C)^{-1}$. Then Q is a solution of the CARE of the system (A, B, C, D).*

Proof We first note that according to Lemma 62 and equation (5.3) that $\tilde{A} = A_P$ where A_P is defined by equation (3.3) with $P = L_B$. We define A_Q by (3.4). Lemma 20 tells us that

$$A_Q = (I + BS^{-1}B^*Q)^{-1}\tilde{A}(I + L_B C^* R^{-1} C). \tag{5.6}$$

Now Lemma 63 tells us that

$$\tilde{A} = (I - L_B L_C)^{-1} A_Q (I - L_B L_C). \tag{5.7}$$

We are now in a position to prove the lemma. We start with the observer Lyapunov equation for $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$

$$\tilde{A}^* L_C \tilde{A} - L_C + \tilde{C}^* \tilde{C} = 0$$

and substitute $\tilde{A}^* = (I + C^* R^{-1} C L_B)^{-1} A_Q^* (I + QBS^{-1}B^*)$ from (5.6) and $\tilde{A} = (I - L_B L_C)^{-1} A_Q (I - L_B L_C)$ from (5.7) to give

$$(I + C^* R^{-1} C L_B)^{-1} A_Q^* (I + QBS^{-1}B^*) L_C (I - L_B L_C)^{-1} A_Q (I - L_B L_C)$$

$$-L_C + \tilde{C}^* \tilde{C} = 0.$$

We multipy by $(I + C^* R^{-1} C L_B)$ from the left and by $(I - L_B L_C)^{-1}$ from the right to give

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - (I + C^* R^{-1} C L_B)Q$$

$$+(I + C^* R^{-1} C L_B)\tilde{C}^* \tilde{C}(I - L_B L_C)^{-1} = 0.$$

According to part 2 of Lemma 61 with $V := L_B$ (we know from the proof of the previous lemma that the assumptions of Lemma 61 are satisfied) this is equivalent to

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - (I + C^* R^{-1} C L_B)Q + C^* R^{-1} C (I - L_B L_C)^{-1} = 0$$

and this is equivalent to

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - Q - C^* R^{-1} C L_B Q + C^* R^{-1} C (I - L_B L_C)^{-1} = 0,$$

which is equivalent to

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - Q + C^*R^{-1}C[I - L_BL_C](I - L_BL_C)^{-1} = 0.$$

This is equivalent to

$$A_Q^*(I + QBS^{-1}B^*)QA_Q - Q + C^*R^{-1}C = 0$$

and this together with the definition of A_Q above shows that Q is a solution of the CARE (3.2) of (A, B, C, D). □

Combining the results of Proposition 58 and Lemmas 62 and 64 we obtain the following.

Proposition 65 *Suppose that $(\tilde{A}, [\tilde{B}_1, \tilde{B}_2], \tilde{C}, [\tilde{D}_1, \tilde{D}_2])$ is an approximately observable and input stable realization of the normalized left factor $\tilde{G} = [\tilde{M}, \tilde{N}]$. Let L_B and L_C be the gramians of this realization. Assume that $I - L_BL_C$ is boundedly invertible. Define*

$$A := \tilde{A} - \tilde{B}_1\tilde{D}_1^{-1}\tilde{C} \quad B := \tilde{B}_2 - \tilde{B}_1\tilde{D}_1^{-1}\tilde{D}_2 \quad C := \tilde{D}_1^{-1}\tilde{C} \quad D := \tilde{D}_1^{-1}\tilde{D}_2$$

and $Q := L_C(I - L_BL_C)^{-1}$. Then (A, B, C, D) is a realization of $\tilde{M}^{-1}\tilde{N}$, L_B is a solution of the FARE of (A, B, C, D) and Q is a solution of the CARE of (A, B, C, D).

A similar result holds for right factors.

Proposition 66 *Suppose that $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$ is an approximately controllable and output stable realization of the normalized right factor $G = [M; N]$. Let L_B and L_C be the gramians of this realization. Assume that $I - L_CL_B$ is boundedly invertible. Define*

$$A := \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 \quad B := \check{B}\check{D}_1^{-1} \quad C := \check{C}_2 - \check{D}_2\check{D}_1^{-1} \quad D := \check{D}_2\check{D}_1^{-1}$$

and $P := L_B(I - L_CL_B)^{-1}$. Then (A, B, C, D) is a realization of NM^{-1}, P is a solution of the FARE of (A, B, C, D) and L_C is a solution of the CARE of (A, B, C, D).

Chapter 6

LQG-balancing

Definition 67 (LQG-balanced realization) *A realization of a transfer function is said to be* LQG-balanced *if there exists a nonnegative operator that is a solution of both the CARE and the FARE of this realization.*

The following Lemma is due to Young [13].

Lemma 68 *Let $G \in \mathbf{H}^\infty$. Then there exists an approximately controllable and approximately observable input and output stable realization of G that has equal gramians.*

We now state and prove the first of our two main results.

Theorem 69 *Let $G \in \mathbf{H}^\infty_\infty$ be such that one of the following four conditions holds*

1. *G has a normalized left coprime factor.*

2. *G has a normalized right coprime factor.*

3. *G has a normalized left factor with compact Hankel Γ such that $I - \Gamma\Gamma^*$ is boundedly invertible.*

4. *G has a normalized right factor with compact Hankel Γ such that $I - \Gamma^*\Gamma$ is boundedly invertible.*

Then G has a LQG-balanced realization.

Proof We only prove the theorem for left factors since the proof for right factors is similar. According to Proposition 48 (for condition one) or Proposition 53 (for condition three) the assumptions of Proposition 65 are satisfied. According to Lemma 68 we can take $L_B = L_C$ in this proposition. We then obtain a realization of G with solution of the FARE L_B and solution of the CARE $L_B(I - L_B^2)^{-1}$. We use Lemma 23 with $V := (I - L_B^2)^{-1/4}$ to obtain another realization of G for which $L_B(I - L_B^2)^{-1/2}$ is a solution of both the FARE and the CARE. □

Chapter 7

The closed-loop system

In this chapter we study systems that have a solution to either Riccati equation. We associate a new system called the closed-loop system to this system. We show that this closed-loop system is a realization of a factor of the transfer function of the original system. We shall also prove that there is a relation between the Riccati equations of the original system and the Lyapunov equations of the closed-loop system.

7.1 The linear quadratic regulator problem

One of the most famous problems in control is the following: Given a well-posed linear system define for $x_0 \in X$ and $u \in l^2(\mathbb{Z}^+, U)$ the sequences x and y by $x_n := A^n x_0 + \mathfrak{B}\tau^n u$ and $y := \mathfrak{C}x + \mathfrak{D}u$. Define

$$J(u, x_0) := ||u||^2 + ||y||^2$$

where the norms are the l^2 norms. The *linear quadratic regulator problem* is: for a given x_0 find the u that minimizes J.

Remark Usually the sequences x and y are defined by $x_{n+1} = Ax_n + Bu_n, y_n = Cx_n + Du_n$, it can be shown that this is equivalent to the definition above.

It can be shown (see Chapter 6 in Curtain & Zwart [1]) that if the system is output stabilizable the CARE of the system has a minimal solution Q and there exists a unique minimizing input $\bar{u}$. This unique minimizing input is given by $\bar{u}_n := F\bar{x}_n$ where $F := -(S + B^*QB)^{-1}(D^*C + B^*QA)$ and $\bar{x}_n$ is defined by $\bar{x}_{n+1} = A\bar{x}_n + B\bar{u}_n$, $\bar{x}_0 = x_0$. The cost function J satisfies $J(x_0, \bar{u}) =< x_0, Qx_0 >$.

It will turn out to be important to consider a new system defined by taking $u := \bar{u} + W^{-1/2}\check{u}$ where $W := S + B^*QB$ and $\check{u}$ is the new input and taking the new output $\check{y} := [u; y]$. To obtain the generating operators of this new system we simply substitute these equations in the difference equations $x_{n+1} = Ax_n + Bu_n, y_n = Cx_n + Du_n$. This leads to

$$\bar{x}_{n+1} = A\bar{x}_n + Bu_n = (A + BF)\bar{x}_n + BW^{-1/2}\check{u}$$

$$\check{y} = [u; y] = [F; C + DF]\bar{x}_n + [W^{-1/2}; DW^{-1/2}]\check{u},$$

that is, the generating operators of this new system are $(A+BF, BW^{-1/2}, [F; C+DF], [W^{-1/2}; DW^{-1/2}])$. This leads to the following definition.

Definition 70 (CARE closed-loop system) *Let (A, B, C, D) be a system that has a solution of its CARE Q. Define the system (A_Q, B_Q, C_Q, D_Q) by*

$$A_Q := A + BF \quad B_Q := BW^{-1/2} \quad C_Q := [F; C + DF] \quad D_Q := [I; D]W^{-1/2} \tag{7.1}$$

where

$$W := S + B^*QB \quad S := I + D^*D \quad F := -W^{-1}(D^*C + B^*QA).$$

The system (A_Q, B_Q, C_Q, D_Q) is called the CARE closed-loop system *associated with (A, B, C, D) and Q.*

The dual concept is the following.

Definition 71 (FARE closed-loop system) *Let (A, B, C, D) be a system that has a solution of its FARE P. Define the system (A_P, B_P, C_P, D_P) by*

$$A_P := A + TC \quad B_P := [T, B + TD] \quad C_P := Z^{-1/2}C \quad D_P := Z^{-1/2}[I, D] \tag{7.2}$$

where

$$Z := R + CPC^* \quad R := I + DD^* \quad T := -(BD^* + APC^*)Z^{-1}.$$

The system (A_P, B_P, C_P, D_P) is called the FARE closed loop system *associated with (A, B, C, D) and P.*

Remark Note that the definitions of A_P and A_Q given above are consistent with the definitions (3.3) and (3.4) given earlier.

The topic of this chapter is the study of these closed-loop systems.

7.2 From Riccati equations to Lyapunov equations

Lemma 72 *Suppose that the FARE of the system (A, B, C, D) has a solution P. Let (A_P, B_P, C_P, D_P) be its FARE closed-loop system. Then*

$$A = A_P - B_{P,1} D_{P,1}^{-1} C_P \quad B = B_{P,2} - B_{P,1} D_{P,1}^{-1} D_{P,2} \quad C = D_{P,1}^{-1} C_P \quad D = D_{P,1}^{-1} D_{P,2}$$

and

$$B_P D_P^* + A_P P C_P^* = 0 \quad C_P P C_P^* + D_P D_P^* = I.$$

The proof is elementary: for the first four equalities just write out the right-hand sides and check that they are equal to the left-hand sides and for the last two equalities just write out the left-hand sides.

Proposition 73 *Suppose that the FARE of the system (A, B, C, D) has a solution P. Let (A_P, B_P, C_P, D_P) be its FARE closed-loop system. Then P is a solution of the control Lyapunov equation of (A_P, B_P, C_P, D_P).*

Proof According to the previous Lemma the assumptions of Lemma 60 with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := (A_P, B_P, C_P, D_P)$ are satisfied and thus (with $X := P$)

$$A_P P A_P^* - P + B_P B_P^* = A_P P (I + C^* R^{-1} C P) A_P - P + B S^{-1} B^*. \tag{7.3}$$

Since P is a solution of the FARE the right-hand side of (7.3) is zero and thus the left-hand side is zero. That is: P is a solution of the control Lyapunov equation of (A_P, B_P, C_P, D_P). $\square$

Proposition 74 *Suppose that the FARE of the system (A, B, C, D) has a solution P and the CARE of this system has a solution Q. Let (A_P, B_P, C_P, D_P) be its FARE closed-loop system. Then $Q(I + PQ)^{-1}$ is a solution of the observer Lyapunov equation of (A_P, B_P, C_P, D_P).*

Proof We start with the expression that we want to be equal to zero

$$\Phi := A_P^* Q (I + PQ)^{-1} A_P - Q(I + PQ)^{-1} + C_P^* C_P,$$

sustitute $A_P^* = (I + C^* R^{-1} C P)^{-1} A_Q^* (I + Q B S^{-1} B^*)$ (this equality comes from Lemma 20) and $(I + PQ)^{-1} A_P = A_Q (I + PQ)^{-1}$ (this equality comes from Lemma 21) to yield

$$\Phi = (I + C^* R^{-1} C P)^{-1} A_Q^* (I + Q B S^{-1} B^*) Q A_Q (I + PQ)^{-1} - Q(I + PQ)^{-1} + C_P^* C_P$$

$$= (I + C^*R^{-1}CP)^{-1}[A_Q^*(I + QBS^{-1}B^*)QA_Q -$$

$$(I + C^*R^{-1}CP)Q + (I + C^*R^{-1}CP)C_P^*C_P(I + PQ)](I + PQ)^{-1}.$$

We now apply Lemma 61 with $V := P$ (according to Lemma 72 the assumptions are satified) to give

$$\Phi = (I + C^*R^{-1}CP)^{-1}[A_Q^*(I + QBS^{-1}B^*)QA_Q -$$

$$Q - C^*R^{-1}CPQ + C^*R^{-1}C(I + PQ)](I + PQ)^{-1},$$

the term $C^*R^{-1}CPQ$ cancels out and the term in square brackets according to the CARE equals zero. Thus $\Phi = 0$ and this proves that $Q(I + PQ)^{-1}$ is a solution of the observer Lyapunov equation of (A_P, B_P, C_P, D_P). □

7.3 Stability of the closed-loop system

In this section we shall prove that the CARE closed-loop system is output stable and input-output stable and that the FARE closed-loop system is input stable and input-output stable.

Proposition 75 *Suppose that the CARE of the system (A, B, C, D) has a solution Q. Then its CARE closed-loop system (A_Q, B_Q, C_Q, D_Q) is output stable.*

Proof We wish to prove that the norm of $\mathfrak{C}_Q$ as an operator from X to $l^2(\mathbb{Z}; U)$ is bounded. Let $x_0 \in X$ and let $\bar{u}$ be the optimal input of the linear quadratic regulator problem associated with the system (A, B, C, D); define $\bar{x}$ to be the corresponding state and $\bar{y}$ to be the corresponding output. Then

$$||\mathfrak{C}_Q x_0||^2 = \sum_{n=0}^{\infty} ||C_Q A_Q^n x_0||^2 = \sum_{n=0}^{\infty} ||C_Q \bar{x}_n||^2 = \sum_{n=0}^{\infty} ||F\bar{x}_n||^2 + ||(C + DF)\bar{x}_n||^2$$

$$= \sum_{n=0}^{\infty} ||\bar{u}_n||^2 + ||\bar{y}_n||^2 = J(x_0, \bar{u}) = < Qx_0, x_0 > .$$

We now take the supremum over all x_0 with norm one on both sides of this expession to obtain $||\mathfrak{C}_Q|| = \sqrt{||Q||}$. Thus $\mathfrak{C}_Q$ is a bounded linear operator from X to $l^2(\mathbb{Z}; U)$ □

The dual result is:

Proposition 76 *Suppose that the FARE of the system (A,B,C,D) has a solution P. Then its FARE closed-loop system (A_P,B_P,C_P,D_P) is output stable.*

Before we prove the input-output stability of the closed loop systems we first need a result about the spectra of A_P, A_Q and A. But first we need the following simple lemma.

Lemma 77 *If W is invertible and $WK = LW$ then K and L have the same spectrum and the same point spectrum.*

Proof This follows from the following equalities

$$(\lambda I - K) = W^{-1}(\lambda W - WK) = W^{-1}(\lambda W - LW) = W^{-1}(\lambda I - L)W$$

□

Lemma 78 *Suppose that the CARE of the system (A,B,C,D) has a solution Q and the FARE of the system has a solution P. Then*

1. *the spectra of A_Q and A_P are equal,*
2. *the spectra of A_Q and A_P in the closed unit disc are contained in the spectrum of A in the closed unit disc.*
3. *If, in addition, the system (A,B,C,D) is approximately controllable and approximately observable, then A_Q and A_P have at most continuous spectrum in the closed unit disc and this part of the spectrum is contained in the continuous spectrum of A.*

Proof 1. This follows from the previous lemma and Proposition 21
2. We prove the second part only for A_Q since $\sigma(A_Q) = \sigma(A_P)$. We first note that for all $x \in X$ and $\lambda \in \mathbb{C}$

$$||(A-\lambda I)x|| \leq ||(A - A_Q)x|| + ||(A_Q - \lambda I)x|| \leq \tag{7.4}$$

$$||B(S + B^*QB)^{-1}||(||D^*Cx|| + ||B^*QAx||) + ||(A_Q - \lambda I)x||.$$

The CARE implies that for every $x \in X$

$$< (I + QBS^{-1}B^*)QA_Qx, A_Qx > - < Qx, x > + < R^{-1/2}Cx, R^{-1/2}Cx >= 0$$

and this implies that for every λ in the closed unit disc

$$< Q(A_Q-\lambda I)x, (A_Q-\lambda I)x > +\lambda < Qx, (A_Q-\lambda I)x > +\bar{\lambda} < (A_Q-\lambda I)x, Qx > \tag{7.5}$$

$$\geq - < QBS^{-1}B^*QA_Qx, A_Qx > - < R^{-1/2}Cx, R^{-1/2}Cx > .$$

We write B^*QA_Q as $S(S + B^*QB)^{-1}B^*QA$ and the right-hand side of (7.5) is thus equal to

$$-||S^{1/2}(S + B^*QB)^{-1}B^*QAx||^2 - ||R^{1/2}Cx||^2.$$

We thus have for every $x \in X$ and λ in the closed unit disc

$$< Q(A_Q-\lambda I)x, (A_Q-\lambda I)x > +\lambda < Qx, (A_Q-\lambda I)x > +\bar{\lambda} < (A_Q-\lambda I)x, Qx > \tag{7.6}$$
$$\geq -||S^{1/2}(S + B^*QB)^{-1}B^*QAx||^2 - ||R^{1/2}Cx||^2.$$

Now assume that λ in the closed unit disc is an eigenvalue of A_Q and that x is the corresponding eigenvector. Then the left-hand side of (7.6) is zero and thus the right-hand side is zero and thus $B^*QAx = 0$ and $Cx = 0$. Then the right-hand side of (7.4) is zero and this implies that the left-hand side is zero. That is: λ is an eigenvalue of A.

Now assume that λ is in the continuous spectrum of A_Q and in the closed unit disc. That is: assume that there exists a sequence x_n with norm one such that $||(A_Q - \lambda I)x_n|| \to 0$. According to (7.6) we then have: $B^*QAx_n \to 0$ and $Cx_n \to 0$. This implies that the right-hand side of (7.4) converges to zero and thus the left-hand side converges to zero. That is: λ is in the continous spectrum of A.

We now turn to the residual spectrum of A_Q. Suppose μ is an element of the closed unit disc and is in the residual spectrum of A_Q. Then $\bar{\mu}$ is in the point spectrum of A_Q^* and thus in the point spectrum of A_P^*. Using this we can show as above, but using the FARE in stead of the CARE, that $\bar{\mu}$ is in the point spectrum of A^*. This implies that μ is in the spectrum of A.
2. We only have to show that the point spectrum and the residual spectrum of A_Q in the closed unit disc are empty. Suppose that the point spectrum of A_Q is not empty. Then there exist λ in the closed unit disc and $x \in X$ such that $A_Qx = \lambda x$ and as above this implies that $Ax = \lambda x$ and $Cx = 0$. Then $(\mathfrak{C}x)_k = CA^kx = 0$ for all $k \in \mathbb{Z}^+$ and this contradicts the approximate observability. Thus the point spectrum of A_Q in the closed unit disc is empty. Similarly we can prove that the residual spectrum of A_Q in the closed unit disc (which is contained in the point spectrum of A_P^* in the closed unit disc) is empty using approximate controllability. □

Lemma 79 *Suppose that the CARE of the system (A, B, C, D) has a solution Q, the FARE of the system has a solution P, the system is approximately* *controllable and approximately observable and that the continuous spectrum of A on the unit circle has measure zero. Then its FARE closed-loop system (A_P, B_P, C_P, D_P) is input-output stable and its transfer function is co-inner.*

Proof Let G be the transfer function of (A_P, B_P, C_P, D_P). According to Lemma 78 $(\frac{1}{z}I - A_P)^{-1}$ exists for almost all z on the unit circle and we then have for almost all z on the unit circle

$$G(z)G(z)^* = (D_P + C_P(\frac{1}{z}I - A_P)^{-1}B_P)(D_P^* + B_P^*(\frac{1}{z}I - A_P)^{-*}C_P^*).$$

If we expand this expression we get

$$D_P D_P^* + D_P B_P^*(\frac{1}{z}I - A_P)^{-*}C_P^* C_P(\frac{1}{z}I - A_P)^{-1}B_P D_P^*$$

$$+C_P(\frac{1}{z}I - A_P)^{-1}B_P B_P^*(\frac{1}{z}I - A_P)^{-*}C_P^*.$$

We use Lemma 72 to write $D_P D_P^* = I - C_P L_B C_P^*$ and $D_P B_P^* = -C_P L_B A_P^*$ to obtain

$$I + C_P[-L_B - L_B A_P^*(\frac{1}{z}I - A_P)^{-*} - (\frac{1}{z}I - A_P)^{-1}A_P L_B$$

$$+(\frac{1}{z}I - A_P)^{-1}B_P B_P^*(\frac{1}{z}I - A_P)^{-*}]C_P^*.$$

Using the control Lyapunov equation (2.1) we see that the term in square brackets is zero and we thus have for almost all z on the unit circle

$$G(z)G(z)^* = I. \tag{7.7}$$

This shows that $G \in \mathbf{L}^\infty$. This together with the fact that $G \in \mathbf{H}^2_{strong}$ (which follows from the output stability) implies that $G \in \mathbf{H}^\infty$ (see lemma 18). Equation (7.7) now tells us that G is co-inner. □

The dual result is:

Proposition 80 *Suppose that the CARE of the system (A, B, C, D) has a solution Q, the FARE of the system has a solution P, the system is approximately controllable and approximately observable and that the continuous spectrum of A on the unit circle has measure zero. Then its CARE closed-loop system (A_Q, B_Q, C_Q, D_Q) is input-output stable and its transfer function is inner.*

7.4 Connection with factorization

Lemma 81 *Suppose that the CARE of the system (A, B, C, D) has a solution Q, the FARE of the system has a solution P, the system is approximately controllable and approximately observable and that the continuous spectrum of A on the unit circle has measure zero. Then the transfer function of its FARE closed-loop system (A_P, B_P, C_P, D_P) is a normalized left factor of the transfer function of (A, B, C, D).*

Proof Let $[\tilde{M}, \tilde{N}]$ be the transfer function of the FARE closed-loop system. Then, according to Lemma 56, $\tilde{M}$ has an inverse in $\mathbf{H}^{\infty}_{\infty}$. According to Lemma 80 we have $[\tilde{M}, \tilde{N}] \in \mathbf{H}^{\infty}$. The relation between (A, B, C, D) and (A_P, B_P, C_P, D_P) is the same as in Lemma 57 (according to Lemma 72) and so $G = \tilde{M}^{-1}\tilde{N}$. □

The dual result is:

Proposition 82 *Suppose that the CARE of the system (A, B, C, D) has a solution Q, the FARE of the system has a solution P, the system is approximately controllable and approximately observable and that the continuous spectrum of A on the unit circle has measure zero. Then the transfer function of its CARE closed-loop system (A_Q, B_Q, C_Q, D_Q) is a normalized right factor of the transfer function of (A, B, C, D).*

We now focus our attention on coprime factorizations. We first define the following concept.

Definition 83 (Pseudo-coprime) *Two functions $M \in \mathbf{H}^{\infty}(U, Z)$ and $N \in \mathbf{H}^{\infty}(U, Y)$ are called* right pseudo-coprime *if there exists a $\mu > 0$ such that for all z with $|z| < 1$*

$$M(z)^* M(z) + N(z)^* N(z) \geq \mu I.$$

If this holds for all z with $|z| < r$ we call M and N right pseudo-coprime over $\mathbb{D}_r$.
Two functions $\tilde{M} \in \mathbf{H}^{\infty}(Z, Y)$ and $\tilde{N} \in \mathbf{H}^{\infty}(U, Y)$ are called left pseudo-coprime *if there exists a $\mu > 0$ such that for all z with $|z| < 1$*

$$\tilde{M}(z)\tilde{M}(z)^* + \tilde{N}(z)\tilde{N}(z)^* \geq \mu I.$$

If this holds for all z with $|z| < r$ we call $\tilde{M}$ and $\tilde{N}$ left pseudo-coprime over $\mathbb{D}_r$.

Lemma 84 *If M and N are right coprime then they are right pseudo-coprime. If $\dim U < \infty$ the converse also holds. If $\tilde{M}$ and $\tilde{N}$ are left coprime then they are left pseudo-coprime. If $\dim Y < \infty$ the converse also holds.*

Proof See [9] Appendix 3 p.288-298. □

Coprimeness over $\mathbb{D}_r$ is defined as follows: the inverse mentioned in Definition 42 should be in $\mathbf{H}_r^\infty$.

Lemma 85 *Suppose that the CARE of the system (A, B, C, D) has a solution Q, the FARE of the system has a solution P, the system is approximately controllable and approximately observable, the continuous spectrum of A on the unit circle has measure zero and that the resolvent set of A is dense in the unit disc. Then the transfer function of the CARE closed-loop system is right coprime over $\mathbb{D}_r$ for every $r < 1$.*

Proof Define $[\tilde{X}(z), \tilde{Y}(z)]$ to be the transfer function of the system

$$\breve{A} := A + TC \quad \breve{B} := [B + TD, T] \quad \breve{C} := -W^{1/2}F \quad \breve{D} := [W^{1/2}, 0]$$

where W, F and T are defined by (7.1) and (7.2). Then it is straightforward to check that

$$\tilde{X}(z)M(z) - \tilde{Y}(z)N(z) = I \text{ for all } z \text{ in the resolvent set of } A. \tag{7.8}$$

Since the FARE closed-loop system is input stable we have that $\breve{B}^*(I - z\breve{A}^*)^{-1}x \in \mathbf{H}^2$ for all $x \in X$ and thus in particular for $x = \breve{C}^*u$, thus $\breve{B}^*(I - z\breve{A}^*)^{-1}\breve{C}^* \in \mathbf{H}^2_{strong}$ and thus (according to Lemma 17) $\breve{B}^*(I - z\breve{A}^*)^{-1}\breve{C}^* \in \mathbf{H}_r^\infty$ for all $r < 1$. Since $g(z) \in \mathbf{H}_r^\infty$ implies $g(\bar{z})^* \in \mathbf{H}_r^\infty$ we have $\breve{C}(I - z\breve{A})^{-1}\breve{B} \in \mathbf{H}_r^\infty$ for all $r < 1$. And thus $z\breve{C}(I - z\breve{A})^{-1}\breve{B} = \breve{C}(\frac{1}{z}I - \breve{A})^{-1}\breve{B} \in \mathbf{H}_r^\infty$ for all $r < 1$. Since $\breve{D} \in \mathbf{H}_r^\infty$ for all $r < 1$ we have $[\tilde{X}(s), \tilde{Y}(s)] = \breve{D} + \breve{C}(\frac{1}{z}I - \breve{A})^{-1}\breve{B} \in \mathbf{H}_r^\infty$ for all $r < 1$.

We already know that $[\tilde{M}; \tilde{N}] \in \mathbf{H}^\infty$ and thus the left-hand side of (7.8) is analytic (and thus continuous) on the open unit disc. Since the equality holds for a dense set the equality must hold on the entire open unit disc. Thus the transfer function $[M; N]$ of the CARE closed-loop system is right coprime over $\mathbb{D}_r$. □

Lemma 86 *A function $G = [M; N] \in \mathbf{H}^\infty$ that satisfies $G^*G = I$ everywhere on the unit circle and is right pseudo-coprime over $\mathbb{D}_r$ for all $r < 1$ is right pseudo-coprime over $\mathbb{D}_1$.*

Proof Let $u \in U$ with $u \neq 0$ and define $f : \mathbb{D}_1 \to \mathbb{R}$ by $f(z) :=< G(z)^* G(z)u, u >= ||G(z)u||^2$. Then by the assumptions $\lim_{r\to 1} f(re^{i\theta})$ exists for all $\theta \in [0, 2\pi)$ and using the symbol f also for this extension we have $f(z) = ||u||^2$ everywhere on the unit circle. Define for $\theta \in [0, 2\pi)$

$$\delta(\theta) := \inf\{R \in [0,1) : \forall r \text{ with } R < r < 1 \;\; f(re^{i\theta}) \geq \frac{||u||^2}{2} \text{ holds}\}.$$

This infimum exists since the set is nonempty and bounded from below by zero. Obviously for all $\theta \in [0, 2\pi)$ and all $r \in [\delta(\theta), 1]$ $f(re^{i\theta}) \geq \frac{||u||^2}{2}$. By continuity $f(\delta(\theta)e^{i\theta}) = \frac{||u||^2}{2}$ if $\delta(\theta) \in (0, 1)$. Suppose it is not equal to $\frac{||u||^2}{2}$, then it must be strictly larger or strictly smaller; if it were strictly larger there would exist a $\eta \in (0, \delta(\theta))$ such that $f(\eta e^{i\theta}) > \frac{||u||^2}{2}$, contradicting that $\delta(\theta)$ is the infimum of the set given above; if it where strictly smaller then there would exist a $\eta \in (\delta(\theta), 1)$ such that $f(\eta e^{i\theta}) < \frac{||u||^2}{2}$, contradicting that $f(re^{i\theta}) \geq \frac{||u||^2}{2}$ for all $r \in (\delta(\theta), 1)$. Now define $\delta := \sup_{\theta\in[0,2\pi)} \delta(\theta)$. Then for all $\theta \in [0, 2\pi)$ and all $r \in [\delta, 1]$ $f(re^{i\theta}) \geq \frac{||u||^2}{2}$. We now prove that $\delta < 1$. Suppose it is not, then there exists a sequence $\theta_n \in [0, 2\pi)$ such that $\delta(\theta_n) \to 1$. Since θ_n is bounded it has a convergent subsequence and we can assume without loss of generality that θ_n is this subsequence and that thus $\theta_n \to \theta$. Now

$$\frac{||u||^2}{2} = f(\delta(\theta_n)e^{i\theta_n}) \to f(e^{i\theta}) = ||u||^2$$

which is a contradiction, thus $\delta < 1$.

We know that G is right pseudo-coprime over $\mathbb{D}_\delta$, thus there exists a $\mu > 0$ such that $G^*G \geq \mu I$ on $\mathbb{D}_\delta$. The above shows that $G(z)^*G(z) \geq \frac{1}{2}I$ for all z with $\delta \leq |z| \leq 1$. Thus if we define $\nu := \min(\mu, \frac{1}{2})$ then $G^*G \geq \nu I$ on $\mathbb{D}_1$ and G is thus right pseudo-coprime over $\mathbb{D}_1$. □

Proposition 87 *Suppose that the system (A, B, C, D) is input and output stabilizable, approximately controllable and approximately observable, the continuous spectrum of A on the unit circle is empty and that the resolvent set of A is dense in the unit disc. Then the transfer function of the CARE closed-loop system is a normalized right pseudo-coprime factor of the transfer function of the system (A, B, C, D). If in addition* $\dim U < \infty$ *then it is coprime.*

Proof We use Lemmas 26 and 27 to conclude that the system (A, B, C, D) has solutions to both its FARE and its CARE. We conclude from Lemma

82 that the CARE closed-loop system is a normalized right factor of the transfer function of the system (A, B, C, D). This leaves just the (pseudo-) coprimeness to be proven. We combine Lemma 78 and the assumption that the intersection of the continuous spectrum of A with the unit circle is empty to conclude that the spectrum of A_Q intersected with the unit circle is empty and that thus $G(z)^*G(z) = I$ for all z on the unit circle. We then use Lemmas 85 and 86 to conclude that the transfer function of the CARE closed-loop system is pseudo-coprime. The assumption that $\dim U < \infty$ together with Lemma 84 then gives that this transfer function is coprime. □

The dual result is:

Proposition 88 *Suppose that the system (A, B, C, D) is input and output stabilizable, approximately controllable and approximately observable, the continuous spectrum of A on the unit circle is empty and that the resolvent set of A is dense in the unit disc. Then the transfer function of the FARE closed-loop system is a normalized left pseudo-coprime factor of the transfer function of the system (A, B, C, D). If in addition $\dim Y < \infty$ then it is coprime.*

Chapter 8

LQG-balancing revisited

If we combine Theorem 69 with Propositions 87 and 88 we obtain the following theorem:

Theorem 89 *Suppose that the system $(A, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is input and output stabilizable, approximately controllable and approximately observable, the continuous spectrum of A on the unit circle is empty, the resolvent set of A is dense in the unit disc and that either the input or the output space is finite-dimensional. Then $\mathfrak{D}$ has a LQG-balanced realization.*

Bibliography

[1] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, New York, 1995.

[2] K. Hoffman. *Banach Spaces of Analytic Functions.* Dover Publications, New York, 1988.

[3] P.A. Fuhrmann and R. Ober. A functional approach to LQG balancing. *Internat. J. Control* 57 no. 3:627–741, 1993.

[4] E.A. Jonckheere, Ph. Opdenacker and L.M. Silverman. Reduced compensator design via LQG-balancing- a case study. *Mathematical theory of networks and systems. Proceedings of the international symposium held at the Ben Gurion University of the Negev, Beer Sheva*, 513–520, 1983.

[5] E.A. Jonckheere and L.M. Silverman. A new set of invariants for linear systems–application to reduced order compensator design. *IEEE Transactions of Automatic Control*, 28:953–964, 1983.

[6] J. Malinen. *Discrete Time H^∞ Algebraic Riccati Equations.* PhD thesis, Helsinki University of Technology, March 2000.

[7] K. Mikkola. *Infinite-Dimensional Linear Systems: Optimal Control and Riccati Equations.* PhD thesis, December 2001.

[8] D. Mustafa and K. Glover. *Minimum entropy H^∞ control.* Springer-Verlag, Berlin, 1990.

[9] Nicolai K. Nicol'skii. *Treatise on the shift operator.* Springer Verlag, Berlin, 1986.

[10] Ph. Opdenacker and E.A. Jonckheere. LQG balancing and reduced LQG compensation of symmetric passive systems. *Internat. J. Control*, 41 no. 1:73–109, 1985.

[11] M. Rosenblum and J. Rovnyak. *Hardy Classes and Operator Theory.* Oxford University Press, New York, 1985.

[12] O. J. Staffans. *Well-Posed Linear Systems I: General Theory.* To be published, 2002. draft available at `http://www.abo.fi/~staffans/publ.htm`.

[13] N.J. Young. Balanced realizations in infinite dimensions. *Operator Theory: Advances and Applications*, 19:449–471, 1986.

[14] N.J. Young. Balanced, normal, and intermediate realizations of nonrational transfer functions. *IMA Journal of Mathematical Control and Information*, 1:43–58, 1986.