On the computation of Lie symmetries

W.H. Ekkelkamp

M.S.Lie

Department of
Mathematics

RuG

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W.H. Ekkelkamp

University of Groningen
Department of Mathematics
P.O. Box 800
9700 AV Groningen

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Introduction

Before telling all kind of details, I would like to tell something in general about Lie symmetries. When you are familiar with algebra, you have probably heard about Lie algebras, but Lie symmetries are not that well known. As you can see by the name, they are invented by the same man, M.S. Lie. He was born in Norway in 1842 and died in 1899, also in Norway. But mentioning only this about Lie, would be a serious shortcoming. Take for example the point that he began to study mathematics (geometry) at the age of 26 and it took him weeks to accomplish the work others did in years. Some of the subjects he worked on are:

- transformations, and he is specially known for his contact transformation.
- the integration theory of partial differential equations.
- invariance.
- transformation groups.
- differential geometry.

But as happened with many other great mathematicians, his life changed. He turned from an open-hearted man into a very sensitive and suspicious man, due to his illness 'neurasthenia'. He had to go to a mental hospital, where he recovered, but his character had changed. This also meant the end of his friendship with Klein. If you want to know more about this interesting mathematician, see [1], [2].

The peculiar thing about Lie symmetries and Lie algebras is that Lie invented his algebra while studying symmetries of differential equations but only the algebra and the groups named after him became well known. For a change, this thesis is about Lie symmetries. First you will find some theory and an example. Later we will use Maple to compute the symmetries for us, according to a new algorithm.

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Ordinary differential equations and Lie symmetries

One would like to solve a given ordinary differential equation (ODE). If you are lucky, you will recognize it as one of a kind you are able to solve with a special integration technique. If you do not recognize it, it is worthwhile to try to compute the Lie symmetry of the ODE. Unfortunately not every ODE has a non-trivial Lie symmetry, but if you find the Lie symmetry, you might be able to solve the ODE and this depends on the number of symmetries you will find.

The definition of a Lie symmetry is as follows:

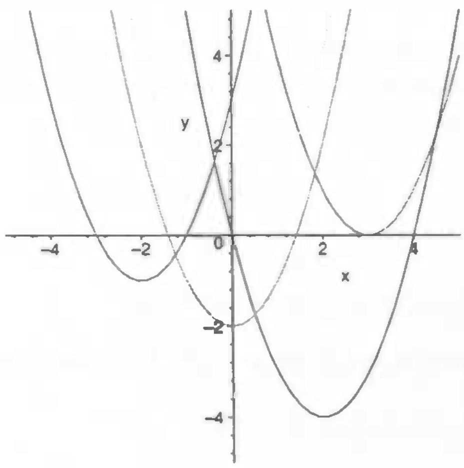
Definition 1 *A point transformation*

$$\begin{cases} \tilde{x} = \tilde{x}(x, y; \epsilon) \\ \tilde{y} = \tilde{y}(x, y; \epsilon) \end{cases}$$

is an infinitesimal transformation of the (x, y)*-plane. It is called a Lie symmetry transformation (= symmetry) of an ordinary differential equation if it maps solutions into solutions.*

This invariance property is used in the solving process, for the more precise details of the solving technique, see [4].

To get a more precise idea of a Lie symmetry, think about an easy differential equation, e.g. $y'' = 2$. The solutions are given by $y(x) = x^2 + ax + b$ with $a, b \in \mathbb{C}$. We can make a graph of some of the solutions:



Now choose a point on one of the solutions. Since a Lie symmetry maps solutions into solutions, we know at least locally how 'far' to go in the x- and y-direction to find a new solution. We can see global symmetries of the (x, y)-plane, for instance translations $(x, y) \to (a_1, a_2) + (x, y)$ and all linear maps.

Instead of writing the word Lie symmetry, we will often use the symbol ∇ and it has the following meaning:

$$\nabla = \nabla_{\xi,\eta} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}.$$

In this expression you can see the infinitesimal movement of a point (x, y) and also that a Lie symmetry is a vector field.
For our equation $y'' = 2$, the Lie symmetry is given by:

$$\nabla = (-ax^3 + (c - 3b)x^2 + ex + f) + (ax + b)y\frac{\partial}{\partial x}$$

$$+(-ax^4 + (c - 4b)x^3 + (2e - d)x^2 + gx + h) + (cx + d)y + ay^2)\frac{\partial}{\partial y}.$$

The space of Lie symmetries is of dimension eight and this is the maximal dimension of a second order equation. We will prove this in a later chapter and the computation of a symmetry will be discussed later on in this chapter.

Lie symmetries and Lie algebra

As remarked in the introduction, Lie invented his algebra while studying symmetries of ordinary differential equations. A Lie algebra is defined as:

Definition 2 *Suppose $\mathcal{L}$ is a vector space over K and there exists a composition $[\ ,\ ] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$. If the following conditions are satisfied, we say that $\mathcal{L}$ is a Lie algebra.*

1. *$[\ ,\ ]$ is a bilinear composition*
2. $[A, A] = 0$
3. $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$ *(Jacobi-identity)*

The second condition can also be written as $[A, B] = -[B, A]$ if the characteristic of the algebra $\neq 2$, because:

$$[A, A] = 0$$

$$\begin{aligned}
&\Rightarrow \quad [A+B, A+B] = 0 \\
&\Rightarrow \quad [A,A] + [B,A] + [A,B] + [B,B] = 0 \\
&\Rightarrow \quad [B,A] + [A,B] = 0 \\
&\Rightarrow \quad [A,B] = -[B,A]
\end{aligned}$$

If the characteristic of the algebra is 2, we cannot use this in the definition, because $[A,B] = -[B,A] \Rightarrow 2[A,A] = 0$ and this is always true in an algebra with characteristic 2.

We will see that the Lie symmetries form a Lie algebra. First we define the composition as: $[\nabla_1, \nabla_2] = \nabla_1\nabla_2 - \nabla_2\nabla_1$ and now we have to check the three points of the definition.

1. bilinearity of the composition

$$\begin{aligned}
[\nabla_1 + \nabla_2, \nabla_3] &= (\nabla_1 + \nabla_2)\nabla_3 - \nabla_3(\nabla_1 + \nabla_2) \\
&= \nabla_1\nabla_3 + \nabla_2\nabla_3 - \nabla_3\nabla_1 - \nabla_3\nabla_2 \\
&= [\nabla_1, \nabla_3] + [\nabla_2, \nabla_3]
\end{aligned}$$

$$\begin{aligned}
[\nabla_1, \nabla_2 + \nabla_3] &= \nabla_1(\nabla_2 + \nabla_3) - (\nabla_2 + \nabla_3)\nabla_1 \\
&= \nabla_1\nabla_2 + \nabla_1\nabla_3 - \nabla_2\nabla_1 - \nabla_3\nabla_1 \\
&= [\nabla_1, \nabla_2] + [\nabla_1, \nabla_3]
\end{aligned}$$

$$\begin{aligned}
\alpha[\nabla_1, \nabla_2] &= \alpha(\nabla_1\nabla_2 - \nabla_2\nabla_1) \qquad \alpha \in K \\
&= \alpha\nabla_1\nabla_2 - \alpha\nabla_2\nabla_1 \\
&= [\alpha\nabla_1, \nabla_2] \\
&= [\nabla_1, \alpha\nabla_2]
\end{aligned}$$

2. $[\nabla_1, \nabla_1] = \nabla_1\nabla_1 - \nabla_1\nabla_1 = 0$

3. Jacoby-identity

$$[\nabla_1, [\nabla_2, \nabla_3]] = \nabla_1\nabla_2\nabla_3 - \nabla_1\nabla_3\nabla_2 - \nabla_2\nabla_3\nabla_1 + \nabla_3\nabla_2\nabla_1,$$

$$[\nabla_3, [\nabla_1, \nabla_2]] = \nabla_3\nabla_1\nabla_2 - \nabla_3\nabla_2\nabla_1 - \nabla_1\nabla_2\nabla_3 + \nabla_2\nabla_1\nabla_3,$$

$$[\nabla_2, [\nabla_3, \nabla_1]] = \nabla_2\nabla_3\nabla_1 - \nabla_2\nabla_1\nabla_3 - \nabla_3\nabla_1\nabla_2 + \nabla_1\nabla_3\nabla_2,$$

$$[\nabla_1, [\nabla_2, \nabla_3]] + [\nabla_3, [\nabla_1, \nabla_2]] + [\nabla_2, [\nabla_3, \nabla_1]] = 0.$$

So indeed the Lie symmetries form a Lie algebra. Furthermore, we can give an explicit form ($\nabla_3 = \xi_3 \frac{\partial}{\partial x} + \eta_3 \frac{\partial}{\partial y}$) of the composition of two symmetries:

$$\nabla_i = \xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y}\ ,\ for\ i = 1, 2$$

$$[\nabla_1, \nabla_2](x) = \nabla_1\nabla_2(x) - \nabla_2\nabla_1(x) = \nabla_1(\xi_2) - \nabla_2(\xi_1) =$$
$$\xi_1\xi_{2,x} + \eta_1\xi_{2,y} - \xi_2\xi_{1,x} - \eta_2\xi_{1,y} = \xi_3$$
$$[\nabla_1, \nabla_2](y) = \nabla_1\nabla_2(y) - \nabla_2\nabla_1(y) = \nabla_1(\eta_2) - \nabla_2(\eta_1) =$$
$$\xi_1\eta_{2,x} + \eta_1\eta_{2,y} - \xi_2\eta_{1,x} - \eta_2\eta_{1,y} = \eta_3$$

Computation of the Lie symmetries

Before we can start with the computation of Lie symmetries, we need an algebraic description of the space we are working in. Therefore we define the following ring R of functions in x and y. (The definitions and lemmas in this section can be found in [3].)

Definition 3 *Let C be a field of characteristic 0 and R a commutative C algebra with a unit $1 \neq 0$. The ring R is equipped with two commuting derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and is supposed to have the following properties:*

1. *The subring $\{f \in R | \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\}$ is equal to C.*
2. *R contains two elements x and y such that $\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1$ and $\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$.*

A very natural example of such a ring R is C^∞-functions on the plane. To be more precise $R = C^\infty(\mathbb{R}^2)$, $C = \mathbb{R}$ and furthermore the usual x, y, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Some other examples are:

- $R = C^\infty(U)$, with U an open connected subset of $\mathbb{R}^2$
- $R = \mathbb{C}[x, y]$
- $R = \mathbb{C}(x)[y]$

Now we know what our ring R looks like, but we want to speak about differential equations, so we define a ring A of polynomial differential equations over R.

Definition 4 *First denote the derivatives of y by y_1, y_2, ... and $y_0 = y$, and let $\{y_i\}_{i\geq 1}$ denote a countable set of variables. Then $A := R[\{y_i\}]$ is the free polynomial ring over R equipped with a differentiation $\frac{d}{dx}$ (the total derivative) defined by:*

1. $\frac{d}{dx}f = f_x + y_1 f_y$ *for* $f \in R$
2. $\frac{d}{dx}y_k = y_{k+1}$ *for* $k \geq 0$

This means that the differential equations are polynomials in the derivatives of y and they have coefficients in R.

We denote an element of the ring A by ω and before we can say something about a solution of ω, we mention that $R^{\frac{\partial}{\partial y}}$ denotes the subring of R consisting of the elements f with $f_y = 0$. Now consider an extension of differential rings $E \supset R^{\frac{\partial}{\partial y}}$ and write again $\frac{d}{dx}$ for the derivation on E. A solution $f \in E$ of ω has the following meaning:

1. A homomorphism of differential rings $\phi : A \to E$, i.e. $\phi \circ \frac{d}{dx} = \frac{d}{dx} \circ \phi$, such that ϕ is the identity on $R^{\frac{\partial}{\partial y}}$ and such that
2. $\phi(y) = f$ and $\phi(\omega) = 0$.

Finally, if $\omega = \sum_{0\leq j\leq n} a_j y_j$ with all $a_j \in R^{\frac{\partial}{\partial y}}$, then ω is called linear homogeneous and if $\omega = a + \sum_{0\leq j\leq n} a_j y_j$ with $a, a_j \in R^{\frac{\partial}{\partial y}}$ it is called linear.

As written in the first part of this chapter $\nabla_{\xi,\eta} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ is defined as operator on R. We want to extend this to an operator on A.

Lemma 1 *For any $\xi, \eta \in R$, the vectorfield $\nabla_{\xi,\eta}$ has a unique extension to a derivation of A (with the same name) such that the Lie bracket $[\frac{d}{dx}, \nabla_{\xi,\eta}]$ is a multiple of $\frac{d}{dx}$.*

Proof. Extending $\nabla_{\xi,\eta}$ to a derivation of A means that we have to choose $\nabla_{\xi,\eta} y_k \in A$ for all $k \geq 1$. Any choice is valid and determines an extension. The Lie bracket $[\frac{d}{dx}, \nabla_{\xi,\eta}]$ is a multiple of $\frac{d}{dx}$ and this means $[\frac{d}{dx}, \nabla_{\xi,\eta}] = h\frac{d}{dx}$. Applying this to x gives:

$$\begin{aligned} [\frac{d}{dx}, \nabla_{\xi,\eta}](x) &= \frac{d}{dx}\nabla_{\xi,\eta}(x) - \nabla_{\xi,\eta}\frac{d}{dx}(x) \\ &= \frac{d}{dx}(\xi) - \nabla_{\xi,\eta}(1) \\ &= \frac{d\xi}{dx} \end{aligned}$$

This means that $h = \frac{d\xi}{dx}$, because $\frac{d}{dx}x = 1$. Applying the same relation to y_k gives:

$$\begin{aligned}
[\frac{d}{dx}, \nabla_{\xi,\eta}](y_k) &= \frac{d\xi}{dx}\frac{d}{dx}(y_k) \\
\frac{d}{dx}\nabla_{\xi,\eta}(y_k) - \nabla_{\xi,\eta}\frac{d}{dx}(y_k) &= \frac{d\xi}{dx}y_{k+1} \\
\nabla_{\xi,\eta}(y_{k+1}) &= \frac{d(\nabla_{\xi,\eta}(y_k))}{dx} - \frac{d\xi}{dx}y_{k+1}
\end{aligned}$$

This determines the extension of $\nabla_{\xi,\eta}$ to A. The expression $D := [\frac{d}{dx}, \nabla_{\xi,\eta}] - \frac{d\xi}{dx}\frac{d}{dx}$ is again a derivation of A. By construction $D(y_k) = 0$ for $k \geq 1$. For $f \in R$, one easily calculates that $D(f) = 0$. This shows that $D = 0$. (Q.E.D.)

We have obtained a recursive expression for $\nabla_{\xi,\eta}(y_{k+1})$ and to see more precisely how this works, we will compute $\nabla(y_1)$ and $\nabla(y_2)$.

$$\begin{aligned}
\nabla(y_1) &= \frac{d(\nabla(y))}{dx} - \frac{d\xi}{dx}y_1 \\
&= \frac{d(\eta)}{dx} - \frac{d\xi}{dx}y_1 \\
&= \eta_x + \eta_y y_1 - \xi_x y_1 - \xi_y y_1^2
\end{aligned}$$

$$\begin{aligned}
\nabla(y_2) &= \frac{d(\nabla(y_1))}{dx} - \frac{d\xi}{dx}y_2 \\
&= \frac{d(\eta_x + \eta_y y_1 - \xi_x y_1 - \xi_y y_1^2)}{dx} - \frac{d\xi}{dx}y_2 \\
&= \eta_y y_2 - 2\xi_x y_2 - 3\xi_y y_1 y_2 + \eta_{xx} + 2\eta_{xy} y_1 - \xi_{xx} y_1 - \\
&\quad 2\xi_{xy} y_1^2 + \eta_{yy} y_1^2 - \xi_{yy} y_1^3
\end{aligned}$$

As one already can see the number of terms will grow larger and larger, so for higher derivatives of y it is wise to use a computer.

The next lemma is important for the computation of a Lie symmetry, because it gives a condition for the existence of a Lie symmetry and with this condition we are able to compute the symmetry.

Lemma 2 *Let $n \geq 2$ and suppose that $\omega \in A$ has the form $y_n +$ terms involving only y_k with $k < n$. Then $\nabla = \nabla_{\xi,\eta} \in \mathcal{L}$ (Lie algebra) is a Lie symmetry for ω if and only if $\nabla(\omega) = (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega$.*

Proof. The expressions $(\frac{d}{dx})^m\omega$ have the form $y_{n+m}+$ terms involving only y_k with $k < n+m$. They generate $\langle\omega\rangle$ (= ideal with $\omega, \frac{d}{dx}\omega, (\frac{d}{dx})^2\omega, \ldots$). Any $f \in R[y_1, \ldots, y_{n+m}]$ can be written uniquely as $f = q(\frac{d}{dx})^m\omega + r$ with $q \in R[y_1, \ldots, y_{n+m}]$ and $r \in R[y_1, \ldots, y_{n+m-1}]$ (Chinese remainder theorem).
It follows easily from this that $\langle\omega\rangle \cap R[\{y_k\}_{k<n}] = 0$.
By induction on N, one easily shows that for $N \geq 2$, the expression ∇y_N is equal to $(\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})y_N+$ terms involving only y_k for $k < N$ (see for the first step ∇y_2 above this lemma). We conclude that $\nabla(\omega) = (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})y_n+$ terms involving only y_k for $k < n$. Then $\nabla(\omega) - (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega \in R[\{y_k\}_{k<n}]$.
If ∇ is a Lie symmetry then $\nabla(\omega) \in \langle\omega\rangle$, and now the statement follows. (Q.E.D.)

In the next chapter we will see some examples of computing the Lie symmetry of an ordinary differential equation.

3

Example of a Lie symmetry

The following example can be found in [4] but the author has not given all the steps that lead to the answer. In order to see how Lie symmetries work I will give the same example in more detail. The differential equation in this example is:

$$y_2 = x^n y^2.$$

One can find the Lie-symmetries with the help of the last lemma in the previous chapter. The lemma gave us an equation for the existence of a Lie symmetry. One computes the left-hand side of the equation, then the right-hand side and by comparing these two, we'll get some conditions on ξ and η. By solving the conditions we get the Lie symmetries of this differential equation.

The left-hand side

We do not have a ω in our differential equation but the ω corresponding to the above equation is (in the terminology of [3]):

$$\omega = y_2 - x^n y^2.$$

∇ is defined as $\nabla_{\xi,\eta} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ and we know that $\nabla(y_{k+1}) = \frac{d(\nabla(y_k))}{dx} - \frac{d\xi}{dx}y_{k+1}$ so we can go on with our example. One has to be careful when calculating $\nabla(y_2 - x^n y^2)$ because it is very easy to make mistakes. But we have already computed ∇y_2 and that gives us a great advantage. We can continue with $\nabla(x^n y^2)$ and put the parts together.

$$\nabla(y_2) = \eta_{xx} + 2\eta_{xy}y_1 + \eta_{yy}y_1^2 + \eta_y y_2 - \xi_{xx}y_1 - 2\xi_{xy}y_1^2 - \xi_{yy}y_1^3 - 3\xi_y y_1 y_2 - 2\xi_x y_2,$$

$$\nabla(x^n y^2) = n\xi x^{n-1}y^2 + 2\eta x^n y,$$

$$\nabla(y_2 - x^n y^2) = \nabla(y_2) - \nabla(x^n y^2) =$$

$$\eta_{xx} + 2\eta_{xy}y_1 + \eta_{yy}y_1^2 + \eta_y y_2 - \xi_{xx}y_1 - 2\xi_{xy}y_1^2 - \xi_{yy}y_1^3 - 3\xi_y y_1 y_2 - 2\xi_x y_2 - n\xi x^{n-1}y^2 - 2\eta x^n y.$$

The right-hand side

We have to compute the following:

$$(\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega.$$

In our example this becomes ($n = 2$):

$$(\xi_x+\eta_y-3\frac{d\xi}{dx})(y_2-x^n y^2) = \xi_x(y_2-x^n y^2)+\eta_y(y_2-x^n y^2)-3\frac{d\xi}{dx}(y_2-x^n y^2) =$$

$$\ldots = -2\xi_x y_2 + 2\xi_x x^n y^2 + \eta_y y_2 - \eta_y x^n y^2 - 3\xi_y y_1 y_2 + 3\xi_y y_1 x^n y^2.$$

Comparison of both sides

In order to compare both sides we have to write them in the same way, that is as an element of $A = R[y_1, y_2, y_3, \cdots]$.

$$\nabla(y_2-x^n y^2) = \eta_{xx}+\eta_y y_2-2\xi_x y_2-n\xi x^{n-1}y^2-2\eta x^n y+(2\eta_{xy}-\xi_{xx}-3\xi_y y_2)y_1+$$
$$(\eta_{yy} - 2\xi_{xy})y_1^2 - \xi_{yy}y_1^3.$$

$$(\xi_x + \eta_y - 3\frac{d\xi}{dx})(y_2 - x^n y^2) = 2\xi_x x^n y^2 - 2\xi_x y_2 + \eta_y y_2 - \eta_y x^n y^2+$$
$$(3\xi_y x^n y^2 - 3\xi_y y_2)y_1.$$

Comparing the coefficients of y_1^n with $n \in \{0, 1, 2, 3\}$ gives the following conditions:

$$\xi_{yy} = 0 \quad (1)$$
$$\eta_{yy} = 2\xi_{xy} \quad (2)$$
$$2\eta_{xy} - \xi_{xx} = 3\xi_y x^n y^2 \quad (3)$$
$$\eta_{xx} - n\xi x^{n-1}y^2 - 2\eta x^n y = 2\xi_x x^n y^2 - \eta_y x^n y^2 \quad (4)$$

From the conditions (1) and (2) we know the general form of ξ and η, with ξ_i and η_i depending on x only:

$$\xi = \xi_0 + \xi_1 y,$$
$$\eta = \eta_0 + \eta_1 y + \eta_2 y^2.$$

We put the general form of ξ and η in condition (2) and get:

$$2\eta_2 = 2\xi_1^{'}.$$

This leads to the condition:

$$\eta_2 = \xi_1'. \tag{5}$$

We do the same again with condition (3):

$$2\eta_1' + 4\eta_2' y - \xi_0'' - \xi_1'' y = 3x^n y^2 \xi_1.$$

Comparing the coefficients of y^i leads to:

$$\begin{aligned} 3x^n\xi_1 &= 0 && (6)\\ 4\eta_2' - \xi_1'' &= 0 && (7)\\ 2\eta_1' - \xi_0'' &= 0 && (8) \end{aligned}$$

From (6) follows that $\xi_1 = 0$. This implies in condition (5) that $\eta_2 = 0$. The general form of ξ and η becomes:

$$\begin{aligned} \xi &= \xi_0, \\ \eta &= \eta_0 + \eta_1 y. \end{aligned}$$

We now move on to condition (4) and substitute the 'new' general form.

$$\eta_0'' + \eta_1'' y - n\xi_0 x^{n-1} y^2 - 2(\eta_0 + \eta_1 y)x^n y = (2\xi_0' - \eta_1)x^n y^2.$$

This leads to the following equations:

$$\begin{aligned} -n\xi_0 x^{n-1} - 2\eta_1 x^n &= (2\xi_0' - \eta_1)x^n && (9)\\ \eta_1'' - 2\eta_0 x^n &= 0 && (10)\\ \eta_0'' &= 0 && (11) \end{aligned}$$

Condition (9) can be rewritten as $2x\xi_0' + n\xi_0 = -x\eta_1$. From (11) it follows that $\eta_0 = a + bx$, $a, b \in \mathbb{C}$. In combination with (10), we get $\eta_1'' = 2ax^n + 2bx^{n+1}$. This leads to:

$$\eta_1 = \frac{2ax^{n+2}}{(n+1)(n+2)} + \frac{2bx^{n+3}}{(n+2)(n+3)} + cx + d.$$

According to (8), this implies:

$$\xi_0'' = \frac{4ax^{n+1}}{n+1} + \frac{4bx^{n+2}}{n+2} + 2c \quad \Rightarrow$$

$$\xi_0 = \frac{4ax^{n+3}}{(n+1)(n+2)(n+3)} + \frac{4bx^{n+4}}{(n+2)(n+3)(n+4)} + cx^2 + ex + f.$$

We have not used condition (9) completely, so let us go on with (9) (rewritten) and substitute ξ_0 and η_1.

$$\frac{8ax^{n+3}}{(n+1)(n+2)} + \frac{8bx^{n+4}}{(n+2)(n+3)} + 4cx^2 + 2ex +$$
$$n(\frac{4ax^{n+3}}{(n+1)(n+2)(n+3)} + \frac{4bx^{n+4}}{(n+2)(n+3)(n+4)} + cx^2 + ex + f) =$$
$$\frac{-2ax^{n+3}}{(n+1)(n+2)} - \frac{2bx^{n+4}}{(n+2)(n+3)} - cx^2 - dx. \quad (12)$$

It would be nice if we could compare the coefficients of the different powers of x. This is possible if we first state that n is not -1, -2, -3 or -4 and look at these cases afterwards. Let us start with the coefficients of x^4:

$$\frac{8b}{(n+2)(n+3)} + \frac{4nb}{(n+2)(n+3)(n+4)} = \frac{-2b}{(n+2)(n+3)}$$
$$\Rightarrow \quad 10b(n+4) + 4nb = 0$$
$$\Rightarrow \quad n = \frac{-20}{7}.$$

This n is apparently special, thereforewe may expect an extra symmetry for this n. We go on with x^3:

$$\frac{8a}{(n+1)(n+2)} + \frac{4na}{(n+1)(n+2)(n+3)} = \frac{-2a}{(n+1)(n+2)}$$
$$\Rightarrow \quad 10a(n+3) + 4na = 0$$
$$\Rightarrow \quad n = \frac{-15}{7}.$$

Comparing the coefficients of x^2 leads to:

$$4c + nc = -c$$
$$\Rightarrow \quad n = -5.$$

The coefficients of x give:

$$2e + ne = -d$$
$$\Rightarrow \quad n = \frac{-d}{e} + 2.$$

And finally, what is left is:

$$nf = 0.$$

Now we know the values of n that are special in our differential equation $y_2 = x^n y^2$. We have used all the conditions on ξ and η, and we are ready to determine all Lie symmetries.

The Lie symmetries

First we start with the symmetry for a general n. The Lie-symmetry has the following form:

$$\nabla = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}.$$

And we already know:

$$\xi = \frac{4ax^{n+3}}{(n+1)(n+2)(n+3)} + \frac{4bx^{n+4}}{(n+2)(n+3)(n+4)} + cx^2 + ex + f,$$

$$\eta = (a + bx) + (\frac{2ax^{n+2}}{(n+1)(n+2)} + \frac{2bx^{n+3}}{(n+2)(n+3)} + cx + d)y.$$

We use again condition (12), but this time we concentrate on a, b, c, d, e and f. Beginning with a and excluding $n = -15/7$, we get:

$$\begin{aligned} & \frac{8ax^{n+3}}{(n+1)(n+2)} + \frac{4nax^{n+3}}{(n+1)(n+2)(n+3)} = \frac{-2ax^{n+3}}{(n+1)(n+2)} \\ \Rightarrow \quad & (14n+30)ax^{n+3} = 0 \\ \Rightarrow \quad & a = 0. \end{aligned}$$

We go on with b (and exclude $n = -20/7$):

$$\begin{aligned} & \frac{8bx^{n+4}}{(n+2)(n+3)} + \frac{4nbx^{n+4}}{(n+2)(n+3)(n+4)} = \frac{-2bx^{n+4}}{(n+2)(n+3)} \\ \Rightarrow \quad & (14n+40)bx^{n+4} = 0 \\ \Rightarrow \quad & b = 0. \end{aligned}$$

We look at c and $n \neq -5$:

$$\begin{aligned} & 4cx^2 + ncx^2 = -cx^2 \\ \Rightarrow \quad & (5+n)cx^2 = 0 \\ \Rightarrow \quad & c = 0. \end{aligned}$$

Until now we only found the trivial solution. But let us see what the following step brings:

$$\begin{aligned} & 2ex + nex = -dx \\ \Rightarrow \quad & (2+n)e = -d. \end{aligned}$$

And finally:

$$\begin{aligned} nf &= 0 \\ \Rightarrow \quad f &= 0. \end{aligned}$$

If we combine all this information, we get the following symmetry:

$$\nabla = (ex)\frac{\partial}{\partial x} - (2+n)ey\frac{\partial}{\partial y}.$$

So we always have this symmetry, but for the 'special' n we calculated above, there may be more symmetries.
Suppose $n = -5$ and we look again at condition (12). We start comparing the coefficients with a, b, c, d, e and f again. For a and b nothing changes. For c we get $-cx^2 = -cx^2$, so any c will do. And we obtain $3e = d$. The symmetry becomes:

$$\nabla = (cx^2 + ex)\frac{\partial}{\partial x} + (cx + 3e)y\frac{\partial}{\partial y}.$$

We can repeat this for the other 'special' n and get the following symmetries:

- $n = -\frac{15}{7}$: $\nabla = (\frac{343}{12}ax^{6/7} + ex)\frac{\partial}{\partial x} + (a + (\frac{49}{4}ax^{-1/7} - \frac{1}{7}e)y)\frac{\partial}{\partial y}$
- $n = -\frac{20}{7}$: $\nabla = (\frac{-343}{12}bx^{8/7} + ex)\frac{\partial}{\partial x} + (bx + (\frac{-49}{3}bx^{1/7} + \frac{6}{7}e)y)\frac{\partial}{\partial y}$

Finally we have to check what happens when $n \in \{-1, -2, -3, -4\}$. If $n = -1$ we get:

$$\begin{aligned} \eta_0 &= a + bx, \\ \eta_1 &= 2a(x\log x - x) + bx^2 + mx + n, \\ \xi &= 2ax^2 \log x - 3ax^2 + \frac{2}{3}bx^3 + mx^2 + px + q. \end{aligned}$$

We use again condition (9) and follow the same procedure as before. So we concentrate on a, b, c, d, e and f. This leads to the same symmetry as for a general n. Repeating this for the other n gives that they all have the symmetry as for a general n.

The only question left, is why $n = 0$ is excluded. Curious as we are, we start computing the Lie symmetry of $\omega = y_2 - y^2$ in the same way as before. This leads to the following symmetry:

$$\nabla = (dx + e)\frac{\partial}{\partial x} - 2dy\frac{\partial}{\partial y}.$$

We can see now that $\xi(x)$ has an extra constant and that is probably the reason why $n = 0$ is excluded in the beginning.

4

On the dimension of the space of Lie symmetries

In this chapter we are going to prove that the space of Lie symmetries of a second order linear differential equation $\omega = y_2 + f(x)y_1 + g(x)y$ has at most dimension 8. This can be done by showing that $\omega = y_2$ has the following symmetry:

$$\nabla = (ax^2 + bx + c + (dx + 2)y)\frac{\partial}{\partial x} + (f + gx + (h + ax)y + dy^2)\frac{\partial}{\partial y}.$$

and by a suitable choice of new variables our old ω will be $\omega = y_2$ in new coordinates.

It can also be shown by using the lemma for the existence of a Lie symmetry. In this lemma we have the following equation:

$$\nabla(\omega) = (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega.$$

If we apply this to $\omega = y_2 + f(x)y_1 + g(x)y$ we get the following:

$$\begin{aligned}
\xi_{yy} &= 0, \\
\eta_{yy} - 2\xi_{xy} &= -2f\xi_y, \\
2\eta_{xy} - \xi_{xx} + \xi f' &= -f\xi_x - 3g\xi_y y, \\
\eta_{xx} + f\eta_x + \xi g' y + g\eta &= g\eta_y y - 2g\xi_x y.
\end{aligned}$$

From the first two equations it follows that ξ and η have the following form:

$$\begin{aligned}
\xi &= \xi_0(x) + \xi_1(x)y, \\
\eta &= \eta_0(x) + \eta_1(x)y + \eta_2(x)y^2.
\end{aligned}$$

And by substitution in the last three equations and comparison of the coefficients we get (for convenience written without (x)):

$$\begin{aligned}
\eta_2 &= \xi_1' - f\xi_1, \\
4\eta_2' - \xi_1'' + \xi_1 f' &= -f\xi_1' - 3g\xi_1, \\
2\eta_1' &= \xi_0'' - \xi_0 f' - f\xi_0', \\
\eta_2'' + f\eta_2' - g\eta_2 &= -\xi_1 g' - 2g\xi_1', \\
\eta_1'' + f\eta_1' &= -\xi_0 g' - 2g\xi_0', \\
\eta_0'' + f\eta_0' + g\eta_0 &= 0.
\end{aligned}$$

We observe that η_0, η_1 and η_2 are arbitrary solutions of a a second order differential equation and ξ_0 and ξ_1 are arbitrary solutions of a first order differential equation. This means that the space of Lie symmetries has dimension 8.

For a higher order n differential equation the dimension of the space of the symmetries is at most $n+4$. We will see this in a later chapter, but we will make a start. Suppose $\omega = y_n + a_{n-1}y_{n-1} + \ldots + a_1y_1 + a_0y \in A$ with all $a_i \in K$ and $n \geq 3$. If we use our, by now famous, equation $\nabla(\omega) = (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega$ we find $\xi_y = 0$ and $\eta_{yy} = 0$. This means that $\xi = \xi_0(x)$ and $\eta = \eta_0(x) + \eta_1(x)y$. As these expressions are smaller, it is not surprising that the dimension of the space of Lie symmetries is relatively smaller.

5

Painlevé equations

A special kind of second-order ordinary differential equations is the six Painlevé equations. These equations possess the so-called Painlevé property, which means that the solutions may have other singularities than poles, but only at certain points and these points are fixed by the equation. Solutions to these equations are called Painlevé transcendents. Recently, new discoveries have led to a renewed interest in these equations. For more information, see [5].
The equations are:

$P_I : \frac{d^2y}{dt^2} = 6y^2 + t$

$P_{II} : \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$

$P_{III} : \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$

$P_{IV} : \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$

$P_V : \frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$

$P_{VI} : \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} +$
$\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$

One can also compute the Lie symmetries of these equations. This can be done by hand, but it is also possible to use Maple. The following code gives the Lie symmetry for the first equation:

```
with(DEtools, symgen);
ode := diff(y(t),t,t) = 6*y(t)*y(t) + t;
sym := symgen(ode,way=formal);
```

Maple gave as answer that there were no symmetries. Not believing this immediately, I did the same computation by hand and it was indeed correct. I've computed the symmetries of the first five equations with Maple and each equation has no symmetries. The computation of the symmetries of the last equation took too much time, but I assume that there are also no symmetries. The reason for believing this is that Painlevé equations cannot be reduced to easier equations. A differential equation which has non-trivial Lie symmetries can be reduced to an easier equation.

6

A different point of view

As we have seen in the examples, the computation of a Lie symmetry takes a lot of time. To improve this for linear differential equations, we will do another computation in a slightly different environment and transform the solution back into a Lie symmetry. Before we can do this new computation we have to define the following:

- $K = R^{\frac{\partial}{\partial y}}$.
- $K[\partial]$ is a skew ring of differential operators, $\partial = \frac{d}{dx}$ and the multiplication is given by $\partial a = a\partial + a'$ with $a' = \frac{da}{dx}$.
- $L \in K[\partial]$, $L \neq 0$ and L has an invertible leading coefficient. A symmetry of the operator L has the form $b\partial + a$ with $a, b \in K$ and $L(b\partial + a)$ is a left multiple of L.

From the last point we get the following lemma:

Lemma 3 *$b\partial + a$ is a symmetry of L if and only if there is an $\tilde{a} \in K$ with $L(b\partial + a) = (b\partial + \tilde{a})L$.*

This can be shown by comparing degrees and coefficients. The comparison of coefficients is as follows:
Suppose $L = c_n\partial^n + c_{n-1}\partial^{n-1} + \ldots + c_1\partial + c_0$. $(b\partial + a)$ is a symmetry if there exists a $\tilde{b}\partial + \tilde{a}$ with $L(b\partial + a) = (\tilde{b}\partial + \tilde{a})L$. Computing both sides gives:

$$(c_n\partial^n + c_{n-1}\partial^{n-1} + \ldots + c_1\partial + c_0)(b\partial + a) = bc_n\partial^{n+1} + (ac_n + nb'c_n + bc_{n-1})\partial^n + \ldots$$

$$(\tilde{b}\partial + \tilde{a})(c_n\partial^n + c_{n-1}\partial^{n-1} + \ldots + c_1\partial + c_0) = \tilde{b}c_n\partial^{n+1} + (\tilde{a}c_n + \tilde{b}c_n' + \tilde{b}c_{n-1})\partial^n + \ldots$$

It follows that $\tilde{b} = b$ and $\tilde{a} = a + nb' - b\frac{c_n'}{c_n}$.

With this lemma we are able to compute a symmetry for L, but it is still a lot of work. Therefore we continue with looking at the degree of L and the corresponding symmetry.

- $n = 0$: $L = c_0$ with c_0 invertible. Any $b\partial + a$ is a symmetry since $(c_0(b\partial + a) = bc_0\partial + ac_0 = (b\partial + \tilde{a})c_0$ with $\tilde{a} = a - b\frac{c_0'}{c_0}$.

- $n = 1$: $L = c_1\partial + c_0 = a_1(\partial + a_0)$. The symmetries $b\partial + a$ are given by $a = a_0 b + c$ with $b \in K$ and $c \in C$.
- $n = 2$: $L = c_2\partial^2 + c_1\partial + c_0 = a_2(\partial^2 + a_1\partial + a_0)$. For a symmetry we have to compute $L(b\partial + a) - (b\partial + \tilde{a})L = 0$. We find for this L the following equations:

$$2a' = -b'' + a_1 b' + a_1' b,$$

$$b''' + (4a_0 - 2a_1' - a_1^2)b' + (2a_0' - a_1 a_1' - a_1'')b = 0.$$

  When we are able to solve these differential equations, we know the symmetry of this operator and we see that the space of the symmetry is of dimension 4.

For higher degrees we could go on in the same way, but that is not the best way. That is why we introduce now the symmetric power of an operator.

The symmetric power of an operator

One can define the symmetric power in different ways. First suppose $L = \partial^2 + a_1\partial + a_0$ and let f_1,f_2 denote a basis over C of the solution space of L. The $(n-1)^{st}$ symmetric power L_n of L is the monic operator of degree n such that its space of solutions has basis $\{f_1^d f_2^e | 0 \leq d, e \; ; \; d + e = n - 1\}$. Another definition of L_n uses a nonzero solution e_0 of L. In this case L_n is the monic operator, of smallest degree, with $L_n e_0^{n-1} = 0$ for every solution e_0 of L.

There are some nice properties of this symmetric power and they are listed in the following proposition. The proof can be found in [3].

Proposition 1 *Let L be the differential operator $\partial^2 + a_1\partial + a_0$.*

1. *The $(n-1)^{st}$ symmetric power L_n of L has the form*

$$\partial^n + b_n a_1 \partial^{n-1} + (c_n a_1^2 + d_n a_1' + e_n a_0)\partial^{n-2} + \ldots,$$

   where $b_n = \frac{n(n-1)}{2}$, $c_n = \frac{n(n-1)(n-2)(3n-1)}{24}$, $d_n = \frac{n(n-1)(n-2)}{6}$ and $e_n = \frac{(n+1)n(n-1)}{6}$ are positive integers.

2. *L_n has the same symmetries as L.*

3. *Suppose that $L_n(b\partial + a) - (b\partial + \tilde{a})L_n$, with $\tilde{a} = a + nb'$, has degree $\leq n - 3$. Then $L_n(b\partial + a) = (b\partial + \tilde{a})L_n$.*

The first part of the proposition is useful if you have given an operator with a high degree, because you can calculate backwards and find out if it is a symmetric power. If it is, the second part tells you that you only have to find the symmetry of L instead of L_n. Finally, the third part says that the first three terms of the symmetric power determine the entire symmetric power.

Another lemma about symmetric powers is the following:

Lemma 4 *If $b\partial + a$ is a symmetry of $L = \partial^2 + a_1\partial + a_0$ then $b\partial + (n-1)a$ is a symmetry of the $(n-1)^{st}$ symmetric power of L.*

Proof. Suppose $L = \partial^2 + a_1\partial + a_0$ and let y_1, y_2 denote a basis over C of the solution space of L. A symmetry maps solutions into solutions, so: $b\partial + a : Cy_1 + Cy_2 \longrightarrow Cy_1 + Cy_2$, $y_i \mapsto by_i' + ay_i$.
A basis of the solution space of L_3 is: $Cy_1^2 + Cy_1y_2 + Cy_2^2$. Applying $b\partial + a$ to this base gives:

$$(b\partial + a)y_1^2 = 2by_1y_1' + ay_1^2 = y_1(2by_1' + ay_1),$$

but $2by_1' + ay_1$ cannot be written in terms of $Cy_1 + Cy_2$. To compensate the 2, we multiply a with 2. So the symmetry becomes $b\partial + 2a$ and this leads to:

$$(b\partial + 2a)y_1^2 = 2by_1y_1' + 2ay_1^2 = 2y_1(by_1' + ay_1) = 2y_1 + (b\partial + a)y_1.$$

For a higher symmetric power it works the same. Take for example the m^{th} symmetric power of L. A basis of the solution space is:
$y_1^m, y_1^{m-1}y_2, \ldots, y_2^m$ and if we apply $b\partial + ma$ we get:

$$(b\partial+ma)y_1^m = mby_1^{m-1}y_1' + may_1^m = my_1^{m-1}(by_1' + ay_1) = my_1^{m-1}(b\partial+a)y_1,$$

$(b\partial + a)y_1 \in Cy_1 + Cy_2$ and this proves the lemma.

For the moment this is all we need to know about the symmetric power of an operator. So far we know how to compute the symmetry of an operator, but we still have to transform it back into the Lie symmetry of the corresponding linear differential equation.

From $b\partial + a$ to $\nabla_{\xi,\eta}$

We start with linking homogeneous $\omega \in A$ and $L \in K[\partial]$. Let $LE \subset A$ denote the set of linear expressions $a_ny_n + a_{n-1}y_{n-1} + \ldots + a_0y + a$ with $n \geq 0$ and $a_n, \ldots, a_0, a \in K$. Then one can identify the homogeneous expressions

with $LE/K \subset A/K$.
Now consider the K-linear isomorphism $\phi : K[\partial] \to LE/K$ given by $\phi(\sum_{i=0}^{n} a_i\partial^i) = \sum_{i=0}^{n} a_i y_i$ modulo K, where y_0 stands for y. The operator $\frac{d}{dx}$ acts on LE/K and corresponds with multiplication on the left by ∂ in $K[\partial]$. This can be written as $\phi(\partial L) = \frac{d}{dx}(\phi(L))$.

We know how to connect $\omega \in A$ and $L \in K[\partial]$ and the following proposition (from [3]) gives us a surjective homomorphism between $b\partial + a$ and $\nabla_{\xi,\eta}$. This proposition works for an operator L of degree $n > 2$ and the situation for an operator of degree 2 is explained after the proof of the proposition.

Proposition 2 *Fix a monic $L \in K[\partial]$ of degree $n > 2$ and let $\omega = \phi(L)$. The map $\nabla_{\xi,\eta} \mapsto \xi\partial - \eta_y$ provides a surjective homomorphism of $\mathcal{L}_\omega$, the algebra of Lie symmetries of ω, to the Lie algebra of the symmetries of the operator L. The kernel of this map consists of the Lie symmetries in $\mathcal{L}_\omega$ of the form $\nabla_{0,\eta}$ with η in the solution space of ω (or L).*

Proof. We take $n > 2$ so $\xi_y = 0$. This implies that the equation $\nabla\omega = (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega$ can be rewritten as $\nabla\omega = (\eta_y - n\xi')\omega$. This leads to $\tilde{\nabla}_{\xi,\eta}L = (\eta_y - n\xi')L$ or equivalently $L(-\xi\partial + \eta_y) = (-\xi\partial + \eta_y - n\xi')L$.
In other words, $\xi\partial - \eta_y$ is a symmetry for the operator L. A straightforward calculation shows that the map of the proposition is a homomorphism from the Lie algebra $\mathcal{L}_\omega$ to the Lie algebra of the symmetries of L. The kernel of this map consists of the $\nabla_{0,\eta} \in \mathcal{L}_\omega$ with $\eta_y = 0$. It is easily seen that the η's with this property are just the elements in K (or in a Picard-Vessiot extension of K) with $L(\eta) = 0$.
Now let $b\partial + a$ be a symmetry of L. Define $\xi = b$ and $\eta = \eta_0 - ay$ with, for the moment, an unknown $\eta_0 \in K$. By construction, the required equation $\nabla_{b,-ay}\omega = (\eta_y - n\xi')\omega$ holds modulo K. Thus we have to choose η_0 such that $\nabla_{0,\eta_0}\omega = f := (\eta_y - n\xi')\omega - \nabla_{b,-ay}\omega \in K$. This is the differential equation $L(\eta_0) = f$ and has a solution (after taking a Picard-Vessiot extension of K). This shows that the map is surjective. (Q.E.D.)

In other words the above proposition tells us that if we have computed $b\partial + a$, we know $\xi = -b$ and $\eta = \int a\ dy + solution\ of\ L$. But this only works for an operator of degree 3 or higher. An operator of degree 1 is not interesting, because there are infinitely many Lie symmetries, so the remaining case is an operator of degree 2. In this case we have $L = \partial^2 + a_1\partial + a_0$, but we also need the transposed of L. This means $\partial \mapsto -\partial$ and the order is changed, so $\partial^2 + a_1\partial + a_0 \mapsto (-\partial)^2 + (-\partial)a_1 + a_0$ and this becomes $\hat{L} = \partial^2 - a_1\partial - a_1' + a_0$. We need this transposed because $Hom(V, V) = V^* \otimes V$.

After some trial and error with the algorithm, comparing with a known Lie symmetry and computations with the equation $\nabla\omega = (\xi_x + \eta_y - (n+1)\frac{d\xi}{dx})\omega$ we found that (with $c = solution\ of\ \hat{L}$)

$$\xi = -b + cy$$

$$\eta = solution\ of\ L + ay + \frac{1}{4}(c' - a_1 c - \int 3a_0 c\ dx)y^2$$

but when $a_0 = 0$ the factor $\frac{1}{4}$ in η vanishes.

Now we could go to the algorithm, but to make the computations easier, we will look again at the dimension of the space of Lie symmetries.

The dimension of the space of symmetries

In the last part of this chapter we will look at the dimension of the space of symmetries. The following theorem (from [3]) gives all the possibilities.

Theorem 1 *Let $L \in K[\partial]$ be a monic operator of degree $n \geq 3$ and let $\omega = \phi(L)$. Then:*

1. *The dimension of $\mathcal{L}_\omega$ can only be $n+1$, $n+2$ or $n+4$.*

2. *The dimension is $n+4$ if and only if L is the $(n-1)^{st}$ symmetric power of a monic operator of degree 2.*

3. *The dimension is $n+2$ if and only if L is not an $(n-1)^{st}$ symmetric power of a monic operator of degree 2 and moreover there is a $\Delta := b\partial + a$ with b invertible and there are constants $c_i \in C$ such that $c_n = 1$ and $L = b^{-n}\sum_{i=0}^{n} c_i \Delta^i$. In the case where K is a differential field, b and a belong to some Picard-Vessiot extension of K.*

Proof. (1) and (2). By the previous proposition we have to show that the dimension of the Lie algebra of the symmetries of L can only be 1,2 or 4. Of course, any constant $c \in C$ is a symmetry for L and this dimension is at least 1. Using the first proposition of this chapter, there is a unique monic operator L_2 of degree 2 such that $L = L_n + R$, where L_n is the $(n-1)^{st}$ symmetric power of L_2 and R is an operator of degree $m \leq n-3$.
If $b\partial + a$ is a symmetry of L, then $L_n(b\partial + a) - (b\partial + \tilde{a})L_n$, with $\tilde{a} = a + nb'$, has degree $\leq n-3$. Thus $b\partial + a$ is a symmetry of both L_n and R. If $R = 0$, then the dimension of the space of symmetries of L is 4. If $R \neq 0$ then $R = A\partial^m + \ldots$ and $A\tilde{a} = A(a + mb') - bA'$ holds. Together with

$\tilde{a} = a + nb'$ this implies $b^{n-m}A$ is a nonzero constant. This property can at most be valid for a single b (up to a multiple). Therefore, the dimension of the symmetries of L is at most 2.
(3). If L has the form of the statement, then clearly Δ is a symmetry of L and the dimension of the Lie symmetries of ω is $\geq n+2$. On the other hand, suppose that L has a symmetry $\Delta := b\partial + a$ with b invertible (b and a are allowed to lie in a Picard-Vessiot extension of K). Then we can of course write $L = b^{-n}M$ with $M := \sum_{i=0}^{n} c_i \Delta^i$, where $c_n = 1$ (and all c_i in some Picard-Vessiot extension of K). The condition $L\Delta = (\Delta + nb')L$ translates into $M\Delta = \Delta M$, and this easily implies that all c_i are constants. (Q.E.D.)

Especially the proof of (1) and (2) is of interest, because if L is not a symmetric power the only possible symmetry is fixed by the leading coefficient of R. This is because R fixes b and a depends on b.

With this in mind we can go to the algorithm for computing the Lie symmetries of a homogeneous linear differential equation.

7

The algorithm

Let $L \in K[\partial]$ be a monic operator of degree n and let $\phi(L) \in A$ denote its equation. The following algorithm computes the symmetries of the operator L and with the help of a proposition in the previous chapter we know what the corresponding Lie symmetries are.

- For $n = 1$ the symmetry is given by $b\partial + a_0 b + c$, with $b \in K$ and $c \in C$.
- For $n = 2$ we have to solve the following equations:

$$2a' = -b'' + a_1 b' + a_1' b,$$

$$b''' + (4a_0 - 2a_1' - a_1^2)b' + (2a_0' - a_1 a_1' - a_1'')b = 0.$$

- For $n > 2$, we will use the symmetric powers of an operator $\partial^2 + a_1\partial + a_0$. L can be written as $sym^{n-1}(\partial^2 + a_1\partial + a_0) + R$, where R has degree $\leq n - 3$.
  1. If $R = 0$, then a and b are given by the solutions of the equations $2a' = -b'' + a_1 b' + a_1' b$ and $b''' + (4a_0 - 2a_1' - a_1^2)b' + (2a_0' - a_1 a_1' - a_1'')b = 0$. The symmetry becomes $b\partial + (n-1)a$.
  2. If $R \neq 0$, then we write $R = AL_1$, where L_1 is monic of degree $n_1 \geq 0$. The only possibility for $b \neq 0$ is a nonzero multiple of $A^{\frac{1}{(n_1 - n)}}$.
  Now we have to verify that $b\partial + a$, with $2a' = -b'' + a_1 b' + a_1' b$, is a symmetry of both $sym^{n-1}(\partial^2 + a_1\partial + a_0)$ and R.
  If this is the case, then $b\partial + a$ is the symmetry, else there are only constant symmetries.
- The last step is the computation of the corresponding Lie symmetry.

The complete Maple-code can be found in appendix A.

8

Some comments

For a scalar linear differential equation there are two theories that associate to this equation a Lie algebra:

1. Lie symmetries.
2. the Lie algebra of the differential Galois group.

The question is whether these two theories are related: The answer is no. (This question is one of the reasons for the writing of [3])
However, if the space of Lie symmetries has dimension $n + 4$, then the equation is so special, that its differential Galois group is also special. Before we continue with this equation, we give a short introduction to differential Galois theory.

Differential Galois theory

We start with the equation $y_n + a_{n-1}y_{n-1} + \ldots + a_0y = 0$ with $a_i \in K$. Here K is a differential field, for example $K = \mathbb{C}(x)$. The following (sloppy) definition gives a good idea about Picard-Vessiot extensions.

Definition 5 *A Picard-Vessiot extension is the smallest extension of differential fields $PV \supset K$, having a full set of solutions, i.e., the solution space V is the n-dimensional space of the field of constants ($= \mathbb{C}$). Moreover, the field of constants of PV is the same as that of K and this is $\mathbb{C}$ in this case.*

Another important notion is the differential Galois group G. It is defined as the group of all the differential automorphisms of PV, that are K-linear. In a formula this is:

$$G = \{\sigma : PV \xrightarrow{\sim} PV \mid \sigma(\lambda) = \lambda \; for \; \lambda \in K; \sigma(f') = \sigma(f)'\}.$$

G embeds into $GL(V)$ and is an algebraic subgroup. Moreover, an algebraic subgroup G has a Lie algebra $Lie(G)$.
Example: $y_2 - xy = 0$, the so-called Airy-equation.
In this case there are two solutions Y_1 and Y_2. The Picard-Vessiot extension is $PV = \mathbb{C}(Y_1, Y_2, Y_1', Y_2')$ and there is only one relation $Y_1Y_2' - Y_1'Y_2 = 1$.

$G = SL_2$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$ acts on $\mathbb{C}Y_1 + \mathbb{C}Y_2$ in the usual way and similarly on $\mathbb{C}Y_1' + \mathbb{C}Y_2'$. The Lie algebra of SL_2 is the following:

$$Lie(SL_2) = \underline{sl}_2 = \{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \mid trace = 0\}.$$

We return to our linear differential equation of order n and the space of its Lie symmetries has dimension $n+4$. This implies that the corresponding operator L can be written as $L = sym^{n-1}(M)$ with $M = \partial^2 + a_1\partial + a_0$. The solution space of M is V and this can be written as $V = \mathbb{C}v_1 + \mathbb{C}v_2$. Moreover $PV(L) \subset PV(M)$. We denote by W the solution space of L and $W = \{v_1^{\alpha} v_2^{\beta} \mid \alpha + \beta = n - 1\}$.
$\sigma \in G$ acts on W by the formula: $\sigma(v_1^{\alpha} v_2^{\beta}) = \sigma(v_1)^{\alpha}\sigma(v_2)^{\beta}$. This means that $sym^{n-1}(G) \subset GL(W)$ and $GL(W)$ is the differential Galois group of L. Therefore this is a rather special differential Galois group and $Lie(G)$ is rather special, too.

Appendix A: Maple code of the algorithm

```
#Description
This algorithm computes the symmetries of a monic linear operator.
It also gives the corresponding linear  differential equation and
the Lie symmetries of that equation.

The algorithm consists of two procedures, the first one computes
the symmetry of a second order equation and
the second procedure computes the symmetries for the other orders
and produces the Lie symmetries.

One calls the procedure with: algorithm(L).
L is of the form: Dx^n + a(n-1)Dx^(n-1) + ... + a(1)Dx + a(0),
where a(i) are functions in x.


degree2 := proc(L2,sym)

local C,A,sys1,sol1,f,g,sym0,a,b:

A:=[Dx,x]:
C:=L2:

sys1:=[2*diff(a(x),x) = -diff(diff(b(x),x),x) + (C[1])*diff(b(x),x)
                        + diff(C[1],x)*b(x) ,
      diff(b(x),x$3) + (4*C[0] - 2*diff(C[1],x) - (C[1])^2)*diff(b(x),x)
      + (2*diff(C[0],x) - (C[1])*diff(C[1],x) - diff(C[1],x,x))*b(x) = 0]:
sol1 := dsolve(sys1,[b(x),a(x)]):
f := x -> subs(sol1[1],a(x)):
g := x -> subs(sol1[2],b(x)):
sym0:=eval(g(x)*Dx + f(x)):
sym:=sym0:
RETURN()
end proc:


algorithm := proc(T)
```

```
local A,c,safe,n,n1,n2,ans,ans1,L,L1,C,i,sym1,sym2,sym3,a0,a1,K,power,
      rest,b,a,h1,h2,h3,h4,h5,h6,h7,h8,h9,f,w,D,w1,g:
with(DEtools):

## some initial conditions#
A:=[Dx,x]:
c := T:
safe:=0:
n:= degree(c,Dx):
ans:='the symmetry is':

## determine the coefficients of L#
for i from 0 to n do
    L[i] := coeff(c,Dx,i):
    end do:

## determine the corresponding ODE#
w:=0:
for i from 1 to n do
D[i]:=L[i]*diff(y(x),x$i):
w:=w+D[i]:
end do:
w:=w+L[0]*y(x):

## protection against a non-monic operator#
## decisions are made by the degree of L#
if L[n]<>1 then
   safe:=1:
   end if:

if n=0 then
   print(ans, b(x)*Dx + a(x)):
   safe:=0:
   end if:

if safe=1 then
   print('this operator is not monic'):
   else
   if n=1 then
      print(ans, b(x)*Dx + L[0]*b(x) + a):
```

```
   elif n=2 then
degree2(L,sym1):
print(ans, sym1):

   elif n>2 then
a1 :=  ((2*L[n-1])/(n*(n-1))):
a0 :=  ((6*L[n-2] - (1/4)*n*(n-1)*(n-2)*(3*n-1)*((a1)^2)
K := Dx^2 + a1*Dx + a0:
power := symmetric_power(K,n-1,A):
rest := collect((c-power),Dx,factor):

## there are two possibilities for rest: rest=0 or rest<>0#
if rest=0 then
   L1[0]:=a0:
   L1[1]:=a1:
   degree2(L1,sym1):
   h1:=coeff(sym1,Dx,1):
   h2:=coeff(sym1,Dx,0):
   sym1:=h1*Dx+(n-1)*h2:
   print(ans,sym1):

   else
   n1:=degree(rest,Dx):
   C[n1] := coeff(rest,Dx,n1):
   b:=_C1*radsimp( ( (C[n1])^(1/(n1-n)))):
           a:=expand((1/2)*(-diff(b,x) + a1*b)) + _C2:
   sym1:=b*Dx + (n-1)*a:
   sym2:=b*Dx + n*diff(b,x) + (n-1)*a:
   sym3:=b*Dx + n1*diff(b,x) + (n-1)*a - (b*diff(C[n1],x)/C[n1]):
           if (mult(power,sym1,A)=mult(sym2,power,A)) and
              (mult(rest,sym1,A)=mult(sym3,rest,A)) then
              print(ans,sym1):
      else
      print(`there are only constant symmetries`):
      sym1:=_C1:
      end if:

## finally one computes the Lie symmetries#
   print(`the corresponding ODE is`,w):
```

```
   if n=1 then
      print(`the number of symmetries is infinite`):
      end if:

   if n=2 then
      h3:=coeff(sym1,Dx,1):
      h4:=coeff(sym1,Dx,0):
      h6:=dsolve(w,y(x)):
      f := x -> subs(h6,y(x)):
      f(x):=subs({_C1=_C7,_C2=_C8},f(x)):
      w1:=diff(y(x),x,x)-L[1]*diff(y(x),x)+(L[0]-diff(L[1],x))*y(x):
      h7:=dsolve(w1,y(x)):
      g := x -> subs(h7,y(x)):
      g(x):=subs({_C1=_C5,_C2=_C6},g(x)):
      h8:=collect((-h3+g(x)*y(x)),y(x),factor):
      if L[0]=0  then
         h9:=collect((h4*y(x)+f(x)+
               (diff(g(x),x)-L[1]*g(x))*y(x)*y(x)),y(x),factor):
         else
         h9:=collect((h4*y(x)+f(x)+(1/4)*(diff(g(x),x)-L[1]*g(x)-
        int(3*L[0]*g(x),x))*y(x)*y(x)),y(x),factor):
         end if:
      print(`xi`,h8):
      print(`eta`,h9):
      end if:

   if n>2 then
      h3:=coeff(sym1,Dx,1):
      h4:=coeff(sym1,Dx,0):
      h5:=-h3:
      h6:=dsolve(w,y(x)):
      f := x -> subs(h6,y(x)):
      f(x):=subs({_C1=_D1,_C2=_D2,_C3=_D3,_C4=_D4},f(x)):
      h7:=h4*y(x)+f(x):
      print(`xi`,h5):
      print(`eta`,h7):
      end if:

end if:
end proc:
```

Bibliography

[1] *Biographical dictionary of mathematicians : reference biographies from the Dictionary of scientific biography.*
New York: Charles Scribner, 1991, (published under the auspices of the American Council of Learned Societies)

[2] E. T. Bell. *The development of mathematics.*
New York [etc.] : McGraw-Hill, 2nd ed, 1945

[3] W.R. Oudshoorn and M. van der Put. *Lie symmetries and differential Galois groups of linear equations.*
Mathematics of Computation, volume 71, number 237, pages 349-361 (article electronically published on October 4, 2001)

[4] H. Stephani. *Differential equations. Their solution using symmetries.*
Cambridge University Press, Cambridge, 1989

[5] A.I. Bobenko and U. Eitner. *Painlevé equations in the differential geometry of surfaces.*
Lecture notes in mathematics(1753), Springer, 2000