Generalized Fresnel Distributions

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Introduction

The wide context of this article is given by the research on the Feynmann integral. The mathematical definition of the Feynmann-integral has been a problem for over 50 years. Because of the analogy of the Feynmann integral with the Wiener measure we first look at that.
We will have a look at the relation between the heat diffusion equation and the Wiener measure. After that we will see the connection between the heat diffusion equation and the Schrödinger equation and discuss the Fresnel distributions.

The heat diffusion equation is:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u \tag{0.1}$$

with $u(t,x) = u_t(x)$, $x \in \mathbb{R}^n$. For simplicity we take $n = 1$. We define

$$G_t(x) = \frac{Y(t)e^{-x^2/2t}}{\sqrt{2\pi t}}$$

We use the physical notation $\delta(x)$ in stead of δ to avoid confusion with the $\delta(t)$. Now G_t is a fundamental solution:

$$(\frac{\partial}{\partial t} - \frac{1}{2}\Delta)G_t = \delta(t)\delta(x).$$

One can see this by recognizing that $\int G_t(x)dx = 1$ for all $t > 0$ and $G_t(x) = \frac{1}{\sqrt{t}}G_1(\frac{x}{\sqrt{t}})$. Therefore $G_t(x)$ is an approximation for $\delta(x)$ for $t \searrow 0$. We define $G_0(x) = \delta(x)$.
The solutions u_t of the Cauchy problem of the heat diffusion equation with initial value u_0 are given by

$$u_t = G_t * u_0$$

for $t > 0$. Then $\lim_{t \searrow 0} u_t = G_0 * u_0 = u_0$.

Now we can define the W_σ. Let σ be a subdivision of $[0,T]$:
$\sigma = 0 \leq t_1 < \ldots < t_n < T$. Then we can write $\mathbb{R}^n = \mathbb{R}^\sigma$ and we define W_σ on $\mathbb{R}^\sigma$ by

$$W_\sigma = G_{t_n - t_{n-1}}(x_n - x_{n-1}) \cdot \ldots \cdot G_{t_2 - t_1}(x_2 - x_1)G_{t_1}(x_1)dx_1 \ldots dx_n. \tag{0.2}$$

We abbreviate: $W_\sigma = G_\sigma(x)dx_1 \ldots dx_n$. Then W_σ is a probability measure on $\mathbb{R}^\sigma$, because $\int_{\mathbb{R}} G_t(x)dx = 1$ for all $t > 0$.

Let $\pi_\sigma : C([0,T]) \to \mathbb{R}^\sigma$ be the projection-map $\pi_\sigma(X) = (X(t_1), \dots, X(t_n)) \in \mathbb{R}^\sigma$.

If $\sigma \leq \sigma'$, we define $\pi_{\sigma\sigma'} : \mathbb{R}^{\sigma'} \to \mathbb{R}^\sigma$ to be the projection. Then $W_\sigma = \pi_{\sigma\sigma'}(W_{\sigma'})$. This means that the set $\{\pi_\sigma\}$ is a projective system. Now we can ask the question: given the projective system of measures W_σ on $\mathbb{R}^\sigma$, does there exist a measure W on $C([0,T])$ such that $W_\sigma = \pi_\sigma(W)$, for all σ? If it does, W is called the **projective limit**. The answer is given by the next theorem.

Theorem [Wiener] *There exists a unique probability measure W on $C([0,T])$ such that*

$$\pi_\sigma W = W_\sigma, \ \forall \sigma.$$

We call W the **Wiener measure**.

Now we switch to the **Schrödinger equation**:

$$\frac{1}{i}\frac{\partial \varphi}{\partial t} = \frac{1}{2}\Delta\varphi \tag{0.3}$$

The difference with the heat diffusion equation is the fraction $\frac{1}{i}$ in front of it. This corresponds with the substitution $t \mapsto it$. The analogue of W_σ is F_σ:

$$F_\sigma = G_{i(t_n - t_{n-1})}(x_n - x_{n-1}) \cdot \ldots \cdot G_{i(t_2-t_1)}(x_2 - x_1) G_{it_1}(x_1) dx_1 \ldots dx_n =$$

$$\prod_{k=1}^{m} \frac{1}{\sqrt{2\pi i(t_k - t_{k-1})}} \; e^{\frac{i}{2}\sum_{k=1}^{m}\frac{(x_k - x_{k-1})^2}{t_k - t_{k-1}}} dx = c_\sigma e^{iQ_\sigma(x)} dx,$$

with Q_σ a quadratic form. If there is a potential in the Schrödinger equation, the Q_σ is replaced by $Q_\sigma + V_\sigma$ with V_σ a potential. In this article we restrict ourselves to polynomial potentials.

We will see in chapter 1 that the F_σ are summable distributions. They are called **Fresnel distributions**. Moreover the set $\{F_\sigma\}$ is a projective system. Now the great question is: does there exist some kind of projective limit

$$F = \lim_{\longleftarrow} F_\sigma$$

such that $\pi_\sigma F = F_\sigma$, for all σ. If it does, F is called the **Feynmann integral**. What is F like? To answer this difficult question, one needs to know more about the F_σ.

In chapter 2 we will get more information about the sum order of the Fresnel distributions. In the case of potentials we get distributions e^{iP} with P more general polynomials, which we call the **Generalized Fresnel Distributions**. Therefore we try to find out for which polynomials P the distribution e^{iP} is summable. And if so, what is the sum order? This is the central question in this article.

Chapter 1

Summable Distributions and Gauss-Fresnel distributions on $\mathbb{R}^n$

In this chapter we will treat the class of summable distributions on $\mathbb{R}^n$: $\mathcal{D}'_L(\mathbb{R}^n)$. This space is in fact the case $p = 1$ of the spaces $\mathcal{D}'_{L^p}(\mathbb{R}^n)$ defined by L. Schwartz for $1 \leq p \leq \infty$ in [TD]. Because we only consider this case, we abbreviate $\mathcal{D}'_{L^1}(\mathbb{R}^n) = \mathcal{D}'_L(\mathbb{R}^n)$. One can find this theory in [TD, Ch.VI, §8] and more detailed in [TH1, Ch1].
Furthermore, we take a look at the Gauss-Fresnel distributions, which turn out to be summable distributions.

1.1 Summable Distributions

Definition We define the spaces $\mathcal{B}$ and $\dot{\mathcal{B}}$ as follows:

$$\mathcal{B} = \mathcal{B}(\mathbb{R}^n) = \{\varphi \in \mathcal{E}(\mathbb{R}^n) : D^k\varphi \in L^\infty(\mathbb{R}^n)\ \forall k \in \mathbb{Z}^n_+\},$$

$$\dot{\mathcal{B}} = \dot{\mathcal{B}}(\mathbb{R}^n) = \{\varphi \in \mathcal{E}(\mathbb{R}^n) : D^k\varphi \in C_0(\mathbb{R}^n)\ \forall k \in \mathbb{Z}^n_+\}.$$

The topologies of these spaces are the natural topologies induced by the (semi-)norms:

$$p_m(\varphi) = \sup_{|k|\leq m} \| D^k\varphi \|_\infty \tag{1.1}$$

Proposition 1.1 *1. The space $\mathcal{B}(\mathbb{R}^n)$ is a Fréchet space (i.e. metrizable and complete).*

2. $\dot{\mathcal{B}}(\mathbb{R}^n)$ is the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}(\mathbb{R}^n)$.

Proof The first statement follows from the completeness of $\mathcal{E}(\mathbb{R}^n)$ and the fact that the semi-norms on $\mathcal{B}(\mathbb{R}^n)$ are norms.
To prove *2.* it is sufficient to verify that $\mathcal{D}$ is dense in $\dot{\mathcal{B}}$. Let $\alpha \in \mathcal{D}$ be a function between 0 and 1, equal to 1 on the unit ball and define $\alpha_n(x) = \alpha(\frac{x}{n})$. Then, if $\varphi \in \dot{\mathcal{B}}$, the functions $\alpha_n\varphi$ converge to φ uniformly on compact sets. Namely, $|\alpha_n(x)\varphi(x) - \varphi(x)| = 0$ for $|x| < n$.

But given $\varepsilon > 0$, $\exists N$ such that $|\alpha_n(x)\varphi(x) - \varphi(x)| \leq 2|\varphi(x)| \leq \varepsilon$ for $|x| > N$, because $\varphi \in C_0$. So $|\alpha_n(x)\varphi(x) - \varphi(x)| \leq \varepsilon$, $\forall x \in \mathbb{R}^n$, for $n > N$, and the convergence is uniform on $\mathbb{R}^n$.
Similarly, for the uniform convergence of the derivatives of φ one uses Leibniz' rule and the fact that $D^k(\alpha_n)$ goes to 0 uniformly for $n \to \infty$. ∎

Definition The space of **summable distributions**, $\mathcal{D}'_L(\mathbb{R}^n)$, is defined to be the dual of $\dot{\mathcal{B}}(\mathbb{R}^n)$, equipped with the strong dual topology.

Because $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{B}}(\mathbb{R}^n)$ densely, $\mathcal{D}'_L(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$. It follows from this definition that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is summable if and only if there exist $m \in \mathbb{N}$ and $M \geq 0$ such that:

$$|\langle T, \varphi\rangle| \leq M p_m(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1.2}$$

Namely, for $\varphi \in \dot{\mathcal{B}}$, there exists a sequence φ_n in $\mathcal{D}$ such that $\varphi_n \to \varphi$ in $\dot{\mathcal{B}}$. So (1.2) implies that $|\langle T, \varphi_n\rangle| \leq M p_m(\varphi_n) \quad \forall n$, and passing to the limit this gives equation (1.2) for $\varphi \in \dot{\mathcal{B}}$. So $T \in \mathcal{D}'$ is a continuous linear form on $\dot{\mathcal{B}}$, i.e. $T \in \mathcal{D}'_L$.

Proposition 1.2 *If $T \in \mathcal{D}'_L$ then*

1. *$D^k T$, defined by*
$$\langle D^k T, \varphi\rangle = (-1)^{|k|}\langle T, D^k\varphi\rangle \text{ for } \varphi \in \mathcal{D} \tag{1.3}$$
is summable.

2. *for $\psi \in \mathcal{B}$, the distribution ψT defined by*
$$\langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle \text{ for } \varphi \in \mathcal{D} \tag{1.4}$$
is summable.

3. *The maps $T \mapsto D^k T$ and $T \mapsto \psi T$ are continuous from $\mathcal{D}'_L(\mathbb{R}^n)$ to $\mathcal{D}'_L(\mathbb{R}^n)$.*

4. *The formulas in 1 and 2 are valid for all $\varphi \in \dot{\mathcal{B}}$ and $\psi \in \mathcal{B}$.*

Proof Take m and M such that $|\langle T, \varphi\rangle| \leq M p_m(\varphi) \quad \forall \varphi \in \mathcal{D}$. Then for $\varphi \in \mathcal{D}$,

$$|\langle D^k T, \varphi\rangle| = |\langle T, D^k\varphi\rangle| \leq M p_m(D^k\varphi) \leq M p_{m+|k|}(\varphi).$$

This implies *1*.
Similar for $\psi \in \mathcal{B}, \varphi \in \mathcal{D}$,

$$|\langle \psi T, \varphi\rangle| = |\langle T, \psi\varphi\rangle| \leq M p_m(\psi\varphi) \leq \tilde{M} p_m(\varphi),$$

where the last inequality follows from Leibniz' rule. This proves the second statement.
For the continuity of the two maps in *3.* take a sequence T_n in $\mathcal{D}'_L$, such that $T_n \to 0$ in $\mathcal{D}'_L$ for $n \to \infty$. Then for all $\varphi \in \dot{\mathcal{B}}$, $\langle D^k T_n, \varphi\rangle = (-1)^{|k|}\langle T_n, D^k\varphi\rangle \to 0$ for $n \to \infty$. And for $\psi \in \mathcal{B}$, $\langle \psi T_n, \varphi\rangle = \langle T_n, \psi\varphi\rangle \to 0$, for $n \to \infty$.

To prove *4.* take, for $\varphi \in \dot{\mathcal{B}}$, a sequence φ_n in $\mathcal{D}$ such that $\varphi_n \to \varphi$ in $\dot{\mathcal{B}}$ for n to ∞. Then $D^k\varphi_n \to \mathcal{D}^k\varphi$ in $\dot{\mathcal{B}}$. Now it follows from the continuity of T on $\dot{\mathcal{B}}$ that $\langle D^kT, \varphi_n\rangle = (-1)^{|k|}\langle T, D^k\varphi_n\rangle \to (-1)^{|k|}\langle T, D^k\varphi\rangle$ and on the other hand $\langle D^kT, \varphi_n\rangle \to \langle D^kT, \varphi\rangle$. So the formula in *1* holds for $\varphi \in \dot{\mathcal{B}}$. One proves the validity of the formula in *2* for $\varphi \in \dot{\mathcal{B}}$ similarly. ∎

Example An important example of summable distributions is the space of distributions with compact support. Because $\dot{\mathcal{B}} \hookrightarrow \mathcal{E}$ with dense image, one has

$$\mathcal{E}' \hookrightarrow \mathcal{D}'_L.$$

Definition The **summability order** of a summable distribution T is the smallest number m such that the inequality (1.2) holds. (Frequently, we will abbreviate this to sum order(T) or even s.o.(T).)

Proposition 1.3 *If T is a summable distribution with sum order m, then s.o.$(\psi T) \leq m$ for all $\psi \in \mathcal{B}$.*

Proof For $\varphi \in \mathcal{D}$ we have $|\langle \psi T, \varphi\rangle| = |\langle T, \psi\varphi\rangle| = Mp_m(\psi\varphi) \leq \tilde{M}p_m(\varphi)$. ∎

One can define spaces $\mathcal{B}^{(m)}$ resp. $\dot{\mathcal{B}}^{(m)}$ to be the spaces of functions whose derivatives up to order m are in L^∞ resp. C_0. These spaces are again Banach spaces when equipped with the topology induced by the norms (1.1). If we define similarly the space $\mathcal{D}^{(m)}$ of functions with compact support whose derivatives up to order m are continuous, then $\mathcal{D}^{(m)}$ is dense in $\dot{\mathcal{B}}^{(m)}$ (one can prove this similar to the proof of proposition 1.1). And because $\mathcal{D}^{(m)} \subset \mathcal{D}$ we also have $\mathcal{D}$ is dense in $\dot{\mathcal{B}}^{(m)}$. When we now define $\mathcal{D}_L'^{(m)}$ as the dual of $\dot{\mathcal{B}}^{(m)}$ and use $\mathcal{D}_L'^{(0)} = \mathcal{M}_b$ (the space of bounded measures) we have the continuous injections:

$$\mathcal{M}_b(\mathbb{R}^n) \hookrightarrow \mathcal{D}_L'^{(m)}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'_L(\mathbb{R}^n)$$

It also follows from (1.2) that $\mathcal{D}'_L$ is the union of the subspaces $\mathcal{D}_L'^{(m)}$, and sum order(T) is the smallest number such that $T \in \mathcal{D}_L'^{(m)}$.

Remember that the order of a distribution in $\mathcal{D}'(\mathbb{R}^n)$ is the smallest number N such that $\forall K \subset \mathbb{R}^n, K$ compact, $\exists M \geq 0$

$$|\langle T, \varphi\rangle| \leq Mp_N(\varphi) \quad \forall \varphi \in \mathcal{D}_K(\mathbb{R}^n).$$

So a summable distribution has finite order and the following inequality holds:

$$\text{order}(T) \leq \text{ sum order}(T) \tag{1.5}$$

Note that for $T \in \mathcal{E}'$ the equality holds.

If T belongs to $\mathcal{D}_L'^{(m)}(\mathbb{R}^n)$ then $D^kT \in \mathcal{D}_L'^{(m+|k|)}(\mathbb{R}^n)$. So if $T = \mu \in \mathcal{M}_b(\mathbb{R}^n) = \mathcal{D}_L'^{(0)}(\mathbb{R}^n)$ then $D^kT = D^k\mu \in \mathcal{D}_L'^{(m)}(\mathbb{R}^n)$ for $|k| \leq m$. Thus the derivatives of bounded measures are again summable distributions. The converse of this is stated in the next theorem.

Theorem 1.4 [TD, Ch.6, §8] *Let $T \in \mathcal{D}'(\mathbb{R}^n)$. Then the following conditions are equivalent:*

1. *T belongs to $\mathcal{D}'_L(\mathbb{R}^n)$.*
2. *T is a finite sum of derivatives of bounded measures on $\mathbb{R}^n$.*
3. *T is a finite sum of derivatives of $L^1(\mathbb{R}^n)$-functions.*
4. *For every $\alpha \in \mathcal{D}(\mathbb{R}^n)$, $\alpha * T$ belongs to $\mathcal{M}_b(\mathbb{R}^n)$.*
5. *For every $\alpha \in \mathcal{D}(\mathbb{R}^n)$, $\alpha * T$ belongs to $L^1(\mathbb{R}^n)$.*

Proof *3.* $\Longrightarrow$ *2.* because $L^1 \subset \mathcal{M}_b$.
2. $\Longrightarrow$ *1.* because of the reasoning above.
1. $\Longrightarrow$ *4.* Let $B = \{\varphi \in \mathcal{D} : \| \varphi \|_\infty \leq 1\}$.

Lemma 1.5 *B_0 is dense in the unit ball of C_0.*

Proof of the lemma
Let $f \in C_0$, $\|f\|_\infty \leq 1$. Then if $\beta \in \mathcal{D}, \beta \geq 0, \int \beta = 1, \beta * f$ belongs to the unit ball of C_0. If we take $\beta_n \in \mathcal{D}$ a standard approximation of δ, then $\beta_n * f$ belongs to the unit ball of C_0 and $\beta_n * f \to f$. Namely,
$|(\beta_n * f)(x) - f(x)| = |\int \beta_n(t) f(x-t)dt - f(x)| = |\int \beta_n(t)[f(x-t) - f(x)]dt|$
$\leq \int_{|x|\leq\delta} \beta_n(t)|f(x-t) - f(x)|dt + \int_{|x|>\delta} \beta_n(t)|f(x-t) - f(x)|dt$
$\leq \varepsilon \int_{|x|\leq\delta} \beta_n(t)dt + \varepsilon \leq 2\varepsilon$, by the uniform continuity of f.
Let $\alpha \in \mathcal{D}$, $0 \leq \alpha \leq 1$, and $\alpha = 1$ on the unit ball, and define $\alpha_n(x) = \alpha(\frac{x}{n})$. Then $\alpha_n(\beta_n * f)$ belongs to B_0 and converges to f uniformly on compact sets. But being dominated by a C_0-function, namely $|f|$, the convergence is uniform. This proves the lemma.

Consider the functions $\check{\alpha} * \varphi$, with $\alpha \in \mathcal{D}$ fixed, and $\varphi \in B_0$. Because $D^k(\check{\alpha} * \varphi) = D^k\check{\alpha} * \varphi$ and $\|D^k(\check{\alpha} * \varphi)\|_\infty \leq \|D^k\check{\alpha}\|_1 \; \|\varphi\|_\infty \leq \|D^k\check{\alpha}\|_1$ for $\varphi \in B$, these functions form a bounded subset of $\mathcal{B}_0$.
Therefore, by the summability of T,

$$\langle \alpha * T, \varphi \rangle = \langle T, \check{\alpha} * \varphi \rangle \tag{1.6}$$

are bounded numbers for all $\varphi \in B_0$, so

$$\sup_{\varphi \in B_0} |\langle \alpha * T, \varphi \rangle| < +\infty \tag{1.7}$$

Now with the use of the lemma, $\alpha * T$ extends to a continuous linear form on the unit ball of C_0, and therefore to the whole space C_0. So $\alpha * T$ belongs to the dual of C_0, that is $\mathcal{M}_b$.
4. $\Longleftrightarrow$ *5.* The inclusion $L^1 \subset \mathcal{M}_b$ gives us the $\Longleftarrow$ direction. For the opposite direction, note that $\alpha * T$ is C^∞ and therefore belongs to L^1.

4. $\Longrightarrow$ *3.* If $\alpha * T \in \mathcal{M}_b$, the numbers:

$$\langle \check{\varphi} * T, \check{\alpha} \rangle = \langle \alpha * T, \varphi \rangle \tag{1.8}$$

are, for fixed $\alpha \in \mathcal{D}$, bounded for $\varphi \in B_0 \subset$ (unit ball of C_0). This implies that the distributions $\check{\varphi} * T$ are bounded in $\mathcal{D}'$ for $\varphi \in B_0$.
Let $A \subset \mathcal{D}'$ be a bounded set, i.e. $\sup_{\alpha \in B} |\langle T, \alpha \rangle| \le M_B$, $\forall T \in A$, $\forall B \subset \mathcal{D}$ bounded (that is $B \subset \mathcal{D}_K$ for some compact K and B bounded in $\mathcal{D}_K$). Then, because $\mathcal{D}_K$ is a Fréchetspace, we can use the Uniform Boundedness Principle for Fréchetspaces: $\sup_{T \in A} |\langle T, \alpha \rangle| < +\infty$, $\forall \alpha \in \mathcal{D}_K$. So the linear forms $T : \mathcal{D}_K \longrightarrow \mathbb{C}$, $T \in A$, with K the unit ball say, are equicontinuous: $\exists m \in \mathbb{N}$, $\exists M \ge 0$, such that

$$|\langle T, \alpha \rangle| \le M p_m(\alpha), \ \forall \alpha \in \mathcal{D}_K, \forall T \in A. \tag{1.9}$$

Therefore T extends to a linear form on $\mathcal{D}_K^{(m)}$ for all $T \in A$ and the extension still satisfies the estimate (1.9).
In the present case $A = \{\check{\varphi} * T, \varphi \in B_0\}$. So the distributions $\check{\varphi} * T$ extend to $\mathcal{D}_K^{(m)}$ and still satisfy the inequality (1.7). This means that the numbers in (1.6) make sense and are bounded for $\varphi \in B_0$ and $\alpha \in \mathcal{D}_K^{(m)}$. But this means that $\alpha * T$ is a bounded linear form on B_0 and therefore on C_0, so it belongs to $\mathcal{M}_b$, even for $\alpha \in \mathcal{D}_K^{(m)}$. And because $\alpha * T$ is $C^{(m)}$ for $\alpha \in \mathcal{D}_K^{(m)}$, $\alpha * T \in L^1$.

For $l \in \mathbb{N}$ sufficiently large ($l > m + \frac{n}{2}$), Δ^l has a fundamental solution E in $\mathcal{E}^{(m)} = C^{(m)}$. Because the Laplace operator is elliptic, so also hypo-elliptic, this solution E is C^∞ on the complement of $\{0\}$. If $\gamma \in \mathcal{D}_K$, is 1 on a neighborhood of the origin, the function $\alpha = \gamma E$ belongs to $\mathcal{D}_K^{(m)}$ and has the property

$$\Delta^l \alpha = \delta + \zeta \tag{1.10}$$

where $\zeta \in \mathcal{D}$ and has support in the complement of a neighborhood of the origin. Therefore,

$$T = \delta * T = \Delta^l \alpha * T - \zeta * T. \tag{1.11}$$

And since $\alpha * T$ and $\zeta * T$ belong to L^1 (this is statement *5.*) T is a sum of derivatives of L^1-functions. This finishes the proof of the theorem. ∎

Remark Using the Hahn-Banach theorem one gets information about the order of the derivatives in the representation as sums of derivatives of measures of a general summable distribution T. In the one dimensional case, let $T \in \mathcal{D}_L'^{(m)}(\mathbb{R})$, i.e. $|\langle T, \varphi \rangle| \le M p_m(\varphi) = M \sup_{k \le m} \|\varphi^{(k)}\|_\infty$, $\forall \varphi \in \mathcal{D}(\mathbb{R})$. Using the map $\varphi \mapsto (\varphi, \varphi', \dots, \varphi^{(m)})$ on $\mathcal{D}(\mathbb{R}) \to (C_0(\mathbb{R}))^{m+1}$, we define the bounded map l on the subspace $A = \{(\varphi, \varphi', \dots, \varphi^{(m)}) \mid \varphi \in \mathcal{D}\}$ of C_0^{m+1} by $l(\varphi, \varphi', \dots, \varphi^{(m)}) = \langle T, \varphi \rangle$. By the Hahn-Banach theorem, this map l can be extended to a map L defined on C_0^{m+1}, such that $L|_A = l$. Now we use Riesz' theorem to conclude that $L \sim (L_{\tilde{\mu}_0}, L_{\tilde{\mu}_1}, \dots, L_{\tilde{\mu}_m})$ with $\tilde{\mu}_k \in \mathcal{M}_b(\mathbb{R})$ i.e. for $\varphi \in \mathcal{D}$,
$\langle T, \varphi \rangle = L(\varphi, \varphi', \dots, \varphi^{(m)}) = \sum_{k=0}^m \langle \tilde{\mu}_k, \varphi^{(k)} \rangle = \sum_{k=0}^m \langle (-1)^k \tilde{\mu}_k^{(k)}, \varphi \rangle =$

$\sum_{k=0}^{m}\langle\mu_k^{(k)},\varphi\rangle, \mu_k \in \mathcal{M}_b(\mathbb{R})$.
For general dimension, a similar argument yields

$$T \in \mathcal{D}_L^{\prime(m)}(\mathbb{R}^n) \quad \Longrightarrow \quad T = \sum_{|k|\le m} D^k\mu_k, \quad \mu_k \in \mathcal{M}_b(\mathbb{R}^n). \tag{1.12}$$

Theorem 1.6 *The topology of $\mathcal{D}'_L(\mathbb{R}^n)$ is the weakest topology for which the maps $T \mapsto \alpha * T \in L^1$ are continuous: $T_i \to T$ in $\mathcal{D}'_L(\mathbb{R}^n)$ if and only if $\alpha * T_i \to \alpha * T$ in $L^1(\mathbb{R}^n)$ for all $\alpha \in \mathcal{D}(\mathbb{R}^n)$.*

The proof of this theorem can be found in [TH1, §1.1].

1.2 The duality with $\mathcal{B}$

Consider a bounded subset of $\mathcal{B}(\mathbb{R}^n)$, i.e. a set $B \subset \mathcal{B}(\mathbb{R}^n)$ such that

$$\sup_{\varphi\in B} p_m(\varphi) < +\infty \ \ \forall m \in \mathbb{N}$$

Then by Ascoli's theorem B is relatively compact in the space $\mathcal{E}(\mathbb{R}^n)$. The compact closure of B in $\mathcal{E}(\mathbb{R}^n)$ is contained in $\mathcal{B}(\mathbb{R}^n)$. Let $\mathcal{K}$ be the set of bounded subsets of $\mathcal{B}(\mathbb{R}^n)$ which are closed in $\mathcal{E}(\mathbb{R}^n)$. Then the sets $K \in \mathcal{K}$ are compact in $\mathcal{E}(\mathbb{R}^n)$ and every bounded set B is $\mathcal{B}(\mathbb{R}^n)$ is contained in a set $K \in \mathcal{K}$.

Theorem 1.7 *Let $f : K \to Y$ be a continuous map, where K is a compact space and Y a Hausdorff space. Assume that f is bijective. Then f is a homeomorphism, i.e. $f^{-1} : Y \to K$ is continuous too.*

Proof Let $g = f^{-1} : Y \to K$, $g^{-1}(F) = f(F)$ for $F \subset K$. Take F a closed set in K, then F is compact. So $f(F)$ is compact because f is continuous. Therefore $g^{-1}(F)$ is closed and it follows that g is continuous, i.e. f is a homeomorphism. ∎

Corollary 1.8 *Let K be a compact space and $X = K$ as linear space with a weaker Hausdorff topology. Then $X = K$ also as topology.*

Proof The identity map $Id : K \to X$ is continuous and bijective on K. So by the theorem the inverse identity map $Id^{-1} : X \to K$ is continuous too. This means that if $O \subset K$ is open, then $Id(O) = O \subset X$ is open. So the topology of K is not stronger than that of X. ∎

It follows from this corollary that on K, and therefore on B, the topology induced by $\mathcal{E}$ coincides with any weaker Hausdorff topology: for instance with the topology of uniform convergence on compact sets or even the topology of pointwise convergence.

Consider a sequence φ_i in $\mathcal{B}(\mathbb{R}^n)$ and let $\varphi \in \mathcal{B}(\mathbb{R}^n)$.

Definition We define $\varphi_i \to \varphi$ **pseudo-topologically** in $\mathcal{B}(\mathbb{R}^n)$ if

a. there exists $B \subset \mathcal{B}(\mathbb{R}^n)$ bounded such that $\varphi_i \in \mathcal{B}$ for all i,

b. $\varphi_i \to \varphi$ in $\mathcal{E}(\mathbb{R}^n)$.

As remarked, the condition *b.* may for instance be replaced by:

b'. $\varphi_i \to \varphi$ uniformly on compact sets.

Lemma 1.9 *$\mathcal{D}$ is dense in $\mathcal{B}$ with respect to the pseudo-topology. Given $\varphi \in \mathcal{B}$, there exists a sequence φ_n in $\mathcal{D}$ such that $\varphi_n \to \varphi$ pseudo-topologically for $n \to \infty$, i.e. $\{\varphi_n\}$ is bounded in $\mathcal{B}$ and $\varphi_n \to \varphi$ in $\mathcal{E}$ for $n \to \infty$.*

Proof Let $\alpha \in \mathcal{D}$ with $\alpha(x) = 1$ on the unit ball, and $0 \le \alpha \le 1$. Let $\alpha_n(x) = \alpha(\frac{x}{n})$ and define $\varphi_n = \alpha_n \varphi$. Then $D^k \alpha_n(x) = \frac{1}{n^{|k|}}(D^k\alpha)(\frac{x}{n})$, therefore $\|D^k\alpha_n\|_\infty \le \frac{1}{n^{|k|}}\|D^k\alpha\|_\infty \le \|D^k\alpha\|_\infty$ for all n.
Using this together with Leibniz' rule one gets that the φ_n are bounded in $\mathcal{B}$. But $\varphi_n(x) = \varphi(x)$, for all x with $|x| \le n$ and so $\varphi_n \to \varphi$ in $\mathcal{E}$. This implies that $\varphi_n \to \varphi$ in $\mathcal{B}$ pseudo-topologically. ∎

If $T \in \mathcal{D}'_L(\mathbb{R}^n)$, and $T = \sum D^k \mu_k$ is a representation of T as sum of derivatives of bounded measures, we have, if $\varphi \in \dot{\mathcal{B}}(\mathbb{R}^n)$

$$\langle T, \varphi \rangle = \sum \langle D^k \mu_k, \varphi \rangle = \sum (-1)^{|k|} \langle \mu_k, D^k \varphi \rangle. \tag{1.13}$$

From this it follows that T is continuous on $\dot{\mathcal{B}}$ equipped with the pseudo-topology. Namely, if $\varphi_n \to \varphi$ pseudo-topologically

$$\langle \mu_k, D^k \varphi_n \rangle \to \langle \mu_k, D^k \varphi \rangle$$

by the Dominated Convergence Theorem of Lebesgue.

Definition $T \in \mathcal{D}'_L(\mathbb{R}^n)$ has the **bounded convergence property** if T is continuous on $\dot{\mathcal{B}}(\mathbb{R}^n)$ equipped with the pseudo-topology, i.e. the restriction of T to bounded subsets of $\dot{\mathcal{B}}(\mathbb{R}^n)$ is continuous for the topology induced by $\mathcal{E}(\mathbb{R}^n)$.

Theorem 1.10 *If T is a summable distribution, then T has a unique linear extension to $\mathcal{B}(\mathbb{R}^n)$ having the bounded convergence property.*

Proof For $\varphi \in \mathcal{B}$, let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\dot{\mathcal{B}}$, such that $\varphi_n \to \varphi$ in the pseudo-topology. We then define

$$\langle T, \varphi \rangle = \lim_{n\to\infty} \langle T, \varphi_n \rangle$$

where we define T by (1.13).
Then T has the bounded convergence property again by the Dominated Convergence Theorem. Because T is uniquely defined on $\dot{\mathcal{B}}$ the right-hand side does not depend on the representation (1.13). Therefore T is uniquely defined. ∎

In particular, one can take $\varphi = 1$, and define the **total mass** of a summable distribution T:

$$\langle T, 1 \rangle.$$

Proposition 1.11 *The extension of $T \in \mathcal{D}'_L(\mathbb{R}^n)$ to $\mathcal{B}(\mathbb{R}^n)$ is compatible with multiplication by functions in $\mathcal{B}(\mathbb{R}^n)$ and with differentiation:*

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle \quad \forall \varphi, \psi \in \mathcal{B}(\mathbb{R}^n) \tag{1.14}$$

$$\langle D^k T, \varphi \rangle = (-1)^{|k|} \langle T, D^k \varphi \rangle \quad \forall \varphi \in \mathcal{B}(\mathbb{R}^n) \tag{1.15}$$

Proof If $\varphi \in \dot{\mathcal{B}}$ these equalities are correct because $\mathcal{D}$ is dense is $\dot{\mathcal{B}}$ and T is continuous. If $\varphi_n \to \varphi \in \mathcal{B}$ pseudo-topologically, i.e. remaining bounded in $\dot{\mathcal{B}}$ and converging in $\mathcal{E}$, we also have in the pseudo-topology $\psi\varphi_n \to \psi\varphi$ and $D^k\varphi_n \to D^k\varphi$. This yields

$$\langle \psi T, \varphi \rangle = \lim_{n\to\infty} \langle \psi T, \varphi_n \rangle = \lim_{n\to\infty} \langle T, \psi\varphi_n \rangle = \langle T, \psi\varphi \rangle.$$

Passing to the limit is justified because ψT belongs to $\mathcal{D}'_L$. Similar for the derivative:

$$\langle D^k T, \varphi \rangle = \lim_{n\to\infty} \langle D^k T, \varphi_n \rangle = \lim_{n\to\infty} (-1)^{-|k|} \langle T, D^k \varphi_n \rangle = (-1)^{-|k|} \langle T, D^k \varphi \rangle$$

because $D^k T \in \mathcal{D}'_L$. ∎

Theorem 1.12 *Let $L : \mathcal{B} \to \mathbb{C}$ be a linear form which has the bounded convergence property. Then there exists a unique $T \in \mathcal{D}'_L(\mathbb{R}^n)$ such that $L(\varphi) = \langle T, \varphi \rangle$ for all $\varphi \in \mathcal{B}$.*

Proof First note that L maps bounded subsets of $\mathcal{B}$ to bounded sets in $\mathbb{C}$. If this were not so there would exist a balanced bounded set $B \subset \mathcal{B}$ and a sequence (φ_n) in B such that $|L(\varphi_n)| \geq n$. But then $\frac{1}{n}\varphi_n$ belongs to B and goes to zero in the pseudo-topology. On the other hand $L(\frac{1}{n}\varphi_n)$ does not go to 0. This contradicts the linearity of L.
Let T be the restriction of L to $\dot{\mathcal{B}}$. Then if $\varphi = \lim_{n\to\infty} \varphi_n$ pseudo-topologically, we have $L(\varphi) = \lim_{n\to\infty} L(\varphi_n) = \lim_{n\to\infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle$, for all $\varphi \in \mathcal{B}$.

Theorem 1.13

1. *If L is a continuous linear form on $\mathcal{D}'_L(\mathbb{R}^n)$ there exists a unique function $\varphi \in \mathcal{B}$, such that $L(T) = \langle T, \varphi \rangle$.*
   Briefly: the dual of $\mathcal{D}'_L(\mathbb{R}^n)$ is $\mathcal{B}(\mathbb{R}^n)$, the bidual of $\dot{\mathcal{B}}(\mathbb{R}^n)$ is $\mathcal{B}(\mathbb{R}^n)$.

2. *The given topology of $\mathcal{B}(\mathbb{R}^n)$ equals the topology of uniform convergence on bounded subsets of $\mathcal{D}'_L(\mathbb{R}^n)$ and the strong dual topology of $\mathcal{D}'_L(\mathbb{R}^n)$ equals the topology of uniform convergence on bounded subsets of $\mathcal{B}(\mathbb{R}^n)$.*

The proof of this theorem can be found in [TH1, §1.2, §1.3].

We have seen now that one can alternatively define the space $\mathcal{D}'_L(\mathbb{R}^n)$ as the space of linear forms on $\mathcal{B}(\mathbb{R}^n)$ having the bounded convergence property. This is the **second definition** of summable distributions.
We can conclude that the linear topological vector space $\mathcal{D}'_L(\mathbb{R}^n)$ can be completely characterized in terms of the space $\mathcal{B}(\mathbb{R}^n)$. Moreover the operations of differentiation and multiplication on $\mathcal{D}'_L(\mathbb{R}^n)$ can be defined by (1.14) and (1.15).

1.3 Operations on summable distributions

In this section we will define image distributions, Fourier transforms, direct products and convolution products of summable distributions.

1.3.1 Image distributions

For linear maps $u : \mathbb{R}^n \to \mathbb{R}^k$, we define the image of a summable distribution $T \in \mathcal{D}'_L(\mathbb{R}^n)$ under u by the formula:

$$\langle u(T), \varphi\rangle = \langle T, \varphi \circ u\rangle \tag{1.16}$$

Let $\varphi \in \mathcal{B}(\mathbb{R}^k)$. Then $\psi = \varphi \circ u$ is bounded, and by the chain rule, has bounded derivatives i.e. ψ belongs to $\mathcal{B}(\mathbb{R}^n)$. That means that $\langle T, \varphi \circ u\rangle$ makes sense. Moreover, if $B \subset \mathcal{B}(\mathbb{R}^k)$ is a bounded subset, the set of composites $B \circ u$ is bounded in $\mathcal{B}(\mathbb{R}^n)$. Thus if φ_i tends to 0 in B (say pointwise) $\varphi_i \circ u$ tends to zero in $B \circ u$ and $\langle T, \varphi_i \circ u\rangle$ tends to 0, because T has the bounded convergence property on $\mathcal{B}(\mathbb{R}^n)$. So $u(T)$ has the bounded convergence property on $\mathcal{B}(\mathbb{R}^k)$. Therefore $u(T) \in \mathcal{D}'_L(\mathbb{R}^k)$ by the second definition.

Note that if $\varphi \in \dot{\mathcal{B}}(\mathbb{R}^k)$, then $\varphi \circ u$ does not in general belong to $\dot{\mathcal{B}}(\mathbb{R}^n)$ so one can not define the image directly by the first definition and transposition. For example $u : \mathbb{R}^n \to \mathbb{R} : u(x) = 0$, then $(\varphi \circ u)(x) = \varphi(0)$, $\forall x \in \mathbb{R}^n$, and $\varphi \circ u \notin \dot{\mathcal{B}}(\mathbb{R}^n)$ if $\varphi(0) \neq 0$.

Since the derivatives of order $\leq m$ of $\varphi \circ u$ only involve derivatives of order $\leq m$ of φ and u, we have:

$$\text{sum order}(u(T)) \leq \text{ sum order}(T) \tag{1.17}$$

1.3.2 Fourier transformations

For $\xi \in \mathbb{R}^n$ let $e_\xi(x) = e^{-ix\xi}$ where $x\xi = \sum_{j=1}^n x_j \xi_j$. Then

$$D^k e_\xi = (-i\xi)^k e_\xi$$

where as usual $\xi^k = \xi_1^{k_1} \cdot \ldots \cdot \xi_n^{k_n}$.
It follows that e_ξ belongs to $\mathcal{B}(\mathbb{R}^n)$. Moreover if ξ remains on a bounded subset of $\mathbb{R}^n$, e_ξ describes a bounded subset of $\mathcal{B}(\mathbb{R}^n)$. It follows that we can define the Fourier-transformation

$\mathcal{F}(T)$ of a summable distribution by:

$$\mathcal{F}(T)(\xi) = \langle T, e_\xi \rangle. \tag{1.18}$$

If ξ_n tends to ξ, e_{ξ_n} remains bounded and converges in the space $\mathcal{E}(\mathbb{R}^n)$ to e_ξ. Thus $\mathcal{F}(T)$ is a continuous function.
Clearly we have $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{B}}(\mathbb{R}^n)$, $\mathcal{S}$ being dense in $\dot{\mathcal{B}}$ because $\mathcal{D}$ is dense. By transposition we get the continuous inclusion:

$$\mathcal{D}'_L(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.19}$$

i.e. summable distributions are temperate. An application of Fubini's theorem shows that $\mathcal{F}(T)$ defined above is also the Fourier-transform in the sense of temperate distributions, i.e.

$$\langle \mathcal{F}(T), \varphi \rangle = \langle T, \mathcal{F}(\varphi) \rangle \quad \forall \varphi \in \mathcal{S}. \tag{1.20}$$

In particular T is uniquely determined by its Fourier transform, or characteristic function. From the representation $T = \sum D^k \mu_k$, with $\mu_k \in \mathcal{M}_b$, it follows that

$$\mathcal{F}(T)(\xi) = \sum (-i\xi)^k \mathcal{F}(\mu_k) \tag{1.21}$$

which shows that $\mathcal{F}(T)$ is a continuous function having at most polynomial growth.
More precisely this shows:

$$T \in \mathcal{D}'^{(m)}_L(\mathbb{R}^n) \Longrightarrow |\mathcal{F}(T)(\xi)| \leq M(1 + |\xi|)^m \quad \forall \xi \in \mathbb{R}^n. \tag{1.22}$$

In the next section we will see that the converse is not true: not every distribution whose Fourier transform is continuous with polynomial growth is summable.
For $u : \mathbb{R}^n \to \mathbb{R}^k$ a linear map, and ${}^t u$ its transpose, we have by this definition:

$$\mathcal{F}(u(T))(\xi) = \langle u(T), e_\xi \rangle = \langle T, e_\xi \circ u \rangle = \langle T, e_{{}^t u(\xi)} \rangle = \mathcal{F}(T)({}^t u(\xi)) \quad \forall \xi \in \mathbb{R}^k.$$

1.3.3 Direct products

Let $X = \mathbb{R}^p$, $Y = \mathbb{R}^q$. Let $T \in \mathcal{D}'_L(X)$ and $S \in \mathcal{D}'_L(Y)$. Then the direct product $T \otimes S$ is summable and one has:

$$\langle T \otimes S, \varphi \otimes \psi \rangle = \langle T, \varphi \rangle \langle S, \psi \rangle \quad \forall \varphi \in \mathcal{B}(X) \quad \forall \psi \in \mathcal{B}(Y). \tag{1.23}$$

For $\Phi \in \mathcal{B}(X \times Y)$ we have:

$$\langle T \otimes S, \Phi \rangle = \langle T, \theta \rangle, \tag{1.24}$$

with $\theta(x) = \langle S, \Phi_x \rangle$ where $\Phi_x(y) = \Phi(x, y)$.

For the proof of the summability of $T \otimes S$ note that the map $x \mapsto \Phi_x$ is C^∞ from X to $\mathcal{E}(Y)$. Since $\Phi \in \mathcal{B}(X \times Y)$, Φ_x belongs to $\mathcal{B}(Y)$ for all $x \in X$. Moreover the functions Φ_x form a bounded subset of $\mathcal{B}(Y)$ and therefore the function $\theta = \langle S, \Phi_x \rangle$ is a well defined function belonging to $\mathcal{B}(X)$. So the formula (1.24) makes sense.

If Φ remains on a bounded subset of $\mathcal{B}(X \times Y)$, the corresponding functions θ remain bounded in $\mathcal{B}(X)$. So if $\Phi_i \to \Phi$ pseudo-topologically the corresponding $\theta_i \to \theta$ pseudo-topologically. This means that $T \otimes S$ has the bounded convergence property, i.e. $T \otimes S$ is summable, satisfying (1.23). Moreover, the restriction to $\mathcal{D}(X \times Y)$ is the tensor product in the usual sense.

If $\Phi \in \mathcal{B}^{(n+m)}(X \times Y)$, it follows that $x \mapsto \Phi_x$ belongs to $\mathcal{B}^{(n)}(X, \mathcal{B}^{(m)}(Y))$. So, if $T \in \mathcal{D}_L'^{(n)}(X)$ and $S \in \mathcal{D}_L'^{(m)}(Y)$, the function $\theta = \langle S, \Phi_x \rangle \in \mathcal{B}^{(n)}(X)$ and $\langle T, \theta \rangle$ make sense. It follows that

$$\text{sum order}(T \otimes S) \leq \text{ sum order}(T) + \text{ sum order}(S). \tag{1.25}$$

The condition that $x \mapsto \Phi_x$ belongs to $\mathcal{B}^{(n)}(X, \mathcal{B}^{(m)}(Y))$ does not involve all derivatives of order $n+m$ of Φ, and so the above inequality may be in some cases a strict inequality. We will see examples where this is the case in the next chapter.

1.3.4 Convolution products

For T, $S \in \mathcal{D}_L'(\mathbb{R}^n)$ we define the convolution product $T * S$ as the image of the direct product $T \otimes S$ under the linear map $(x, y) \mapsto x + y$. Thus, for $\Phi \in \mathcal{B}(\mathbb{R}^n)$

$$\langle T * S, \Phi \rangle = \langle T \otimes S, \Psi \rangle \tag{1.26}$$

where $\Psi(x, y) = \Phi(x + y)$. It follows that $T * S$ is summable. Note that there is no condition on the supports of T and S.

Since $e_\xi(x+y) = e_\xi(x) e_\xi(y)$ we have by (1.18) $\mathcal{F}(T * S)(\xi) = \langle T * S, e_\xi \rangle =$
$\langle T \otimes S, e_\xi(x+y) \rangle = \langle T \otimes S, e_\xi \otimes e_\xi \rangle = \langle T, e_\xi \rangle \, \langle S, e_\xi \rangle =$
$\mathcal{F}(T)(\xi) \; \mathcal{F}(S)(\xi)$. This yields

$$\mathcal{F}(T * S) = \mathcal{F}(T) \, \mathcal{F}(S). \tag{1.27}$$

1.4 The class $\mathcal{F}(\mathcal{O}_M)$

Definition We define the class $\mathcal{O}_M(\mathbb{R}^n)$ to be the space of functions $f \in \mathcal{E}(\mathbb{R}^n)$ such that f and all its derivatives have at most polynomial growth.

Theorem 1.14 *Every $T \in \mathcal{F}(\mathcal{O}_M)$ is summable. More precisely we have the continuous inclusion:*

$$\mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n) \hookrightarrow \mathcal{D}_L'(\mathbb{R}^n). \tag{1.28}$$

Moreover, if P is a polynomial and T belongs to $\mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n)$, we have $PT \in \mathcal{D}_L'(\mathbb{R}^n)$. Conversely, if PT belongs to $\mathcal{D}_L'(\mathbb{R}^n)$ for all polynomials P, we have $T \in \mathcal{F}(\mathcal{O}_M)$.

Proof It is clear that $T \in \mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n) \iff \overline{\mathcal{F}}(T) \in \mathcal{O}_M \iff \mathcal{F}(T) = \hat{T} \in \mathcal{O}_M$. Let $\alpha \in \mathcal{D}(\mathbb{R}^n)$, and assume $\hat{T} \in \mathcal{O}_M$. Then $\mathcal{F}(\alpha * T) = \hat{\alpha}\hat{T} \in \mathcal{S}(\mathbb{R}^n)$. It follows that $\alpha * T \in \mathcal{S}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$, which implies that $T \in \mathcal{D}'_L(\mathbb{R}^n)$ by theorem 1.4.
If T_i goes to 0 in $\mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n)$, i.e. $\hat{T_i}$ goes to 0 in $\mathcal{O}_M$, $\hat{\alpha}\hat{T_i}$ goes to 0 in $\mathcal{S}$ for all $\alpha \in \mathcal{D}$, and so $\alpha * T_i$ goes to 0 in $\mathcal{S}$ and in L^1. Therefore by theorem 1.6 T_i goes to 0 in $\mathcal{D}'_L(\mathbb{R}^n)$. This proves the continuous inclusion (1.28).
Let P be a polynomial and $T \in \mathcal{F}(\mathcal{O}_M)$, then $\mathcal{F}(PT) = D\hat{T}$ where D is a differential operator with constant coefficients. If follows that $\mathcal{F}(PT)$ is a derivative of a distribution in $\mathcal{O}_M$, so it belongs to $\mathcal{O}_M$. Therefore $PT \in \mathcal{F}(\mathcal{O}_M) \subset \mathcal{D}'_L(\mathbb{R}^n)$.
Conversely, if PT belongs to $\mathcal{D}'_L(\mathbb{R}^n)$ for all polynomials P, it follows that $D\hat{T}$ is continuous with polynomial growth for all differential operators D. This implies that $\hat{T}$ belongs to $\mathcal{O}_M$. So we have $T \in \mathcal{F}(\mathcal{O}_M)$. ∎

Remark The Fourier transform of a summable distribution is continuous, but not in general C^∞. Thus $\mathcal{F}(\mathcal{O}_M)(\mathbb{R}^n)$ is a proper subset of $\mathcal{D}'_L(\mathbb{R}^n)$.

1.5 Gauss-Fresnel distributions

Let $G_\lambda = \frac{1}{(2\pi\lambda)^{n/2}} e^{-|x|^2/(2\lambda)} dx$ for $\Re(\lambda) \geq 0$ and $\lambda \neq 0$, the root being determined so as to coincide with the positive root if $\lambda > 0$. Let $G_0 = \delta$. We know that if $\lambda > 0$

$$\mathcal{F}(\frac{1}{(2\pi\lambda)^{n/2}} e^{-|x|^2/(2\lambda)}) = e^{-\lambda|\xi|^2/2}. \tag{1.29}$$

Now for all λ with $\Re(\lambda) \geq 0$, $e^{-\lambda|\xi|^2/2}$ belongs to $\mathcal{O}_M$.

Theorem 1.15 *G_λ belongs to $\mathcal{D}'_L(\mathbb{R}^n)$ for all λ with $\Re(\lambda) \geq 0$. Moreover the map $\lambda \mapsto G_\lambda \in \mathcal{D}'_L(\mathbb{R}^n)$ is continuous on the closed half plane, and holomorphic in the interior of $\mathbb{C}_+$.*

Proof $\mathcal{F}(G_\lambda) = e^{-\lambda|\xi|^2/2} \in \mathcal{O}_M$ for all λ with $\Re(\lambda) \geq 0$. Using theorem 1.14 this implies that $G_\lambda = \overline{\mathcal{F}}(\mathcal{F}(G_\lambda)) \in \mathcal{F}\mathcal{O}_M \subset \mathcal{D}'_L(\mathbb{R}^n)$.
Moreover it is clear that the map $\lambda \mapsto e^{-\lambda|\xi|^2/2} = \mathcal{F}(G_\lambda)$ is continuous and holomorphic on $\mathbb{C}_+$. Therefore the composition with the linear map $\overline{\mathcal{F}}$, i.e. $\lambda \mapsto G_\lambda$ is holomorphic too. ∎

Corollary 1.16 *If P is any polynomial on $\mathbb{R}^n$, then PG_λ belongs to $\mathcal{D}'_L(\mathbb{R}^n)$ and the map $\lambda \mapsto PG_\lambda$ is holomorphic for $\Re(\lambda) > 0$ and continuous for $\Re(\lambda) \geq 0$.*

Proof We have $\mathcal{F}(PG_\lambda) = De^{-\lambda|\xi|^2/2} \in \mathcal{O}_M$, where D is a differential operator with constant coefficients, which maps $\mathcal{O}_M$ continuously into itself. ∎

We will calculate the sum order of several Fresnel distributions in Chapter 2. In the third chapter we try to find out which polynomials P lead to summable distributions e^{iP}.

Chapter 2

Fresnel distributions

In this chapter we will compute the sum order of several Fresnel distributions, as preparation to compute the summability order of the distributions e^{iP} (with P a general polynomial) which maybe infinite. Some theorems and special cases of this chapter are already written down in [TH1, Ch. 2].

2.1 The one dimensional case

In this section we exclusively deal with $\mathbb{R}$, so instead of writing $\mathcal{D}(\mathbb{R})$, we just write $\mathcal{D}$, etc.

Proposition 2.1 [TH1] *The sum order of* e^{ix^2} *is exactly* 2.

Proof $\int e^{ix^2}\varphi(x)dx = (\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{+\infty})(e^{ix^2}\varphi(x)dx)$. The middle integral can be estimated by $2\|\varphi\|_\infty$. The first integral is equivalent to the last one, by reflection $x \mapsto -x$. So it is sufficient to consider only $\int_1^{+\infty} e^{ix^2}\varphi(x)dx$.

$$\int_1^{+\infty} e^{ix^2}\varphi(x)dx = \int_1^{+\infty} xe^{ix^2}\frac{\varphi(x)}{x}dx = c^{st}\int_1^{+\infty}\frac{d}{dx}[e^{ix^2}]\frac{\varphi(x)}{x}dx =$$

$$c^{st}\int_1^{+\infty} e^{ix^2}\frac{d}{dx}(\frac{\varphi(x)}{x})dx + c^{st}\varphi(1) = c^{st}\int_1^{+\infty} xe^{ix^2}\frac{1}{x}\frac{d}{dx}(\frac{\varphi(x)}{x})dx + c^{st}\varphi(1) =$$

$$c^{st}\int_1^{+\infty} e^{ix^2}\frac{d}{dx}[\frac{1}{x}\frac{d}{dx}(\frac{\varphi(x)}{x})]dx + c^{st}\varphi(1) + c^{st}\varphi'(1) =$$

$$c^{st}\int_1^{+\infty} e^{ix^2}\frac{d}{dx}[\frac{1}{x}(\frac{\varphi'(x)}{x} - \frac{\varphi(x)}{x^2})]dx + c^{st}\varphi(1) + c^{st}\varphi'(1) =$$

$$c^{st}\int_1^{+\infty} e^{ix^2}(\frac{\varphi''(x)}{x^2} - 2\frac{\varphi'(x)}{x^3} - \frac{\varphi'(x)}{x^3} + \frac{3\varphi(x)}{x^4})dx + c^{st}\varphi(1) + c^{st}\varphi'(1) =$$

$$c^{st}\int_1^{+\infty} e^{ix^2}(\frac{\varphi''(x)}{x^2} - 3\frac{\varphi'(x)}{x^3} + \frac{3\varphi(x)}{x^4})dx + c^{st}\varphi(1) + c^{st}\varphi'(1)$$

Because $\frac{1}{x^k}$ is integrable over $[1,+\infty)$ for $k>1$,

$$|\int_1^{+\infty} e^{ix^2}\varphi(x)dx| \leq M_0(\|\varphi\|_\infty + \|\varphi'\|_\infty + \|\varphi''\|_\infty) = Mp_2(\varphi).$$

And

$$|\int e^{ix^2}\varphi(x)dx| \leq Mp_2(\varphi), \ \ \forall\varphi \in \mathcal{D}(\mathbb{R}). \tag{2.1}$$

So s.o.$(e^{ix^2}) \leq 2$.

For the converse estimate, we must contradict the inequality:

$$|\int e^{ix^2}\varphi(x)dx| \leq Mp_1(\varphi) \tag{2.2}$$

for some $\varphi \in \mathcal{D}$.
Let $\alpha \in \mathcal{D}$ be such that $0 \leq \alpha(x) \leq 1$, $\alpha(x) = 1$ for $|x| \leq 1$ and $\alpha(x) = 0$ for $|x| \geq 2$. Let $\beta \in \mathcal{B}$ with support in $[\frac{1}{2}, +\infty)$, $0 \leq \beta(x) \leq 1$, $\beta(x) = 1$ on $[1,+\infty)$. Let $\alpha_n(x) = \alpha(\frac{x}{n})\beta(x)$, then $\alpha_n \in \mathcal{D}$. Now define $\varphi(x) = \frac{1}{x}e^{-ix^2}$ and $\varphi_n = \alpha_n\,\varphi$, then $\varphi_n \in \mathcal{D}$ and $\varphi_n(x) = \varphi(x)$ on $[1,n]$.
Then $\|\varphi_n\|_\infty \leq M_1 < +\infty$. Furthermore, because $\varphi_n'(x) = \alpha_n'(x)\varphi(x) + \alpha_n(x)\varphi'(x) = \alpha_n'(x)\frac{1}{x}e^{-ix^2} + \alpha_n(x)(\frac{-1}{x^2}e^{-ix^2} - 2ie^{-ix^2})$, it also follows that $\|\varphi_n'\|_\infty \leq M_2 < +\infty$. So $p_1(\varphi_n) = \max(\|\varphi_n\|_\infty, \|\varphi_n'\|_\infty) \leq M < +\infty$, i.e. the $p_1(\varphi_n)$ are uniformly bounded in n.

On the other hand, the integral

$$\int e^{ix^2}\varphi_n(x)dx = \int_{\frac{1}{2}}^{+\infty} e^{ix^2}\varphi_n(x) = \int_{\frac{1}{2}}^{+\infty} e^{ix^2}\alpha_n(x)\frac{1}{x}e^{-ix^2}dx =$$

$$\int_{\frac{1}{2}}^{+\infty} \alpha_n(x)\frac{1}{x}dx \geq \int_1^n \frac{1}{x}dx \geq \log(n) + c^{st}$$

diverges to $+\infty$, for $n \to \infty$. So the left-hand side in (2.2) with φ_n in stead of φ diverges for $n \to \infty$, while the right-hand side is uniformly bounded in n. This means that the inequality (2.2) does not hold for all $\varphi \in \mathcal{D}$, i.e. s.o.$(e^{ix^2}) > 1$.
This finishes the proof of the proposition: s.o.$(e^{ix^2}) = 2$. ∎

Proposition 2.2 *For $E_k(x) = x^k e^{ix^2}$, the following holds for all $\varphi \in \mathcal{D}$,*

1. $|\int_1^\infty E_k(x)\varphi(x)dx| \leq Mp_0(\varphi)$ *for* $k \leq -2$.
2. $|\int_1^\infty E_{-1}(x)\varphi(x)dx| \leq Mp_1(\varphi)$.
3. [TH1] *s.o.*$(E_k) = k+2$, *for* $k \geq 0$.

Proof The first statement follows directly from the fact that $x^k \in L^1([1,+\infty))$, for $k < -1$.
For *2.*, take $\varphi \in \mathcal{D}$, then

$$\int_1^\infty x^{-1}e^{ix^2}\varphi(x)dx = \int_1^\infty xe^{ix^2}\frac{\varphi(x)}{x^2}dx = c^{st}\int_1^\infty \frac{d}{dx}[e^{ix^2}]\frac{\varphi(x)}{x^2}dx =$$

$$c^{st}\int_1^\infty e^{ix^2}\frac{d}{dx}[\frac{\varphi(x)}{x^2}]dx + c^{st}\varphi(1) = c^{st}\int_1^\infty e^{ix^2}(\frac{\varphi'(x)}{x^2} - 2\frac{\varphi(x)}{x^3})dx + c^{st}\varphi(1)$$

So

$$|\int_1^\infty x^{-1}e^{ix^2}\varphi(x)dx| \leq Mp_1(\varphi).$$

For the proof of *3.* we again write $\int E_k(x)\varphi(x)dx =$
$(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_1^{+\infty})(E_k(x)\varphi(x)dx)$, and note that the second integral can be estimated by $M\|\varphi\|_\infty$ and the first and last are equivalent.
Because *3.* holds for $k = 0$ it is sufficient to prove by induction that

$$|\int_1^\infty x^k e^{ix^2}\varphi(x)dx| \leq Mp_{k+2}(\varphi), \ \ \forall\varphi \in \mathcal{D} \tag{2.3}$$

and that

$$\text{s.o.}(E_k) > k+1 \text{ for } k \geq 0. \tag{2.4}$$

Let's assume (2.3) for k, then for $k+1$:

$$\int_1^\infty x^{k+1}e^{ix^2}\varphi(x)dx = \int_1^\infty xe^{ix^2}x^k\varphi(x)dx =$$

$$c^{st}\int_1^\infty \frac{d}{dx}[e^{ix^2}]x^k\varphi(x)dx = c^{st}\int_1^\infty e^{ix^2}\frac{d}{dx}[x^k\varphi(x)]dx + c^{st}\varphi(1) =$$

$$c^{st}\int_1^\infty e^{ix^2}(kx^{k-1}\varphi(x) + x^k\varphi'(x))dx + c^{st}\varphi(1) =$$

$$c^{st}\int_1^\infty x^{k-1}e^{ix^2}\varphi(x)dx + c^{st}\int_1^\infty x^k e^{ix^2}\varphi'(x)dx + c^{st}\varphi(1)$$

Now, the induction hypothesis yields:

$$|c^{st}\int_1^\infty x^{k-1}e^{ix^2}\varphi(x)dx| \leq M_1p_{k-1+2}(\varphi)$$

and

$$|c^{st}\int_1^\infty x^k e^{ix^2}\varphi'(x)dx| \leq M_2p_{k+2}(\varphi') \leq M_2p_{k+2+1}(\varphi)$$

So, we have

$$|\int_1^\infty x^{k+1}e^{ix^2}\varphi(x)dx| \leq Mp_{k+1+2}(\varphi)$$

which finishes the proof of the inequality (2.3) for all $k \geq 0$.

For the proof of (2.4), define $\varphi(x) = \frac{1}{x^{k+1}} e^{-ix^2}$ and α_n as in the proof of proposition 2.1. Then, if we define $\varphi_n(x) = \alpha_n(x)\varphi(x)$, $\varphi_n \in \mathcal{D}$.
By Leibniz' rule

$$(\frac{d}{dx})^m \varphi_n = \sum_{l=0}^{m} \left(\begin{array}{c} m \\ l \end{array} \right) (\frac{d}{dx})^l \alpha_n \, (\frac{d}{dx})^{k-l} \varphi.$$

And

$$(\frac{d}{dx})^l \alpha_n(x) = \frac{1}{n^l}((\frac{d}{dx})^l \alpha)(\frac{x}{n})$$

implies that

$$\|(\frac{d}{dx})^l \alpha_n\|_\infty \leq \|(\frac{d}{dx})^l \alpha\|_\infty$$

thus the derivatives of α_n are uniformly bounded in n. It follows that

$$\|(\frac{d}{dx})^m \varphi_n\|_\infty \leq M_0 \sup_{|k| \leq m} \sup_{x \in [\frac{1}{2}, \infty)} |\varphi^{(k)}(x)|.$$

Now

$$(\frac{d}{dx})^l \varphi = (\frac{d}{dx})^l [\frac{1}{x^{k+1}} e^{-ix^2}] = \sum_{j=0}^{l} (\frac{d}{dx})^j [\frac{1}{x^{k+1}}] (\frac{d}{dx})^{l-j} [e^{-ix^2}].$$

This is a finite linear combination of terms

$$\frac{1}{x^r} e^{-ix^2}$$

where $k+1-l \leq r \leq k+1+l$.
If $0 \leq l \leq k+1$, then $0 \leq r \leq 2k+2$ and $\frac{1}{x^r} e^{-ix^2}$ remains bounded on $[\frac{1}{2}, \infty)$. From this it follows:

$$\sup_{|l| \leq k+1} \sup_{x \in [\frac{1}{2}, \infty)} |\varphi^{(l)}(x)| < M_1 < +\infty$$

and therefore

$$p_{k+1}(\varphi_n) = \sup_{l \leq k+1} \|(\frac{d}{dx})^l \varphi_n\|_\infty < M_2 < +\infty$$

uniformly in n.
If s.o.$(E_k) \leq k+1$ then

$$|\int_1^\infty x^k e^{ix^2} \varphi(x) dx| \leq M p_{k+1}(\varphi), \quad \forall \varphi \in \mathcal{D}$$

in particular for the φ_n. The right-hand side of this inequality is, for φ_n, uniformly bounded in n, while the left-hand side,

$$\int_1^\infty x^k e^{ix^2} \varphi_n(x) dx = \int_1^\infty x^k e^{ix^2} \alpha_n(x) \frac{1}{x^{k+1}} e^{-ix^2} dx =$$

$$\int_1^\infty \alpha_n(x) \frac{1}{x} dx \geq \int_1^n \frac{1}{x} dx = \mathcal{O} \log(n)$$

diverges. So (2.4) has been proved now. This finishes the proof of *3*. ∎

2.2 Two dimensions

In this section we wish to calculate the sum order of $e^{i(x^2-y^2)}$ and $P(x,y)e^{i(x^2-y^2)}$ with P a polynomial.

Definition We define on $\mathbb{R}^2$ the differential operator $D_{a,b} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$.

Proposition 2.3 *For f, g general functions on $\mathbb{R}^2$ and $\varphi \in \mathcal{D}(\mathbb{R}^2)$ the following holds:*

1. $D_{a,b}(fg) = (D_{a,b}f)g + f(D_{a,b}g)$ *(product rule)*
2. $\int_\alpha^\infty dx \int_\beta^\infty D_{a,b}\varphi(x,y)dy = -a\int_\beta^\infty \varphi(\alpha,y)dy - b\int_\alpha^\infty \varphi(x,\beta)dx$ *(integration rule)*

The simple proof is left to the reader.

Proposition 2.4 *The sum order of the distribution T defined by $\langle T,\varphi\rangle = \iint e^{i(x^2-y^2)}\varphi(x,y)dxdy$ is 3.*

Proof The proof is in two steps. We first prove that

$$\text{s.o.}(e^{i(x^2-y^2)}) \leq 3. \tag{2.5}$$

And after that we will prove that

$$|\iint e^{i(x^2-y^2)}\varphi(x,y)dxdy| \leq Mp_2(\varphi) \tag{2.6}$$

does not hold for all $\varphi \in \mathcal{D}$. Then it follows that the summability order of T is exactly 3. Let's abbreviate $E(x,y) = e^{i(x^2-y^2)}$. Then $D_{1,-1}E(x,y) = 2i(x+y)E(x,y)$. We first consider the quadrant $[1,\infty)\times[0,\infty)$, where $x+y$ is not zero. So we can write:

$$\int_0^\infty dy \int_1^\infty E(x,y)\varphi(x,y)dx = \int_0^\infty dy \int_1^\infty (x+y)E(x,y)\frac{\varphi(x,y)}{x+y}dx =$$

$$c^{st}\int_0^\infty dy \int_1^\infty D_{1,-1}E(x,y)\frac{\varphi(x,y)}{x+y}dx =$$

$$c^{st}\int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)\frac{\varphi(x,y)}{x+y}]dx \; -$$

$$c^{st}\int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1}[\frac{\varphi(x,y)}{x+y}]dx =$$

(we abbreviate: B_1 is the first integral)

$$B_1 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1}[\frac{\varphi(x,y)}{x+y}]dx =$$

$$B_1 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)(\frac{D_{1,-1}\varphi(x,y)}{x+y} - \frac{\varphi(x,y)D_{1,-1}(x+y)}{(x+y)^2})dx =$$

$$B_1 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)\frac{D_{1,-1}\varphi(x,y)}{x+y}dx =$$

$$B_1 + c^{st}\int_0^\infty dy \int_1^\infty (x+y)E(x,y)\frac{D_{1,-1}\varphi(x,y)}{(x+y)^2}dx =$$

$$B_1 + c^{st}\int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)]\frac{D_{1,-1}\varphi(x,y)}{(x+y)^2}dx =$$

$$B_1 + c^{st}\int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)\frac{D_{1,-1}\varphi(x,y)}{(x+y)^2}]dx -$$

$$c^{st}\int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1}[\frac{D_{1,-1}\varphi(x,y)}{(x+y)^2}]dx =$$

(again, B_2 is the first integral)

$$B_1 + B_2 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1}[\frac{D_{1,-1}\varphi(x,y)}{(x+y)^2}]dx =$$

$$B_1 + B_2 +$$

$$c^{st}\int_0^\infty dy \int_1^\infty E(x,y)(\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^2} - \frac{2D_{1,-1}\varphi(x,y)D_{1,-1}(x+y)}{(x+y)^3})dx =$$

$$B_1 + B_2 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^2}dx =$$

$$B_1 + B_2 + c^{st}\int_0^\infty dy \int_1^\infty (x+y)E(x,y)\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^3}dx =$$

$$B_1 + B_2 + c^{st}\int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)]\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^3}dx =$$

$$B_1 + B_2 + c^{st}\int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^3}]dx -$$

$$c^{st}\int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1}[\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^3}]dx =$$

(B_3 is an abbreviation for the first integral)

$$B_1 + B_2 + B_3 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)D_{1,-1}[\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^3}]dx =$$

$$B_1 + B_2 + B_3 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)(\frac{(D_{1,-1})^3\varphi(x,y)}{(x+y)^3} -$$

$$\frac{3(D_{1,-1})^2\varphi(x,y)D_{1,-1}(x+y)}{(x+y)^4})dx =$$

$$B_1 + B_2 + B_3 + c^{st}\int_0^\infty dy \int_1^\infty E(x,y)\frac{(D_{1,-1})^3\varphi(x,y)}{(x+y)^3}dx.$$

Now, because $x+y > \sqrt{x^2+y^2} = r$ on this quadrant, $\frac{1}{(x+y)^3} \leq \frac{1}{r^3} \in L^1([1,\infty) \times [0,\infty))$. This implies that we can estimate:

$$|\int_0^\infty dy \int_1^\infty E(x,y)\varphi(x,y)dx| \leq$$

$$|B_1| + |B_2| + |B_3| + c^{st}\|(D_{1,-1})^3\varphi\|_\infty \int_0^\infty dy \int_1^\infty \frac{1}{(x+y)^3}dx \leq$$

$$|B_1| + |B_2| + |B_3| + M_0 p_3(\varphi)$$

For the calculation of the B_k's, we use the integration rule from the proposition above.

$$B_1 = c^{st} \int_0^\infty dy \int_1^\infty D_{1,-1}[E(x,y)\frac{\varphi(x,y)}{x+y}]dx =$$

$$c^{st} \int_0^\infty E(1,y)\frac{\varphi(1,y)}{1+y}dy + c^{st} \int_1^\infty E(x,0)\frac{\varphi(x,0)}{x}dx =$$

$$c^{st} \int_0^\infty e^{i(1+y^2)}\frac{\varphi(1,y)}{1+y}dy + c^{st} \int_1^\infty e^{ix^2}\frac{\varphi(x,0)}{x}dx =$$

$$c^{st} \int_0^\infty \frac{1}{1+y}e^{iy^2}\varphi(1,y)dy + c^{st} \int_1^\infty \frac{1}{x}e^{ix^2}\varphi(x,0)dx.$$

From proposition 2.2 we know that $\text{s.o.}(\frac{1}{1+y}e^{iy^2}) = \text{s.o.}(\frac{1}{x}e^{ix^2}) = 1$. This implies that

$$|B_1| \leq M_1 p_1(\varphi).$$

For B_2 we have

$$B_2 = c^{st} \int_0^\infty \int_1^\infty D_{1,-1}[E(x,y)\frac{D_{1,-1}\varphi(x,y)}{(x+y)^2}]dxdy =$$

$$c^{st} \int_0^\infty E(1,y)\frac{D_{1,-1}\varphi(1,y)}{(1+y)^2}dy + c^{st} \int_1^\infty E(x,0)\frac{D_{1,-1}\varphi(x,0)}{x^2}dx =$$

$$c^{st} \int_0^\infty e^{i(1+y^2)}\frac{D_{1,-1}\varphi(1,y)}{(1+y)^2}dy + c^{st} \int_1^\infty e^{ix^2}\frac{D_{1,-1}\varphi(x,0)}{x^2}dx =$$

$$c^{st} \int_0^\infty e^{iy^2}\frac{D_{1,-1}\varphi(1,y)}{(1+y)^2}dy + c^{st} \int_1^\infty e^{ix^2}\frac{D_{1,-1}\varphi(x,0)}{x^2}dx.$$

So

$$|B_2| \leq M_1 p_0(\varphi).$$

And the last one:

$$B_3 = c^{st} \int_0^\infty \int_1^\infty D_{1,-1}[E(x,y)\frac{(D_{1,-1})^2\varphi(x,y)}{(x+y)^3}]dxdy =$$

$$c^{st} \int_0^\infty E(1,y)\frac{(D_{1,-1})^2\varphi(1,y)}{(1+y)^3}dy + c^{st} \int_1^\infty E(x,0)\frac{(D_{1,-1})^2\varphi(x,0)}{x^3}dx =$$

$$c^{st}\int_0^\infty e^{i(1+y^2)}\frac{(D_{1,-1})^2\varphi(1,y)}{(1+y)^3}dy + c^{st}\int_1^\infty e^{ix^2}\frac{(D_{1,-1})^2\varphi(x,0)}{x^3}dx =$$

$$c^{st}\int_0^\infty e^{iy^2}\frac{(D_{1,-1})^2\varphi(1,y)}{(1+y)^3}dy + c^{st}\int_1^\infty e^{ix^2}\frac{(D_{1,-1})^2\varphi(x,0)}{x^3}dx.$$

It follows that

$$|B_3| \leq M_3 p_2(\varphi).$$

This yields

$$|\int_0^\infty \int_1^\infty E(x,y)\varphi(x,y)dxdy| \leq Mp_3(\varphi).$$

For the integral over $[0,\infty) \times [1,\infty)$ we can use the reflection $x \mapsto y, y \mapsto x$ and write:

$$\int_1^\infty dy \int_0^\infty e^{i(x^2-y^2)}\varphi(x,y)dx = \int_0^\infty dy \int_1^\infty e^{-i(x^2-y^2)}\varphi(y,x)dx.$$

Similar to the calculations above, the last integral can be estimated by $Mp_3(\varphi)$. Namely, the minus sign in the exponent only causes minus signs in the constants in front of the integrals, so the absolute values remain the same.
We have now proved that

$$|\iint_{([0,\infty))^2\setminus([0,1])^2} e^{i(x^2-y^2)}\varphi(x,y)dxdy| \leq Mp_3(\varphi).$$

For the other three quadrants we use reflections $x \mapsto -x$ and $y \mapsto -y$. This then results in:

$$|\iint_{\mathbb{R}^2\setminus([-1,1])^2} e^{i(x^2-y^2)}\varphi(x,y)dxdy| \leq Mp_3(\varphi).$$

Because $([-1,1])^2$ is compact, the integral over this square can be estimated by $Mp_0(\varphi)$, so we have now proved:

$$|\langle T,\varphi\rangle| \leq Mp_3(\varphi) \tag{2.7}$$

i.e. s.o.$(T) \leq 3$ and this is (2.5).

To prove that (2.6) does not hold for all $\varphi \in \mathcal{D}$, we need a sequence φ_n in $\mathcal{D}$ such that

$$p_2(\varphi_n) < M, \text{ uniformly in } n \tag{2.8}$$

and

$$\iint e^{i(x^2-y^2)}\varphi_n(x)dxdy \nearrow +\infty \text{ for } n \to \infty. \tag{2.9}$$

When we have found such a sequence, it is clear that (2.6) does not hold for the φ_n, so s.o.$(T) = 3$.

Let $\alpha \in \mathcal{D}(\mathbb{R}^2)$ be such that $0 \leq \alpha \leq 1$, $\alpha(x,y) = 1$ for $(x,y) \in B(1) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ and $\alpha(x,y) = 0$ for $(x,y) \in (B(2))^c$. Take $\beta \in \mathcal{B}(\mathbb{R}^2)$, with $\text{supp}(\beta) \subset Q =$

$([\frac{1}{2},\infty))^2$, $0 \leq \beta \leq 1$, $\beta(x,y) = 1$ for $(x,y) \in ([1,\infty))^2$.
Define $\alpha_n(x,y) = \alpha(\frac{x}{n}, \frac{y}{n})\beta(x,y)$, then $\alpha_n \in \mathcal{D}$.
When we now define

$$\varphi(x,y) = \frac{e^{i(y^2-x^2)}}{(x+y)^2},$$

the sequence

$$\varphi_n(x,y) = \alpha_n(x,y)\varphi(x,y)$$

satisfies (2.8) and (2.9).

To see (2.8) note that $\alpha \in \mathcal{D}$, so the derivatives $D^k(\alpha_n(x,y)) = D^k(\alpha(\frac{x}{n}, \frac{y}{n})) = \frac{1}{n^{|k|}} D^k\alpha(x,y)$ are uniformly bounded in n.

From this it follows that

$$p_2(\varphi_n) \leq M \sup_{|k|\leq 2} \sup_{(x,y)\in Q} |D^k\varphi(x,y)| \tag{2.10}$$

by Leibniz' rule. To majorize this we need to consider the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial y \partial x}$ of φ.

$$\frac{\partial}{\partial x}\varphi(x,y) = \frac{-2ixe^{i(y^2-x^2)}}{(x+y)^2} - \frac{2e^{i(y^2-x^2)}}{(x+y)^3}$$

So $\sup_{(x,y)\in Q} |\frac{\partial}{\partial x}\varphi(x,y)| < +\infty$. The $\frac{\partial}{\partial y}$-derivative is equivalent to this one, and therefore also bounded on Q.

$$\frac{\partial^2}{\partial x^2}\varphi(x,y) = c_1\frac{x^2e^{i(y^2-x^2)}}{(x+y)^2} + c_2\frac{e^{i(y^2-x^2)}}{(x+y)^2} + c_3\frac{xe^{i(y^2-x^2)}}{(x+y)^3} + c_4\frac{e^{i(y^2-x^2)}}{(x+y)^4}.$$

And because all these functions are bounded on Q, $\sup_{(x,y)\in Q} |\frac{\partial^2}{\partial x^2}\varphi(x,y)| < +\infty$. And similar for the $\frac{\partial^2}{\partial y^2}$-derivative.

$$\frac{\partial^2}{\partial y \partial x}\varphi(x,y) = c_1\frac{xye^{i(y^2-x^2)}}{(x+y)^2} + c_2\frac{xe^{i(y^2-x^2)}}{(x+y)^3} + c_3\frac{ye^{i(y^2-x^2)}}{(x+y)^3} + c_4\frac{e^{i(y^2-x^2)}}{(x+y)^4}.$$

Because $\frac{xy}{(x+y)2} \leq \frac{1}{2}$ on Q, all these functions are bounded on Q.
So (2.8) holds.
The integral sequence is

$$\iint e^{i(x^2-y^2)}\varphi_n(x)dxdy = \int_{\frac{1}{2}}^{\infty}\int_{\frac{1}{2}}^{\infty} e^{i(x^2-y^2)}\alpha_n(x,y)e^{i(y^2-x^2)}\frac{1}{(x+y)^2}dxdy =$$

$$\int_{\frac{1}{2}}^{\infty}\int_{\frac{1}{2}}^{\infty} \alpha_n(x,y)\frac{1}{(x+y)^2}dxdy \geq \int_1^{\frac{n}{2}}\int_1^{\frac{n}{2}} \frac{1}{(x+y)^2}dxdy \geq \log(n) + c^{st}.$$

So (2.9) holds.
This finishes the proof of the proposition. ∎

Theorem 2.5 *For $p \in \mathbb{Z}_+$, $q \in \mathbb{Z}_+$, the summability order of the distribution $T_{p,q}$, defined by*

$$\langle T_{p,q}, \varphi \rangle = \iint x^p y^q e^{i(x^2-y^2)} \varphi(x,y) dx dy$$

is $p+q+3$.

Proof To prove that s.o.$(T_{p,q}) \leq p+q+3$, we use induction on $p+q$.
If $p+q=0$, this is exactly the proposition above.

Assume that s.o.$(T_{p,q}) \leq p+q+3$ for $0 \leq p+q \leq m$. Then if $p+q = m+1$, (we may assume that $p \neq 0$), it follows

$$\langle T_{p,q}, \varphi \rangle = \iint x^p y^q e^{i(x^2-y^2)} \varphi(x,y) dx dy = \iint x e^{i(x^2-y^2)} x^{p-1} y^q \varphi(x,y) dx dy =$$

$$c^{st} \iint \frac{\partial}{\partial x} e^{i(x^2-y^2)} x^{p-1} y^q \varphi(x,y) dx dy =$$

$$c^{st} \iint e^{i(x^2-y^2)} \frac{\partial}{\partial x} [x^{p-1} y^q \varphi(x,y)] dx dy =$$

$$c^{st} \iint e^{i(x^2-y^2)} x^{p-2} y^q \varphi(x,y) dx dy + c^{st} \iint e^{i(x^2-y^2)} x^{p-1} y^q \frac{\partial}{\partial x} \varphi(x,y) dx dy \tag{2.11}$$

for $p \neq 1$. For $p=1$ (2.11) reduces to the second integral, which is dominating in sum order. By the induction hypothesis

$$|\langle T_{p,q}, \varphi \rangle| \leq M_1 \; p_{p-2+q+3}(\varphi) + M_2 \; p_{p-1+q+3}(\frac{\partial \varphi}{\partial x}) \leq M \; p_{p+q+3}(\varphi)$$

So s.o.$(T_{p,q}) \leq p+q+3$.

To prove that the sum order is exactly 3, we consider the sequence φ_n in $\mathcal{D}$, defined by $\varphi_n(x,y) = \alpha_n(x,y)\varphi(x,y)$. With α_n the same as in the proof of the proposition above and

$$\varphi(x,y) = \frac{e^{i(y^2-x^2)}}{(x+y)^{p+q+2}}.$$

Now we have to prove (similar to (2.8) and (2.9)) that

$$p_{p+q+2}(\varphi_n) < M, \text{ uniformly in } n \tag{2.12}$$

and

$$\int\int x^p y^q e^{i(x^2-y^2)} \varphi_n(x) dx dy \nearrow +\infty \text{ for } n \to \infty. \tag{2.13}$$

By a similar argument the analogue of (2.10) holds:

$$p_{p+q+2}(\varphi_n) \leq M \sup_{|k| \leq p+q+2} \sup_{(x,y) \in Q} |D^k \varphi(x,y)|$$

Using Leibniz' rule again we find

$$D^k\varphi(x,y) = D^k\left(\ \tfrac{1}{(x+y)^{p+q+2}}e^{i(y^2-x^2)}\ \right) =$$

$$\sum_{0\le l\le k}\left(\begin{array}{c}k\\ l\end{array}\right)D^l\frac{1}{(x+y)^{p+q+2}}\ D^{k-l}e^{i(y^2-x^2)} =$$

And the terms

$$D^l\frac{1}{(x+y)^{p+q+2}}\ D^{k-l}e^{i(y^2-x^2)}$$

are of the form

$$\sum_{t+s\le|k-l|}\frac{c_{t,s}}{(x+y)^{p+q+2+|l|}}x^ty^se^{i(y^2-x^2)}.$$

Now because

$$\frac{x^ty^s}{(x+y)^u}\le\frac{r^tr^s}{r^u}=r^{t+s-u},$$

where $r=\sqrt{x^2+y^2}$, the terms in the last sum are uniformly bounded if $t+s-(p+q+2+|l|)\le 0$. We have

$$t+s-(p+q+2+|l|)\le|k-l|-(p+q+2+|l|)\le|k|-(p+q+2)\le 0$$

if $|k|\le p+q+2$. So $D^k\varphi$ is uniformly bounded for $|k|\le p+q+2$, i.e.

$$p_{p+q+2}(\varphi_n)<M.$$

This proves (2.12). The integral sequence in (2.13) is divergent, namely

$$\int\int x^py^qe^{i(x^2-y^2)}\varphi_n(x)dxdy =$$

$$\int_{\frac{1}{2}}^{\infty}\int_{\frac{1}{2}}^{\infty}x^py^qe^{i(x^2-y^2)}\alpha_n(x,y)\frac{1}{(x+y)^{p+q+2}}e^{i(y^2-x^2)}dxdy =$$

$$\int_{\frac{1}{2}}^{\infty}\int_{\frac{1}{2}}^{\infty}x^py^q\alpha_n(x,y)\frac{1}{(x+y)^{p+q+2}}dxdy \ge$$

$$\int_1^{\frac{n}{2}}\int_1^{\frac{n}{2}}x^py^q\frac{1}{(x+y)^{p+q+2}}dxdy =$$

(we substitute $x=yt$, and $dx=ydt$)

$$\int_1^{\frac{n}{2}}dy\int_{\frac{1}{y}}^{\frac{n}{2y}}(yt)^py^q\frac{1}{(yt+y)^{p+q+2}}ydt = \int_1^{\frac{n}{2}}dy\int_{\frac{1}{y}}^{\frac{n}{2y}}\frac{t^py^{p+q+1}}{y^{p+q+2}(t+1)^{p+q+2}}dt =$$

$$\int_1^{\frac{n}{2}}\frac{1}{y}dy\int_{\frac{1}{y}}^{\frac{n}{2y}}\frac{t^p}{(t+1)^{p+q+2}}dt\nearrow+\infty$$

for $n\to\infty$. This ends the proof of the theorem. ∎

Corollary 2.6 *The summability order of $P(x,y)e^{i(x^2-y^2)}$ is $d^o(P)+3$.*

In the next section we use these results together with the desintegration formula to calculate the sum order of e^{iP} with P a non-degenerate quadratic form on $\mathbb{R}^n$.

2.3 Non-degenerate quadratic forms on $\mathbb{R}^m$

We will determine the sum order of e^{iP} with P a non-degenerate quadratic form on $\mathbb{R}^m$, $P(x) = (Ax, x)$, $\det(A) \neq 0$. First we reduce to the case that A is positive definite, later on we take A general. For the integral calculations we will need the desintegration formula. This is stated in the first proposition of this section.

Proposition 2.7 (Desintegration Formula) *Let f be a radial function on $\mathbb{R}^m$. Then*

$$\int_{\mathbb{R}^m} f(|x|)dx = \int_0^\infty dr \int_{|x|=r} f(|x|)ds_r(x) = \omega_m \int_0^\infty r^{m-1} f(r)dr$$

where

$$\omega_m = 2\frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}.$$

One can prove this by using polar coordinates.

Proposition 2.8 *Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be an invertible and linear map and T a summable distribution on $\mathbb{R}^m$ with sum order m_0. Then the distribution $T \circ L$ is summable and has sum order m_0.*

Proof Let $\varphi \in \mathcal{D}(\mathbb{R}^m)$. Then $\langle T \circ L, \varphi\rangle = J^{-1}\langle T, \varphi \circ L^{-1}\rangle$, where J is the determinant of the Jacobi matrix of L. So $|\langle T \circ L, \varphi\rangle| \leq M_0\ p_{m_0}(\varphi \circ L^{-1})$. Now for D of first order $D(\varphi \circ L^{-1}) = (D\varphi \circ L^{-1})L^{-1}$, and by induction, using the productrule, it follows that $D^k(\varphi \circ L^{-1}) = (D^k\varphi \circ L^{-1})(L^{-1})^{|k|}$.
This implies that $\|D^k(\varphi \circ L^{-1})\|_\infty \leq c\ \|D^k\varphi \circ L^{-1}\|_\infty = c\ \|D^k\varphi\|_\infty$.
So $p_{m_0}(\varphi \circ L^{-1}) \leq M\ p_{m_0}(\varphi)$ and

$$|\langle T \circ L, \varphi\rangle| \leq M_1\ p_{m_0}(\varphi) \ \ \forall \varphi \in \mathcal{D},$$

i.e. $T \circ L$ is summable and has sum order smaller or equal to m_0, the sum order of T. Because this holds for all linear and invertible maps L and all summable distributions T, the converse also holds. Namely write $S = T \circ L$, then S is summable and s.o.$(T) =$ s.o.$(S \circ L^{-1}) \leq$ s.o.$(S) =$ s.o.$(T \circ L)$. ∎

Theorem 2.9 [TH1] *Let P be a positive definite quadratic form on $\mathbb{R}^m$. Then the summability order of $T : \varphi \mapsto \int e^{iP(x)}\varphi(x)dx$ is $m+1$.*

Proof There exists a positive definite matrix A such that $P(x) = (Ax, x)$. Thus there exists an orthogonal matrix U, such that $U^tAU = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_k > 0$, for $k = 1, \ldots, m$. If $W = \mathrm{diag}(\frac{1}{\sqrt{(\lambda_1)}}, \frac{1}{\sqrt{(\lambda_2)}}, \ldots, \frac{1}{\sqrt{(\lambda_m)}})$, then $W^tDW = I$. So $W^tU^tAUW = I$. Define $L = UW$, $J = \det(L)$ then

$$\langle T, \varphi\rangle = \int_{\mathbb{R}^m} e^{i(Ax,x)}\varphi(x)dx = J\int_{\mathbb{R}^m} e^{i(ALy,Ly)}\varphi(Ly)dy =$$

$$J\int_{\mathbb{R}^m} e^{i(L^tALy,y)}\varphi(Ly)dy = J\int_{\mathbb{R}^m} e^{i(Iy,y)}\varphi(Ly)dy =$$

$$J\int_{\mathbb{R}^m} e^{i|y|^2}\varphi(Ly)dy = J\,\langle T_0, \varphi\circ L\rangle$$

where T_0 is the distribution $\varphi \mapsto \int e^{i|x|^2}\varphi(x)dx$. Because L is an invertible linear transformation, the previous proposition applies: s.o.(T) = s.o.(T_0).
Let B be the unit ball in $\mathbb{R}^m$.

$$\langle T_0, \varphi\rangle = \left(\int_B + \int_{B^c}\right)e^{i|x|^2}\varphi(x)dx.$$

Because B is compact the first integral can be estimated by $Mp_0(\varphi)$. The second integral is equal to

$$\int_{B^c} e^{i|x|^2}\varphi(x)dx = \omega_m \int_1^\infty r^{m-1}\Phi(r)dr$$

where $\Phi(r) = \int_{|x|=r}\varphi(x)ds_r(x) = \int_{|y|=1}\varphi(ry)ds_1(y)$. Moreover

$$\Phi^{(n)}(r) = \int_{|y|=1}\left(\sum_{k=1}^m y_k\frac{\partial}{\partial x_k}\right)^n\varphi(ry)ds_1(y)$$

implies that Φ belongs to $\mathcal{D}(\mathbb{R})$ and $p_n(\Phi) \le c^{st}p_n(\varphi)$. By proposition 2.2

$$\left|\int_{B^c} e^{i|x|^2}\varphi(x)dx\right| \le Mp_{m+1}(\Phi) \le \tilde{M}p_{m+1}(\varphi)\ \ \forall\varphi\in\mathcal{D}(\mathbb{R}^m).$$

Therefore the sum order of T_0 is at most $m+1$. To see that it is not smaller than $m+1$, consider a sequence of radial functions φ_n with support in $(B(\frac{1}{2}))^c$. Then $\varphi_n(x) = \Phi_n(r)$ if $|x| = r$, $\operatorname{supp}(\Phi_n) \subset [\frac{1}{2}, \infty)$. Let $\alpha_n \in \mathcal{D}(\mathbb{R})$ be as in the proof of proposition 2.1: $0 \le \alpha_n \le 1$ and $\alpha_n(x) = 1$ for $1 \le x \le n$. Define

$$\Phi_n(r) = \alpha_n(r)\frac{e^{-ir^2}}{r^m}.$$

Then the derivatives $D^k\varphi_n$ are linear combinations of derivatives of Φ_n up to order $|k|$ with bounded coefficients. For example

$$\frac{\partial\varphi_n}{\partial x_i}(x) = \Phi_n'(r)\frac{\partial r}{\partial x_i} = \Phi_n'(r)\frac{x_i}{r} = \mathcal{O}(\Phi_n'(r)).$$

The derivatives up to order m of Φ_n are uniformly bounded in n, so $p_m(\varphi_n)$ is uniformly bounded in n. On the other hand, the integral sequence diverges:

$$\int_{\mathbb{R}^n} e^{i|x|^2}\varphi_n(x)dx \ge \int_{1\le|x|\le n} e^{i|x|^2}\frac{e^{-i|x|^2}}{|x|^m}dx =$$

$$\omega_m\int_1^n r^{m-1}e^{ir^2}\frac{e^{-ir^2}}{r^m}dr = \omega_m\int_1^n \frac{1}{r}dr \nearrow +\infty$$

for $n \to \infty$. This implies that the summability order of T_0 is exactly $m+1$. ∎

Corollary 2.10 *Let P be a negative definite quadratic form on $\mathbb{R}^m$. Then the summability order of $T : \varphi \mapsto \int e^{iP(x)}\varphi(x)dx$ is $m+1$.*

Remark Because $e^{i|x|^2} = e^{ix_1^2} \cdot \ldots \cdot e^{ix_n^2} = e^{ix^2} \otimes \ldots \otimes e^{ix^2}$, we know from the previous chapter that $\text{s.o.}(e^{i|x|^2}) \leq \text{s.o.}(e^{ix^2}) + \ldots + \text{s.o.}(e^{ix^2}) = m\ \text{s.o.}(e^{ix^2}) = 2m$. But we have proved in the theorem that $\text{s.o.}(e^{i|x|^2}) = m+1 < 2m$ for $m > 1$. So we see that we have found examples of the strict inequality in (1.25).

Theorem 2.11 *The summability order of the distribution $T : \varphi \mapsto \int e^{iP(x)}\varphi(x)dx$ with P a non-degenerate quadratic form on $\mathbb{R}^m$ is $m+1$.*

Proof For P positive or negative definite this is the previous theorem. Assume P not to be definite. Then by a similar argument as in the previous theorem, $\text{s.o.}(T) = \text{s.o.}(T_0)$, where $T_0 : \varphi \mapsto \int e^{i(Dx,x)}\varphi(x)dx$, with $D = \text{diag}(1,\ldots,1,-1,\ldots,-1)$ (k times 1 and l times -1).
Denote $\eta = (x_1,\ldots,x_k)$, $\xi = (x_{k+1},\ldots,x_m)$ and $x = (\eta,\xi)$, $dx = d\eta\ d\xi$. Furthermore, $r^2 = \|\eta\|^2 = \sum_{i=1}^k |x_i|^2$ and $\rho^2 = \|\xi\|^2 = \sum_{i=k+1}^m |x_i|^2$. We again use the desintegration formula, but now successively on two variables.

$$\langle T_0, \varphi\rangle = \int_{\mathbb{R}^m} e^{i(Dx,x)}\varphi(x)dx = \int_{\mathbb{R}^l} d\xi \int_{\mathbb{R}^k} e^{i(\|\eta\|^2 - \|\xi\|^2)}\varphi(\eta,\xi)d\eta =$$

$$\int_{\mathbb{R}^l} e^{-i\|\xi\|^2} d\xi \int_0^\infty r^{k-1} e^{ir^2} \Phi_\xi(r)dr,$$

where

$$\Phi_\xi(r) = \int_{\|t\|=r} \varphi(t,\xi)ds_r(t) = \int_{\|t\|=1} \varphi(rt,\xi)ds_1(t).$$

Continuing,

$$\langle T_0, \varphi\rangle = \int_0^\infty \int_0^\infty r^{k-1}\rho^{l-1} e^{ir^2} e^{-i\rho^2} \Phi(r,\rho)dr d\rho,$$

with

$$\Phi(r,\rho) = \int_{\|u\|=\rho} \Phi_u(r)ds_\rho(u) = \int_{\|u\|=1} \Phi_{\rho u}(r)ds_1(u) =$$

$$\int_{\|u\|=1}\int_{\|t\|=1} \varphi(rt,\rho u)ds_1(t)ds_1(u).$$

Now we use the results from the previous section, and know that $r^{k-1}\rho^{l-1}e^{i(r^2-\rho^2)}$ has sum order $k-1+l-1+3 = k+l+1 = m+1$ on $\mathbb{R}^2$, which yields an estimate:

$$|\langle T_0, \varphi\rangle| \leq M p_{m+1}(\Phi) \tag{2.14}$$

Now we need a relation between $p_{m+1}(\Phi)$ and $p_{m+1}(\varphi)$.

$$\frac{\partial}{\partial r}\Phi(r,\rho) = \int_{\|u\|=1}\int_{\|t\|=1} \left(\sum_{i=1}^k t_i \frac{\partial}{\partial \eta_i}\right)\varphi(rt,\rho u)ds_1(t)ds_1(u),$$

$$(\frac{\partial}{\partial r})^n\Phi(r,\rho) = \int_{\|u\|=1}\int_{\|t\|=1}(\sum_{i=1}^{k} t_i\frac{\partial}{\partial\eta_i})^n\varphi(rt,\rho u)ds_1(t)ds_1(u),$$

And this gives us

$$(\frac{\partial}{\partial r})^{n_1}(\frac{\partial}{\partial\rho})^{n_2}\Phi(r,\rho) =$$

$$\int_{\|u\|=1}\int_{\|t\|=1}(\sum_{i=1}^{k} t_i\frac{\partial}{\partial\eta_i})^{n_1}(\sum_{j=1}^{l} u_j\frac{\partial}{\partial\xi_j})^{n_2}\varphi(rt,\rho u)ds_1(t)ds_1(u).$$

From this the relation follows:

$$p_{m+1}(\Phi) \leq Mp_{m+1}(\varphi).$$

So (2.14) can be replaced by

$$|\langle T_0,\varphi\rangle| \leq Mp_{m+1}(\varphi) \ \ \forall\varphi \in \mathcal{D}(\mathbb{R}^m),$$

i.e. s.o.$(T) \leq m+1$.

To see that the sum order of T_0 is not smaller than $m+1$, consider the sequence of 'biradial' functions $\varphi_n = \varphi_n(\eta,\xi)$ in $\mathcal{D}(\mathbb{R}^m)$ defined by $\varphi_n(\eta,\xi) = \Phi_n(\|\eta\|,\|\xi\|)$, where Φ_n is similar to the function φ_n used in the proof of theorem 2.5:

$$\Phi(r,\rho) = \frac{1}{(r+\rho)^m}\, e^{i(\rho^2-r^2)}, \ \ \Phi_n(r,\rho) = \alpha_n(r,\rho)\ \Phi(r,\rho),$$

where α_n is the same function as before. We also use $\varphi(\eta,\xi) = \Phi(\|\eta\|,\|\xi\|)$. We know from that proof that $p_m(\Phi_n)$ is uniformly bounded in n and that the integrals diverge:

$$\langle T_0,\varphi_n\rangle = \iint r^{k-1}\rho^{l-1}e^{i(r^2-\rho^2)}\Phi_n(r,\rho)drd\rho =$$

$$\iint r^{k-1}\rho^{l-1}e^{i(r^2-\rho^2)}\frac{\alpha_n(r,\rho)}{(r+\rho)^m}e^{i(\rho^2-r^2)}drd\rho \nearrow +\infty,$$

for $n\to\infty$. So it remains to prove that

$$p_m(\Phi_n) < M \Longrightarrow p_m(\varphi_n) < \tilde{M}.$$

Define again $Q = ([\frac{1}{2},\infty))^2$, then

$$p_m(\varphi_n) = \sup_{|k|\leq m}\ \sup_{(\|\eta\|,\|\xi\|)\in Q}|D^k\varphi_n(\eta,\xi)| \leq M\sup_{|k|\leq m}\ \sup_{(\|\eta\|,\|\xi\|)\in Q}|D^k\varphi(\eta,\xi)| \leq$$

$$M\sup_{|k|\leq m}\ \sup_{(\|\eta\|,\|\xi\|)\in Q}|D^k\Phi(\|\eta\|,\|\xi\|)|.$$

The derivatives $D^k\Phi(\|\eta\|,\|\xi\|)$ with $|k|\leq m$ are linear combinations of

$$\frac{\partial^s}{\partial r^s}\frac{\partial^t}{\partial\rho^t}\Phi(\|\eta\|,\|\xi\|)\ D^\alpha_\eta\|\eta\|\ D^\beta_\xi\|\xi\|$$

with $t + s \leq m$ and $0 < |\alpha| \leq m$, $0 < |\beta| \leq m$. Because $p_m(\Phi)$ is bounded on Q, the first term can be estimated by a constant. The normderivatives are treated in the following lemma.

Lemma 2.12 *For* $|k| > 0$,

$$D^k \|\eta\| = \sum_i c_i \frac{\eta^{l_i}}{\|\eta\|^{n_i}},$$

where the sum is finite and the multi-index $l_i \geq 0$, $n_i \geq 0$, $|l_i| \leq n_i$, *for all* i.

Proof of the lemma
We prove the lemma with the use of induction. For $|k| = 1$, $D^k = \frac{\partial}{\partial \eta_j}$ and

$$\frac{\partial}{\partial \eta_j} \|\eta\| = \frac{\eta_j}{\|\eta\|}.$$

Assume that for $|k| \leq k_0$ the lemma holds, then for $|k| = k_0 + 1$, $D^k = \frac{\partial}{\partial \eta_j} D^{k-e_j}$ for some j, $1 \leq j \leq m$. We apply the induction hypothesis to D^{k-e_j}:

$$D^k \|\eta\| = \frac{\partial}{\partial \eta_j} D^{k-e_j} \|\eta\| = \frac{\partial}{\partial \eta_j} \sum_i c_i \frac{\eta^{l_i}}{\|\eta\|^{n_i}} =$$

($|l_i| \leq n_i$ for all i)

$$\sum_i c_i \frac{\partial}{\partial \eta_j} \frac{\eta^{l_i}}{\|\eta\|^{n_i}} = \sum_i \tilde{c_i} \frac{\eta^{l_i - e_j}}{\|\eta\|^{n_i}} + \tilde{\tilde{c}}_i \frac{\eta^{l_i + e_j}}{\|\eta\|^{n_i + 2}}$$

where $\tilde{c_i} = 0$ if $(l_i)_j = 0$. In this sum we compare: $|l_i - e_j| = |l_i| - 1 \leq n_i - 1 < n_i$ and $|l_i + e_j| = |l_i| + 1 \leq n_i + 1 < n_i + 2$. This proves the lemma.

It follows from the lemma that $D^\alpha \|\eta\|$ and $D^\beta \|\xi\|$ are bounded on Q for $|\alpha| > 0$ and $|\beta| > 0$.
We have now proved that the derivatives $D^k \Phi(\|\eta\|, \|\xi\|)$ are bounded on Q for $|k| \leq m$ and therefore $p_m(\varphi_n)$ is uniformly bounded in n.
This proves that the summability order of T_0 is exactly $m + 1$. So the sum order of T is $m + 1$. ∎

Chapter 3

Generalized Fresnel Distributions

In this chapter we try to find out which polynomials P lead to summable distributions e^{iP} on $\mathbb{R}^m$. We first consider the one dimensional case.

3.1 One dimension

For general polynomials P in one dimension we have the following result:

Proposition 3.1 [E.G.F. Thomas] *Let $P : \mathbb{R} \longrightarrow \mathbb{R}$ be a real polynomial, then*

1. *e^{iP} is summable $\Longleftrightarrow$ $d^o(P) \geq 2$.*
2. *if $d^o(P) = 2$, $s.o.(e^{iP}) = 2$.*
3. *if $d^o(P) \geq 3$, $s.o.(e^{iP}) = 1$.*

Proof If $d^o(P) = 0$, then $e^{iP} = c^{st}$, so $\mathcal{F}(e^{iP}) = c\delta$ is not continuous, therefore e^{iP} is not summable. Similarly, if $d^o(P) = 1$, $\mathcal{F}(e^{iP}) = \mathcal{F}(e^{i(a+bx)}) = c\mathcal{F}(e^{ibx}) = c\delta_{-b}$ is also not continuous, and therefore not summable. For polynomials P of degree 2 e^{iP} can be written as $ce^{ia(x+b)^2}$ by completing the square. By proposition 2.1, using a translation over $-b$, for these polynomials e^{iP} has sum order 2.
For P with $d^o(P) \geq 3$, take a large enough, such that $|P'(x)| > 0$, for $|x| > a$. Because $[-a, a]$ is compact the integral of $e^{iP}\varphi$ over this interval can again be estimated by $Mp_0(\varphi)$, and we need only consider the integral over $[a, \infty)$. For $\varphi \in \mathcal{D}$, we have

$$\int_a^\infty e^{iP(x)}\varphi(x)dx = \int_a^\infty P'(x)e^{iP(x)}\frac{\varphi(x)}{P'(x)}dx =$$

$$c^{st}\int_a^\infty \frac{d}{dx}[e^{iP(x)}]\frac{\varphi(x)}{P'(x)}dx = c^{st}\int_a^\infty e^{iP(x)}\frac{d}{dx}[\frac{\varphi(x)}{P'(x)}]dx + c^{st}\varphi(a) =$$

$$c^{st}\int_a^\infty e^{iP(x)}(\frac{\varphi'(x)}{P'(x)} - \frac{\varphi(x)P''(x)}{(P'(x))^2})dx + c^{st}\varphi(a).$$

If $d^o(P) = n \geq 3$, $d^o(\frac{1}{P'}) = 1 - n \leq -2$, and $d^o(\frac{P''}{(P')^2}) = n - 2 + 2(1-n) = -n \leq -3$. So $\frac{1}{P'}$ and $\frac{P''}{(P')^2}$ are integrable on $[a, \infty)$, therefore

$$|\int_a^\infty e^{iP(x)}\varphi(x)dx| \leq Mp_1(\varphi)$$

and s.o.$(e^{iP(x)}) = 1$, because it cannot be zero (that would imply that $e^{iP(x)}$ is integrable in the usual sense, taking $\varphi_n(x) = \alpha(\frac{x}{n})$ with α from the proof of proposition 2.1). This proves the proposition. ∎

Definition We define the round up map $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$ by $\lceil a \rceil = n$ with $n - 1 < a \leq n$.

Theorem 3.2 *Let $E_{k,P}(x) = x^k e^{iP(x)}$ and $n = d^o(P) \geq 2$. Then*

1. $|\int_1^\infty E_{k,P}(x)\varphi(x)dx| \leq Mp_0(\varphi)$ *for* $k \leq -2$.
2. $|\int_1^\infty E_{-1,P}(x)\varphi(x)dx| \leq Mp_1(\varphi)$.
3. *s.o.*$(E_{k,P}) = \lceil\frac{k+2}{n-1}\rceil$ *for* $k \geq 0$.

Proof The first statement is clear: $E_{k,P}$ is integrable over $[1, \infty)$ in that case.
For *2.* take $\varphi \in \mathcal{D}$ and a large enough such that $|P'(x)| > 0$ for $|x| > a$. Then

$$\int_a^\infty E_{-1,P}(x)\varphi(x)dx = \int_a^\infty \frac{1}{x}e^{iP(x)}\varphi(x)dx = \int_a^\infty P'(x)e^{iP(x)}\frac{\varphi(x)}{xP'(x)}dx =$$

$$c^{st}\int_a^\infty \frac{d}{dx}(e^{iP(x)})\frac{\varphi(x)}{xP'(x)}dx = c^{st}\int_a^\infty e^{iP(x)}\frac{d}{dx}[\frac{\varphi(x)}{xP'(x)}]dx + c^{st}\varphi(a) =$$

$$c^{st}\int_a^\infty e^{iP(x)}(\frac{\varphi'(x)}{xP'(x)} - \frac{\varphi(x)}{x^2P'(x)} - \frac{\varphi(x)P''(x)}{x(P'(x))^2})dx + c^{st}\varphi(a).$$

Now considering the degrees: $d^o(\frac{1}{xP'(x)}) = -n \leq -2$, $d^o(\frac{1}{x^2P'(x)}) = -n-1 \leq -3$ and $d^o(\frac{P''(x)}{x(P'(x))^2}) = n-2-1-2(n-1) = -n-1 \leq -3$, we see that all these rational functions are integrable on $[a, \infty)$. Because $[1, a]$ is compact the integral over this interval can be estimated by $Mp_0(\varphi)$. This results in:

$$|\int_1^\infty E_{-1,P}(x)\varphi(x)dx| \leq Mp_1(\varphi).$$

This proves *2.*
We prove the third statement using induction on k. For $k = 0$, we know that s.o.$(e^{iP}) = 2$ if $n = 2$ and s.o.$(e^{iP}) = 1$ if $n \geq 3$ from proposition 3.1. So we have to check whether $\lceil\frac{2}{n-1}\rceil = 2$ for $n = 2$ and $\lceil\frac{2}{n-1}\rceil = 1$ for $n \geq 3$. The first equality is obvious. For the second note that $0 < \frac{2}{n-1} \leq \frac{2}{2} = 1$ if $n \geq 3$. So then $\lceil\frac{2}{n-1}\rceil = 1$. Now, assume that s.o.$(E_{l,P}) = \lceil\frac{l+2}{n-1}\rceil$ for $0 \leq l \leq k-1$. Then we also have that s.o.$(Q(x)e^{iP(x)}) = \lceil\frac{d^o(Q)+2}{n-1}\rceil$

if $d^o(Q) \leq k-1$. Again, we only consider the interval $[a,\infty)$ which is sufficient by compactness of $[-a,a]$ and equivalence to $(-\infty,-a]$. We now have for $\varphi \in \mathcal{D}$

$$\int_a^\infty E_{k,P}\varphi(x)dx = \int_a^\infty e^{iP(x)}x^k\varphi(x)dx = \int_a^\infty P'(x)e^{iP(x)}\frac{x^k\varphi(x)}{P'(x)}dx =$$

$$c^{st}\int_a^\infty \frac{d}{dx}(e^{iP(x)})\frac{x^k\varphi(x)}{P'(x)}dx = c^{st}\int_a^\infty e^{iP(x)}\frac{d}{dx}[\frac{x^k\varphi(x)}{P'(x)}]dx + c^{st}\varphi(a) =$$

$$c^{st}\int_a^\infty e^{iP(x)}(\frac{kx^{k-1}\varphi(x)}{P'(x)} + \frac{x^k\varphi'(x)}{P'(x)} - \frac{x^k\varphi(x)P''(x)}{(P'(x))^2})dx + c^{st}\varphi(a).$$

The degrees of the rational functions are $d^o(\frac{x^{k-1}}{P'(x)}) = k-1-(n-1) = k-n < k$, $d^o(\frac{x^k}{P'(x)}) = k-(n-1) = k-n+1 < k$ and $d^o(\frac{x^kP''(x)}{(P'(x))^2}) = k+n-2-2(n-1) = k-n < k$. So we can apply the induction hypothesis to all three:

$$\text{s.o.}(e^{iP(x)}\frac{x^{k-1}}{P'(x)}) = \lceil\frac{k-n+2}{n-1}\rceil$$

$$\text{s.o.}(e^{iP(x)}\frac{x^k}{P'(x)}) = \lceil\frac{k-n+1+2}{n-1}\rceil$$

$$\text{s.o.}(e^{iP(x)}\frac{x^kP''(x)}{(P'(x))^2}) = \lceil\frac{k-n+2}{n-1}\rceil.$$

Therefore $\text{s.o.}(E_{k,P}) = \max(\lceil\frac{k-n+2}{n-1}\rceil, \lceil\frac{k-n+1+2}{n-1}\rceil + 1) =$
$\lceil\frac{k-n+1+2}{n-1}\rceil + 1 = \lceil\frac{k+2-(n-1)}{n-1}\rceil + 1 = \lceil(\frac{k+2}{n-1} - 1)\rceil + 1 = \lceil\frac{k+2}{n-1}\rceil$. So the statement is true for k. This proves *3.* ∎

3.2 Symmetric polynomials in the two dimensional case

Proposition 3.3 *For $E_a(x,y) = e^{i(x^2+y^2+axy)}$ the following holds:*

1. *$s.o.(E_a) = 3$, $|a| \neq 2$.*
2. *E_{-2} and E_2 are not summable.*

Proof Consider the matrix $A = \begin{pmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{pmatrix}$ Then

$$E_a(x,y) = e^{i(A(\frac{x}{y}),(\frac{x}{y}))}.$$

We can apply theorem 2.11 if $\det(A) \neq 0$, that is if $|a| \neq 2$. So *1.* follows directly from theorem 2.11.
For the proof of *2.* we introduce new coordinates: $u = x+y$, $v = x-y$. If $a = 2$, then

$E_a(x,y) = e^{i(x+y)^2} = e^{iu^2}$. If $a=-2$, then $E_a(x,y) = e^{i(x-y)^2} = e^{iv^2}$. So (in the first case for example)

$$\iint E_a(x,y)\varphi(x,y)dxdy = \iint e^{i(x+y)^2}\varphi(x,y)dxdy =$$
$$2\iint e^{iu^2}\varphi(\frac{1}{2}(u+v), \frac{1}{2}(u-v))dudv$$

We see that by a linear transformation of coordinates the distribution only depends on one variable. So it cannot be summable. ∎

Theorem 3.4 *Let $k \in \mathbb{R}$ and let T_k be the distribution $\varphi \mapsto \int_{\mathbb{R}^2} e^{i(x^2+y^2+kx^2y^2)}\varphi(x,y)dxdy$ for $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Then*

1. *$s.o.(T_k) = 1$ for $k > 0$.*
2. *$s.o.(T_0) = 3$.*
3. *$s.o.(T_k) = 1$ for $k < 0$.*

Proof It is sufficient to consider $[1,\infty) \times [0,\infty)$ because the polynomial is symmetric. Namely, using the reflections $x \mapsto -x$, $y \mapsto -y$ and $x \mapsto y$ and $y \mapsto x$ one gets the same result for the other quadrants in $\mathbb{R}^2 \setminus [-1,1]^2$. We abbreviate $E_k(x,y) = e^{i(x^2+y^2+kx^2y^2)}$ and $B = (x+y)(1+xy)$. Then $D_{1,1}E_1 = c^{st}BE_1$.
To prove the first statement, consider the case $k=1$. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$.

$$\int_0^\infty dy \int_1^\infty E_1(x,y)\varphi(x,y)dx = \int_0^\infty dy \int_1^\infty BE_1(x,y)\frac{\varphi(x,y)}{B}dx =$$
$$c^{st}\int_0^\infty dy \int_1^\infty D_{1,1}(E_1(x,y))\frac{\varphi(x,y)}{B}dx =$$
$$c^{st}\int_0^\infty dy \int_1^\infty D_{1,1}(E_1(x,y)\frac{\varphi(x,y)}{B})dx + c^{st}\int_0^\infty dy \int_1^\infty E_1(x,y)D_{1,1}(\frac{\varphi(x,y)}{B})dx =$$
$$c^{st}\int_0^\infty E_1(1,y)\frac{\varphi(1,y)}{B(1,y)}dy + c^{st}\int_1^\infty E_1(x,0)\frac{\varphi(x,0)}{B(x,0)}dx +$$
$$c^{st}\int_0^\infty dy \int_1^\infty E_1(x,y)D_{1,1}(\frac{\varphi(x,y)}{B})dx =$$
$$c^{st}\int_0^\infty e^{i(1+2y^2)}\frac{\varphi(1,y)}{(1+y)^2}dy + c^{st}\int_1^\infty e^{ix^2}\frac{\varphi(x,0)}{x}dx +$$
$$c^{st}\int_0^\infty dy \int_1^\infty E_1(x,y)D_{1,1}(\frac{\varphi(x,y)}{B})dx.$$

We call these integrals respectively I, II and III.

$$|I| \le c^{st}\int_0^\infty \frac{|\varphi(1,y)|}{(1+y)^2}dy \le Mp_0(\varphi)$$

and

$$|II| \leq Mp_1(\varphi)$$

by proposition 2.2. To estimate III we first have to calculate $D_{1,1}\frac{\varphi}{B}$.

$$D_{1,1}\frac{\varphi}{(x+y)(1+xy)} = \frac{D_{1,1}\varphi}{(x+y)(1+xy)} - \frac{\varphi D_{1,1}((x+y)(1+xy))}{(x+y)^2(1+xy)^2} =$$

$$\frac{D_{1,1}\varphi}{(x+y)(1+xy)} - \frac{2\varphi(1+xy)}{(x+y)^2(1+xy)^2} - \frac{\varphi(x+y)^2}{(x+y)^2(1+xy)^2} =$$

$$\frac{D_{1,1}\varphi}{(x+y)(1+xy)} - \frac{2\varphi}{(x+y)^2(1+xy)} - \frac{\varphi}{(1+xy)^2}.$$

This yields

$$III = c^{st}\int_0^\infty dy \int_1^\infty E_1(x,y)D_{1,1}(\frac{\varphi(x,y)}{B})dx =$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1(x,y)\frac{D_{1,1}\varphi(x,y)}{(x+y)(1+xy)}dx +$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1(x,y)\frac{\varphi(x,y)}{(x+y)^2(1+xy)}dx +$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1(x,y)\frac{\varphi(x,y)}{(1+xy)^2}dx.$$

We will call these integrals respectively III_1, III_2 and III_3.
For III_1 we calculate the integral:

$$\int_0^\infty dy\int_1^\infty \frac{1}{(x+y)(1+xy)}dx = \int_0^\infty dt \int_1^\infty \frac{1}{(x+\frac{t}{x})(1+t)}\frac{1}{x}dx =$$

(using the substitution $y = \frac{t}{x}$)

$$\int_0^\infty dt\int_1^\infty \frac{1}{1+t}\frac{1}{x^2+t}dx = \int_0^\infty dt\int_1^\infty \frac{1}{t(1+t)}\frac{1}{\frac{x^2}{t}+1}dx =$$

(substitute $u = \frac{x}{\sqrt{t}}$)

$$\int_0^\infty dt\int_{\frac{1}{\sqrt{t}}}^\infty \frac{1}{t(1+t)}\frac{1}{u^2+1}\sqrt{t}du \leq \int_0^\infty \frac{1}{\sqrt{t}(1+t)}dt\int_0^\infty \frac{1}{u^2+1}du < \infty.$$

This gives the estimate

$$|III_1| \leq Mp_1(\varphi).$$

Now because

$$|\frac{1}{(x+y)^2(1+xy)}| \leq |\frac{1}{(x+y)(1+xy)}|$$

on $[1,\infty)\times[0,\infty)$ we have the estimate

$$|III_2| \leq Mp_0(\varphi).$$

For III_3 we need an extra integration by parts, because $\frac{1}{(1+xy)^2}$ is not integrable on $[1,\infty)\times[0,\infty)$.

$$III_3 = c^{st}\int_0^\infty dy \int_1^\infty E_1(x,y)\frac{\varphi(x,y)}{(1+xy)^2}dx =$$

$$c^{st}\int_0^\infty dy\int_1^\infty (2x+2xy^2)e^{i(x^2+y^2+x^2y^2)}\frac{\varphi(x,y)}{2x(1+xy)^2(1+y^2)}dx =$$

$$c^{st}\int_0^\infty dy\int_1^\infty D_{1,0}E_1\frac{\varphi(x,y)}{x(1+xy)^2(1+y^2)}dx =$$

$$c^{st}\int_0^\infty dy\int_1^\infty D_{1,0}(E_1\frac{\varphi(x,y)}{x(1+xy)^2(1+y^2)})dx+$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1 D_{1,0}(\frac{\varphi(x,y)}{x(1+xy)^2(1+y^2)})dx.$$

Now we first calculate $D_{1,0}\frac{1}{x(1+xy)^2(1+y^2)}$.

$$D_{1,0}\frac{1}{x(1+xy)^2(1+y^2)} = \frac{(1+xy)^2(1+y^2)+2xy(1+xy)(1+y^2)}{x^2(1+xy)^4(1+y^2)^2} =$$

$$\frac{(1+xy)^2(1+y^2)}{x^2(1+xy)^4(1+y^2)^2}+\frac{2xy(1+xy)(1+y^2)}{x^2(1+xy)^4(1+y^2)^2} =$$

$$\frac{1}{x^2(1+xy)^2(1+y^2)}+\frac{2y}{x(1+xy)^3(1+y^2)}.$$

This yields

$$III_3 = c^{st}\int_0^\infty dy\int_1^\infty D_{1,0}(E_1\frac{\varphi(x,y)}{x(1+xy)^2(1+y^2)})dx+$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1\frac{D_{1,0}\varphi(x,y)}{x(1+xy)^2(1+y^2)})dx+$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1\frac{\varphi(x,y)}{x^2(1+xy)^2(1+y^2)}dx+$$

$$c^{st}\int_0^\infty dy\int_1^\infty E_1\frac{y\varphi(x,y)}{x(1+xy)^3(1+y^2)}dx.$$

Again we call these integrals III_{31}, III_{32}, III_{33} and III_{34} respectively.

$$III_{31} = c^{st}\int_0^\infty e^{i(1+2y^2)}\frac{\varphi(1,y)}{(1+y)^2(1+y^2)}dy.$$

Because $\frac{1}{(1+y)^2(1+y^2)}$ is integrable on $[0,\infty)$ this part can be estimated by

$$|III_{31}| \leq Mp_0(\varphi).$$

For III_{32} we use the substitution $y = \frac{t}{x}$:

$$\int_0^\infty dy \int_1^\infty \frac{1}{x(1+xy)^2(1+y^2)}dx = \int_0^\infty dt \int_1^\infty \frac{1}{x(1+t)^2(1+\frac{t^2}{x^2})}\frac{1}{x}dx =$$

$$\int_0^\infty dt \int_1^\infty \frac{1}{(1+t)^2(x^2+t^2)}dx \leq \int_0^\infty dt \int_1^\infty \frac{1}{(1+t)^2}\frac{1}{x^2}dx < \infty.$$

Therefore

$$|III_{32}| \leq Mp_1(\varphi).$$

Furthermore

$$\frac{1}{x^2(1+xy)^2(1+y^2)} \leq \frac{1}{x(1+xy)^2(1+y^2)}$$

on $[1,\infty) \times [0,\infty)$. Also

$$\frac{y}{x(1+xy)^3(1+y^2)} \leq \frac{1}{x(1+xy)^2(1+y^2)}$$

on $[1,\infty) \times [0,\infty)$ because $\frac{y}{1+xy} \leq 1$ on this area. These two inequalities then yield

$$|III_{33}| \leq Mp_0(\varphi), \quad |III_{34}| \leq Mp_0(\varphi).$$

We have now proved that

$$|\int_0^\infty dy \int_1^\infty e^{i(x^2+y^2+x^2y^2)}\varphi(x,y)dx| \leq Mp_1(\varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2). \tag{3.1}$$

And therefore, as remarked in the beginning of the proof:

$$\text{s.o.}(e^{i(x^2+y^2+x^2y^2)}) = \text{s.o.}(T_1) = 1.$$

Let $k > 0$, then we use the linear transformation $(x,y) \mapsto (u,v) = (\sqrt{k}x, \sqrt{k}y)$. Now we can write $x^2 + y^2 + kx^2y^2 = \frac{u^2}{k} + \frac{v^2}{k} + \frac{u^2v^2}{k} = \frac{1}{k}(u^2+v^2+u^2v^2)$.
Therefore

$$\int_0^\infty dy \int_1^\infty e^{i(x^2+y^2+kx^2y^2)}\varphi(x,y)dx = \frac{1}{\sqrt{k}}\int_0^\infty dv \int_{\frac{1}{\sqrt{k}}}^\infty e^{\frac{i}{k}(u^2+v^2+u^2v^2)}\varphi(u,v)du$$

by substituting u and v.
The proof of an estimate similar to (3.1) is the same as for $k = 1$. Namely, the factor $\frac{1}{k}$ in the exponent causes some factor in B and can be put in the constant in front of the integral. This completes the proof of *1*.

The statement *2.* is a special case of theorem 2.11.

To prove *3.* it suffices to prove the case $k = -1$ by the reasoning above.
This proof is in two steps. We first consider the quadrants and after that we will have a look at the neighborhoods of the axes.

We first consider the area $[2, \infty) \times [2, \infty)$. For this area we abbreviate $B = (x+y)(xy-1)$, so $D_{1,1}E_{-1} = c^{st}BE_{-1}$.

$$\int_2^\infty \int_2^\infty E_{-1}(x,y)\varphi(x,y)dxdy = \int_2^\infty \int_2^\infty BE_{-1}(x,y)\frac{\varphi(x,y)}{B}dxdy =$$

$$c^{st}\int_2^\infty \int_2^\infty D_{1,1}(E_{-1}(x,y))\frac{\varphi(x,y)}{B}dxdy =$$

$$c^{st}\int_2^\infty \int_2^\infty D_{1,1}(E_{-1}(x,y)\frac{\varphi(x,y)}{B})dxdy +$$

$$c^{st}\int_2^\infty \int_2^\infty E_{-1}(x,y)D_{1,1}(\frac{\varphi(x,y)}{B})dxdy.$$

We call these integrals respectively I and II. For I we use the integration rule for $D_{1,1}$:

$$I = c^{st}\int_2^\infty \int_2^\infty D_{1,1}(E_{-1}(x,y)\frac{\varphi(x,y)}{B})dxdy =$$

$$c^{st}\int_2^\infty (E_{-1}\frac{\varphi}{B})(x,2)dx + c^{st}\int_2^\infty (E_{-1}\frac{\varphi}{B})(2,y)dy =$$

$$c^{st}\int_2^\infty e^{i(4-3x^2)}\frac{\varphi(x,2)}{(x+2)(2x-1)}dx + c^{st}\int_2^\infty e^{i(4-3y^2)}\frac{\varphi(2,y)}{(2+y)(2y-1)}dy.$$

Because the rational functions in these integrals are integrable on $[2, \infty)$, we have the estimate:

$$|I| \leq Mp_0(\varphi).$$

We can write the integral II as a sum of integrals

$$II = c^{st}\int_2^\infty \int_2^\infty E_{-1}(x,y)D_{1,1}(\frac{\varphi(x,y)}{B})dxdy =$$

$$c^{st}\int_2^\infty \int_2^\infty E_{-1}(x,y)\frac{D_{1,1}\varphi(x,y)}{B}dxdy + c^{st}\int_2^\infty \int_2^\infty E_{-1}(x,y)\frac{\varphi(x,y)D_{1,1}B}{B^2}dxdy.$$

Let II_1 be the first integral and II_2 the second. Then for II_1 we estimate the following integral:

$$\int_2^\infty \int_2^\infty \frac{1}{(x+y)(xy-1)}dxdy \leq \int_2^\infty \int_2^\infty \frac{4}{3}\frac{1}{(x+y)xy}dxdy$$

$$\leq c^{st}\int_2^\infty \int_2^\infty \frac{1}{x^{\frac{3}{2}}y^{\frac{3}{2}}}dxdy < \infty$$

where we have used the inequalities $xy - 1 \geq \frac{3}{4}xy$ and $x + y \geq \sqrt{xy}$. This yields

$$|II_1| \leq Mp_1(\varphi).$$

In II_2 we need the derivative $D_{1,1}B$:

$$D_{1,1}B = D_{1,1}((x+y)(xy-1)) = 2(xy-1) + (x+y)^2.$$

Therefore

$$II_2 = c^{st} \int_2^\infty \int_2^\infty E_{-1}(x,y) \frac{\varphi(x,y) D_{1,1}B}{B^2} dxdy =$$

$$c^{st} \int_2^\infty \int_2^\infty E_{-1}(x,y) \frac{\varphi(x,y)(xy-1)}{(x+y)^2(xy-1)^2} dxdy +$$

$$c^{st} \int_2^\infty \int_2^\infty E_{-1}(x,y) \frac{\varphi(x,y)(x+y)^2}{(x+y)^2(xy-1)^2} dxdy =$$

$$c^{st} \int_2^\infty \int_2^\infty E_{-1}(x,y) \frac{\varphi(x,y)}{(x+y)^2(xy-1)} dxdy +$$

$$c^{st} \int_2^\infty \int_2^\infty E_{-1}(x,y) \frac{\varphi(x,y)}{(xy-1)^2} dxdy.$$

We call these integrals respectively II_{21} and II_{22}. For II_{21} we mention that $\frac{1}{(x+y)^2(xy-1)} < \frac{1}{(x+y)(xy-1)}$ on this area and the last function is integrable on $([2,\infty))^2$ as we have seen in II_1. This gives

$$|II_{21}| \leq Mp_0(\varphi).$$

The function in II_{22} is also integrable:

$$\int_2^\infty \int_2^\infty \frac{1}{(xy-1)^2} dxdy = \int_2^\infty \frac{dy}{y} \int_2^\infty y(xy-1)^{-2} dx = -\int_2^\infty (xy-1)^{-1}|_2^\infty \frac{dy}{y} =$$

$$\int_2^\infty (2y-1)^{-1} \frac{dy}{y} = \int_2^\infty \frac{1}{y(2y-1)} dy < \infty.$$

From this it follows that

$$|II_{22}| \leq Mp_0(\varphi).$$

Now we have proved that

$$\left| \int_2^\infty \int_2^\infty E_{-1}(x,y)\varphi(x,y) dxdy \right| \leq Mp_1(\varphi).$$

By the reflections $x \mapsto -x$, $y \mapsto -y$, $x \mapsto y$ and $y \mapsto x$ we have

$$\left| \int\int_R E_{-1}(x,y)\varphi(x,y) dxdy \right| \leq Mp_1(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2) \tag{3.2}$$

where $R = [\pm 2, \infty) \times [\pm 2, \infty)$. This finishes the first step.

For the neighborhoods of the axes we consider $[0,2] \times [3,\infty)$. We now use the differential operator $D_{1,-1}$ and $B = (y-x)(xy+1)$. Then we have $D_{1,-1}E_{-1} = c^{st}BE_{-1}$.

$$\int_3^\infty dy \int_0^2 E_{-1}(x,y)\varphi(x,y)dx = \int_3^\infty dy \int_0^2 BE_{-1}(x,y)\frac{\varphi(x,y)}{B}dx =$$

$$c^{st}\int_3^\infty dy \int_0^2 D_{1,-1}(E_{-1}(x,y))\frac{\varphi(x,y)}{B}dx =$$

$$c^{st}\int_3^\infty dy \int_0^2 D_{1,-1}(E_{-1}(x,y)\frac{\varphi(x,y)}{B})dx+$$

$$c^{st}\int_3^\infty dy \int_0^2 E_{-1}(x,y)D_{1,-1}(\frac{\varphi(x,y)}{B})dx.$$

We call the first integral I and the second II. By the integration rule we have

$$I = c^{st}\int_3^\infty e^{iy^2}\frac{\varphi(0,y)}{y}dy + c^{st}\int_3^\infty e^{i(4-3y^2)}\frac{\varphi(2,y)}{(y-2)(2y+1)}dy+$$

$$c^{st}\int_0^2 e^{i(9-8x^2)}\frac{\varphi(x,3)}{(3-x)(3x+1)}dx.$$

By proposition 2.2 we can estimate these integrals:

$$|I| \leq Mp_1(\varphi).$$

The integral II is equal to

$$II = c^{st}\int_3^\infty dy \int_0^2 E_{-1}(x,y)D_{1,-1}(\frac{\varphi(x,y)}{B})dx =$$

$$c^{st}\int_3^\infty dy \int_0^2 E_{-1}(x,y)\frac{D_{1,-1}\varphi(x,y)}{B}dx+$$

$$c^{st}\int_3^\infty dy \int_0^2 E_{-1}(x,y)\frac{\varphi(x,y)D_{1,-1}B}{B^2}dx.$$

We need to calculate this derivative of B.

$$\frac{D_{1,-1}B}{B^2} = \frac{D_{1,-1}(y-x)(xy+1)}{(y-x)^2(xy+1)^2} = \frac{-(xy+1)-(xy+1)+(y-x)(y-x)}{(y-x)^2(xy+1)^2} =$$

$$\frac{-2(xy+1)+(y-x)^2}{(y-x)^2(xy+1)^2} = \frac{-2}{(xy+1)(y-x)^2} + \frac{1}{(xy+1)^2}.$$

This yields

$$II = c^{st}\int_3^\infty dy \int_0^2 E_{-1}(x,y)\frac{D_{1,-1}\varphi(x,y)}{B}dx+$$

$$c^{st}\int_3^\infty dy \int_0^2 E_{-1}(x,y)\frac{\varphi(x,y)}{(xy+1)^2}dx+$$

$$c^{st}\int_3^\infty dy\int_0^2 E_{-1}(x,y)\frac{\varphi(x,y)}{(xy+1)(y-x)^2}dx.$$

We will abbreviate: II_1 is the first integral, II_2 the second and II_3 the last one. Then for II_1 we have a look at the following integral.

$$\int_3^\infty dy\int_0^2 \frac{1}{B}dx = c^{st}\int_3^\infty dy\int_0^2 \frac{1}{(y-x)(xy+1)}dx \leq$$

$$\int_3^\infty dy\int_0^2 \frac{1}{(y-2)(xy+1)}dx = \int_3^\infty \frac{dy}{y(y-2)}\int_0^2 \frac{ydx}{xy+1} =$$

$$\int_3^\infty \frac{1}{y(y-2)}\log(xy+1)|_{x=0}^2 dy =$$

$$\int_3^\infty \frac{1}{y(y-2)}(\log(2y+1)-\log 1)dy =$$

$$\int_3^\infty \frac{\log(2y+1)}{y(y-2)}dy \leq \int_3^\infty \frac{\sqrt{2y+1}}{y(y-2)}dy \leq$$

$$c^{st}\int_3^\infty \frac{\sqrt{y}}{y(y-2)}dy = c^{st}\int_3^\infty \frac{1}{\sqrt{y}(y-2)}dy < \infty.$$

This gives

$$|II_1| \leq Mp_1(\varphi).$$

By the inequality $\frac{1}{(y-x)^2(xy+1)} \leq \frac{1}{(y-x)(xy+1)}$ on $[0,2]\times[3,\infty)$ we also have:

$$|II_3| \leq Mp_0(\varphi).$$

For II_2 we need to split up the integration area in $[0,\frac{1}{2}]\times[3,\infty)$ and $[\frac{1}{2},2]\times[3,\infty)$.

$$II_2 = II_{21} + II_{22} =$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} E_{-1}(x,y)\frac{\varphi(x,y)}{(xy+1)^2}dx + c^{st}\int_3^\infty dy\int_{\frac{1}{2}}^2 E_{-1}(x,y)\frac{\varphi(x,y)}{(xy+1)^2}dx.$$

On the first area we have an extra integration by parts. For this we use $D_{0,1}E_{-1} = c^{st}y(1-x^2)E_{-1}$.

$$II_{21} = c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} y(1-x^2)E_{-1}(x,y)\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2}dx =$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} D_{0,1}(E_{-1}(x,y))\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2}dx =$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} D_{0,1}(E_{-1}(x,y)\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2})dx+$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} E_{-1}(x,y)D_{0,1}(\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2})dx.$$

We call the first integral II_{211} and the second II_{212}. Then by the integration rule we have

$$|II_{211}| = |c^{st}\int_0^{\frac{1}{2}} E_{-1}(x,3)\frac{\varphi(x,3)}{3(1-x^2)(3x+1)^2})dx| =$$

$$|c^{st}\int_0^{\frac{1}{2}} e^{i(9-8x^2)}\frac{\varphi(x,3)}{(1-x^2)(3x+1)^2})dx| \leq Mp_0(\varphi).$$

For the integral II_{212} we first calculate the derivative:

$$D_{0,1}(\frac{1}{y(1-x^2)(xy+1)^2}) = \frac{(1-x^2)(xy+1)^2+2xy(1-x^2)(xy+1)}{y^2(1-x^2)^2(xy+1)^4} =$$

$$\frac{1}{y^2(1-x^2)(xy+1)^2}+\frac{2x}{y(1-x^2)(xy+1)^3}.$$

This yields

$$II_{212} = c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} E_{-1}(x,y)D_{0,1}(\frac{\varphi(x,y)}{y(1-x^2)(xy+1)^2})dx =$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} E_{-1}(x,y)\frac{D_{0,1}\varphi(x,y)}{y(1-x^2)(xy+1)^2}dx+$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} E_{-1}(x,y)\frac{\varphi(x,y)}{y^2(1-x^2)(xy+1)^2}dx+$$

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}} E_{-1}(x,y)\frac{x\varphi(x,y)}{y(1-x^2)(xy+1)^3}dx.$$

We abbreviate: II_{2121} is the first one, II_{2122} the second and II_{2123} the last integral. For II_{2121} we calculate the following integral:

$$c^{st}\int_3^\infty dy\int_0^{\frac{1}{2}}\frac{1}{y(1-x^2)(xy+1)^2}dx = c^{st}\int_3^\infty\frac{dy}{y^2}\int_0^{\frac{1}{2}}\frac{ydx}{(1-x^2)(xy+1)^2} \leq$$

$$c^{st}\int_3^\infty\frac{dy}{y^2}\int_0^{\frac{1}{2}}\frac{ydx}{\frac{3}{4}(xy+1)^2} = c^{st}\int_3^\infty\frac{dy}{y^2}\int_0^{\frac{1}{2}}\frac{ydx}{(xy+1)^2} =$$

$$c^{st}\int_3^\infty\frac{1}{y^2}\frac{1}{xy+1}|_0^{\frac{1}{2}}dy = c^{st}\int_3^\infty\frac{1}{y^2}(\frac{1}{\frac{1}{2}y+1}-1)dy =$$

$$c^{st}\int_3^\infty\frac{1}{\frac{1}{2}y^3+y^2}dy + c^{st}\int_3^\infty\frac{1}{y^2}dy.$$

Because these integrals are finite we have the following estimate

$$|II_{2121}| \leq Mp_1(\varphi).$$

For II_{2122} we use the inequality

$$\frac{1}{y^2(1-x^2)(xy+1)^2} \leq \frac{1}{y(1-x^2)(xy+1)^2} \text{ on } [0,\frac{1}{2}] \times [3,\infty).$$

Now the calculation of II_{2121} gives

$$|II_{2122}| \leq Mp_0(\varphi).$$

Similarly for II_{2123} we use

$$\frac{x}{y(1-x^2)(xy+1)^3} \leq \frac{1}{y(1-x^2)(xy+1)^2}$$

because $\frac{x}{1+xy} \leq 1$ on this area. This yields

$$|II_{2123}| \leq Mp_0(\varphi).$$

In all we now have

$$|II_{21}| \leq Mp_1(\varphi).$$

For II_{22} we look at

$$\int_3^\infty dy \int_{\frac{1}{2}}^2 \frac{1}{(xy+1)^2} dx = \int_3^\infty \frac{dy}{y} \int_{\frac{1}{2}}^2 \frac{y dx}{(xy+1)^2} =$$

$$\int_3^\infty \frac{1}{y} \frac{1}{(xy+1)} |_{\frac{1}{2}}^2 dy = \int_3^\infty \frac{1}{y} (\frac{1}{(2y+1)} - \frac{1}{(\frac{1}{2}y+1)}) dy =$$

$$\int_3^\infty \frac{1}{y} \frac{1}{(2y+1)} dy + c^{st} \int_3^\infty \frac{1}{y} \frac{1}{(\frac{1}{2}y+1)} dy.$$

These integrals are finite so

$$|II_{22}| \leq Mp_0(\varphi).$$

We have now proved that

$$|\int_3^\infty dy \int_0^2 E_{-1}(x,y)\varphi(x,y)dx| \leq Mp_1(\varphi).$$

By the reflections this yields

$$|\int\int_R E_{-1}(x,y)\varphi(x,y)dx| \leq Mp_1(\varphi) \tag{3.3}$$

where R is one of the areas $[0,2] \times [\pm 3,\infty)$, $[-2,0] \times [\pm 3,\infty)$, $[\pm 3,\infty) \times [0,2]$ and $[\pm 3,\infty) \times [-2,0]$.

Now the inequalities (3.2) and (3.3) and the compactness of the cross we have avoided finish the proof of *3*. ∎

3.3 Symmetric polynomials as exponents on $\mathbb{R}^m$

In this section we consider the generalized Fresnel distributions $e^{i(P+V)}$ with P a polynomial of degree at most 2 (e.g. a quadratic form) and V a special kind of polynomial potential.

Theorem 3.5 *The summability order of the distribution T on $\mathbb{R}^m$ defined by*

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^m} e^{i(P(x)+\sum_{k=1}^{m} \alpha_k x_k^{2l})} \varphi(x) dx$$

where P is a polynomial of degree at most 2 and $\alpha_k \neq 0 \, \forall k$, is smaller or equal to $\lceil \frac{m+1}{2l-1} \rceil$ for $l \in \mathbb{N}$ and $l \geq 2$.

Lemma 3.6 *Denote on $\mathbb{R}^m$: $r = \| x \|$ and $A = \mathbb{R}^m \setminus B(1)$. Then the function $\frac{1}{r}$ belongs to $\mathcal{B}(A)$.*

Proof of the lemma
To prove this lemma we will prove the following statement:

$$D^k \frac{1}{\| x \|} = \sum c_i \frac{x^{l_i}}{\| x \|^{n_i}}$$

where the sum is finite and the l_i are multi-indices with $l_i \geq 0$ and $n_i \geq 0$ and $|l_i| < n_i$ for all i.
We prove this statement by induction on $|k|$. For $k = 0$ we have $D^k \frac{1}{\|x\|} = \frac{1}{\|x\|}$. Therefore $l_1 = 0$ and $n_1 = 1$ and clearly we have $|l_1| < n_1$.
Now assume the statement is true for integers strictly smaller than k. Then for k we have: $D^k = \frac{\partial}{\partial x_j} D^{k-e_j}$ for some j. It now follows by the induction hypothesis on D^{k-e_j} that

$$D^k \frac{1}{\| x \|} = \frac{\partial}{\partial x_j} D^{k-e_j} \frac{1}{\| x \|} = \frac{\partial}{\partial x_j} \sum c_i \frac{x^{l_i}}{\| x \|^{n_i}} =$$

(where $|l_i| < n_i$ for all i)

$$\sum \tilde{c}_i \frac{x^{l_i - e_j}}{\| x \|^{n_i}} + \tilde{\tilde{c}}_i \frac{x^{l_i + e_j}}{\| x \|^{n_i + 1}}$$

where $\tilde{c}_i = 0$ if $(l_i)_j = 0$. In this sum we again check the inequality: $|l_i - e_j| = |l_i| - 1 < n_i - 1 < n_i$ and $|l_i + e_j| = |l_i| + 1 < n_i + 1$. This proves the statement for k.
Now the lemma follows from the statement:

$$\left| \frac{x^{l_i}}{\| x \|^{n_i}} \right| \leq 1 \quad \forall x \in A.$$

Therefore $\frac{1}{\|x\|} \in \mathcal{B}(A)$.

Proof of the theorem First we abbreviate $E_m = e^{i(P(x)+\sum_{k=1}^m \alpha_k x_k^{2l})}$, $D_1 = \sum_{k=1}^m \frac{\partial}{\partial x_k}$ and $l_0 = 2l-1$. The differential operator D_1 has similar properties to those of $D_{a,b}$ in proposition 2.3. Then $D_1 E_m = c^{st}(c_0 + \sum_{k=1}^m c_k x_k + 2l\alpha_k x_k^{l_0})$, where the $c_k \in \mathbb{R}$, for $k = 0, 1, \dots, m$. We write $B_m = c_0 + \sum_{k=1}^m c_k x_k + 2l\alpha_k x_k^{l_0}$.

Now we need to construct an integration area Q_m. Define $I_k = [0, \infty)$ for k with $\alpha_k > 0$ and $I_k = (-\infty, 0]$ for k with $\alpha_k < 0$, $k = 1, \dots, m$. Similarly, we define $J_k = [0, R]$ for k with $\alpha_k > 0$ and $J_k = [-R, 0]$ for k with $\alpha_k < 0$, $k = 1, \dots, m$.
Now we take $Q_m = (\prod_{k=1}^m I_k) \setminus (\prod_{k=1}^m J_k)$. To determine R we look at the conditions: $B_m(x) \geq 1$ on Q_m and $R \geq 1$. Given $c_0, \dots, c_m$ and $\alpha_1, \dots \alpha_m$, there exists an R which satisfies these conditions.

Proposition 3.7 *s.o.*$_{Q_m}(\frac{E_m}{B_m^k}) \leq \max(\lceil \frac{m+1}{l_0} \rceil - k, 0)$*, for* $k \in \mathbb{Z}_+$.

Proof of the proposition
We use induction on m to prove this proposition. For $m = 1$,

$$\frac{E_1}{B_1^k} = \frac{e^{i(P(x)+\alpha_1 x^{2l})}}{(c_0 + c_1 x + 2l\alpha_1 x^{l_0})^k}.$$

If $k = 0$, we have by theorem 3.2 s.o.$(E_1) = 1 = \max(\lceil \frac{2}{l_0} \rceil, 0)$. If $k > 0$ s.o.$_{Q_1}(\frac{E_1}{B_1^k}) =$ s.o.$_{Q_1}(\frac{E_1}{(c_0+c_1x+2l\alpha_1 x^{l_0})^k}) = 0 = \max(\lceil \frac{2}{l_0} \rceil - k, 0)$.
From this the proposition follows for the case $m = 1$.

Now we assume that the proposition holds for all dimensions strictly smaller than m.

Lemma 3.8 *s.o.*$_{Q_m}(E_m) \leq \max(\lceil \frac{m}{l_0} \rceil - 1, s.o._{Q_m}(\frac{E_m}{B_m}) + 1)$.

Proof of the lemma
For $\varphi \in \mathcal{D}(\mathbb{R}^m)$ we have

$$\int_{Q_m} E_m \varphi(x) dx = \int_{Q_m} B_m E_m \frac{\varphi(x)}{B_m} dx = c^{st} \int_{Q_m} D_1(E_m) \frac{\varphi(x)}{B_m} dx =$$

$$c^{st} \int_{Q_m} D_1(E_m \frac{\varphi(x)}{B_m}) dx + c^{st} \int_{Q_m} E_m D_1(\frac{\varphi(x)}{B_m}) dx.$$

We call the first integral I and the second II. For I we first look at the last term in the sum: $I^{(m)} = \int_{Q_m} \frac{\partial}{\partial x_m}(E_m \frac{\varphi(x)}{B_m}) dx$. We can write

$$Q_m = \prod_{k=1}^{m-1} J_k \times [R, \infty) \bigcup (\prod_{k=1}^{m-1} I_k) \setminus (\prod_{k=1}^{m-1} J_k) \times [0, \infty) =$$

$$\prod_{k=1}^{m-1} J_k \times [R, \infty) \bigcup Q_{m-1} \times [0, \infty)$$

where the union is disjunct.
Writing $\tilde{x} = (x_1, \dots, x_{m-1})$, this implies by Fubini's theorem

$$I^{(m)} = \int_{Q_m} \frac{\partial}{\partial x_m}(E_m \frac{\varphi(x)}{B_m})dx =$$

$$\int_{\prod_{k=1}^{m-1} J_k} d\tilde{x} \int_R^\infty \frac{\partial}{\partial x_m}(E_m \frac{\varphi(x)}{B_m})dx_m + \int_{Q_{m-1}} d\tilde{x} \int_0^\infty \frac{\partial}{\partial x_m}(E_m \frac{\varphi(x)}{B_m})dx_m =$$

$$c^{st} \int_{\prod_{k=1}^{m-1} J_k} (E_m \frac{\varphi(x)}{B_m})(\tilde{x}, R)d\tilde{x} + c^{st} \int_{Q_{m-1}} (E_m \frac{\varphi(x)}{B_m})(\tilde{x}, 0)d\tilde{x} =$$

$$c^{st} \int_{\prod_{k=1}^{m-1} J_k} (E_m \frac{\varphi(x)}{B_m})(\tilde{x}, R)d\tilde{x} + c^{st} \int_{Q_{m-1}} E_{m-1}(\tilde{x}) \frac{\varphi(\tilde{x}, 0)}{B_{m-1}(\tilde{x})} d\tilde{x}.$$

We abbreviate $I_1^{(m)}$ is the first integral and $I_2^{(m)}$ the second. We can estimate now:

$$|I_1^{(m)}| \leq M p_0(\varphi)$$

because the integration area here is compact and $\frac{|E_m|}{|B_m|} \leq 1$ on this area. In $I_2^{(m)}$ we apply the induction hypothesis of the proposition. This yields: the sum order of $I_2^{(m)}$ is smaller or equal to $\text{s.o.}_{Q_{m-1}}(\frac{E_{m-1}}{B_{m-1}}) \leq \max(\lceil \frac{m}{l_0} \rceil - 1, 0)$. In all we have that the summability order of $I^{(m)}$ is smaller or equal to $\max(\lceil \frac{m}{l_0} \rceil - 1, 0)$.
For the other $m-1$ terms of I $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{m-1}})$ we can do the same. In all we have that the sum order of I is smaller or equal to $\max(\lceil \frac{m}{l_0} \rceil - 1, 0)$.

The integral II can be written

$$II = c^{st} \int_{Q_m} E_m D_1(\frac{\varphi(x)}{B_m})dx =$$

$$c^{st} \int_{Q_m} E_m \frac{D_1 \varphi(x)}{B_m} dx + c^{st} \int_{Q_m} E_m \frac{\varphi(x) D_1 B_m}{B_m^2} dx.$$

The first integral has sum order smaller or equal to $\text{s.o.}_{Q_m}(\frac{E_m}{B_m}) + 1$. We write the second integral in a slightly different way:

$$c^{st} \int_{Q_m} E_m \frac{\varphi(x)}{B_m} \frac{D_1 B_m}{B_m} dx.$$

Now we consider the last fraction:

$$\frac{D_1 B_m}{B_m} = \frac{\sum_{k=1}^m c_k + 2 l l_0 x_k^{l_0 - 1}}{c_0 + \sum_{k=1}^m c_k x_k + 2 l x_k^{l_0}} \sim \frac{1}{\| x \|} \in \mathcal{B}(Q_m)$$

by lemma 3.6. Now it follows by proposition 1.3 that the sum order of the second integral of II is smaller than the sum order of the first one.
We can conclude $\text{s.o.}_{Q_m}(E_m) \leq \max(\lceil \frac{m}{l_0} \rceil - 1,\ \text{s.o.}_{Q_m}(\frac{E_m}{B_m}) + 1,\ 0) =$
$\max(\lceil \frac{m}{l_0} \rceil - 1,\ \text{s.o.}_{Q_m}(\frac{E_m}{B_m}) + 1)$.
This finishes the proof of the lemma.

Lemma 3.9 *For $k \in \mathbb{Z}_+$ such that s.o.$(\frac{E_m}{B_m^k}) > 0$, we have s.o.$_{Q_m}(\frac{E_m}{B_m^k}) \leq \max(\lceil \frac{m}{l_0} \rceil - (k+1), \text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k+1}}) + 1)$.*

Proof of the lemma
For $\varphi \in \mathcal{D}(\mathbb{R}^m)$ we have

$$\int_{Q_m} E_m \frac{\varphi(x)}{B_m^k} dx = \int_{Q_m} B_m E_m \frac{\varphi(x)}{B_m^{k+1}} dx = c^{st} \int_{Q_m} D_1(E_m) \frac{\varphi(x)}{B_m^{k+1}} dx =$$

$$c^{st} \int_{Q_m} D_1(E_m \frac{\varphi(x)}{B_m^{k+1}}) dx + c^{st} \int_{Q_m} E_m D_1(\frac{\varphi(x)}{B_m^{k+1}}) dx.$$

We abbreviate: I is the first integral and II the second. Then as in the proof of the previous lemma the last term of I (the $\frac{\partial}{\partial x_m}$-term) is

$$I^{(m)} = \int_{\prod_{k=1}^{m-1} J_k} (E_m \frac{\varphi(x)}{B_m^{k+1}})(\tilde{x}, R) d\tilde{x} + \int_{Q_{m-1}} E_{m-1}(\tilde{x}) \frac{\varphi(\tilde{x}, 0)}{B_{m-1}^{k+1}(\tilde{x})} d\tilde{x}.$$

We call the first integral $I_1^{(m)}$ and the second $I_2^{(m)}$. By arguments similar to those in the previous lemma we can estimate $|I_1^{(m)}| \leq M p_0(\varphi)$. Moreover we again apply the induction hypothesis of the proposition and get that the sum order of $I_2^{(m)}$ is smaller or equal to $\max(\lceil \frac{m}{l_0} \rceil - (k+1), 0)$. Together with the other $m-1$ terms we get that the sum order of I is smaller or equal to $\max(\lceil \frac{m}{l_0} \rceil - (k+1), 0)$.

The integral II can be written

$$II = c^{st} \int_{Q_m} E_m \frac{D_1 \varphi(x)}{B_m^{k+1}} dx + c^{st} \int_{Q_m} E_m \frac{\varphi(x) D_1 B_m}{B_m^{k+2}} dx.$$

The first integral has sum order smaller or equal to s.o.$_{Q_m}(\frac{E_m}{B_m^{k+1}}) + 1$ and the summability order of the second integral is smaller.
Now we have proved s.o.$_{Q_m}(\frac{E_m}{B_m^k}) \leq \max(\lceil \frac{m}{l_0} \rceil - (k+1), \text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k+1}}) + 1, 0) = \max(\lceil \frac{m}{l_0} \rceil - (k+1), \text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k+1}}) + 1)$.
This finishes the proof of the lemma.

We can now prove the proposition for general dimension m.
We have $B_m^k = (c_0 + \sum_{k=1}^m c_k x_k + 2l\alpha_k x_k^{l_0})^k \sim \| x \|^{l_0 k}$. Therefore, if $k\ l_0 \geq m+1$ then $\frac{1}{B_m^k} \in L^1(Q_m)$. Define $k_0 = \lceil \frac{m+1}{l_0} \rceil$. Then $k_0\ l_0 \geq m+1$ and k_0 is the smallest integer k for which $\frac{1}{B_m^k} \in L^1(Q_m)$. Therefore s.o.$_{Q_m}(\frac{E_m}{B_m^k}) = 0$ for $k \geq k_0$.

To prove the proposition for dimension m, we first write it in a different form:

$$\text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k_0-k}}) \leq \begin{cases} 0 & k < 0 \\ k & 0 \leq k \leq k_0. \end{cases} \tag{3.4}$$

We prove (3.4) by induction on k. For $k = 0$ we have already proved this above. Assume that for integers strictly smaller than k with $0 < k \leq k_0$ (3.4) holds. Then we have for k, using the second lemma,

$$\begin{aligned} \text{s.o.}_{Q_m}\left(\frac{E_m}{B_m^{k_0-k}}\right) &\leq \max(\lceil \tfrac{m}{l_0} \rceil - (k_0 - k + 1),\ \text{s.o.}_{Q_m}(\tfrac{E_m}{B_m^{k_0-k+1}}) + 1) \\ &\leq \max(\lceil \tfrac{m}{l_0} \rceil - (k_0 - k + 1),\ k - 1 + 1) \\ &= \max(\lceil \tfrac{m}{l_0} \rceil - (\lceil \tfrac{m+1}{l_0} \rceil - k + 1),\ k) \\ &= \max(\lceil \tfrac{m}{l_0} \rceil - \lceil \tfrac{m+1}{l_0} \rceil + k - 1,\ k) \\ &= k. \end{aligned}$$

For $k < 0$ we have that $|\frac{1}{B_m^{k_0-k}}| \leq |\frac{1}{B_m^{k_0}}| \in L^1(Q_m)$, so $\text{s.o.}_{Q_m}(\frac{E_m}{B_m^{k_0-k}}) = 0$. This proves (3.4) for all k and therefore the proposition.

Taking $k = 0$ in the proposition yields $\text{s.o.}(E_m) \leq \lceil \frac{m+1}{l_0} \rceil$ on Q_m. For the other quadrants we use the reflections $x_k \mapsto -x_k$. These reflections cause minus signs in the polynomial P, but because the proposition holds for any polynomial P of degree at most 2, the sum order remains the same. This yields the theorem:

$$\text{s.o.}(E_m) \leq \lceil \frac{m+1}{l_0} \rceil \tag{3.5}$$

by compactness of $[-R, R]^m$. ∎

Remark 1 The result of this theorem is not a precise equality yet. To prove the equality in (3.5) one needs to construct a sequence of test functions as we did before. By comparing this result with the other results in this article one would expect the equality in (3.5).

Remark 2 The case of Fresnel distributions with a potential of this form is a special case of this theorem. We see that the presence of the quadratic form makes no difference in the sum order. The part of the exponent with the highest order is dominating. This means that also degenerate quadratic forms with such a potential have this sum order.

Conclusions

In section 3.1 we have stated the most general result for the one dimensional case in theorem 3.2. We know for which polynomials P on $\mathbb{R}$ the distribution e^{iP} is summable and if so, we know the sum order. Moreover we know the sum order of $x^k e^{iP}$. This means that the one dimensional case is finished now.

For higher dimensions we now know the sum order of the Fresnel distributions. The sum order of e^{iQ} on $\mathbb{R}^m$ is exactly $m+1$ for Q a non-degenerate quadratic form on $\mathbb{R}^m$. For more general polynomials we have considered $e^{i(P+V)}$ with P a polynomial of degree at most 2 (for example a quadratic form) and V a potential of the form $\sum \alpha_k x_k^{2l}$. These distributions are summable. Moreover, we have seen: the higher the degree of V the lower the sum order. This means that adding such a potential reduces the sum order. Furthermore, after addition the sum order does not depend on the quadratic form anymore. The potential is dominating.

For the case of two dimensions we have considered $P = x^2 + y^2 + kx^2y^2$ as a special case. We have seen that the sum order is equal to 1 for $k \neq 0$. For $k = 0$ the polynomial is a non-degenerate quadratic form and has therefore sum order 3. So, also in this special case we see: a polynomial of higher degree causes a lower sum order.

The sum order of Fresnel distributions on $\mathbb{R}^m$ diverges for $m \to \infty$. Also when we add a potential of the given form the sum order goes to ∞. A precise conclusion from this about the Feynmann-integral can be found in [TH2, Thm 3.1].

It is clear that we have not given a full answer to the question: for which polynomials P is e^{iP} summable? We know that for a 'degenerate' polynomial P (i.e. P can be written as polynomial in less coordinates by a linear transformation), e^{iP} is not summable, because its Fourier-transform is not continuous in that case. Conversely, we can ask: if P is at least quadratic in all coordinates and P is not degenerate, is e^{iP} automatically summable then? A precise answer to this question has not been given yet.

Finally, to get more information about the Feynmann integral one should try an other kind of potentials. For example $V = \sum e^{|x_k|}$. Maybe one can get the sum order independent of the dimension by using other potentials. Then the sum order remains bounded for $m \to \infty$, which is important for taking the limit.

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