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The Schwarz-Christoffel transformation and elliptic functions

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Summary

An important result from complex analysis, the Riemann mapping theorem, states that there exists a conformal bijective mapping $f : A \to B$ between any two simply connected open sets $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$, both not equal to the whole complex-plane $\mathbb{C}$. In the case where the upper half-plane is conformally mapped onto an open set which is the inside of a simple polygon, the mapping has the form of a Schwarz-Christoffel transformation (SCT).
The SCT will be discussed in detail, and will be used to define Jacobi elliptic functions. Jacobi elliptic functions form a special set of elliptic functions in general. Elliptic functions are doubly periodic meromorphic functions. Basic properties of elliptic functions will be discussed.

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Chapter 1

Introduction

Elliptic functions are doubly periodic meromorphic functions. The Schwarz-Christoffel transformation describes an expression in integral form, to map the upper half-plane conformally to a domain in the shape of a bounded or unbounded polygon in the complex plane. Anthony Osbourne describes a way to define certain elliptic functions with use of the Schwarz-Christoffel transformation, these elliptic functions are called jacobi elliptic function [6]. This paper takes a closer look on this process and concepts.

Chapter 2

The Schwarz-Christoffel transformation

2.1 Useful Tools

The following results will be used throughout the text, and it will be assumed the reader is familiar with them.

Lemma 2.1.1 (Schwarz Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function from the open unit disc into itself such that $f(0) = 0$. Then:*

- $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.
- *If* $|f(z_0)| = |z_0|$ *for some* $z_0 \in \mathbb{D}, z_0 \neq 0$, *then there exists some* $\varphi \in \mathbb{R}$ *such that:*

$$f(z) = e^{i\varphi} z.$$

Theorem 2.1.2 (Schwarz reflection principle). *Let U^+ be an open set in the upper half-plane, where its boundary contains an open interval I of the real numbers. Let U^- be the reflection of U^+ along the real axis.*

If $f(z)$ is a function on $U^+ \cup I$, analytic on U^+, and continuous and real valued on I, then $f(z)$ has a unique analytic continuation $F(z)$ defined on $U^- \cup U^+ \cup I$ which satisfies $F(z) = \overline{f(\bar{z})}$.

A direct consequence of Theorem 2.1.2 is given below.

Corollary 2.1.3 (Horizontal Schwarz reflection). *Let V^+ be an open subset of $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, where its boundary contains an open interval J of purely imaginary numbers. Let V^- be the reflection of V^+ along the imaginary axis.*

If g is a function on $V^+ \cup J$, analytic on V^+, and continuous and real valued on J, then $g(z)$ has a unique analytic continuation $G(z)$ defined on $V^+ \cup V^- \cup J$ which satisfies $G(z) = \overline{g(-\bar{z})}$.

Proofs of the above results can be found, for instance, in [3] or [5]. The following results concerning logarithms will be useful.

Lemma 2.1.4. *Let $\varphi(z)$ be analytic and non-zero in the simply connected open set R. Then a single-valued and analytic branch of $\log \varphi(z)$ can be defined in R.*

Proof. It suffices to shows the existence of function $g(z)$, analytic in R such that

$$e^{g(z)} = \varphi(z). \tag{2.1.1}$$

Since $\varphi(z) \neq 0$ in open set R, φ'/φ is analytic in R. Define h such that

$$h'(z) = \frac{\varphi'(z)}{\varphi(z)}.$$

For $z_0 \in R$, select $\log \varphi(z_0)$ arbitrary. Define $g(z)$ as

$$g(z) = h(z) - h(z_0) + \log \varphi(z_0),$$

which is analytic in R. Define $d(z)$ as

$$d(z) = e^{-g(z)}\varphi(z),$$

which has $d(z_0) = 1$. Since

$$d'(z) = \frac{\varphi'(z)}{\varphi(z)}e^{-g(z)}\varphi(z) + \varphi'(z)e^{-g(z)},$$

$d(z) = 1$, and indeed $g(z)$ satisfies (2.1.1). □

2.2 Conformal transformations

A mapping f from a set D in the complex plane to a set D' in the complex plane will be denoted by:

$$w = f(z), \qquad z \in D;$$

i.e. f takes a point from D in the z-plane into a unique point w in the w-plane.

The inverse function theorem states that if f is analytic and $f'(a) \neq 0$, then there exists an open neighbourhood U of a and an open neighbourhood V of $f(a)$ such that f is one-on-one from U onto V (and $f'(z) \neq 0$ for all $z \in U$).

Definition 2.2.1 (Conformal transformations)**.** Let f be analytic on a set D. Then f is called a conformal mapping on D if $f'(z) \neq 0$ for all $z \in D$.

Lemma 2.2.2. *Let f be a conformal mapping on an open connected set D. Let $\gamma(t)$ and $\eta(t)$ be two differentiable curves mapping inside D, with:*

$$z_0 = \gamma(t_0) = \eta(t_1).$$

Then

$$\arg(\gamma'(t_0)) - \arg(\eta'(t_1)) = \arg((f \circ \gamma)'(t_0)) - \arg((f \circ \eta)'(t_1)).$$

Proof. Note that $(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$ and $(f \circ \eta)'(t_1) = f'(z_0)\eta'(t_1)$. Hence

$$\arg(\gamma'(t_0)) - \arg(\eta'(t_1)) = \arg(f'(z_0)(\gamma'(t_0)) - \arg(f'(z_0)\eta'(t_1)).$$

which holds since $f'(z_0) \neq 0$. □

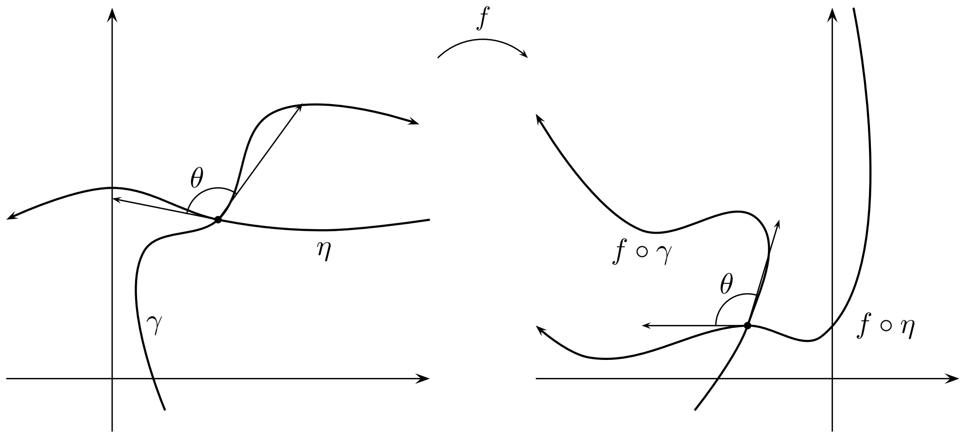

Figure 2.1: A conformal transformation f is angle-preserving

This lemma has an important geometric interpretation. The angle θ between the tangent vectors of two curves at an intersection point z_0 remains the same after conformal transformation f, as shown in Fig. 2.1. Let A and B be open sets in $\mathbb{C}$. An analytic function $f : A \to B$ is called an analytic isomorphism if f maps A one-to-one onto B and there exists an analytic inverse f^{-1} on B. If $A = B$ then f is called an automorphism.

Lemma 2.2.3. *With $\mathbb{D}$ denoted as the open unit disc, let $f : \mathbb{D} \to \mathbb{D}$ be an analytic automorphism. Let $p \in \mathbb{D}$ with $f(p) = 0$. Then f must be of the form:*

$$f(z) = e^{i\varphi}\frac{p-z}{1-\bar{p}z}, \qquad \varphi \in \mathbb{R}.$$

Proof. Consider $g(z) = \frac{p-z}{1-\bar{p}z}$, which is analytic inside the closed unit disc since $|p| < 1$. Now observe

$$g(e^{i\theta}) = \frac{p - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \bar{p})} = \frac{p - e^{i\theta}}{-e^{i\theta}\overline{(p - e^{i\theta})}}, \qquad \theta \in \mathbb{R}.$$

Hence if $|z| = 1$ then $|g(z)| = 1$, which shows 1 is the maximum of $|g(z)|$ in $\overline{\mathbb{D}}$ by the maximum modules principle.

Therefore $|g(z)| \leq 1$ for $|z| \leq 1$. Calculations show that $g^{-1} = g$, i.e. g is the inverse of itself, and therefore is an analytic automorphism.

Consider $h = f \circ g$. Note that $h(0) = 0$ and that h is an analytic automorphism since f and g are. Therefore by the Schwarz lemma $|h(z)| \leq |z|$ if $|z| < 1$. Also note that $h^{-1} = g \circ f^{-1}$ is also an analytic automorphism with $h^{-1}(0) = 0$, hence $|z| \leq |h(z)|$. Hence there exists a $z_0 \in \mathbb{D}, z_0 \neq 0$ with $|h(z_0)| = |z_0|$. By the Schwarz lemma, h is of the form $h(z) = e^{i\varphi}z$ for some $\varphi \in \mathbb{R}$ which proofs the lemma. ∎

Given two open connected sets F and G, the question may arise if there exists a conformal transformation which maps F onto G.

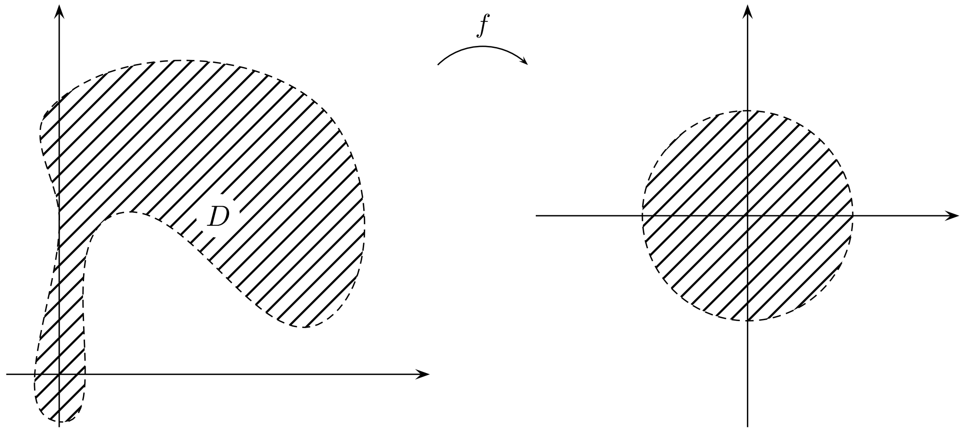

Figure 2.2: By the Riemann mapping theorem there exists a conformal mapping from D onto the inside of the unit disc.

Theorem 2.2.4 (Riemann mapping theorem). *Let R be a simply connected open set which is not the whole complex plane and let $z_0 \in R$. Then there exists a unique injective conformal mapping $f : R \to \mathbb{D}$, mapping R onto $\mathbb{D}$, the open unit disc, such that*

$$f(z_0) = 0, \qquad \arg(f'(z_0)) = 0.$$

In particular, if D and D' are simply connected open sets both not equal to the complex-plane, then there exists an injective conformal mapping from D onto D'. A proof of the Riemann mapping theorem can be found in [3] or [5].

Closely related to the Riemann mapping theorem is the Osgood-Carathéodory theorem, wherefore the notion of a Jordan curve and a Jordan region will be introduced.

A Jordan curve is a closed curve, which has no multiple points. Hence, if $\gamma : [a, b] \to \mathbb{C}$ is a Jordan curve then for $t_0 < t_1$, $\gamma(t_0) = \gamma(t_1)$ if and only if $t_0 = a$ and $t_1 = b$. A region is a Jordan region if it is the interior of a Jordan curve.

Theorem 2.2.5 (Osgood-Carathéodory theorem). *Let D and E be two Jordan regions. Any function f mapping D conformal and one-to-one onto E can be extended to a injective continuous map of the closure of D onto the closure of E.*

The above results will be useful in presenting the Schwarz-Christoffel transformation. In the following section, two examples of conformal mappings will be discussed in detail.

2.3 Two examples of conformal mappings

Example 2.3.1 (The triangle). Define the complex function $f(z)$ by

$$f(z) = \int_0^z u^{\alpha-1}(u-1)^{\beta-1}\, du, \qquad \alpha, \beta \in \mathbb{R}.$$

It will be assumed that $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. As a complex-valued function the integrand is in general multi-valued, since it is the product of the multi-valued functions $u^{\alpha-1}$ and $(u-1)^{\beta-1}$ with branch-points 0 and 1 respectively. Both factors can be interpreted as single-valued functions by means of branch-cuts as shown in Fig. 2.3. Note that with

Figure 2.3: Branch-cuts $\{-|a|i\}$ and $\{1-|a|i\}$, for $a \in \mathbb{R}$.

the indicated choice of branch-cuts, the integrand is single-valued and analytic in the upper half-plane.

Since $f'(z) = z^{\alpha-1}(z-1)^{\beta-1}$, the function has a non-zero derivative for z not equal to 0 or 1, and therefore is conformal in the upper half-plane.

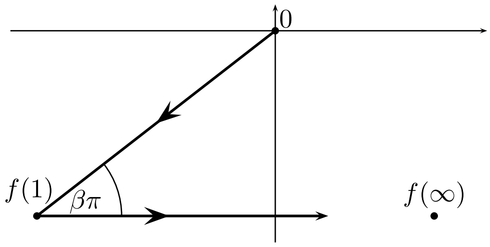

Figure 2.4: The image of $f(x)$ as $x \to \infty$.

Note that $f(0) = 0$. Consider the behaviour for $x \in \mathbb{R}$:

- For $0 < x < 1$, $f(x)$ can be rewritten as

$$e^{i\pi(\beta-1)} \int_0^x u^{\alpha-1}(1-u)^{\beta-1}\, du.$$

  The integrand is real, positive and finite. Therefore the integral is real and strictly increasing. Hence the image $f(x)$ will travel from 0 in a straight line to the finite point $f(1)$ in the complex plane, as $x \to 1$. $e^{i\pi(\beta-1)}$ indicates the direction, as shown in Fig. 2.4.

- For $x > 1$ the integral can be rewritten as

$$\begin{aligned} f(x) &= f(1) + \int_1^x u^{\alpha-1}(u-1)^{\beta-1}\, du \qquad (u = 1/t) \\ &= f(1) + \int_{1/x}^1 t^{\beta-1}(1-t)^{1-\alpha-\beta-1}\, dt. \end{aligned}$$

  The last integral is real and increases as $x \to 0$. Note that $f(\infty)$ is a finite point in the complex plane (since $\alpha + \beta < 1$). Hence, the image $f(x)$ will travel from $f(1)$ to $f(\infty)$ along a straight horizontal line as x increases from 1 to ∞, as shown in Fig. 2.4.

- For $x < 0$ the integral can be rewritten as

$$\begin{aligned} f(x) = & -e^{i\pi(\alpha+\beta-1)} \int_0^x (-u)^{\alpha-1}(1-u)^{\beta-1}\,du \qquad (u = 1-1/t) \\ &= e^{i\pi(\alpha+\beta-1)} \int_{\frac{1}{1-x}}^1 t^{\alpha-1}(1-t)^{1-\alpha-\beta-1}\,dt. \end{aligned}$$

The last integral is real and increases as $x \to -\infty$. Since $f(0) = 0$, the image $f(x)$ travels along a straight line from 0 to $f(-\infty)$ as x decreases from 0 to $-\infty$. $e^{i\pi(\alpha+\beta-1)}$ indicates the direction of this straight line.

With the assumptions $\alpha, \beta > 0$ and $\alpha + \beta < 1$, the location of the limits $f(\infty)$ and $f(-\infty)$ can be expressed in terms of the Beta-function. For real $\zeta > 0$, $\eta > 0$ the Beta-function $B(\zeta, \eta)$ is defined as:

$$B(\zeta,\eta) = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1}dt.$$

Recall that $B(\zeta, \eta) = B(\eta, \zeta)$ and that, with Γ as the Gamma-function:

$$B(\zeta,\eta) = \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)},\ \zeta > 0,\ \eta > 0, \qquad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)},\ 0 < a < 1.$$

Now observe

- Clearly $f(1) = e^{i\pi(\beta-1)}B(\alpha, \beta)$. $f(1)$ is located in the lower half-plane since $\beta \in (0, 1)$.
- With $\gamma = 1 - \alpha - \beta$, the expression for $f(\infty)$ can be rewritten as:

$$\begin{aligned} f(\infty) &= e^{i\pi(\beta-1)}B(\alpha,\beta) + B(\gamma,\beta) \\ &= \frac{1}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)[\sin(\pi\alpha) + e^{i\pi(\beta-1)}\sin(\pi\gamma)]. \end{aligned}$$

  The last factor $\sin(\pi\alpha) + e^{i\pi(\beta-1)}\sin(\pi\gamma)$ is equal to

$$\begin{aligned} &\sin\pi\alpha + \cos\pi(\beta-1)\sin\pi\gamma + i\sin\pi(\beta-1)\sin\pi\gamma \\ &= \sin\pi(1-\beta)\cos\pi\gamma - \cos\pi(1-\beta)\sin\pi\gamma \\ &\quad + \cos\pi(\beta-1)\sin\pi\gamma + i\sin\pi(\beta-1)\sin(\pi\gamma) \\ &= \sin\pi(1-\beta)\cos\pi\gamma + i\sin\pi(\beta-1)\sin(\pi\gamma) \\ &= \sin(\pi\beta)e^{-i\gamma\pi}. \end{aligned}$$

  Hence $f(\infty) = \sin(\pi\beta)e^{-i\gamma\pi}\frac{1}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$.

- Finally, $f(-\infty) = e^{i\pi(\alpha+\beta-1)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}\,dt$, which can be rewritten as:

$$e^{-i\gamma\pi}B(\alpha,\gamma) = \sin(\pi\beta)e^{-i\gamma\pi}\frac{1}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma).$$

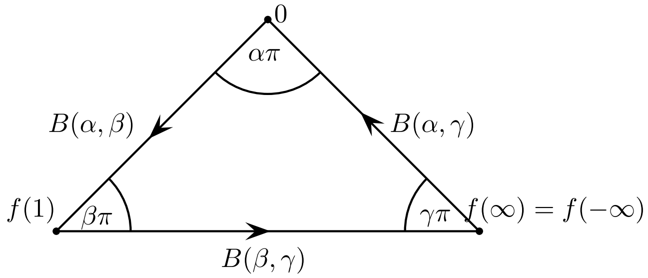

Figure 2.5: f maps the real line to a triangle, with vertices at 0, $f(1)$, and $f(\infty)$.

This shows $f(-\infty) = f(\infty)$, hence f maps the real line to a triangle, as shown in Fig. 2.5.

Note that the triangle has sides $B(\beta,\gamma)$, $B(\alpha,\gamma)$, and $B(\alpha,\beta)$, with corresponding opposite angles $\alpha\pi$, $\beta\pi$, and $\gamma\pi$, respectively. It follows from the formulas

$$\begin{aligned} B(\beta,\gamma) &= \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin(\pi\alpha)}{\pi}, \\ B(\alpha,\gamma) &= \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin(\pi\beta)}{\pi}, \\ B(\alpha,\beta) &= \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin(\pi\gamma)}{\pi}, \end{aligned}$$

that

$$\frac{\sin\alpha\pi}{B(\beta,\gamma)} = \frac{\sin\beta\pi}{B(\alpha,\gamma)} = \frac{\sin\gamma\pi}{B(\alpha,\beta)}.$$

In other words, these identities stand for the so-called sine-rule for the triangle.

Points in the upper half-plane are mapped inside the triangle. Numerical approximations show that horizontal lines in the upper half-plane which approach the real line are mapped to expanding balloon-shaped curves inside the triangle as indicated in Fig. 2.6.

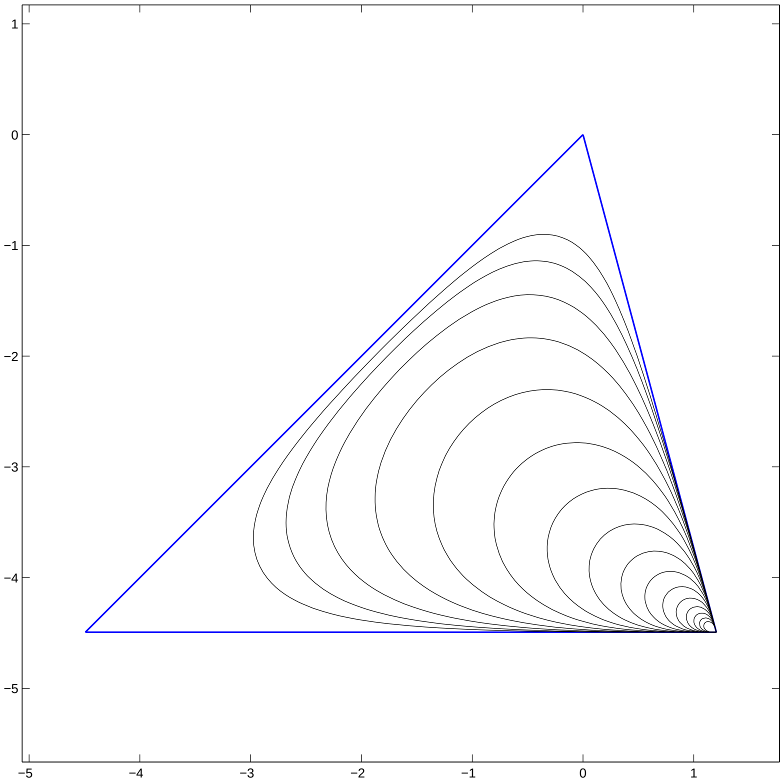

Figure 2.6: Plot of f where $\alpha = 1/3$ and $\beta = 1/4$. Horizontal lines h_n with Im $h_n = 2^n$ have been plotted for $n = -5, -4, -3, \cdots, 8, 9, 10$, where the smallest balloon-shaped curve is the image of the horizontal line h_{10}. The plot has been made with the Schwarz-Christoffel toolbox for MATLAB.

Figure 2.7: Branch-cuts for $f(z)$

Example 2.3.2 (The rectangle)**.** For $\rho > 1$, define the complex function $f(z)$ by:

$$f(z) = \int_0^z t^{(-1/2)}(t-1)^{(-1/2)}(t-\rho)^{(-1/2)}\, dt.$$

The integral has similarities with the integral from the previous example. The integrand is multi-valued, but with branch-cuts as in Fig. 2.7, f is single-valued. The function f is conformal in the upper half-plane and conformal on the real line for $z \neq 0, 1, \rho$.

Note that $f(0) = 0$, and that for real z the square roots are either positive or purely imaginary with a positive imaginary part. Consider the behaviour for $x \in \mathbb{R}$:

- For $0 < x < 1$, there is one real, and two imaginary square roots. Therefore $f(z)$ travels from 0 to point $-K$ on the negative real axis as x increases from 0 to 1, with

$$\int_0^1 \frac{dt}{\sqrt{t(1-t)(\rho-t)}} = K.$$

- For $1 < x < \rho$, there is only one imaginary square root. Hence the image travels from $-K$ to $-K - iK'$, as x increases from 1 to ρ, with

$$f(\rho) = -K - iK' = f(1) - i\int_1^\rho \frac{dt}{\sqrt{t(t-1)(\rho-t)}}.$$

- For $x > \rho$, the integrand is positive, and $f(x)$ will travel from $-K - iK'$ into positive horizontal direction, with length K. This can be seen from:

$$\begin{aligned} f(\infty) - f(\rho) &= \int_\rho^\infty \frac{dt}{\sqrt{t(t-1)(t-\rho)}} \qquad \left(t = \frac{\rho - u}{1-u}\right) \\ &= \int_0^1 \frac{du}{\sqrt{u(1-u)(\rho-u)}} = K. \end{aligned}$$

- For $x < 0$, there are three imaginary square roots, and $f(x)$ will therefore be purely imaginary with a negative imaginary part.

$$\begin{aligned} f(-\infty) &= \int_{-\infty}^0 \frac{dt}{\sqrt{t(t-1)(t-\rho)}} \qquad \left(t = \frac{\rho - u}{1-u}\right) \\ &= -i\int_1^\rho \frac{du}{\sqrt{u(u-1)(\rho-u)}} = -iK'. \end{aligned}$$

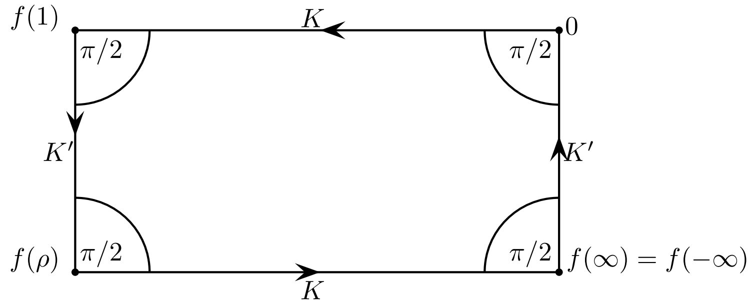

Figure 2.8: f maps the real line to a rectangle, with vertices at $0, f(1), f(\rho)$, and $f(\infty)$.

Hence the function f maps the real line to a rectangle as shown in Fig. 2.8. Points in the upper half-plane are mapped inside the rectangle. Numerical approximations show that horizontal lines in the upper half-plane which approach the real line are mapped to expanding balloon-shaped curves inside the rectangle as indicated in Fig. 2.9.

2.4 Improper path integrals

Let $h(t)$ be analytic inside a disc around 0, denoted by D. Let $t^{\alpha-1}, \alpha \in \mathbb{R}^+$ be single-valued via branch-cut c for the branch-point 0. Then for $z_0 \in D\backslash\{\{0\} \cup c\}$, the integral

$$i(z) = \int_{z_0}^{z} t^{\alpha-1}h(t)\, dt$$

is well-defined and analytic as a function of z in $D\backslash\{\{0\}\cup c\}$. Moreover, the improper Riemann integral

$$j(z) = \lim_{z_0 \to 0} \int_{z_0}^{z} t^{\alpha-1}h(t)dt$$

is well-defined and analytic as a function of z in $D\backslash\{\{0\} \cup c\}$, and continuous at $z = 0$.

Proof. $i(z)$ is analytic in the simply connected open set $D\backslash\{\{0\} \cup c\}$. since the integrand is analytic in $D\backslash\{\{0\} \cup c\}$.

Since $h(t)$ is analytic inside the disc D, it can be written as a power series around 0, which converges for all t inside D.

$$h(t) = \sum_{n=0}^{\infty} a_n t^n.$$

It remains to prove that

$$j(z) = \lim_{z_0 \to 0} \int_{z_0}^{z} t^{\alpha-1}h(t)dt$$

is well-defined and analytic in $D\backslash\{\{0\} \cup c\}$. Define

$$G(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{a_n z^n}{n+\alpha},$$

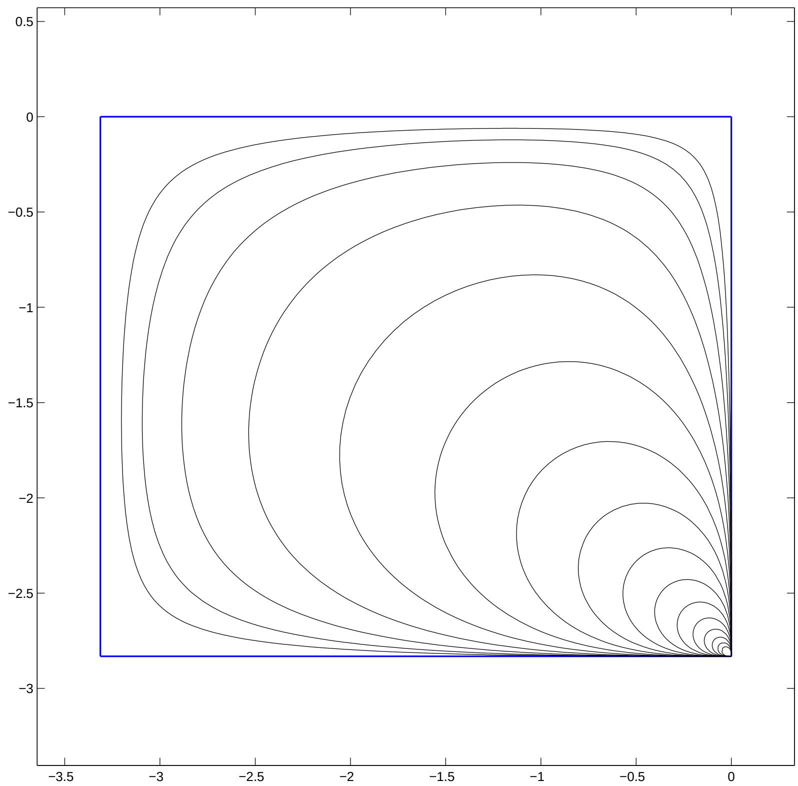

Figure 2.9: Plot of f, with $\rho = 1.5$. Horizontal lines h_n with Im $h_n = 2^n$ have been plotted for $n = -5, -4, -3, \cdots, 8, 9, 10$, where the smallest balloon-shaped curve is the image of the horizontal line h_{10}. The plot has been made with the Schwarz-Christoffel toolbox for MATLAB.

which is an anti-derivative of $z^{\alpha-1}h(z)$ in $D\backslash\{\{0\}\cup c\}$ since

$$\begin{aligned} j(z) = \int_{z_0}^{z} t^{\alpha-1}h(t)\, dt &= \int_{z_0}^{z} t^{\alpha-1}\sum_{n=0}^{\infty} a_n t^n\, dt \\ &= \sum_{n=0}^{\infty}\int_{z_0}^{z} t^{\alpha-1}a_n t^n\, dt \\ &= \sum_{n=0}^{\infty}\frac{a_n z^{n+\alpha}}{n+\alpha} \\ &= z^{\alpha}\sum_{n=0}^{\infty}\frac{a_n z^n}{n+\alpha}. \end{aligned}$$

Note that the power series in the definition of $G(z)$ has the same radius of convergence as $h(z)$. Therefore $G(z)$ is analytic in $D\backslash\{\{0\}\cup c\}$.

$$\begin{aligned} j(z) &= \lim_{z_0\to 0}\int_{z_0}^{z} t^{\alpha-1}h(t)dt \\ &= \lim_{z_0\to 0}(G(z)-G(z_0)) \\ &= G(z), \end{aligned}$$

where the last equality holds since

$$\lim_{z_0\to 0} z_0^{\alpha}\frac{a_n z_0^n}{n+\alpha} = 0.$$

Hence $G(z)$ is analytic in $D\backslash\{\{0\}\cup c\}$ and continuous for $z = 0$, which proves the last assertion. □

If $h(t)$ is analytic in the upper half-plane and z is a point outside the disc of convergence around 0, then there exists a w inside $D\backslash\{\{0\}\cup c\}$ such that

$$\lim_{z_0\to 0}\int_{z_0}^{z} t^{\alpha-1}h(t)\, dt = \lim_{z_0\to 0}\int_{z_0}^{w} t^{\alpha-1}h(t)\, dt + \int_{w}^{z} t^{\alpha-1}h(t)\, dt.$$

2.5 The main principle of the Schwarz-Christoffel transformation

By the Riemann mapping theorem, for every two simply open connected sets D and E both not equal to the whole complex plane, there exists a conformal mapping from D onto E. Only in certain cases the explicit form of the mapping is known.

In the case where the upper half-plane needs to be mapped conformally to the inside of a polygon shaped region, the Schwarz-Christoffel transformation (SCT) is used. The SCT gives an expression in integral form for the relevant conformal map. The two previous examples, the triangle and the rectangle map, are both examples of Schwarz-Christoffel transformations.

The SCT is constructed by creating a map f which has piecewise constant $\arg(f')$ on the real line, since then, by the following lemma, f maps the real line to straight lines.

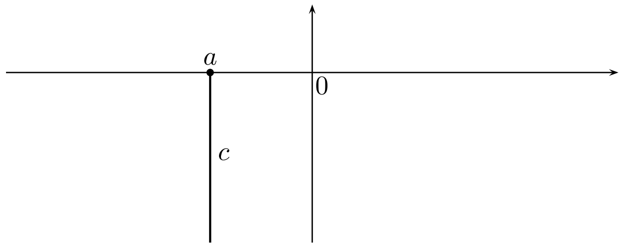

Figure 2.10: Branch-cut $c = \{z \in \mathbb{C} : z = a - |\rho|i,\ \rho \in \mathbb{R}\}$

Lemma 2.5.1 (function f with constant $\arg(f')$)**.** *Let f be analytic on the real line and let $x_0 \in \mathbb{R}$. If $f'(x) \neq 0$ for $x \in \mathbb{R}$ and $\arg(f'(x)) =$ constant for $x \in \mathbb{R}$, then f maps the real line to a straight line through $f(x_0)$ with slope $\arg(f'(x))$.*

Proof. The proof is geometric. Interpret $f'(x_0)$ as the tangent vector at $f(x_0)$ on the curve $\{f(x) : x \in \mathbb{R}\}$. Since the tangent vector is not zero and remains constant along the curve, the curve must be a straight line. □

2.6 An illustrative example

Let $\alpha \in \mathbb{R},\ -1 < \alpha < 1$. Define function $f(z)$ by:

$$f(z) = (z - a)^\alpha. \tag{2.6.1}$$

The function $f(z)$ is in general multi-valued with a branch-point at a. Define the branch-cut c as the straight line parallel to the imaginary axis, starting in point a going downwards to infinity (Fig. 2.10). With branch-cut c, $f(z)$ is single-valued:

$$f(z) = (z - a)^\alpha = |z - a|^\alpha e^{i\alpha\theta} \text{ where } -\pi/2 < \theta < 3\pi/2. \tag{2.6.2}$$

When referring to $f(z)$, the single-valued definition in (2.6.2) is used. f is analytic for all $z \neq a$. $\arg(f'(z))$ is piecewise-constant for $z \in \mathbb{R}$:

$$\arg(f'(z)) = \begin{cases} 0 & \text{for} \quad \{z \in \mathbb{R} :\ z > a\} \\ \alpha & \text{for} \quad \{z \in \mathbb{R} :\ z < a\} \end{cases} \tag{2.6.3}$$

f will map $(-\infty, a) \subset \mathbb{R}$ to a halfline with slope $\alpha\pi$ to the real axis, by Lemma 2.5.1. The same arguments show that f maps $(a, \infty) \subset \mathbb{R}$ to a halfline parallel to the real axis. Moreover, the two halflines meet in $f(a)$, and $f(z)$ maps the upper half-plane to the region shown in Fig. 2.11. This can be shown by varying θ and $|z - a|$ in $f(z) = |z - a|^\alpha e^{i\theta\alpha}$:

- For $\theta = 0$; $f(z)$ can be rewritten to $f(z) = |z - a|^\alpha$ where $|z - a|$ takes values in $\{r \in \mathbb{R} :\ r > 0\}$. This shows the set $(a, \infty) \subset \mathbb{R}$ will be mapped to $\{\omega \in \mathbb{R} :\ \omega > 0\}$. In Fig. 2.11 this is shown where L_2 is mapped to L_2'
- For $\theta = \pi$; $f(z)$ can be rewritten top $f(z) = |z - a|^\alpha e^{i\theta\alpha}$ where $|z - a|$ takes values in $\{r \in \mathbb{R} :\ r > 0\}$. This shows the set $(-\infty, a) \subset \mathbb{R}$ will be mapped to $\{re^{i\alpha} : r \in \mathbb{R} : r > 0\}$. In Fig. 2.11 this is shown where L_1 is mapped to L_1'.

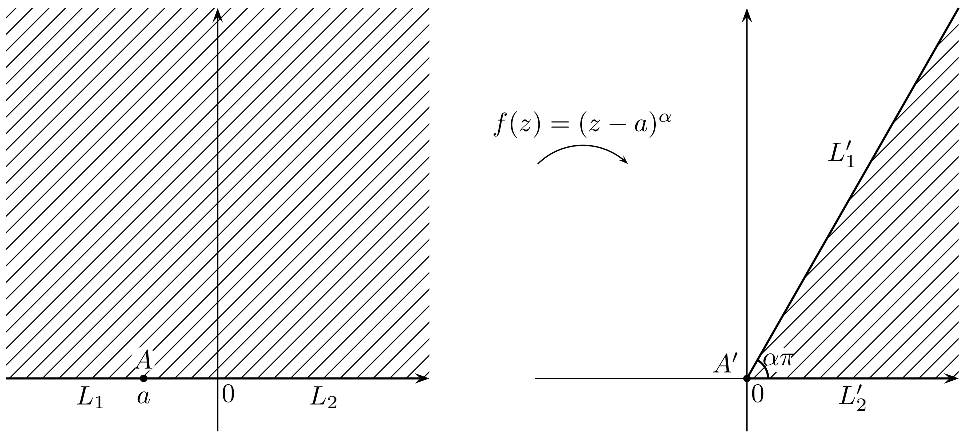

Figure 2.11: $f(z) = (z-a)^{\varphi/\pi}$

- For $0 < \theta < \pi$; $f(z) = |z-a|^\alpha e^{i\theta\alpha}$, where $|z-a|$ takes values in $\{r \in \mathbb{R} : \; r > 0\}$ and $0 < \theta < \pi$. In Fig. 2.11 this is shown where the lined upper half gets mapped to the lined region.

For $h(z) = f'(z)$, $\arg(h(z))$ for $z \in \mathbb{R}$ is a step-function, with an image set consisting of two values. This process can be turned around; integration of a function $g(z)$, where $\arg(g(z))$ is a step function for $z \in \mathbb{R}$, will lead to a function to which Lemma 2.5.1 can be applied.

A product of n functions of the form $(z-a)^{\alpha-1}$ will define a function whose argument is a step-function (for the real domain) with an image set of n values. Integration will lead to a function f with piecewise constant $\arg(f'(z))$ for $z \in \mathbb{R}$. This will lead to the SCT.

2.7 The Schwarz Christoffel transformation

Definition 2.7.1. Let $\alpha_r \in \mathbb{R}, a_r \in \mathbb{R}$ with $a_{r-1} < a_r$, for $r = 1, 2, \ldots, n$. Define

$$g(z) = \prod_{r=1}^{n} (z - a_r)^{\alpha_r}. \tag{2.7.1}$$

In general, g defines a multi-valued function with branch-points in $\{a_r\}$. Note that g can be made single-valued by writing g as the product of single-valued factors $(z-a_r)^{\alpha_r}$:

$$g(z) = \prod_{r=1}^{n} (z - a_r)^{\alpha_r} = \prod_{r=1}^{n} h_r(z), \tag{2.7.2}$$

where each $h_r(z)$ is interpreted as

$$h_r(z) = (z-a_r)^{\alpha_r} = |z-a_r|^{\alpha_r} e^{i\alpha_r\theta} \text{ where } -\pi/2 < \theta < 3\pi/2,$$

with branch-cuts $\{c_r\}$ as shown in Fig. 2.12.

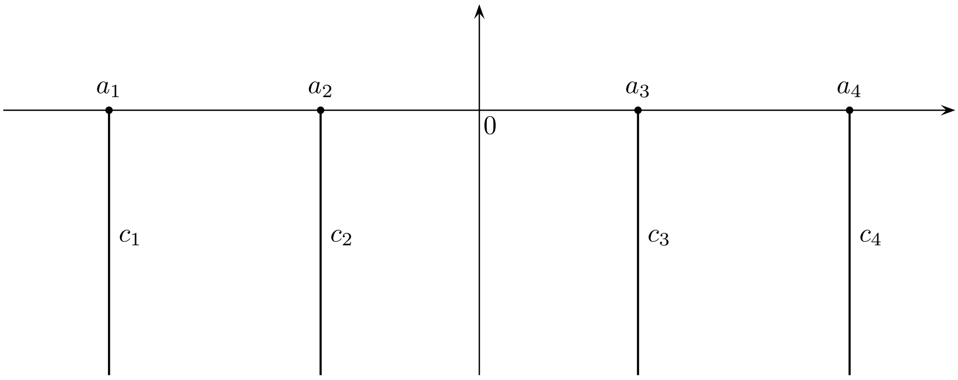

Figure 2.12: Branch-cuts $c_r = \{z \in \mathbb{C} : z = a_r - |p|i, p \in \mathbb{R}\}$

Lemma 2.7.2. *The function in (2.7.2) is single-valued and analytic in the upper half-plane* $\{z \in \mathbb{C} : \text{ Im } z > 0\}$*, and single-valued and analytic on the real line segments* $\{x \in \mathbb{R} : a_{r-1} < x < a_r\}$*.*

Proof. The function g is a product of h_r's. Each h_r is analytic and single-valued in the upper half-plane and on the real line segments $\{x \in \mathbb{R} : a_{r-1} < x < a_n\}$. Therefore g analytic and single-valued in the upper half-plane. □

Integrating g will result in a function f with piecewise constant $\arg(f')$. This leads to the Schwarz-Christoffel Transformation; see [3] and [6].

Definition 2.7.3 (Schwarz-Christoffel Transformation)**.** Let $a_r \in \mathbb{R}$ with $a_{r-1} < a_r$, and $\alpha_r \in \mathbb{R}$, where $0 < \alpha_r \leq 2$, with $r = 1, 2, \dots, n$, and $z_0 \in \mathbb{C}$ with Im $z_0 \geq 0$. With constants $K, C \in \mathbb{C}$, where $K \neq 0$, the SCT is defined as:

$$w = f(z) = K \int_{z_0}^{z} \prod_{r=1}^{n} (u - a_r)^{\alpha_r - 1}\, du + C \qquad (2.7.3)$$

Since $\prod_{r=1}^{n}(u - a_r)^{\alpha_r - 1}$ is defined as a single-valued function, which is analytic in the upper half-plane (Lemma 2.7.2), the SCT is well defined in the upper half-plane and on the real line segments $\{x \in \mathbb{R} : a_{r-1} < x < a_n\}$. The points $\{a_n\}$ are referred to as pre-vertices.

Theorem 2.7.4. *Let $\mathcal{P} \subset \mathbb{C}$ be a simply connected bounded open set, such that its boundary is a simple polygon with $n+1$ vertices at $\omega_0, \omega_1, \dots, \omega_n$, where each vertex ω_i has an inner angle of $\alpha_i \pi$ for $i = 1, \dots n$, see Fig. 2.13. A function f such that:*

- *f maps the real line continuous to the boundary of $\mathcal{P}$,*
- *$\omega_i = f(a_i), i = 1, \dots, n$, where $a_1 < a_2 < \cdots < a_n$; and $\omega_0 = f(\infty) = f(-\infty)$,*
- *f maps the upper half-plane $\{z : \text{ Im } z > 0\}$ one-to-one and conformally onto $\mathcal{P}$,*

must be of the form (2.7.3) and therefore is a SCT.

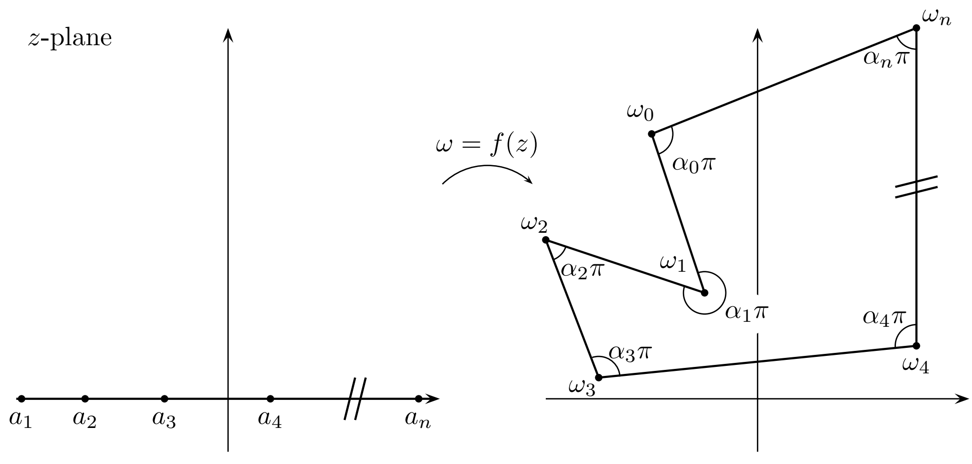

Figure 2.13: Schwarz-Christoffel transformation

Since $\mathcal{P}$ is a simply connected open set with a Jordan curve as boundary, the Riemann mapping theorem, together with the Osgood-Carathéodory theorem proves the existence of a function f as claimed in the theorem. It remains to show that f has the form of (2.7.3). The function $\log f'$ can be well defined as a single-valued analytic function in the upper half-plane, since $f'(z) \neq 0$ for z with $\text{Im } z > 0$ (Lemma 2.1.4).

Let $f(z)$ be the function obtained via the Riemann mapping theorem and the Osgood-Carathéodory theorem. Define function $F(z)$ by:

$$F(z) = \log f'(z),$$

Lemma 2.7.5. *The function $F'(z) = f''(z)/f'(z)$ can be extended by analytic continuation to a function which is analytic $\mathbb{C}\backslash\{a_1, a_2, \ldots, a_n\}$.*

Proof. f is conformal in the upper half-plane, and for all $0 < i < n$ maps the real open intervals $(a_i, a_{i+1}), (-\infty, a_1)$, and (a_n, ∞) continuous to a straight line. By the Schwarz Reflection principle, Theorem 2.1.2, it can be extended to a function f which is analytic on the whole plane with the possible exception of points $\{a_1, \ldots, a_n\}$. Since $f' \neq 0$ in the upper half-plane and for z in the lower half-plane $f(z) = \overline{f(\bar{z})}$, $f' \neq 0$ in the lower half-plane and therefore F can be extended to the lower half-plane.

The extended function f maps the intervals $(a_i, a_{i+1}), (-\infty, a_1)$, and (a_n, ∞) one-to-one onto straight lines, and therefore cannot have $f' = 0$. Therefore $F = \log f'$ can be defined analytically on $(a_i, a_{i+1}), (-\infty, a_1)$, and (a_n, ∞). On these open intervals, $\arg f'(z) = \text{constant}$, by Lemma 2.5.1.

$$\text{Im } F(z) = \text{constant}, \qquad z \in (a_i, a_{i+1}) \cup (-\infty, a_1) \cup (a_n, \infty).$$

By differentiation:

$$\text{Im } F'(z) = 0, \qquad z \in (a_i, a_{i+1}) \cup (-\infty, a_1) \cup (a_n, \infty).$$

This shows that F' is analytic in the upper half-plane, lower half-plane, and the real line segments $(a_i, a_{i+1}), (-\infty, a_1)$, and (a_n, ∞). □

Lemma 2.7.6. *The function F' is rational, it has simple poles at the points a_i with residue $\alpha_i - 1$, and vanishes at ∞. Therefore it must be of the form*

$$F'(z) = \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k}.$$

Proof. For $1 \le k \le n$, let Im $z \ge 0$, For $|z - a_k| < \epsilon$ with $\epsilon > 0$ sufficiently small define

$$h(z) = (f(z) - \omega_k)^{1/\alpha_k}.$$

Note that a branch of h can be chosen such that h is continuous for z in the upper half-plane. h is clearly analytic for Im $z > 0$ and continuous and injective for Im $z \ge 0$. Moreover $h(a_k) = 0$. As z increases on the real line across a_k, $\arg(z - a_k)$ increases with π, $\arg[f(z) - \omega_k]$ increases by $\alpha_k \pi$, hence $\arg h(z)$ increases by π. Therefore the straight line segment $(a_k - \epsilon, a_k + \epsilon)$ is mapped onto a straight line segment through 0. By the Schwarz reflection principle, theorem 2.1.2, h can be defined analyticly on the whole disc $|z - a_k| < \epsilon$, and there exists an expansion:

$$h(z) = c_1(z - a_k)[1 + c_2(z - a_k) + \cdots],$$

with $c_1 \ne 0$, hence f can be written as

$$f(z) - \omega_k = \tilde{c}_1(z - a_k)^{\alpha_k}[1 + \tilde{c}_2(z - a_k) + \cdots],$$

with $\tilde{c}_1 \ne 0$. Now F' can be written as

$$F' = \frac{f''}{f'} = \frac{\alpha_k - 1}{z - a_k} + A(z),$$

where $A(z)$ is analytic. Since the above derivation holds for every a_k,

$$F'(z) - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k} \tag{2.7.4}$$

is an entire function.

Now it remains to show that the function in (2.7.4) is bounded. To see this, consider with $\alpha_0 = n - 1 - \sum_{k=1}^{n} \alpha_k$:

$$g(z) = (f(1/z) - \omega_0)^{1/\alpha_0},$$

for $|z| < \epsilon$, with ϵ sufficiently small. Now g is analytic for z with Im $z < 0$ and if $g(0) = 0$, continuous for Im $z \le 0$. As before, g can be extended by the Schwarz reflection principle and maps the entire disc conformally:

$$g(z) = d_1 z(1 + d_2 z + \cdots),$$

with $d_1 \ne 0$, hence

$$f(1/z) = \omega_0 + \tilde{d}_1 z^{\alpha_0}(1 + \tilde{d}_2 z + \cdots).$$

For $|z|$ sufficiently large

$$f(z) = \omega_0 + \tilde{d}_1 z^{-\alpha_0}(1 + \tilde{d}_2 z^{-1} + \cdots).$$

Calculate f' and f'' to get

$$F'(z) = \frac{f''}{f'} = \frac{-\alpha_0 - 1}{z} + O(z^{-2}), \qquad z \to \infty. \tag{2.7.5}$$

Since $\alpha_0 = n - 1 - \sum_{k=1}^{n} \alpha_k$,

$$\sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k} = \frac{\sum_{k=1}^{n} a_k - 1}{z} + O(z^{-2}) = \frac{-\alpha_0 - 1}{z} + O(z^{-2}).$$

Hence, by (2.7.5),

$$F'(z) - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k} = F'(z) - \frac{-\alpha_0 - 1}{z} + O(z^{-2}) = O(z^{-2}), \qquad z \to \infty.$$

Therefore the entire function in (2.7.4) is bounded, and vanishes as $z \to \infty$. By Liouville's theorem F' has the form as stated in the lemma. □

Proof of Theorem 2.7.4. Recall that $f(z)$ denoted the function obtained via the Riemann mapping theorem and the Osgood-Carathéodory theorem. By means of $f(z)$ the function $F(z) = \log f'(z)$ was defined. Above it has been shown that

$$F'(z) = \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k}.$$

Integrating this identity gives:

$$F(z) = \log \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1} + D,$$

which exists in the upper half-plane. Hence

$$f'(z) = \tilde{D} \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1},$$

which leads to

$$f = K \int_{z_0}^{z} \prod_{k=1}^{n} (u - a_k)^{\alpha_k - 1}\, du + C.$$

This completes the proof. □

It can be proved that the theorem still holds when $\mathcal{P}$ is a simply open connected set with an unbounded boundary, in the shape of a polygon, with a finite number of vertices. The proof, which is variation of the case where the boundary of $\mathcal{P}$ is a finite polygon, can be found in [3].

It can also be proved that any function of form (2.7.3) must map the upper half-plane to a bounded or unbounded polygon, with vertices at $f(a_i)$ and inner angles $\alpha_i\pi$ (with possibly an additional vertex at $f(\infty)$ with inner angle $\alpha_0\pi$). A proof of this statement can be found in [6].

There exists a variation on the SCT where instead of the upper half-plane, the unit disc is conformally mapped to the inside of a bounded (or a unbounded) polygon. This is also considered as a SCT, and the expression for such a SCT can be found in [3].

For a polygon with n vertices, the sum of the outer angles must be 2π. With $\alpha_0\pi$ as the inner angle at vertex $f(\infty)$, the following equation holds:

$$\left(n + 1 + \alpha_0 + \sum_{k=1}^{n} \alpha_k\right)\pi = 2\pi.$$

Hence

$$\alpha_0 = n - 1 - \sum_{k=1}^{n} \alpha_k.$$

The identity above shows that the choice of $\{\alpha_k : k > 0\}$ determine if there is a vertex at ω_0, see Fig. 2.14. Note that the formula of the SCT (2.7.3) does not explicitly contain α_0.

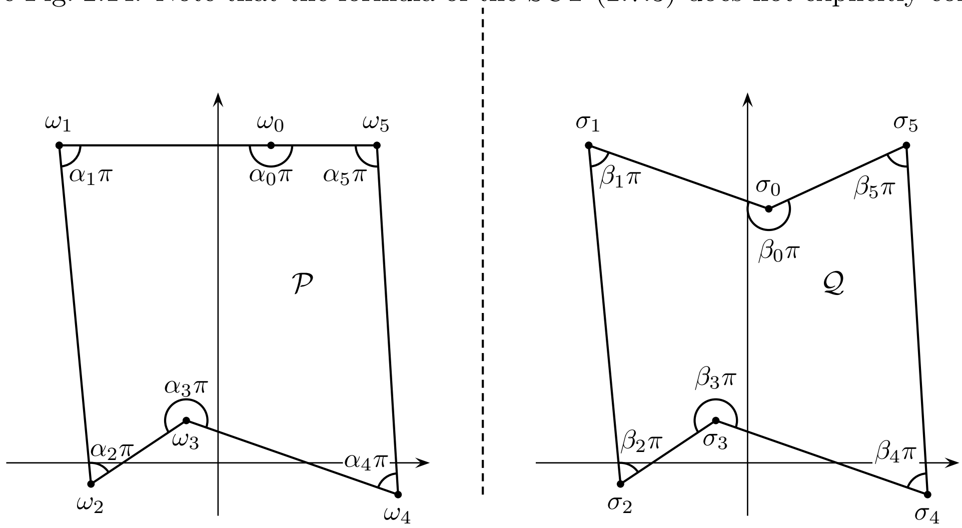

Figure 2.14: Two polygons $\mathcal{P}$ and $\mathcal{Q}$. Polygon $\mathcal{P}$ has $\alpha_0 = 1$ and therefore has no vertex at ω_0. Polygon $\mathcal{Q}$ has $\beta_0 \neq 1$ and therefore has a vertex at σ_0.

Both functions in Example 2.3.1 and Example 2.3.2 are SCT's with $\alpha_0 \neq 1$.

2.8 More examples of SCT's

Example 2.8.1 (Rectangle transformation)**.** For $w \in \mathbb{C}$ with $\operatorname{Im} w \geq 0$, and $k \in \mathbb{R}$, $0 < k < 1$, and $\alpha = -1/2$, define function f by:

$$f(w) = -\int_0^w (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du \tag{2.8.1}$$

The branch-cuts for f are chosen as in Fig. 2.12, so all the branch-cuts lie in the lower half-plane. f has the form of a SCT, with pre-vertices $1, -1, 1/k$, and $-1/k$. Hence f will map the upper half-plane to a polygon $\mathcal{P}$ with vertices $f(-1), f(1), f(1/k)$, and $f(-1/k)$. Note that by the choice of inner-angles, the image of f must be a polygon with the shape of a rectangle.

Consider the behaviour for $x \in \mathbb{R}, x > 0$:

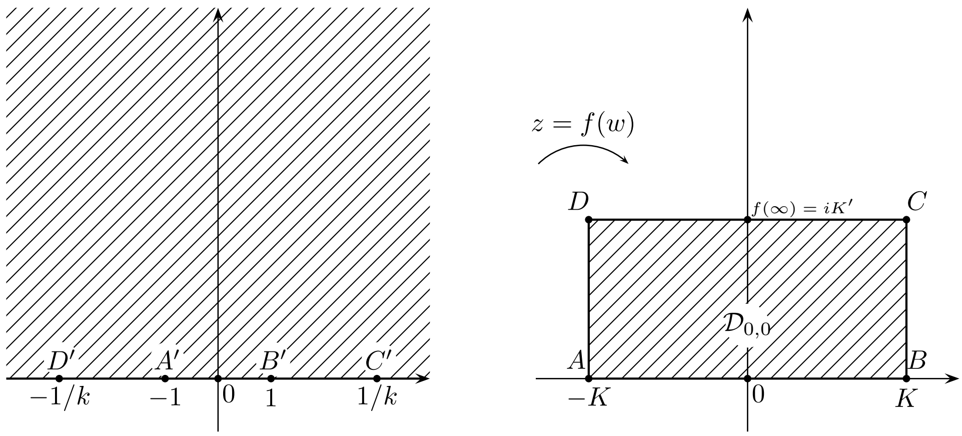

Figure 2.15: SCT to Rectangle $ABCD$.

- For $0 < x < 1$, note that $f(x)$ is real and positive since the integrand in (2.8.1) is negative. Hence for some $K > 0$:
$$f(1) = K.$$

- For $1 < x < 1/k$, note that the integrand in (2.8.1) is strictly imaginary with negative imaginary part. Hence for some $K' > 0$
$$f(1/k) = f(1) - \int_1^{1/k} (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du = K + iK'.$$

- For $1/k < x$, the integrand in (2.8.1) is real and positive. Note that
$$\begin{aligned} iK' &= f(1/k) - \int_0^1 (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du \\ & \qquad\qquad (u = 1/kv) \\ &= f(1/k) - \int_{1/k}^\infty (v-1)^\alpha (v+1)^\alpha (v-1/k)^\alpha (v+1/k)^\alpha \, dv. \end{aligned}$$
This shows that
$$f(\infty) = iK'.$$

To calculate the location of $f(-1)$ and $f(-1/k)$ observe that for $a > 0$
$$\begin{aligned} f(a) &= -\int_0^a (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du \qquad (u = -v) \\ &= \int_0^{-a} (-v-1)^\alpha (-v+1)^\alpha (-v-1/k)^\alpha (-v+1/k)^\alpha \, dv \\ &= \int_0^{-a} (v-1)^\alpha (v+1)^\alpha (v-1/k)^\alpha (v+1/k)^\alpha \, dv. \end{aligned}$$

Here the last integrand has different branch-cuts than the original integrand of $f(z)$, since each factor of the integrand has been multiplied by $(e^{i\pi})^\alpha$. Therefore the branch-cuts for the last integrand all lie in the upper half-plane. Hence for $x < 0$, $f(x)$ can be rewritten to:

$$f(x) = \int_0^{-x} (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du, \tag{2.8.2}$$

where the branch-cuts of the integrand lie in the upper half-plane.

Now consider the behaviour of $f(x)$ for $x \in \mathbb{R}$, $x < 0$:

- For $-1 < x < 0$, it follows from (2.8.2) that $f(x) = -f(-x)$. Hence the integrand for $f(-x)$ in (2.8.2) is real and
$$f(-1) = -K.$$
- For $-1/k < x < -1$, note that the integrand of $f(x)$ in (2.8.2) is strictly imaginary with positive imaginary part. Therefore
$$f(-1/k) = -K + iK'.$$
- For $x < -1/k$, the integrand of $f(x)$ in (2.8.2) is real and positive. Note that
$$\begin{aligned} iK' &= f(-1/k) - \int_0^{-1} (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du \\ & \qquad (u = 1/kv) \\ &= f(-1/k) - \int_{-1/k}^{-\infty} (v-1)^\alpha (v+1)^\alpha (v-1/k)^\alpha (v+1/k)^\alpha \, dv. \end{aligned}$$
Hence,
$$f(-\infty) = iK'.$$

This shows f maps the upper half-plane to a rectangle, as shown in Fig. 2.15.

Example 2.8.2 (Bar transformation)**.** Now a SCT $f(z)$ will be determined which maps the upper half-plane to a bar with vertices $A = ai$ and 0 such that $f(0) = 0, f(1) = A$, as in Fig. 2.16.

Note that this is a SCT mapping to an unbounded polygon. The SCT must have the following form:

$$\begin{aligned} f(z) &= K \int_0^{u=z} u^{-1/2}(1-u)^{-1/2} \, du + C \qquad (u = \sin^2(t)) \\ &= K \int_0^{t=\sin^{-1}(z^{1/2})} (\sin^2(t))^{-1/2} (\cos^2(t))^{-1/2} 2 \sin(t)\cos(t) \, dt + C \\ &= \tilde{K} \int_0^{t=\sin^{-1}(z^{1/2})} dt + C \\ &= \tilde{K} \sin^{-1}(z^{1/2}) + C. \end{aligned}$$

The constants $\tilde{K}, C$ need to be chosen such that $f(0) = 0$ and $f(1) = A$. $C = 0$ since $f(0) = 0$. $f(1) = \tilde{K}\pi/2 = A \Longrightarrow \tilde{K} = A2/\pi$. For $A = i\pi/2 \Longrightarrow \tilde{K} = i$. The SCT

$$f(z) = i \sin^{-1}(z^{1/2})$$

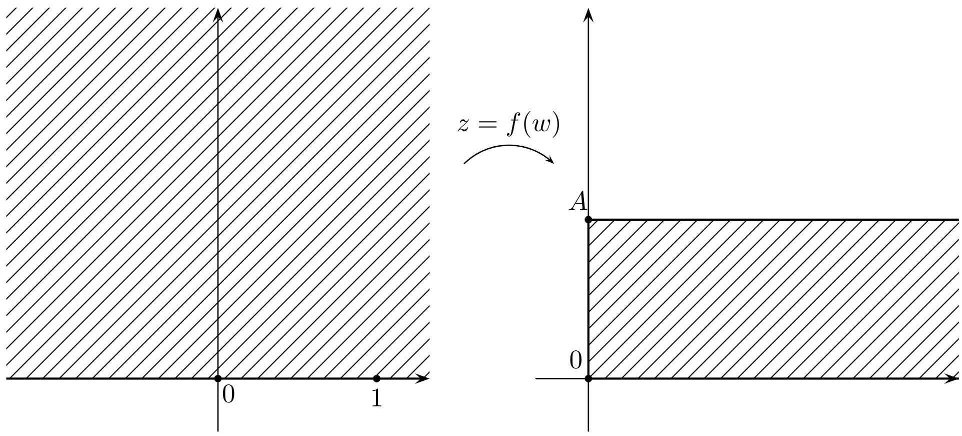

Figure 2.16: SCT to bar shaped region.

will map the upper half-plane to the bar in Fig. 2.16, where $A = i\pi/2$. The inverse of function f, denoted as g, will map the bar shaped region to the upper half-plane:

$$g(w) = \sin^2(iz).$$

2.9 Numerical approximations of the SCT

In the formula of the SCT, the values of the pre-vertices $\{a_r\}$ are responsible for the length of the sides of polygon $\mathcal{P}$. Finding the right set of pre-vertices $\{a_r\}$ for a given polygon $\mathcal{P}$ is known as the parameter-problem.

In Appendix A, Mathematica source-code has been provided for a simple program; given $\{a_r, \alpha_r\}$ for a SCT, the program numerically approximates the location of the vertices. Since the Mathematica program can only draw the vertices for a SCT, given the inner-angles and pre-vertices, finding the right set of pre-vertices for a given polygon is left as a problem for the user.

The program has been used in an attempt to try find the set $\{a_r, \alpha_r\}$ for the SCT which maps the upper half-plane to a polygon in the shape of Escher's reptile, Fig. 2.17. The pre-vertices had to be guessed.

It turns out to be difficult to guess the right set of pre-vertices for a polygon in the shape of Escher's reptile. The set of pre-vertices have not been found. The program does give some insight in how the polygon depends on the values of $\{a_r\}$. Increasing one $a_i \in \{a_r\}$ can change the resulting polygon severely on all sides.

Driscoll [1] made a MATLAB package which numerically approximates the needed $\{a_r\}$. With his package, the $\{a_r\}$ for a SCT which maps the the upper half-plane to a polygon in the shape of Escher's reptile have been found, see Fig. 4.3. The input and resulting $\{\alpha_r\}, \{a_r\}$ can be found in Appendix B.

Figure 2.17: Escher's drawing of reptiles.

Chapter 3

Jacobian Elliptic Functions

The Schwarz reflection principle in Theorem 2.1.2 and Corollary 2.1.3 will be used to define Jacobian elliptic functions. For $k \in \mathbb{R}, 0 < k < 1$, define $f(w)$ by:

$$f(w) = -\int_0^w (u-1)^\alpha (u+1)^\alpha (u-1/k)^\alpha (u+1/k)^\alpha \, du, \tag{3.0.1}$$

so that f maps the upper half-plane into the rectangle as in Fig. 3.1, with length $2K$ and height K'. The open set inside the rectangle is denoted by $\mathcal{D}_{0,0}$, as shown in the figure.

The translation of $\mathcal{D}_{0,0}$, by $n\,2K + m\,iK'$, for $n, m \in \mathbb{Z}$, is an open set denoted by $\mathcal{D}_{n,m}$, see Fig. 3.3. Since f is one-to-one and onto $\mathcal{D}_{0,0}$, the inverse g can be analytically defined

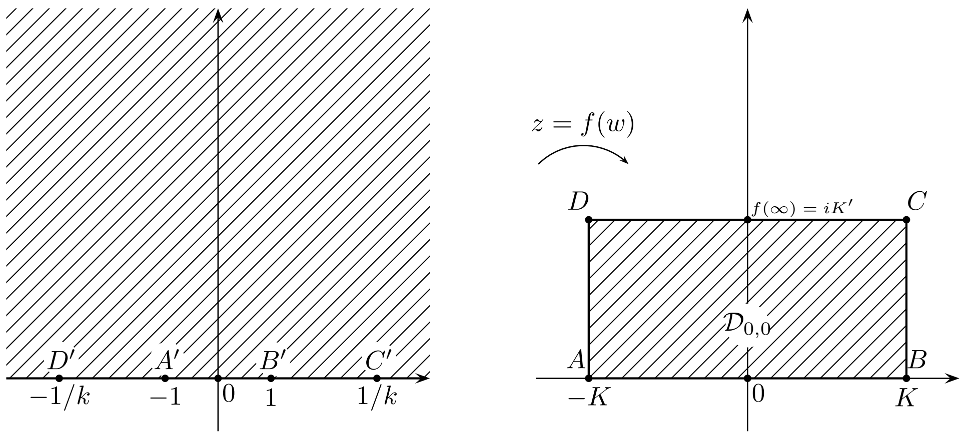

Figure 3.1: SCT to rectangle $ABCD$.

inside $\mathcal{D}_{0,0}$. Note that g can also be defined on the closure of $\mathcal{D}_{0,0}$, with the exception of the point iK', since $f(\infty) = iK'$. Hence for $z \in \overline{\mathcal{D}_{0,0}} \backslash \{iK'\}$:

$$g(z) = w \Longleftrightarrow z = f(w). \tag{3.0.2}$$

Note that g meets all the requirements for the Schwarz reflection principle in Theorem 2.1.2 and Corollary 2.1.3. Via the Schwarz reflection principle in Theorem 2.1.2, g can be defined in

$\mathcal{D}_{0,-1}$. When $g_{0,-1}$ denotes the analytic continuation of g inside $\mathcal{D}_{0,-1}$, and when $g_{0,0}$ denotes the function g from (3.0.2), the following relation holds:

$$g_{0,-1}(z) = \overline{g_{0,0}(\bar{z})}, \qquad z \in \mathcal{D}_{0,-1}. \tag{3.0.3}$$

Since $g_{0,0}(z)$ maps $z \in \mathcal{D}_{0,0}$ into the upper half-plane, it follows from (3.0.3) that $g_{0,-1}$ maps into the lower half-plane, see Fig. 3.2.

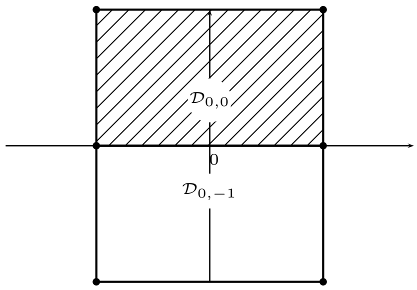

Figure 3.2: $\mathcal{D}_{0,0}$ and $\mathcal{D}_{0,-1}$. The lined region is mapped to the upper half-plane, by g.

This process can be continued inductively. Let $g_{n,m}$ denote the analytic continuation of g in $\mathcal{D}_{n,m}$, for $n, m \in \mathbb{Z}$, then for $g_{n,m}$ the following expression holds:

$$g_{n,m}(z) = \begin{cases} g_{0,0}(z - n2K - miK') & \text{if} \quad n + m \equiv 0 \mod 2 \\ g_{0,-1}(z - n2K - (m+1)iK') & \text{if} \quad n + m \equiv 1 \mod 2 \end{cases}$$

The general expression for the function $g_{n,m}$ as above is a consequence of iteratively applying the Schwarz reflection principle together with Corollary 2.1.3 to the function $g_{0,0}(z)$. This process is tedious to describe. However, as an example, the derivation for $g_{0,1}$ will be given.

Define $g'_{0,0}(z)$ by:

$$g'_{0,0}(z) = g_{0,0}(z + iK'), \qquad z \in \mathcal{D}_{0,-1}.$$

Via the Schwarz reflection theorem, $g'_{0,0}$ can be analytically extended to domain $\mathcal{D}_{0,0}$. Let $g'_{0,1}$ denote the analytic continuation of $g'_{0,0}$ on domain $\mathcal{D}_{0,0}$, then:

$$g'_{0,1}(z) = \overline{g'_{0,0}(\bar{z})}, \qquad z \in \mathcal{D}_{0,0},$$

hence,

$$\begin{aligned} g_{0,1}(z) &= g'_{0,1}(z - iK') \\ &= \overline{g'_{0,0}(\bar{z} + iK')} \\ &= \overline{g_{0,0}(\bar{z} + 2iK')} \\ &= g_{0,-1}(z - 2iK'), \qquad z \in \mathcal{D}_{0,1}. \end{aligned}$$

Note that $g(z)$ depends on the parameter k, since g is the inverse of f in (3.0.1) where a constant k is used. Hence g_k can be written instead of g, to emphasise the dependence of the parameter k.

Definition 3.0.1 (Jacobian elliptic function)**.** The Jacobian elliptic function $\text{sn}(z, k)$ is the analytic continuation of g_k in the whole complex-plane, with the exception of countable many points (where g_k cannot be extended analytically).

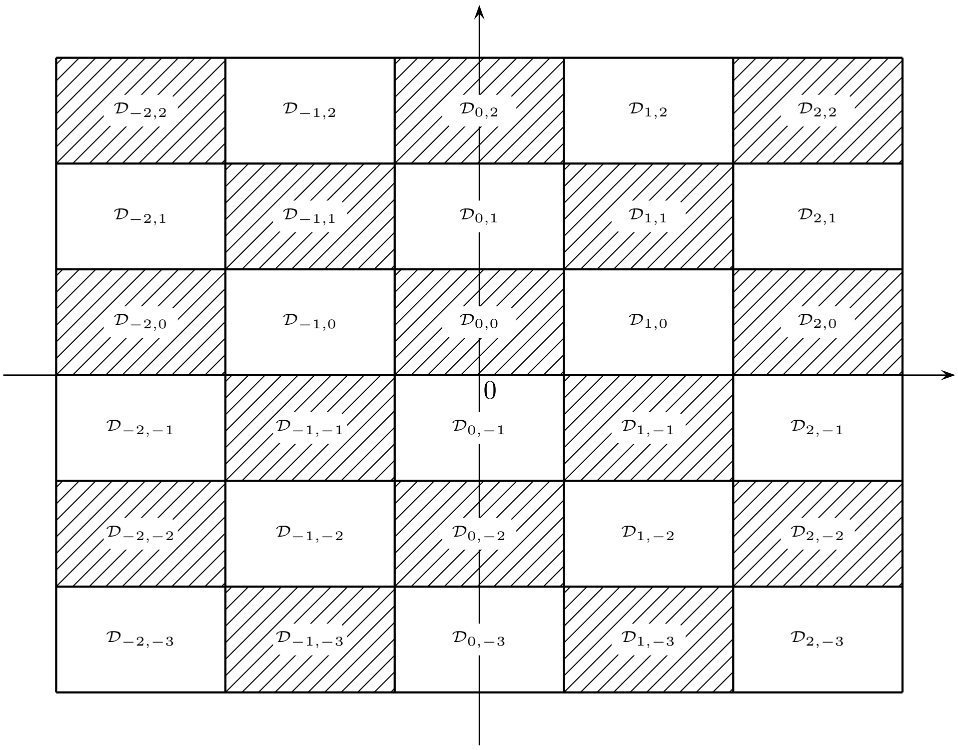

Figure 3.3: Schwarz reflection principle applied to g. The marked area's will be mapped to the upper half-plane.

From the analytic continuation for g is can be seen that g_k is periodic. Hence for sn:

$$\mathrm{sn}(z + 2iK') = sn(z + 4K) = \mathrm{sn}(z), \qquad z \in \mathbb{C},$$

so that sn is a doubly periodic function. Note that in the imaginary direction, $2iK'$ is the smallest period, and in the real direction $4K$ is the smallest period.

Recall that sn defined on $\mathcal{D}_{0,0}$ and its boundary has a singular point at iK' and a zero at 0. Analytic continuation with the Schwarz reflection principle will therefore add zero's and singular points to the definition of sn on $\mathbb{C}$.

Definition 3.0.2. The Jacobian elliptic functions cn and dn are defined as:

$$\mathrm{cn}(z) = (1 - \mathrm{sn}^2(z))^{1/2} \tag{3.0.4}$$

$$\mathrm{dn}(k, z) = (1 - k^2\mathrm{sn}^2(z))^{1/2} \tag{3.0.5}$$

The branch-cut for both square-roots in the upper definition is $(-\infty, 1)$. Like sn, cn and dn are doubly periodic, which will be proved later by Lemma 4.0.4.

On domain $\mathcal{D}_{0,0}$, the inverse of sn is $f(w)$ from (3.0.1). If $z = f(w)$, with $f^{-1}(z)$ as the inverse of $f(w)$, then

$$\left[f^{-1}\right]'(z) = \frac{1}{f'(f^{-1}(z))}.$$

Using the expression above and (3.0.1), the derivative of sn can be calculated:

$$\mathrm{sn}'(z) = (1 - \mathrm{sn}^2(z))^{1/2}(1 - k^2\mathrm{sn}^2(z))^{1/2} = \mathrm{cn}(z)\mathrm{dn}(z).$$

With this expression, the derivatives of cn and dn can be calculated:

$$\mathrm{cn}'(z) = \frac{-\mathrm{sn}(z)\mathrm{cn}(z)\mathrm{dn}(z)}{(1 - \mathrm{sn}^2(z))^{1/2}} \qquad = -\mathrm{sn}(z)\mathrm{dn}(z)$$

$$\mathrm{dn}'(z) = \frac{-k^2\mathrm{sn}(z)\mathrm{cn}(z)\mathrm{dn}(z)}{(1 - k^2\mathrm{sn}^2(z))^{1/2}} \qquad = -k^2\mathrm{sn}(z)\mathrm{cn}(z)$$

Chapter 4

Elliptic Functions in General

A lattice $\mathcal{L}$ in the complex plane is a subgroup of $\mathbb{C}$, consisting of all points:

$$z_1\mathbb{Z} + z_2\mathbb{Z}, \qquad z_1, z_2 \in \mathbb{C}\backslash\{0\}, \qquad z_1/z_2 \notin \mathbb{R}.$$

Such a lattice is generated by the points z_1 and z_2, and denoted by $\mathcal{L}(z_1, z_2)$ or, simply, by $\mathcal{L}$. The lattice $\mathcal{L}(z_1, z_2)$ and a point $a \in \mathbb{C}$ give rise to a fundamental parallelogram $P_a(\mathcal{L})$, or just P_a, defined by:

$$a + b_1 z_1 + b_2 z_2, \qquad 0 \leq b_i \leq 1, \qquad a \in \mathbb{C}.$$

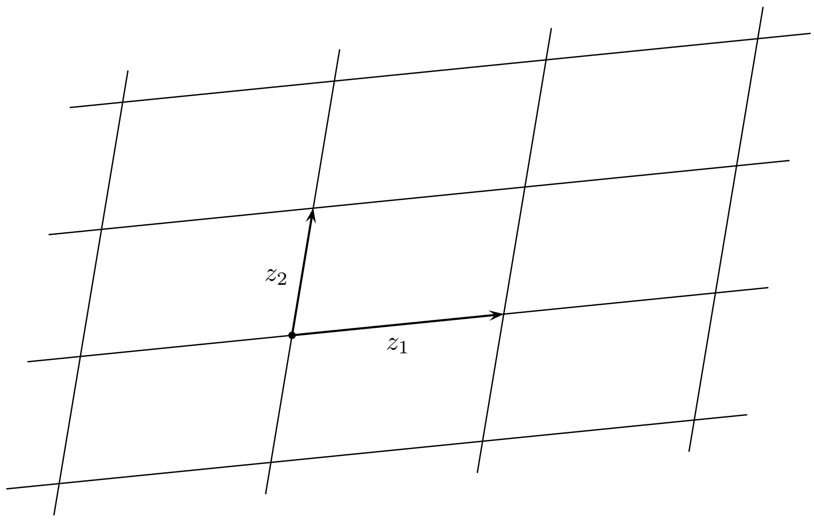

Figure 4.1: Lattice $\mathcal{L}(z_1, z_2)$.

Definition 4.0.3 (Elliptic functions)**.** A meromorphic function $f(z)$ is called an elliptic function if there exists a lattice $\mathcal{L}$, such that:

$$f(z + l) = f(z),$$

for all $z \in \mathbb{C}$ and $l \in \mathcal{L}$.

An elliptic function f always has a finite number of poles inside any arbitrary fundamental parallelogram P_x. Therefore for an elliptic function f, there always exists a fundamental parallelogram P_y such that f doesn't have pole in the boundary of P_y. The boundary of P_y is denoted by ∂P_y. Due to Liouville's theorem, entire elliptic functions (analytic everywhere) must be constant. So interesting elliptic functions must have at least one pole.

Lemma 4.0.4. *Let f, g be elliptic functions with the lattice $\mathcal{L}(z_1, z_2)$, and let ψ be a meromorphic function. Then the following statements are true:*

- *$af + bg$, $f \cdot g$ and f/g are elliptic (with constants $a, b \in \mathbb{C}$),*
- *$\psi \circ f$ is elliptic,*
- *f' is elliptic.*

Proof. That $af + bg$, $f \cdot g$, and f/g are elliptic follows directly from the definition. Let $l \in \mathcal{L}$ and $z \in \mathbb{C}$ be chosen arbitrarly, then

$$f(z + l) = f(z), \tag{4.0.1}$$

and

$$\psi(f(z + l)) = \psi(f(z)).$$

i.e. $\psi \circ f$ is elliptic. Both sides in (4.0.1) can be differentiated, which shows f' is elliptic. □

In particular, elliptic functions, with respect to a lattice $\mathcal{L}$, form a field.

Theorem 4.0.5. *Let P_a be a fundamental parallelogram for the elliptic function $f(z)$, such that f has no poles on the boundary ∂P_a of P_a (so that all the poles of $f(z)$ are inside ∂P_a). Then the sum of the residues of $f(z)$ in P_a is zero.*

Proof. Observe that by Cauchy's theorem

$$\int_{\partial P} f(z)\, dz = 2\pi i \sum \mathrm{Res} f.$$

Due to periodicity the integrals over opposite sides cancel and the result follows. □

Hence an non-constant elliptic function has at least two poles (counting multiplicities) inside P_a.

Theorem 4.0.6. *Let P_a be a fundamental parallelogram for elliptic function $f(z)$, such that f has no poles on the boundary ∂P_a of P_a. Let $\{a_i\}$ be all poles and zeros of $f(z)$ inside P_a, with order m_i at a_i. Then*

$$\sum m_i = 0.$$

Moreover,

$$\sum m_i a_i = 0 \mod \mathcal{L}.$$

Proof. For a_i a zero or pole of order m_i observe that:

$$\mathrm{Res}_{a_i}\frac{f'}{f} = m_i.$$

To see this, let $f(z)$ have the expansion:

$$f(z) = (z-a_i)^m(b_0 + b_1(z-a_i) + \cdots), \qquad b_0 \neq 0,$$

around $z = a_i$. Hence,

$$f'(z) = m(z-a_i)^{m-1}(\tilde{b}_0 + \tilde{b}_1(z-a_i) + \cdots), \qquad \tilde{b}_0 \neq 0.$$

The statement follows from:

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m(z-a_i)^{m-1}}{(z-a_i)^m}(c_0 + c_1(z-a_i) + \cdots), \qquad c_0 \neq 0\\ &= \frac{m}{z-a_i}(c_0 + c_1(z-a_i) + \cdots), \qquad c_0 \neq 0.\end{aligned}$$

Since f'/f is elliptic:

$$0 = \int_{\partial P_a} f(z)'/f(z)dz = 2\pi i \sum \mathrm{Res} f'/f = 2\pi i \sum m_i.$$

This proves the first statement of the theorem. For the second part consider the integral

$$\int_{\partial P_a} z\frac{f'(z)}{f(z)}dz = 2\pi i \sum m_i a_i,$$

which is valid since $\mathrm{Res}_{a_i} zf'(z)/f(z) = m_i a_i$. Now consider the integral and integrate over the indicated opposite sides:

$$\int_{\alpha}^{\alpha+\omega_1} z\frac{f'(z)}{f(z)}\,dz - \int_{\alpha+\omega_2}^{\alpha+\omega_1+\omega_2} z\frac{f'(z)}{f(z)}\,dz.$$

With the substitution $u = z - \omega_2$ for the last integral, both integrals are taken from α to $\alpha + \omega_1$:

$$\int_{\alpha}^{\alpha+\omega_1} (-\omega_2 + z - z)\frac{f'(z)}{f(z)}\,dz = -\omega_2 \int_{\alpha}^{\alpha+\omega_1} \frac{f'(u)}{f(u)}\,du = 2\pi i k\omega_2,$$

for $k \in \mathbb{Z}$, since the last integral can be interpreted as a path integral over ω_2/z for some closed path with a winding number denoted by k around 0, and therefore must be $2\pi i k\omega_2$ by Cauchy's theorem. Integration over the remaining opposite sides is done in the same way. Hence,

$$2\pi i \sum m_i a_i = 2\pi i k\omega_2 + 2\pi i l\omega_1, \qquad k, l \in \mathbb{Z},$$

which proves the theorem. □

Definition 4.0.7 (Weierstrass function)**.** Let $\mathcal{L}(z_1, z_2)$ be a lattice. The Weierstrass function, corresponding to the lattice $\mathcal{L}(z_1, z_2)$, is defined as a series:

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\mathcal{L}'}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right),$$

where $\mathcal{L}'$ defines the lattice $\mathcal{L}$ without the point 0.

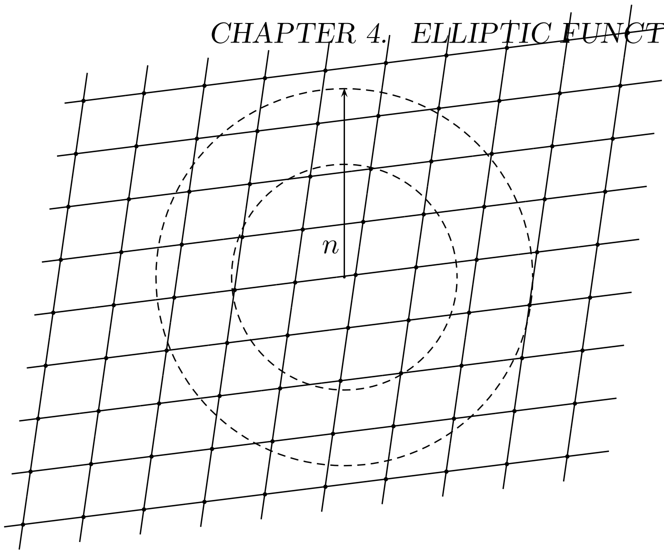

Figure 4.2: Annulus which contains lattice point of order n.

To show that the series converges uniformly on compact sets not including lattice points, the following lemma will be used. Observe that for a fixed non-lattice point z, the difference $\frac{1}{(z-w)^2} - \frac{1}{w^2}$ has the order of magnitude of $1/|w|^3$, which converges by the following lemma [4].

Lemma 4.0.8. *If* $\lambda > 2$, *then* $\sum_{w \in \mathcal{L}'} \frac{1}{|w|^\lambda}$ *converges.*

Proof. Consider all $w \in \mathcal{L}$ such that $n-1 \leq |w| \leq n$ (i.e. all $w \in \mathcal{L}$ in the corresponding annulus), Fig. 4.2. Note that the annulus has area of $\pi(2n-1)$. Solutions of Gauss's circle problem show that in each annulus, the number of lattice points has order n, and therefore there exists a a such that each annulus contains maximum an lattice points [7]. The partial sum for $|w| \leq N$ can now be decomposed into a sum for w with $n-1 \leq |w| \leq n$:

$$\sum_{|w| \leq N} \frac{1}{|w|^\lambda} \leq \sum_{1}^{N} \frac{an}{n^\lambda} \leq a \sum_{1}^{N} \frac{1}{n^{\lambda-1}}.$$

The last sum converges for $N \to \infty$ if $\lambda > 2$. □

For $|z| < R$ with $R > 0$, the partial sum:

$$\sum_{w \in \mathcal{L}', |w| < R} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

defines a meromorphic function, with a double pole at each lattice point. Since for $|z| < R$,

$$\sum_{w \in \mathcal{L}', |w| \geq R} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

converges by the previous lemma, $\wp(z)$ defines a meromorphic function with a double pole at each lattice point.

Clearly $\wp(z) = \wp(-z)$, because summing over lattice points is the same as summing over their negatives. Furthermore,

$$\wp'(z) = \frac{-2}{z^3} + \sum_{w \in \mathcal{L}'} \frac{-2}{(z-w)^3} = \sum_{w \in \mathcal{L}} \frac{-2}{(z-w)^3}.$$

From the definition it follows directly that $\wp'(z)$ is periodic, and that $\wp'(-z) = -\wp'(z)$. From the definition of $\wp$ it cannot be directly seen that $\wp$ is periodic. Since $\wp'(z+w_1) = \wp'(w)$, it follows that

$$\wp(z + w_1) = \wp(z) + C,$$

for some constant C. For $z = -w_1/2$, $\wp(w_1/2) = \wp(-w_1/2) + C$, hence $C = 0$ since $\wp$ is even. This shows that $\wp$ is periodic as well. The next theorem shows that any elliptic function corresponding to the lattice $\mathcal{L}$, can be written as a rational function of $\wp$ and $\wp'$, where both $\wp$ and $\wp'$ correspond to the lattice $\mathcal{L}$. To see this the following lemma is useful.

Lemma 4.0.9. *If the elliptic function f satisfies:*

$$f(z) = f(-z),$$

and has a zero or pole of order m at some point u, then f also has a zero or pole of the same order at $-u$.

Moreover, if $u \equiv -u \mod \mathcal{L}$, then f has a zero or pole of even order at u.

Proof. Since

$$f^{(k)}(u) = (-1)^k f^{(k)}(-u),$$

indeed f has a zero or pole of the same order at $-u$, which proves the first statement.

If $u \equiv -u \mod \mathcal{L}$, then $2u \equiv 0 \mod \mathcal{L}$. Note that there are only four points with this property inside a fundamental parallelogram P_0:

$$0, \frac{w_1}{2}, \frac{w_2}{2}, \frac{w_1 + w_2}{2}.$$

If f is even, f' is odd, and $f'(u) = -f'(-u)$. Since $u \equiv -u \mod \mathcal{L}$ and f' is periodic, $f'(u) = 0$. Hence f has a zero of order 2 or greater at u.

First assume $u \not\equiv 0 \mod \mathcal{L}$, then

$$g(z) = \wp(z) - \wp(u)$$

has a zero of order 2 or greater. But by Theorem 4.0.6, the order of $g(z)$ has to be 2, since $\wp$ has only one pole of order 2 inside a fundamental parallelogram.

Note that f/g is even and elliptic. If $f(u)/g(u) \neq 0$, then order of f at $u = 2$, if $f(u)/g(u) = 0$ then f/g again has a zero of order 2 or greater at u and the process can be repeated.

If $u \equiv 0 \mod \mathcal{L}$, the same arguments can be used with $g = 1/\wp$, proofing f has a zero of even order at u. □

Theorem 4.0.10. *The field of elliptic functions with respect to the lattice $\mathcal{L}$, is generated by $\wp$ and $\wp'$.*

Proof. Let f be an elliptic function and let $f(z) = o(z) + e(z)$, where the odd and even parts $o(z)$ and $e(z)$ of $f(z)$ are defined by

$$o(z) = \frac{f(z) - f(-z)}{2}, \qquad e(z) = \frac{f(z) + f(-z)}{2}.$$

Then $e(z)$ and $o(z)$ are elliptic since $f(-z)$ is elliptic. It will suffice to prove that $\wp$ generates the field of even elliptic functions. Since the odd part always be written as $o(z) = o(z)\wp'(z)/\wp'(z)$, where $o(z)\wp'(z)$ is even.

To prove that if f is even, it can be written as a rational function of $\wp$, it will be shown that a rational function of $\wp$ can be constructed with the same zero's and poles as f.

Let $u_i, i = 1, 2, \ldots, r$ be the set of points containing one representative from each class $(u, -u) \mod \mathcal{L}$, where f has a zero or pole, other than the class of $\mathcal{L}$ itself. Let

$$m_i = \begin{cases} \text{ord}_{u_i} f & \text{if} \quad 2u_i \not\equiv 0 \mod \mathcal{L} \\ 1/2 \, \text{ord}_{u_i} f & \text{if} \quad 2u_i \equiv 0 \mod \mathcal{L} \end{cases} \tag{4.0.2}$$

By Lemma 4.0.9, $\text{ord}_{u_i} f$ is even if $2u_i \equiv 0 \mod \mathcal{L}$, hence $m_i \in \mathbb{Z}$. Lemma 4.0.9 also show that for $a \in \mathbb{C}, a \not\equiv 0 \mod \mathcal{L}$, the function

$$\wp(z) - \wp(a),$$

has a zero of order 2 at a if and only of $2a \equiv 0 \mod \mathcal{L}$, and has distinct zeros of order 1 at a and $-a$ otherwise. For all $z \neq 0 \mod \mathcal{L}$, define $h(z)$ by:

$$h(z) = \prod_{i=1}^{r} (\wp(z) - \wp(u_i))^{m_i}.$$

By construction, $h(z)$ has the same order at z as f, for all $z \neq 0 \mod \mathcal{L}$. f and h cannot have a zero or pole of different order at $0 \mod \mathcal{L}$, since this would contradict with Theorem 4.0.6. This shows f/h is an elliptic function without zeros or poles, and must be constant by Liouville's theorem. Theferfore $f(z)$ is a rational of $\wp$, which proves the theorem. □

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Appendices

APPENDIX A: Mathematica Source code

The following Mathematica source code has been used to numerically approximate the location of the vertices of the image of a Schwarz Christoffel transformation. Variable **a** contains the inner-angles of the polygon.

```
a = {3π/5, 3π/5, 3π/5, 3π/5, 3π/5};
d = {0, 3, 0.8, 0.3, 0.7};
f[u_,i_]:=Piecewise[{{(u-x[[i]])^(a[[i]]/π-1),u¡x[[i]]},{Re[(u-x[[i]])^(a[[i]]/π-1)],u¿x[[i]]},{0,u==x[[i]]}}];
K = 10;
h[u_]:=NIntegrate[K*Product[f[z,i],{i,1,Length[a]}],{z,0,u}];
hp[j_]:=NIntegrate[K*Product[f[z,i],{i,1,Length[a]}],{z,x[[j]],x[[j+1]]}];
hs[k_]:=Sum[hp[p],{p,1,k}];
CalcFirst[p_]:=A=Table[hp[i],{i,1,p}];
x = Table[Sum[d[[j]],{j,1,i}],{i,1,Length[a]}];
CalcFirst[Length[a]-1];
Cs[p_]:=Sum[A[[n]],{n,1,p}]; X[p_]:=Re[Cs[p]]; Y[p_]:=Im[Cs[p]];
MakeCoor[p_]:= Table[{X[i],Y[i]},{i,0,p}];
MakeCoor[4];
ListPlot[MakeCoor[Length[a]-1],PlotStyle → {Thick}, AspectRatio → Automatic]
```

APPENDIX B: SCT to Escher's reptile

To approximate the set's $\{a_r\}, \{\pi\alpha_r\}$ for the SCT which maps the real line to a polygon in the shape of Escher's reptile, the Schwarz-Christoffel toolbox for MATLAB version 241 have been used. This package is made by Toby Driscoll and can be found on the web. `http://www.math.udel.edu/~driscoll/software/SC/`

With the following command's from the toolbox, the list of $\{a_r\}, \{\pi\alpha_r\}$ can be calculated:

- `polygon()`: To construct a polygon object: *input:* 1 dimensional vector containing complex coordinates of desired polygon. *output:* 2 dimensional vector, containing vertices and inner angles as rows.

- `hplmap()`: To construct generic Schwarz-Christoffel map object: *input:* polygon object *output:* MATLAB generic Schwarz-Christoffel map object.

- `plot()`: Plot SCT: *input:* generic Schwarz-Christoffel map object. *output:* Plot of the image of the SCT.

The following input created the polygon object, stored in variable p. The complex location of the vertices have been measures from a digital picture (approximating the location of the pixel in the middle of the vertex) and are not accurate.

```
p = polygon([0+i*5.33 0.63+i*3.97 1.73+i*3.59 2.95+i*3.51 4.05+i*2.43 2.95+i*0.83 3.77+i*0.01 4.08+i*0.77 5.07+i*1.67
4.79+i*2.98 4.29+i*3.65 5.41+i*3.93 6.06+i*2.31 7.73+i*2.61 8.16+i*3.36 7.14+i*3.87 6.85+i*3.48 6.47+i*3.44 6.28+i*4.46
6.45+i*4.82 7.85+i*4.30 8.94+i*5.12 8.99+i*5.61 7.74+i*6.85 6.04+i*6.40 5.63+i*6.68 5.47+i*8.04 6.63+i*9.01 6.38+i*9.34
5.42+i*9.67 5.25+i*9.30 5.45+i*8.89 4.51+i*8.25 4.84+i*6.78 3.98+i*5.57 4.21+i*6.74 3.76+i*7.73 3.24+i*8.07 2.19+i*7.65
1.68+i*7.80 0.93+i*7.23 1.94+i*6.58 3.00+i*7.12 3.11+i*6.19 2.72+i*5.81 2.57+i*4.70 2.02+i*4.42 0.92+i*4.53 0+i*5.33])
```

The following input created the generic Schwarz-Christoffel map object stored in variable f.

```
f = hplmap(p)
```

A plot of the map object can be seen in Fig. 4.3. The list of resuling $\{a_r\}, \{\pi\alpha_r\}$ are listed below. Note that most a_r have small distance to its neigbours. The next two pages contain the lists of $\{a_r\}$ and $\{\pi\alpha_r\}$.

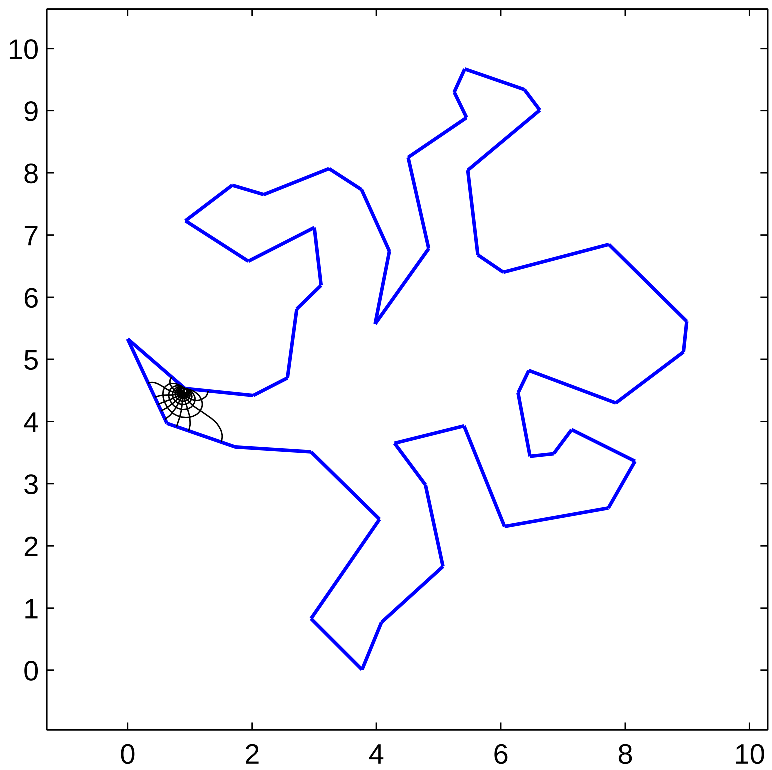

Figure 4.3: MATLAB output of the `plot()` commando

$a_1 = -1.000000000000$	$a_2 = 2.111369026161 \cdot 10^{-1}$
$a_3 = 9.723481817987 \cdot 10^{-1}$	$a_4 = 9.916906780873 \cdot 10^{-1}$
$a_5 = 9.920882424490 \cdot 10^{-1}$	$a_6 = 9.920898977561 \cdot 10^{-1}$
$a_7 = 9.920899040471 \cdot 10^{-1}$	$a_8 = 9.920899138244 \cdot 10^{-1}$
$a_9 = 9.920901957994 \cdot 10^{-1}$	$a_{10} = 9.920946306244 \cdot 10^{-1}$
$a_{11} = 9.921571108651 \cdot 10^{-1}$	$a_{12} = 9.922711396164 \cdot 10^{-1}$
$a_{13} = 9.922763745565 \cdot 10^{-1}$	$a_{14} = 9.922764157620 \cdot 10^{-1}$
$a_{15} = 9.922764176722 \cdot 10^{-1}$	$a_{16} = 9.922764209670 \cdot 10^{-1}$
$a_{17} = 9.922764268741 \cdot 10^{-1}$	$a_{18} = 9.922765032143 \cdot 10^{-1}$
$a_{19} = 9.922815256005 \cdot 10^{-1}$	$a_{20} = 9.922871104032 \cdot 10^{-1}$
$a_{21} = 9.922900570658 \cdot 10^{-1}$	$a_{22} = 9.922902992817 \cdot 10^{-1}$
$a_{23} = 9.922903344673 \cdot 10^{-1}$	$a_{24} = 9.922907505051 \cdot 10^{-1}$
$a_{25} = 9.922944384657 \cdot 10^{-1}$	$a_{26} = 9.922985998787 \cdot 10^{-1}$
$a_{27} = 9.923001857948 \cdot 10^{-1}$	$a_{28} = 9.923001897838 \cdot 10^{-1}$
$a_{29} = 9.923001897925 \cdot 10^{-1}$	$a_{30} = 9.923001898363 \cdot 10^{-1}$
$a_{31} = 9.923001898446 \cdot 10^{-1}$	$a_{32} = 9.923001899441 \cdot 10^{-1}$
$a_{33} = 9.923001970151 \cdot 10^{-1}$	$a_{34} = 9.923010540415 \cdot 10^{-1}$
$a_{35} = 9.923631972537 \cdot 10^{-1}$	$a_{36} = 9.924150286796 \cdot 10^{-1}$
$a_{37} = 9.924166405449 \cdot 10^{-1}$	$a_{38} = 9.924167691190 \cdot 10^{-1}$
$a_{39} = 9.924168651228 \cdot 10^{-1}$	$a_{40} = 9.924168683213 \cdot 10^{-1}$
$a_{41} = 9.924168690765 \cdot 10^{-1}$	$a_{42} = 9.924168705866 \cdot 10^{-1}$
$a_{43} = 9.924169827729 \cdot 10^{-1}$	$a_{44} = 9.924261041648 \cdot 10^{-1}$
$a_{45} = 9.924561402988 \cdot 10^{-1}$	$a_{46} = 9.929732869941 \cdot 10^{-1}$
$a_{47} = 1.000000000000$	$a_{48} = \infty$

$\pi\alpha_1 = 0.13409$	$\pi\alpha_2 = 0.74396$
$\pi\alpha_3 = 0.91497$	$\pi\alpha_4 = 1.22624$
$\pi\alpha_5 = 1.44463$	$\pi\alpha_6 = 0.55829$
$\pi\alpha_7 = 0.37328$	$\pi\alpha_8 = 1.14187$
$\pi\alpha_9 = 0.66783$	$\pi\alpha_{10} = 0.86296$
$\pi\alpha_{11} = 1.62609$	$\pi\alpha_{12} = 1.45652$
$\pi\alpha_{13} = 0.56488$	$\pi\alpha_{14} = 0.72228$
$\pi\alpha_{15} = 0.48188$	$\pi\alpha_{16} = 0.55594$
$\pi\alpha_{17} = 1.26309$	$\pi\alpha_{18} = 1.47476$
$\pi\alpha_{19} = 1.19905$	$\pi\alpha_{20} = 1.47277$
$\pi\alpha_{21} = 0.68150$	$\pi\alpha_{22} = 0.73767$
$\pi\alpha_{23} = 0.71635$	$\pi\alpha_{24} = 0.66891$
$\pi\alpha_{25} = 1.27309$	$\pi\alpha_{26} = 1.27200$
$\pi\alpha_{27} = 1.31560$	$\pi\alpha_{28} = 0.51531$
$\pi\alpha_{29} = 0.81176$	$\pi\alpha_{30} = 0.53170$
$\pi\alpha_{31} = 0.71844$	$\pi\alpha_{32} = 1.45419$
$\pi\alpha_{33} = 0.61998$	$\pi\alpha_{34} = 1.26698$
$\pi\alpha_{35} = 1.86510$	$\pi\alpha_{36} = 0.80241$
$\pi\alpha_{37} = 0.82012$	$\pi\alpha_{38} = 0.69456$
$\pi\alpha_{39} = 1.21217$	$\pi\alpha_{40} = 0.70209$
$\pi\alpha_{41} = 0.38888$	$\pi\alpha_{42} = 0.66800$
$\pi\alpha_{43} = 1.61250$	$\pi\alpha_{44} = 1.29161$
$\pi\alpha_{45} = 0.78862$	$\pi\alpha_{46} = 1.30735$
$\pi\alpha_{47} = 1.18162$	$\pi\alpha_{48} = 1.19610$