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# Skin curves in geometric modeling and morphing

Nico Kruithof

Supervisor: Dr. G. Vegter

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## Abstract

Skin curves form a class of smooth curves and surfaces introduced by Edelsbrunner in [Ede99b], mainly to model large molecules. Like the traditional spline models, skin curves are composed of piecewise polynomial (quadratic) patches. However, the control parameters are weighted points having a direct geometric interpretation that allows for convenient manipulation of skin curves in applications like molecular modeling.

Another application of skin curves is in computer graphics, especially in (shape) morphing of smooth surfaces at the object level. Due to the direct geometric link between the control parameters and the shape of the curves, the morphing process is easily controlled, either by direct user intervention or by imposing geometric constraints (like minimizing the number of topological changes during the morphing process).

Given the flexibility in manipulating skin curves, it would be convenient to have an algorithm that approximates an arbitrary curve by a skin curve. Then it would be possible to use the advantages of morphing skin curves for general shapes. As an easy case, I designed an algorithm that reconstructs the control points for a given skin curve. For approximation, we propose a method using the medial axis.

## Acknowledgements

I would like to thank Gert Vegter for his guidance and many critical notes. Without his encouragement and advice, I would not have achieved this thesis. I would also like to thank my wife Irene for her support.

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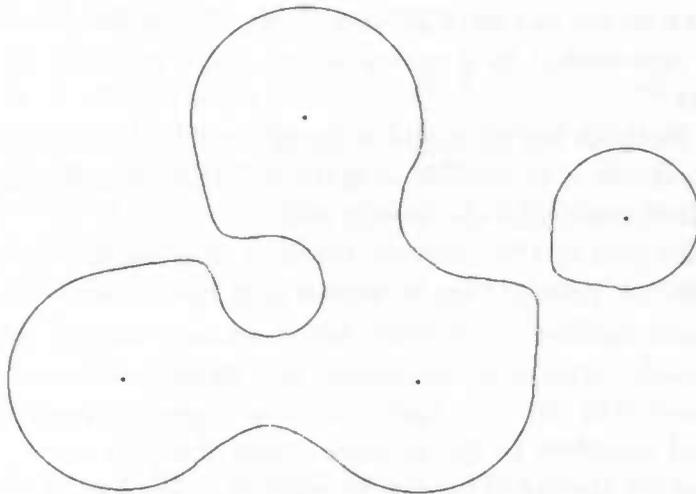


Figure 1: The skin curve defined by four weighted points in  $\mathbb{R}^3$ . The centers of the weighted points are marked.

## Introduction

Object morphing, in contrast with image morphing, is an area of interest in computational geometry. The main goal of object morphing is to transform one object into another. This can be done in infinitely many ways. Many approaches have been suggested. First, there is volume morphing. The main advantage of this class is the ease with which they support changes in topology. On the other hand, memory and computational cost for this class of methods are very high. An alternative is to work directly on the boundary, and to find a correspondence between the two objects. Attempts to find such a correspondence fail for morphs between objects with different topology.

A totally different approach has been suggested by Edelsbrunner in [Ede99b]. He introduces a class of curves, called skin curves and surfaces, that can be changed by manipulating the input set. The control parameters for a skin curve consist of a set of weighted points (circles or spheres) and a factor  $s$  with which the radius of these points are shrunken. The skin curve consists of the union of these shrunken weighted points and patches between the connected points so that the skin curve is one time continuously differentiable, see Figure 1 for a two dimensional example. Because skin curve and surfaces

consist of connected circles, they can be used for molecular modeling. There an atom is represented by a spherical ball and a molecule by the union of a set of atoms.

Several methods are proposed to morph one skin curve into another. Using these methods, it is possible to make restrictions on the way the first skin curve is transformed into the second one.

The main goal of this Masters thesis is to construct an algorithm that approximates an arbitrary curve with a skin curve. Such an approximation scheme would facilitate morphing two arbitrary smooth curves: first approximate each curve by a skin curve, and subsequently construct a morph between these skin curves. Approximation proves to be difficult. Instead, we restricted ourselves to the reconstruction of a skin curve, which is easier since we restrict the set of curves we want to approximate.

## Outline

This paper is organized as follows. Section 1 gives the formal definition of a skin curve and derives a method to visualize skin curves fast. Section 2 discusses some problems encountered in implementing skin curves. Section 3 addresses the problem of reconstructing a skin curve. Section 4 handles three methods of morphing found in the literature. Section 5 concludes the paper with a global overview and future work.

# 1 Definition of a skin curve

In this section we will describe the input set for a skin curve, i.e. weighted points, how they can be transformed and how two or three points relate to each other in terms of affine and convex hulls. Then we will proceed with a formal definition of the skin curve, expressed in terms of convex hulls and we will give some properties. Finally we will make the relationship between this implicit definition and a representation of the skin curve in terms of the mixed complex.

## 1.1 Basic concepts

In this section we will have a closer look at the input set for a skin curve, a set of weighted points  $P$ . We will discuss some of their properties, and operations that can be performed upon them. Also a short definition of the regular triangulation, which is the weighted counterpart of the Delaunay triangulation, will be given. Finally we will give the definition of the power diagram, which is the dual of the regular triangulation.

### 1.1.1 Weighted points

Definitions and the outline are very similar to [Ede99b, pag. 90 and further]. Each weighted point  $p$  is a pair  $(z_p, w_p)$  of location and weight, where  $z_p \in \mathbb{R}^d$  and  $w_p \in \mathbb{R}$ . Its main attribute is its distance to other (weighted) points. The (*weighted*) distance from a weighted point  $p$  to a point  $x$  is defined by

$$\pi_p(x) = \|x - z_p\|^2 - w_p \quad (1)$$

where  $\|v\|$  is the Euclidean norm of a vector in  $\mathbb{R}^d$ .

The zero-set of  $\pi_p$  is the set of points for which  $\pi_p = 0$ . In two and three dimensions, these are the circle and sphere with center  $z_p$  and radius  $\sqrt{w_p}$  respectively. For convenience's sake, we will make no difference between a weighted point and this sphere. Weighted points can also have negative weights, which correspond to circles with imaginary radii.

For a sphere with zero radius (a point), the weighted distance to a point is equal to the square of the Euclidean distance. When the distance from a point to a weighted point is negative (positive), this point lies within (outside) the circle.

From now on, we will restrict ourselves to weighted points in  $\mathbb{R}^2$  (i.e.,  $d = 2$ ). In order to be able to calculate with weighted points, we will lift a two dimensional weighted point to a three dimensional space, where we will use the usual vector space operations to add two weighted points and multiply a weighted point with a scalar. For the transformation we will use the bijection  $\Pi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^3$ , defined by

$$\Pi(p) = (\xi_1, \xi_2, \|z_p\|^2 - w_p), \text{ where } z_p = (\xi_1, \xi_2), \quad (2)$$

and we have:  $\Pi(P) = \{\Pi(p) | p \in P\}$ . When we want to add two weighted points, first we project them to the three dimensional vector space, there we can add the two projected points, and then that point can be transformed back, in a formula  $p + q = \Pi^{-1}(\Pi(p) + \Pi(q))$ . For multiplication by scalars there exists a similar formula:  $\gamma \cdot p = \Pi^{-1}(\gamma * \Pi(p))$ .

Assume that  $z_p = (\zeta_1, \zeta_2)$  and  $z_q = (\xi_1, \xi_2)$ , when we expand the formula above, we get:

$$\begin{aligned} p + q &= \Pi^{-1}(\Pi(p) + \Pi(q)) \\ &= \Pi^{-1}((\zeta_1, \zeta_2, \|z_p\|^2 - w_p) + (\xi_1, \xi_2, \|z_q\|^2 - w_q)) \\ &= \Pi^{-1}((\zeta_1 + \xi_1, \zeta_2 + \xi_2, \|z_p\|^2 + \|z_q\|^2 - w_p - w_q)) \\ &= (z_{p+q}, \|p + q\|^2 - \|z_p\|^2 - \|z_q\|^2 + w_p + w_q) \\ &= (z_{p+q}, 2 \langle z_p, z_q \rangle + w_p + w_q) \end{aligned}$$

where  $\langle v, w \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is the usual scalar product. Similar calculations for scalar multiplication lead to  $\gamma \cdot p = (\gamma \cdot z_p, \gamma w_p + (\gamma^2 - \gamma) \|z_p\|^2)$ . Notice that the usual vector space operations in  $\mathbb{R}^2$  hold for the centers of weighted points.

The *upward closure* of a weighted point, consists of all points with same center, but less or equal weight,  $\text{ucl}(p) = \{q | z_q = z_p \wedge w_q \leq w_p\}$ . In the transformed vector space, this corresponds to the ray  $\{\Pi(p) + \gamma \cdot (0, 0, 1) | \gamma \geq 0\}$ . The upward closure of  $p$  can also be seen as the disc, with boundary  $p$ .

All circles with zero radius (i.e. all points) form a paraboloid  $\varpi = \{\Pi(p) | p = (z_p, 0) \in \mathbb{R}^2 \times \mathbb{R}\}$  in the projected three dimensional vector space. All points above  $\varpi$  correspond to circles with imaginary radius.

A point  $p$  can be shrunken or grown with a factor  $s$ . Let  $p'$  be the circle with the same center as  $p$  and zero weight, then  $p^s = s * p + (1 - s) * p'$ . This is the same as shrinking the radius with a factor  $\sqrt{s}$  and leaving the center constant. When we generate a skin curve,  $s$  lies on the interval  $[0, 1]$ . Notice that in this case imaginary circles stay imaginary and real circles stay real.

We can extend the weighted distance function to pairs of weighted points, say  $p$  and  $q$ :

$$\pi_{p,q} = \|z_p - z_q\|^2 - w_p - w_q. \quad (3)$$

Notice, that if  $p$  or  $q$  is a point (i.e., its weight is zero), the familiar distance function appears. Furthermore, if  $\pi_{p,q} = 0$  then we say that the spheres  $p$  and  $q$  are *orthogonal*, denoted  $p \perp q$ . If both circles have positive radii, then tangent lines in the intersection point meet in a right angle. This can be seen, using Pythagoras's Theorem. If  $\pi_{p,q} > 0$  then  $p$  and  $q$  are said to be *further than orthogonal*.

Assume that two spheres  $p$  and  $q$  are orthogonal and have positive weight. Let  $s, t \geq 0$  and  $s + t = 1$  then

$$p^s \cap q^t = \begin{cases} x & \text{if } s \cdot w_q = t \cdot w_p \\ \emptyset & \text{otherwise} \end{cases} \quad (4)$$

where  $x = (w_q \cdot z_p + w_p \cdot z_q) / (w_p + w_q)$ .

*Proof.* Since  $p \perp q$  and  $s + t = 1$  we have

$$\begin{aligned} \|z_p - z_q\|^2 &= w_p + w_q \\ &\geq (s+t)(w_p + w_q) - (\sqrt{sw_q} - \sqrt{tw_p})^2 \\ &= (\sqrt{sw_p} - \sqrt{tw_q})^2 \end{aligned}$$

This implies that the radii of  $p^s$  and  $q^t$  add up to at most the Euclidean distance between the centers, and do so when  $\sqrt{s \cdot w_q} = \sqrt{t \cdot w_p}$ . This is equivalent to  $s \cdot w_p = t \cdot w_q$ , as claimed above. It follows that, in that case  $s = w_p / (w_p + w_q)$  and  $t = w_q / (w_p + w_q)$ , and the two spheres meet in the point  $x = s \cdot z_p + t \cdot z_q$ .

□

### 1.1.2 Flats and focuses

In the previous section we described the vector space of weighted points, now we will use this vector space structure to define affine and convex hulls. Furthermore, we will give the formal definition of a focus.

An *affine hull* of a set of points  $P = \{p_0, p_1, \dots, p_n\}$  is defined as

$$\text{aff}(P) = \left\{ \sum_{i=0}^n \gamma_i p_i \mid \sum_{i=0}^n \gamma_i = 1 \right\}. \quad (5)$$

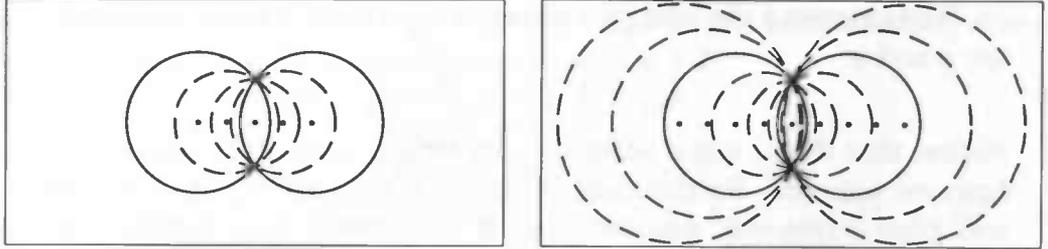


Figure 2: The convex and affine hull of two weighted points.

A set  $P$  is *affinely independent* (a.i.) if  $p \notin \text{aff}(P - \{p\})$  for all  $p \in P$ . The maximum cardinality for any a.i. set of weighted points in two dimensions is four. If  $P$  has cardinality  $k + 1$  and is affinely independent then  $\Pi(\text{aff}(P))$  is a  $k$ -dimensional subspace of the transformed vector space  $\mathbb{R}^3$ . If  $P$  is affinely independent then the affine hull forms the affine subspace spanned by the points in  $P$ . For example, if  $P = \{p, q\}$  then the affine hull consists of all points on the line through  $\Pi(p)$  and  $\Pi(q)$ , for three points it is a plane, and for four points it spans  $\mathbb{R}^3$ .

A  $k$ -flat is the affine hull of an affinely independent set  $P$  of cardinality  $k + 1$ . Its dimension is  $k$ . The 0-flat is a finite set of points and has one element. Flats of dimension 1 or higher are infinite sets of spheres.

The *convex hull* is a subset of the affine hull and is defined by:

$$\text{conv}(P) = \left\{ \sum_{i=0}^n \gamma_i p_i \mid \sum_{i=0}^n \gamma_i = 1 \wedge \gamma_i \geq 0 \text{ for all } i \right\}. \quad (6)$$

Let  $\eta_p(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ , the function that gives the weight for which a point  $x$  is orthogonal to a circle  $p$ , defined by:

$$\begin{aligned} \eta_p(x) &= \|x\|^2 - \pi_p(x) \\ &= 2 \langle x, z_p \rangle - \|z_p\|^2 + w_p \end{aligned}$$

Every weighted point  $(x, \eta_p(x))$  has zero distance to  $p$ . Its graph is a plane in the transformed vector space  $\mathbb{R}^3$ . Calculations show that the plane is parallel to the tangent plane of the paraboloid  $\varpi$  in  $(z_p, \|z_p\|)$ .

Previously, we introduced a flat of a set  $P$  of weighted points. Similarly, we can introduce an orthogonal flat  $P^\perp$  as the flat of points that are

orthogonal to every point in  $P$ :

$$\begin{aligned} P^\perp &= \{q \mid q \perp p \text{ for all } p \in P\} \\ &= \bigcap_{p \in P} p^\perp \\ &= \bigcap_{p \in P} \{(x, \eta_p(x)) \mid x \in \mathbb{R}^2\} \end{aligned}$$

We also have  $P^\perp = \text{aff}(P)^\perp$ . If  $P$  has cardinality one this is trivial, because  $\text{aff}(P) = P$ . We will show that it also holds when  $P$  has cardinality two. Assume that  $P = \{p_0, p_1\}$ . Let  $\{(x, \eta_{p_0}(x)) \mid x \in \mathbb{R}^2\}$  and  $\{(x, \eta_{p_1}(x)) \mid x \in \mathbb{R}^2\}$  be two planes in  $\mathbb{R}^3$  of points that are orthogonal to  $p_0$  and  $p_1$ , respectively. They intersect in a line  $k$ . Let  $q_0, q_1 \in k$  and  $q_0 \neq q_1$  then  $\text{aff}(\{q_0, q_1\}) = k$ . Similarly to the points in  $P$  we can define two planes of spheres that are orthogonal to  $q_0$  and  $q_1$ . Let the intersection of these two planes be the line  $l$ . Notice that  $p_0, p_1 \in l$  because of the commutativity of orthogonality and we have  $\text{aff}(\{p_0, p_1\}) = l$ . This leads us to the conclusion that  $P^\perp = \text{aff}(\{q_0, q_1\}) = \text{aff}(P)^\perp$ . We will omit the proof for the case that  $P$  has cardinality three, because it is similar to the second case, but then the three planes meet in point, which is a 0-flat. Another result is  $\text{aff}(P) = (P^\perp)^\perp$ .

If  $P$  and  $P' = \{z_p \mid p \in P\}$  are both affinely independent sets, we can define the *focus* of  $P$ ,  $p_f = \text{foc}(P)$ . The focus is the weighted point on the affine hull,  $F = \text{aff}(P)$ , with minimum weight. This definition is slightly different from [Ede99b], in the sense that it is a weighted point instead of a normal point. Its center is the same as the focus in [Ede99b].

For example, let us determine the focus of two weighted points  $p_0$  and  $p_1$ ,  $z_{p_0} \neq z_{p_1}$ . The focus lies on the affine hull  $F = \text{aff}(P) = \gamma p_1 + (1 - \gamma)p_0$  and we can rewrite the weight in terms of  $\gamma$ . This leads to a quadratic function, which has its minimum in

$$\gamma = \frac{w_{p_0} - w_{p_1} + \|z_{p_1} - z_{p_0}\|^2}{2\|z_{p_1} - z_{p_0}\|^2}.$$

Notice that the focus doesn't necessarily have to lie between  $p_0$  and  $p_1$ . E.g., if  $w_{p_1} > w_{p_0} + \|z_{p_1} - z_{p_0}\|^2$ ,  $\gamma$  becomes negative and the focus lies outside the segment  $\overline{p_0 p_1}$ .

For a set of weighted points  $P$  let  $P'$  be the set of their centers,  $P' = \{z_p \mid p \in P\}$ . We will show that the center of a focus is an element of  $G' = \{z_q \mid q \in G\}$  where  $G = P^\perp$ . Assume again that  $P = \{p_0, p_1\}$ . Let  $x_\gamma \in F'$ ,

so it is of the form  $x_\gamma = \gamma \cdot z_{p_1} + (1 - \gamma) \cdot z_{p_0}$ . We will calculate the value of  $\gamma$  for which  $x_\gamma$  has the same distance to each  $p \in P$ :

$$\begin{aligned}
\pi_{p_0}(x_\gamma) &= \pi_{p_1}(x_\gamma) \\
\|z_{p_0} - x_\gamma\|^2 - w_{p_0} &= \|z_{p_1} - x_\gamma\|^2 - w_{p_1} \\
\|z_{p_0} - (\gamma z_{p_1} + (1 - \gamma)z_{p_0})\|^2 - w_{p_0} &= \|z_{p_1} - (\gamma \cdot z_{p_1} + (1 - \gamma) \cdot z_{p_0})\|^2 - w_{p_1} \\
\gamma^2 \|z_{p_1} - z_{p_0}\|^2 - w_{p_0} &= (1 - \gamma)^2 \|z_{p_1} - z_{p_0}\|^2 - w_{p_1} \\
2\gamma \|z_{p_1} - z_{p_0}\|^2 &= \|z_{p_1} - z_{p_0}\|^2 - w_{p_1} + w_{p_0} \\
\gamma &= \frac{\|z_{p_1} - z_{p_0}\|^2 - w_{p_1} + w_{p_0}}{2\|z_{p_1} - z_{p_0}\|^2}
\end{aligned}$$

which is exactly the value of  $\gamma$  in the focus. Furthermore we have  $x_\gamma \in G'$ , because the weighted point  $(x_\gamma, \eta_{p_0}(x_\gamma))$  is orthogonal to both  $p_0$  and  $p_1$ , so it is an element of  $G$ . Similar calculations can be done when  $P$  has cardinality one or three. Notice that  $w_{p_f} + w_{p_g} = 0$ , because  $p_f$  and  $p_g$  have the same center and are orthogonal.

For a given focus  $f$  we can define the *orthogonal focus*  $g$  as the point with the same location in the orthogonal flat,  $g = \{q | z_f = z_q \wedge q \in f^\perp\} = (z_f, -w_f)$ .

It can also be seen that the center of the focus and of the orthogonal focus have to be the same. We know that every  $p \in F$  is orthogonal to every  $q \in G$ ; this means that  $\pi_{p,q} = \|z_p - z_q\|^2 - w_p - w_q = 0$ . Let  $q$  be any arbitrary fixed point, in order to minimize the weight of  $p$  we need to minimize the distance between  $p$  and  $q$ . With  $F'$  orthogonal to  $G'$  we know this is the case when  $z_p \in G'$ .

For every set  $P$  the dimension of the projected flat and the dimension of the projected orthogonal flat add up to two.

For a set  $P$  of weighted points, let  $F = \text{aff}(P)$  and  $G = \text{aff}(P^\perp)$ . Let  $F^s = \{p^s | p \in F\}$ ,  $G^s = \{q^s | q \in G\}$ , for  $s, t > 0$  and  $s + t = 1$ . Then:

$$\begin{aligned}
\bigcup \text{ucl}(F^s) \cup \bigcup \text{ucl}(G^t) &= \mathbb{R}^2 \\
\bigcup \text{ucl}(F^s) \cap \bigcup \text{ucl}(G^t) &= \text{env}(F^s) \\
&= \text{env}(G^t)
\end{aligned} \tag{7}$$

where  $\text{env}(F)$  is the boundary of  $F$ . For the proof, we refer to [Ede99b, pag. 95]. In Figure 4, we visualized the shrunken flats of Figure 3 for  $s = 1 - t = 0.5$

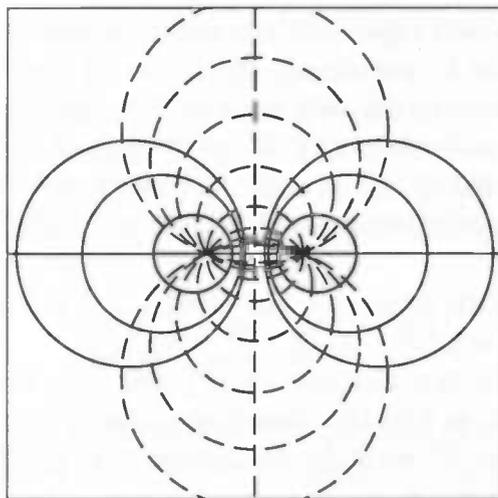


Figure 3: Two orthogonal 1-flats.

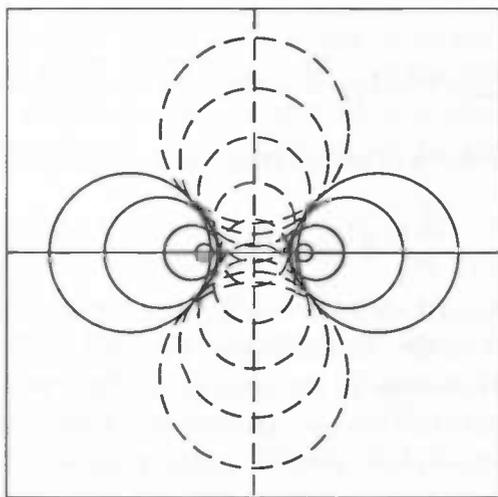


Figure 4: Two orthogonal shrunken flats.

### 1.1.3 Envelope

In this section, we will show that the envelope of a family of circles forming  $k$ -flat is the zero-set of a quadratic function. To give an intuitive picture on how this formula can be derived, we will do so for a 1-flat.

Assume that the centers of this flat  $F$  form the  $x_1$ -axis and the centers of the orthogonal flat  $G$  the  $x_2$ -axis. Let the focus of  $F$  be  $p_f$ . Notice that  $z_{p_f} = \vec{0}$ . In the transformed space,  $F$  is the line parallel to the  $x_1$ -axis through  $(0, 0, -w_{p_f})$ .

We shrink  $F$  with a factor  $s$  and  $G$  with  $t = 1 - s$ ,  $s, t > 0$ . The weight in a point  $x \in F^s$  is  $w(x) = s \cdot (\|x\|^2 - w_{p_f})$ .

In order to find the distance from point  $x = (x_1, x_2)$  to the shrunken affine hull, we need to find the minimum distance from each point  $p \in F^s$  to  $x$ . We parameterize  $F^s$  with the parameter  $\tau$ :  $p_\tau \in F^s$  with  $z_{p_\tau} = (\tau, 0)$ . We get the distance function:

$$\begin{aligned} f(\tau, x) &= \pi_{p_\tau}(x) \\ &= \|x - z_{p_\tau}\|^2 - w(p_\tau) \\ &= (x_1 - \tau)^2 + x_2^2 - s(\tau^2 + w_{p_f}). \end{aligned}$$

In order to find the minimum we need to solve  $\frac{\partial f}{\partial \tau} = 0$  and we get  $\tau = \frac{x_1}{1-s}$ . This leads to

$$f_{\min}(x) = f\left(\frac{x_1}{1-s}, x\right) = \frac{-sx_1^2 + tx_2^2 - stw_{p_f}}{t}.$$

With simple formula manipulation, the envelope is the zero-set of:

$$\frac{1}{s}f_{\min}(x) = -\frac{1}{t}x_1^2 + \frac{1}{s}x_2^2 - w_{p_f}. \quad (8)$$

A 0-flat consists of only one weighted point and its zero-set is the corresponding shrunken circle. According to Formula (7), the envelope of a 2-flat is the same as the envelope of its orthogonal flat, which is a 0-flat.

Now, we can write the zero-set in a general form: let  $s, t > 0$  and  $s+t = 1$ . The common envelope of  $F^s$  and  $G^t$  is the zero-set of

$$f_{\min}(x) = -\frac{1}{t} \sum_{i=1}^k x_i^2 + \frac{1}{s} \sum_{j=k}^2 x_j^2 - w_{p_f}, \text{ with } x = (x_1, x_2), \quad (9)$$

where  $k$  is the dimension of  $F^s$  and  $p_f = \text{foc}(F)$ .

## 1.2 Formal definition

In this section we will give the formal definition of the skin curve using the definitions and concepts of the previous sections. This definition holds for any dimension, but we will restrict ourselves again to the plane.

A *skin curve* (skn) is defined as the boundary of a *s-body*. This *s-body* (bdy) is the union of the discs defined by the weighted points in the convex hull of a, possibly infinite, set  $P$  of weighted points after they are shrunken with a factor  $s$ . Formally:

$$\begin{aligned} \text{bdy}^s P &= \bigcup \text{ucl}(\text{conv}(P)^s), \\ \text{skn}^s P &= \text{bd bdy}^s P. \end{aligned} \tag{10}$$

Each skin curve can be defined as the envelope of two different *s-bodies*. The intersection of these two *s-bodies* is exactly the skin curve and the union forms the whole  $\mathbb{R}^2$ . Therefore it makes sense to make a distinction between these two *s-bodies*. Assume that a skin curve is formed by a set of weighted points  $P$ . We say that this set defines the skin curve from the inside and the orthogonal set defines the skin from the outside.

For an arbitrary skin curve defined by a set of weighted point  $P$  and a shrinking factor  $s$ , we would like to generate an orthogonal set  $Q$  that defines the skin curve from the outside. To do this, first we will generate the set of spheres that are (further than) orthogonal to all spheres  $p \in P$ , formally  $\{q | \pi_{p,q} \geq 0, \text{ for all } p \in P\} = \bigcap_{p \in P} \text{ucl}(p^\perp)$ . This infinite set of weighted points define, when shrunken with a factor  $1 - s$ , the skin curve from the outside. We define the set  $Q$  as the smallest set for which  $\text{ucl}(\text{conv}(Q)) = \bigcap_{p \in P} \text{ucl}(p^\perp)$ .

$Q$  may contain points with arbitrarily large radii. More precisely, every such sphere is orthogonal to two spheres in  $P$  and further than orthogonal to all other spheres. Similarly, every finite sphere in  $Q$  is orthogonal to three spheres in  $P$  and further than orthogonal to all other spheres.

We can also define  $Q$  in terms of the Delaunay triangulation and the Voronoi diagram introduced in Section 1.3. We divide  $Q$  in two subsets, one with spheres with finitely large weights and the other of spheres with infinitely large spheres. The first set correspond to the set of vertices of the Voronoi diagram, which is the set of focuses of Delaunay faces. Every point in the second set corresponds to the point that lies on the infinite end of a (half-) open edge. It is orthogonal to two Delaunay vertices that form an edge on the convex hull.

### 1.3 Delaunay triangulation and Voronoi diagram

In this section we will introduce the Delaunay complex and Voronoi diagram of a set of weighed points in the plane. Furthermore, we will show that a mixture between these two, the mixed complex, subdivides the skin curve in patches of degree two.

#### 1.3.1 Voronoi diagram

We will discuss the Delaunay complex and Voronoi diagram only briefly here. For a more profound definition, we suggest [dBvKOS00, ch. 7 and 9], where the non-weighted Delaunay triangulation and Voronoi diagram are discussed. The only difference is the distance function, which is given by Formula (1) and Formula (3). These diagrams are also called the regular triangulation and power diagram, respectively.

The *Voronoi cell* of a weighted point  $p \in P$  consists of all points that are closer to  $p$  than to any other point in  $P$ , formally:

$$V_p = \{x \in \mathbb{R}^2 \mid \pi_p(x) \leq \pi_q(x), \text{ for all } q \in P\}. \quad (11)$$

Notice that  $V_p$  and  $V_q$  intersect in a line formed by the centers of their orthogonal flat,  $\{z_r \mid r \in \{p, q\}^\perp\}$ , which is the line  $\{x \mid \pi_p(x) = \pi_q(x)\}$ . When we intersect this line with  $\overline{pq}$ , we will find the location of the focus.

In contrast with the regular Voronoi cells described in [dBvKOS00], weighted Voronoi cells can be empty. A non-empty Voronoi cell  $V_p$  doesn't necessarily have to contain  $p$ . In Section 1.1.2 we saw that the focus of two points does not necessarily have to lie between these points. We also saw that the intersection of two Voronoi-cells is the line defined by the centers of their orthogonal flat, which goes through the center of the focus. If the focus of  $p$  and  $q$  lies outside  $pq$  then  $p \notin V_p$  or  $q \notin V_q$ .

A non-empty Voronoi cell is a 2-dimensional convex polyhedron. This can be seen by slightly manipulating its definition. We can define  $V_p = \bigcap_{q \in P \wedge q \neq p} H(p, q)$ , where  $H(p, q) = \{x \mid \pi_p(x) \leq \pi_q(x)\}$  is a half-plane. We know that the intersection of any convex polygon with a half-plane is either empty or a convex polygon. This finishes our proof

For a subset  $X \subseteq P$  we have:

$$\nu_X = \bigcap_{p \in X} V_p$$

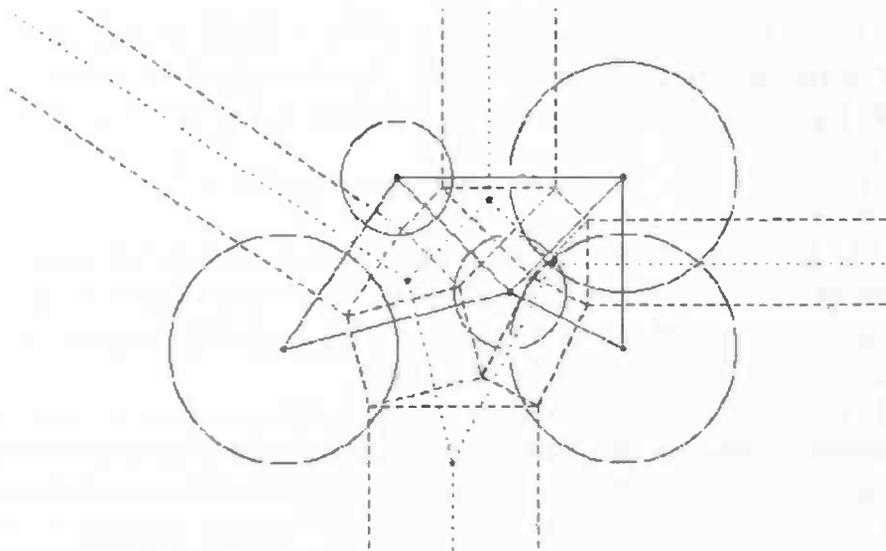


Figure 5: The Delaunay triangulation (solid), Voronoi complex (dotted) and the Mixed complex (dashed).

which we will call a *Voronoi  $\ell$ -cell* ( $\ell = \dim \nu_X = 3 - |X|$ ), if it is not empty and  $X$  is in general position. In the nondegenerate case, every 0-cell is a point, which is the intersection of three adjacent cells. It corresponds to the focus of the three weighted points. A 1-cell is a line segment, which is the common border of two 2-cells, goes through the focus of the two points and ends in a 0-cell. A 2-cell is a nonempty Voronoi-cell of some  $p \in P$ . All vertices of this polyhedron are 0-cells and all edges are 1-cells.

The *Voronoi diagram* consists of all non-empty Voronoi  $\ell$ -cells:

$$\text{Vor } P = \{\nu_X \neq \emptyset \mid X \subseteq P\}. \quad (12)$$

For an intuitive picture, we refer to Figure 5. Notice that all (extended) Voronoi edges of overlapping weighted points, go through the intersection points of their circles. These are the points where both distances are zero, and therefore equal.

### 1.3.2 Delaunay triangulation

The definition of the Delaunay complex can be based upon the Voronoi-diagram, and is defined in terms of the Delaunay cells. There is a *Delaunay*

$(2 - \ell)$ -cell for every Voronoi  $\ell$ -cell  $\nu_X \in \text{Vor } P$  and is formed by the convex hull of the centers of the points in  $X$ . In a formula  $\delta_X = \text{conv}(X')$ , where  $X' = \{z_p | p \in X\}$ . The *Delaunay triangulation* is the set of all these cells:

$$\text{Del } P = \{\delta_X | \nu_X \in \text{Vor } P\}. \quad (13)$$

The Delaunay triangulation consists thus of triangles (2-cells), line segments (1-cells) and points (0-cells). Notice that every vertex is the center of a sphere in  $P$  and that every Delaunay edge meets the Voronoi edge in the focus of the weighted points by definition.

In the previous section we introduced an orthogonal set  $Q$ , with the same skin curve. How do the Voronoi diagram and the Delaunay triangulation relate to  $Q$ ? We saw that a Voronoi 1-cell is part of the set of centers of its orthogonal flat. Similarly, every 0-cell of the Voronoi diagram is the center of its orthogonal flat, which consists of one point, the focus. Also notice that a Voronoi  $\ell$ -cell  $\nu_X$  is not empty iff  $\delta_X$  is an element of the Delaunay triangulation. So, for a  $\delta_X$  we have  $\text{foc}(X) \in Q$ . Now we only need to expand this definition in order to accompany the infinitely large spheres in  $Q$ . These are the infinite endpoints of a (half-) open Voronoi edge.

These two graphs form the base for the skin curve, like the affine hull and its orthogonal flat formed it for two points.

### 1.3.3 Mixed complex

The *mixed complex* is a mixture between the Delaunay complex and the Voronoi diagram. More precisely, for every  $\nu_X$  and  $\delta_X$  a *mixed cell* is defined by  $\mu_X^s = s \cdot \nu_X + t \cdot \delta_X$ , where  $s, t \geq 0$  and  $s + t = 1$  as usual. For  $s = 0$ , the mixed cell is the Delaunay cell. When  $s$  increases, it deforms linearly into the Voronoi cell for  $s = 1$ . We will call a mixed cell corresponding to a Delaunay  $\ell$ -cell a mixed  $\ell$ -cell.

For a Delaunay vertex, the mixed cell (a mixed 0-cell) consists of all points on the corresponding Voronoi cell, shrunken towards the Delaunay vertex with a factor  $s$ . An example is drawn in the left image of Figure 6.

The mixed cell (a mixed 1-cell) of a Delaunay edge, and its corresponding Voronoi edge, is a rectangle. This rectangle can be open on one or both sides, depending on whether the Voronoi edge is infinite on that side. The middle image of Figure 6 is an example of the generation of such a cell for a finite Voronoi edge.

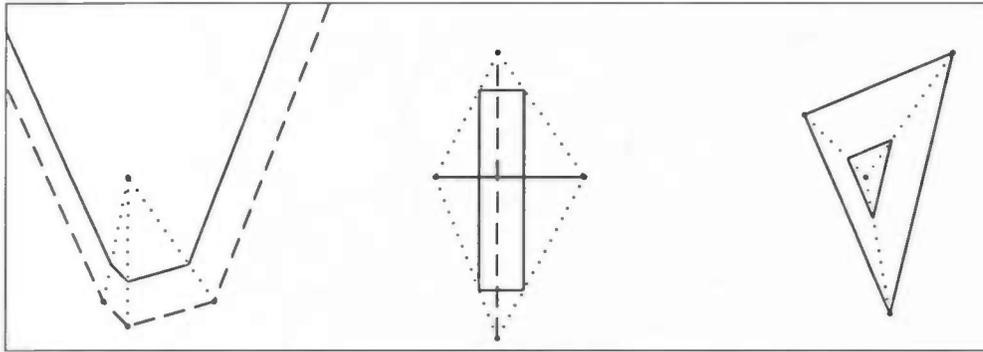


Figure 6: The generation of a mixed cell (solid) from a Delaunay  $\ell$ -cell (also solid) and Voronoi  $(2 - \ell)$ -cell (dashed). For  $\ell$  from 0 to 2 and  $s = 0.7$ .

For a Delaunay triangle, the mixed cell (a mixed 2-cell) consists of a shrunken triangle. Their endpoints lie on the line from the corresponding Voronoi vertex to each of the endpoints of the Delaunay triangle, see the right image of Figure 6.

The mixed complex is the set of all mixed cells:

$$\text{Mix}^s P = \{\mu_X^s \mid \nu_X \in \text{Vor } P\}. \quad (14)$$

An example is given in Figure 7. The mixed  $\ell$ -cells are marked with their numbers. Notice that mixed 0- and 2-cells contain arcs of circles and mixed 1-cells contain parts of hyperbolic sheets. These curves correspond to the envelope of the affine hull of the points in the Delaunay cell.

Now, we want to prove that the mixed complex subdivides the mixed complex into patches of degree two. We will do this by computing the point where one affine hull stops contributing to the skin curve, and the contribution of another affine hull starts. We will do so for the skin curve of two points, and omit the proof for the other cases, because of similarity.

Assume that  $P = \{p_0, p_1\}$  then the skin curve consists of  $\text{ucl}(\text{conv}(P)^s)$ . This is formed by a part of  $p_0^s$ , a part of  $p_1^s$  and a part of  $\text{aff}(P)^s$ . See also Figure 8. Notice that these are all zero sets of functions of degree 2. In order to find the point where the affine hull changes into  $p_0^s$ , we need to find the intersection point  $x$  of  $p_0^s$  with the affine hull. Formula (4) says that a point  $p$  on the affine hull, intersects with a point  $q$  on the orthogonal flat if  $s \cdot w_q = t \cdot w_p$ . This implies that  $p_0^s$  meets a point  $q_0^s \in (P^s)^\perp$  with

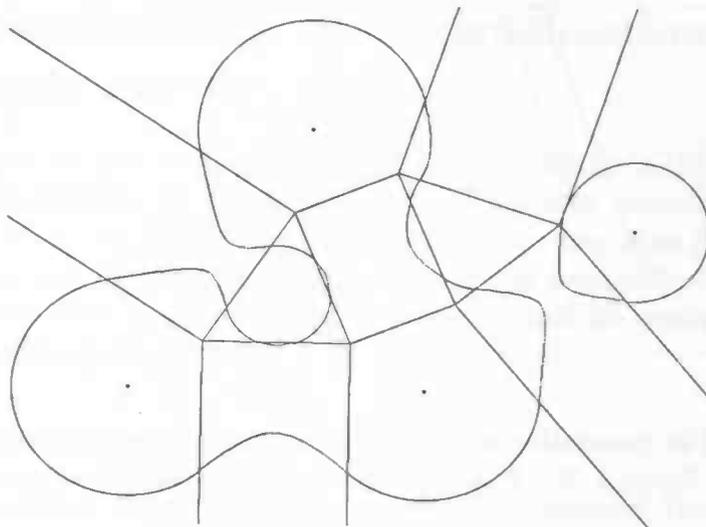


Figure 7: The skin curve of four weighted points. Each mixed cell contains parts of the envelope of a flat.

$w_{q_0} = t/s \cdot w_{p_0}$ . It follows that  $\|p_0 - x\|/\|q_0 - x\| = s/t$ . The same reasoning can be applied for  $p_1$ . Thus every change of flats lies on an edge of the mixed cell. This is a technical statement that the skin curve is effectively decomposable into patches of degree 2.

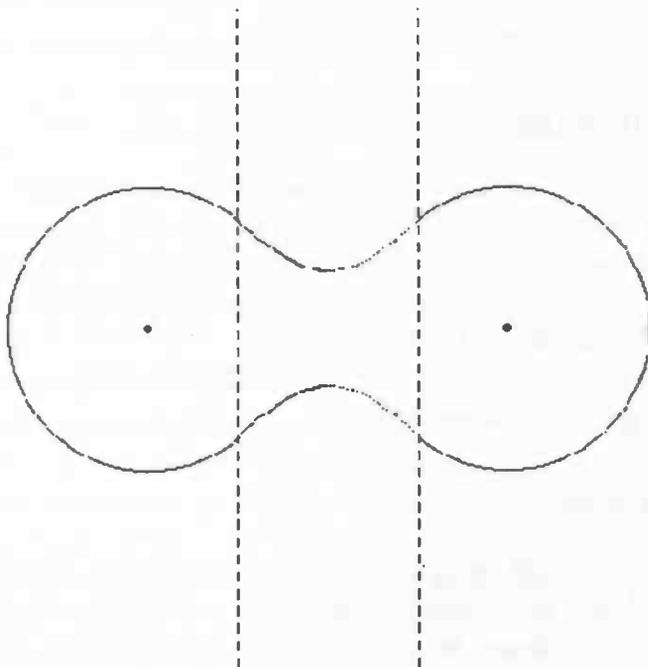


Figure 8: The skin curve of two weighted points. The mixed complex divides the skin curve into patches of degree 2.

## 2 Implementation of a visualization toolkit for skin curves

We implemented skin curves in two dimensions using CGAL [CGA] and LEDA [LED],[MN00]. In this section we will describe globally how this is done. First of all, we will discuss how we built our complexes (the Delaunay triangulation, the Voronoi diagram and the mixed complex). In the second section, we will show how we selected only the parts of the quadratic function that lies within a mixed cell.

### 2.1 Complexes

For our complexes we used CGAL, in which a weighted Delaunay triangulation has been implemented. Both the Voronoi diagram and the Mixed complex are based upon the Delaunay triangulation. In the previous chapter, we discussed how Voronoi  $\ell$ -cells could be derived from Delaunay  $(2 - \ell)$ -cells using focuses of adjacent Delaunay cells.

Once we have constructed a Voronoi cell from a Delaunay cell we can use these two to generate mixed cells.

### 2.2 Clipping

This section consists of two parts. First we will discuss the clipping for mixed cells that are based on Delaunay edges. After that, we will discuss clipping of a mixed cell that is based on Delaunay vertices and faces, which in principle is the same.

A mixed cell associated with a Delaunay edge is always a rectangle, possibly, open at one or two sides. Therefore it is easy to compute the endpoints of the zero set of the affine hull corresponding to the Delaunay edge (the intersection points of a quadratic function with the clipping rectangle) and generate the proper segments.

The second case is a little more difficult. The envelope of the flat corresponds to a circle and the mixed cell is a triangle. Only the part of the circle inside the triangle has to be drawn. So we need to clip a circle against a triangle. Because of the large number of possible cases in which the triangle can lie over the circle, we used a more general method.

We are going to generate a set of line segments. The part of the circle that

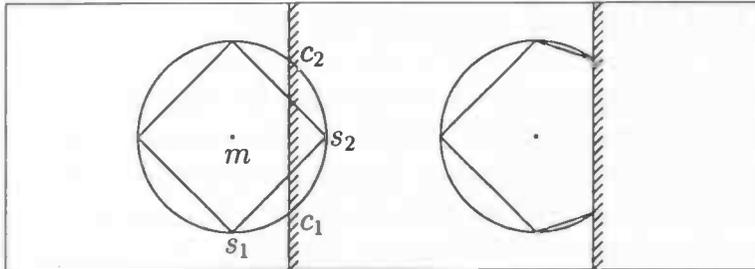


Figure 9: Initial circle (left), and the circle after clipping (right).

$\det(c_1 - m, s_1 - m)$	$\det(c_2 - m, s_2 - m)$	new segment
negative	negative	$(s_1, c_1)$
positive	negative	none
negative	positive	$(s_1, c_1)$ and $(c_2, s_2)$
positive	positive	$(c_2, s_2)$

Table 1: Four cases that a line segment can be clipped.

has to be drawn is generated by projecting every segment from the midpoint onto the circle, e.g. initially we have four segments in a square around the midpoint. Each segment is projected onto a quarter of the circle. Together they form the whole circle. See Figure 9, the left image. There are two restrictions on the set of line segments: first, all segments generate less than half a circle (the smallest arc is always drawn). Secondly, each segment is directed and the center of the circle is located to its left.

Instead of clipping the whole triangle at once, we clip three half-spaces. Each clipping operation takes time linear in the number of segments. First of all, determine the two intersection points  $c_1$  and  $c_2$  of the circle with the edge of the halfplane. The case that the circle does not touch the edge of the halfplane is trivial and the case that they only touch in 1 point is a degenerate case. Assume that the halfplane lies to the right of the directed line through  $c_1$  and  $c_2$

We call the center of the circle  $m$ . For each segment  $S = (s_1, s_2)$  we can determine whether it has to be clipped and what the new segment will be using Table 1. See e.g., the right image of Figure 9.

### 3 Reconstruction of a skin curve

In this section we aim to an approximation algorithm for an arbitrary curve. Having such an algorithm we can use the advantages of morphing skin curves for any pair of curves. This can be done by approximating both curves with a skin curve and then morping the first skin curve into the second.

Approximation of an arbitrary curve proved very difficult. Therefore, as an easy case, we will reconstruct the input set from a skin curve. Reconstruction is an easy case of approximation, because we restrict the arbitrary curve to the class of skin curves. Therefore, reconstruction can be seen as a first step towards approximation.

Different sets of weighted points may generate the same skin curve. To see this, let us analyse the definition of a skin curve, Formula (10). Adding a point  $p \in \text{conv}(P)$  to the set of weighted points  $P$  will not change the skin curve, because that point would already be generated by the convex hull. The same holds for points in the upward closure of the convex hull. Even, when we do not add points to the input set, but only vary the points in the input set, this does not necessarily affect the skin curve. For reconstruction this means that the reconstructed set of weighted points does not have to be the same as the original set the skin curve was constructed from.

Because the set of weighted points is not defined uniquely by its skin curve, therefore we do not have to find the weighted points that construct the skin curve, but we need to find the regions in the space of weighted points that contain points that can be added without changing the skin curve.

In the next section we determine conditions under which addition of a weighted point does not affect a certain part of the current skin curve. These conditions depend on the part of the skin curve that may not be changed and on the weight and the center of the weighted point to be added. In Section 3.2 we will use these restrictions to define a global function. This function will return the set of weighted points that do not change any part of the skin curve. We will conclude this section with an algorithm that reconstructs the shrinking factor and the input set.

#### 3.1 Points that do not change the skin curve

In this section we will define sets of weighted points that do not affect a part of a skin curve. We will subdivide a skin curve into parts of three different types, which correspond to the envelope of parts of the envelope of a 0-,

1- and 2- flat. Each part of the skin curve can be subdivided into smaller patches that only contain one of these three types.

Assume that the skin curve is divided into patches with maximal size that belong to only one type. Each such patch is bounded by a mixed cell. For this subdivision into patches, see also Section 1.3.3. For every type we will discuss where weighted points can be added without changing the patch of the skin curve. This will be done in the next three sections. In Section 3.2 we will use these restrictions on certain parts of the skin curve to define a global function. This function will return the set of weighted points that do not change any part of the skin curve.

In the last section we will move points in the input set of a skin curve and derive restrictions under which the skin curve does not change.

The convention for weighted points in this section is:  $p, p_0, p_1, \dots$  for points already contained in the input set,  $q$  for the point we will add to the input set and  $g$  the orthogonal focus of some points in the input set and  $q$ .

### 3.1.1 Adding points to a 0-flat

Each patch corresponding to the envelope of a 0-flat in a skin curve is generated by only one weighted point in the input set. In this section we will show that it is possible to add other weighted points to the input set without changing such a patch of the skin curve. At first we will restrict ourselves to the skin curve of one point, which corresponds to only one 0-flat. We will conclude this section with remarks on how to generalize these restrictions so that they will also hold for parts of 0-flats.

**Lemma 1** *Assume that a skin curve is generated from one point  $p$  with positive weight and is shrunken by a factor  $s$ . Let  $q$  be a weighted point that we will add to the inputset.*

*We have  $\text{skn}^s\{p\} = \text{skn}^s\{p, q\}$  if and only if*

$$w_q \leq l^2 - 2\sqrt{\frac{w_p}{s}}l + w_p \quad (15)$$

and

$$l < \frac{1 + \sqrt{1 - s}}{\sqrt{s \cdot w_p}} \quad (16)$$

where  $l = \|z_p - z_q\|$

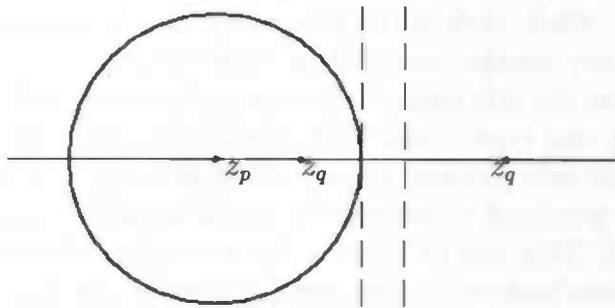


Figure 10: The skin curve of  $\{p\}$  is the same as the skin curve of  $\{p, q\}$  as long as  $p^s$  lies entirely within the mixed cell  $\mu_{\{p\}}^s$ . The dashed lines are the edges of the mixed complex.

*Proof.* The skin curve of  $\{p\}$  is the circle  $p^s$ . This skin curve lies entirely within mixed 0-cell  $\mu_{\{p\}}^s$  (which is  $\mathbb{R}^2$ ). After adding points to the input set,  $\mu_{\{p\}}^s$  becomes smaller, but the skin curve will not change as long as it is contained in the mixed cell of  $p$ .

Let  $g$  be the orthogonal focus of  $p$  and  $q$ . We will fix  $q$  so, that  $\mu_{\{p\}}^s$  in the skin curve of  $\{p, q\}$  touches  $p^s$ . This happens when  $\|z_p - z_g\| = \sqrt{w_p/s}$ . It can be seen, because the edge of the mixed cell  $\mu_{\{p\}}^s$  is shrunk from the orthogonal focus  $g$  towards  $p$  with a factor  $s$ . The skin curve of  $\{p\}$  and thus also of  $\{p, q\}$  consists of a circle with radius  $\sqrt{sw_p}$  and center  $z_p$ . Therefore, the distance  $\|z_p - z_g\|$  has to be at least  $\sqrt{sw_p}/s$ .

In order to fix  $g$  at this distance, both  $p$  and  $q$  must be orthogonal to  $g$ . Using this fact, we will first determine the weight of  $g$  and then the relationship between the the position and weight of  $q$ .

$$\begin{aligned} \pi_{p,g} &= 0 \\ \|z_p - z_g\|^2 - w_p - w_g &= 0 \\ w_g &= \frac{1-s}{s}w_p \end{aligned}$$

Now, assume that  $\|z_p - z_q\| \leq \sqrt{w_p/s}$  then, using orthogonality, we have

$$\begin{aligned}
 w_q &= \|z_g - z_q\|^2 - w_g \\
 &= (\|z_g - z_p\| - l)^2 - w_g \\
 &= (\sqrt{w_p/s} - l)^2 - \frac{1-s}{s}w_p \\
 &= w_p - 2\sqrt{w_p/s} \cdot l + l^2
 \end{aligned} \tag{17}$$

where  $l = \|z_p - z_q\|$ . On the other hand, if  $\|z_p - z_q\| \geq \sqrt{w_p/s}$ , we have

$$\begin{aligned}
 w_q &= \|z_g - z_q\|^2 - w_g \\
 &= (l - \|z_g - z_p\|)^2 - w_g \\
 &= (l - \sqrt{w_p/s})^2 - \frac{1-s}{s}w_p \\
 &= w_p - 2\sqrt{w_p/s} \cdot l + l^2
 \end{aligned} \tag{18}$$

Combining Equation (17) and Equation (18) we get the function for  $w_q$  as defined in Equation (15). This restriction guarantees that  $p^s$  is also contained in the skin curve of  $\{p, q\}$ . This means that  $\text{skn}^s\{p\} \subseteq \text{skn}^s\{p, q\}$ . It is still possible to add a weighted point  $q$  that does not interfere with  $\text{skn}^s\{p\}$ , but does add a new component. To guarantee that this does not happen, we restrict  $q$  so that if  $q$  lies outside the skin curve, we have  $w_q < 0$ . We will show that this results in Formula (16)

$$\begin{aligned}
 0 &= w_q \\
 0 &= w_p - 2\sqrt{w_p/s} \cdot l + l^2 \\
 l &= \frac{2\sqrt{w_p/s} \pm \sqrt{4w_p/s - 4w_p}}{2} \\
 l &= \sqrt{w_p/s} \pm \sqrt{\frac{1-s}{s}w_p}
 \end{aligned}$$

Since  $l = \sqrt{w_p/s} - \sqrt{\frac{1-s}{s}w_p}$  lies inside the skin curve, our boundary is  $l = \sqrt{w_p/s} + \sqrt{\frac{1-s}{s}w_p} = (1 + \sqrt{1-s})\sqrt{w_p/s}$ .

□

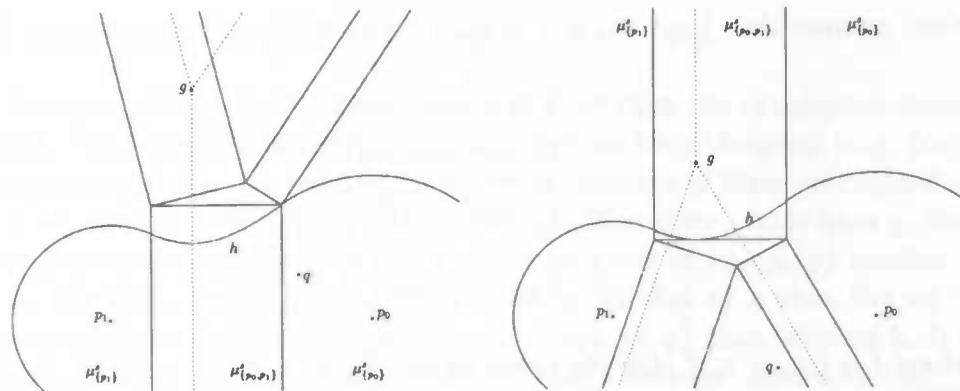


Figure 11: How points can be added without changing the part of a 1-flat.

Notice that this lemma is too restrictive if only a part of the 0-flat is contained in a skin curve. In this case, Formula (16) is not that strict, because it is possible that the newly generated part already belonged to the skin curve.

### 3.1.2 Adding points to a 1-flat

In the previous section we defined a set of points that can be added to the input set without changing the 0-flat in the skin curve. In this section we will derive a formula for a part of a 1-flat. The derivation is similar to the one described in the previous section. The point with largest weight for a certain location, that can be added to the input set without changing this part of the skin curve, minimizes the mixed cell. Therefore we will first determine the restrictions on the size of the mixed cell and then generate these by adding a new point to the input set.

Let  $P = \{p_0, p_1\}$ . Assume that these two points generate a hyperbolic patch  $h$  in the skin curve. Analyzing the mixed complex leads us to two different cases. In both cases, after adding a point to the input set of the skin curve, a new Voronoi cell is created and the Voronoi edge of  $\{p_0, p_1\}$  is shortened. Shortening the Voronoi edge leads to a smaller mixed cell of  $\{p_0, p_1\}$ .

The difference between the two cases is the way in which the mixed cell  $\mu_{\{p_0, p_1\}}^s$  is shrunken. In the first case, we shrink the mixed cell towards the line  $l$  through the centers of  $p_0$  and  $p_1$ . Therefore we need to add a point to

the input set that lies on the same side of  $l$  as  $h$ . If  $h$  intersects  $l$ , this case holds for both sides. See also Figure 11, left.

The second case, only occurs if  $h$  lies entirely on one side of  $l$ . We can then shrink the mixed cell  $\mu_{\{p,q\}}^s$ , so that the edge of  $\mu_{\{p,q\}}^s$  is moved from  $l$  towards  $h$ . This is done by adding a point that lies on the other side of  $l$  as  $h$  lies. See also Figure 11, right.

We will handle the first case in Lemma 2 and the second case in Lemma 3.

**Lemma 2** *Let  $h$  be a part of the envelope of a 1-flat in the skin curve of  $P = \{p_0, p_1\}$ . Assume that  $h$  is maximal, i.e. the circles corresponding to  $p_0^s$  and  $p_1^s$  touch  $h$  in the endpoints. We will add a weighted point  $q$  to the input set and make restrictions on  $q$  such that the  $h \subset \text{skn}^s\{p_0, p_1, q\}$ .*

*Without loss of generality assume that  $w_{p_0} \geq w_{p_1}$ . Let  $g \in \{p_0, p_1\}^\perp$  be the point with  $\|z_g - z_{p_0}\| = \sqrt{w_{p_0}/s}$  that lies on the same side of the line  $l$  through  $z_{p_0}$  and  $z_{p_1}$  as  $h$ . This defines  $w_g$  and therefore  $g$  uniquely.*

*Assume that  $z_q$  also lies on that side of  $l$  and that  $q$  is orthogonal to  $g$ . Then  $h \subset \text{skn}^s\{p_0, p_1, \tilde{q}\}$ , for all  $\tilde{q} \in \text{ucl}(q)$ .*

*Proof.* The proof is similar to the proof of Lemma 1. First we will specify the limit situation, in which the mixed cell of  $\{p_0, p_1\}$  touches  $h$ . Therefore we will generate a Voronoi vertex  $g$  orthogonal to  $p_0, p_1$  and  $q$ . We will choose  $g$  so that its mixed cell touches  $h$ .

If the mixed cell of  $\{p_0, p_1\}$  touches  $h$  this happens in one of the endpoints of  $h$ . More specifically, when it touches the endpoint in which the point with largest weight contributes, i.e.  $p_0$ . In this case the mixed cell touches the skin curve exactly in the point where  $h$  transforms into the next part of a flat on the skin curve. So if the mixed cell is shrunken further not only  $h$  is modified, but also the point where  $p_0$  touches the skin curve, which is part of  $p_0^s$ . In Lemma 1 we computed at which distance  $g$  had to lie from  $p_0$ , so that the mixed cell  $\mu_{p_0}^s$  touches the 0-flat. Because the point where  $p_0$  touches  $h$  belongs both to  $h$  and to the adjacent 0-flat, this holds also for  $g$  as defined in this lemma. Therefore if the mixed cell of  $\{p_0, p_1\}$  touches the skin curve in a point of  $h$ ,  $g$  has to lie at a distance  $\sqrt{w_{p_0}/s}$  from  $p_0$ .

This defines  $g$  uniquely. To summarize,  $g$  has to:

1. lie on the same side of  $l$  as  $h$ ,
2. be an element of  $\{p_0, p_1\}^\perp$ ,

3. lie at a distance  $\sqrt{w_{p_0}/s}$  from  $p_0$ .

The second step is to generate a point  $q$ , so that the orthogonal focus  $g$  is fixed. Therefore the points  $p_0$ ,  $p_1$  and  $q$  have to be orthogonal to  $g$ . Notice that  $p_0$  and  $p_1$  are orthogonal because  $g$  is an element of their orthogonal set. So, if we would add a point  $q$  that is also orthogonal to  $g$ , this fixes  $g$ . Since  $q$  also lies on the same side of  $l$  as  $g$ , the mixed cell of  $\{p_0, p_1, q\}$  touches  $h$ .

To conclude, every point  $q$  in the orthogonal flat of  $g$  that lies on the same side of  $l$  as  $g$  defines the mixed cell of  $\{p_0, p_1, q\}$  that touches  $h$ . If we decrease the weight of  $q$ ,  $g$  moves away from  $l$  and the mixed cell  $\mu_{\{p_0, p_1\}}^s$  that contains  $h$  becomes larger. Therefore  $h$  will not change after addition of  $\tilde{q} \in \text{ucl}(q)$ .

□

Now, let us have a closer look at the other case, viz Figure 11, right. In this case we have a part of the envelope of a 1-flat  $h$  that is defined by two weighted endpoints  $p_0$  and  $p_1$ . The restriction on  $h$  is that it lies entirely on one side of the line  $l$  through  $z_{p_0}$  and  $z_{p_1}$ . Want to add a point  $q$ , that lies on the other side of  $l$  as  $h$ , to the input set of the skin curve without changing the part of the 1-flat  $h$ . This corresponds to the right part of Figure 11.

Calculations are very similar, except for the generation of the mixed cell.

**Lemma 3** *Let  $P = \{p_0, p_1\}$  and let  $h$  be the hyperbolic patch in the skin curve generated by these two points. Assume that  $h$  lies entirely on one side of the line  $l$  through  $z_{p_0}$  and  $z_{p_1}$ . We will restrict ourselves to adding points to the input set that lie on the other side of  $l$  as  $h$ . Let  $s$  be the shrinking factor.*

*Let  $x_{\min}$  be the point on  $h$  with the minimal distance to  $l$  and let  $g \in \{p_0, p_1\}^\perp$  be the weighted point with  $y$ -coordinate  $s \cdot x_{\min}$*

*If a weighted point  $q$  lies on the other side of  $l$  as  $h$  and is (further than) orthogonal to  $g$ , then  $h \subset \text{skn}^s P \cup \{r\}$ .*

*Remark:* Notice that  $x_{\min}$  is either the center of the focus of  $p_0$  and  $p_1$  or one of the extreme points of  $h$ .

*Proof.* We will reconstruct the minimal mixed cell for this case. Calculations where new points can be added are similar to Lemma 1 and Lemma 2.

The hyperbolic patch  $h$  does not change as long as it is entirely contained in the mixed cell  $\mu_{\{p_0, p_1\}}^s$ . After adding a third point  $q$  on the other side of

$l$  as  $h$ , the mixed cell of  $\{p_0, p_1\}$  is shrunken from  $l$  towards  $h$ . In the limit situation, the edge of  $\mu_{\{p_0, p_1\}}^s$  touches  $h$  in the point closest to  $l$ , which is  $x_{min}$ .

Because the mixed cell of  $\{p_0, p_1\}$  is shrunken with a factor  $s$  from the Voronoi vertex  $g$  towards the Delaunay edge of  $\{p_0, p_1\}$ ,  $g$  has to lie at a distance  $y_{min}/s$  from  $l$ . We also know that  $g$  is orthogonal to  $p_0, p_1$  and lies at the same side of  $l$  as  $h$ . These three restrictions define  $g$  uniquely.

Each point  $q$  orthogonal to  $g$  on the opposite side of  $l$  as  $h$  causes the mixed cell of  $\{p_0, p_1, q\}$  to touch the hyperbolic patch. When the weight is decreased,  $\mu_{\{p_0, p_1, q\}}^s$  grows and  $h$  is not affected.

□

Notice that the affine hull of the two orthogonal focuses  $g$  defined in the lemmas form the orthogonal set of  $p_0$  and  $p_1$ . Therefore every point on the affine hull of  $p$  and  $q$  can be added without changing the hyperbolic patch. More specific, the points  $p$  and  $q$  can be added without changing the hyperbolic patch.

### 3.1.3 Adding points to a 2-flat

We will discuss the last case, the envelope 2-flat, only briefly. A part of a the envelope of a 2-flat  $h$  is formed by three points, say  $p_0, p_1$  and  $p_2$ . If the focus  $\text{foc}(\{p_0, p_1, p_2\})$  has negative weight, the envelope of the 2-flat consists of the circle corresponding to  $g^s$ , where  $g$  is the orthogonal focus of  $p_0, p_1$  and  $p_2$ .

The orthogonal set that also defines the 2-flat is the 0-flat formed by  $g$ . Thus, as long as  $g$  does not change, the 0-flat formed by  $g$  does not change and therefore the 2-flat does not change. This leads us to the conclusion that, when we add points to the input set that do not affect  $g$ , the 2-flat does not change. All points that are (further than) orthogonal to  $g$ , do not influence  $g$  and can be added to the input set without changing the 2-flat.

More precisely,  $g$  will only be modified if a point  $q$  with  $\pi_{g,q} < 0$  is added to the input set. Without further proof, this leads us to the following lemma.

**Lemma 4** *Let  $P$  be the input set of a skin curve. Let  $g$  be an orthogonal focus with a mixed cell that contains a part of the skin curve. This part consists of one or more circular arcs. These arcs will also be contained in the skin curve after adding a point  $q$ , as long as  $q$  is (further than) orthogonal to  $g$ .*

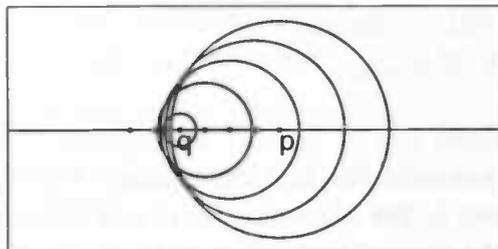


Figure 12: Not all points in the convex hull contribute to the skin curve.

### 3.1.4 Moving points without changing the skin curve

On the shrunken convex hull of the two points  $p_0$  and  $p_1$  in Figure 12 there are points that do not contribute to the skin curve of the convex hull. We can divide these points into two categories. First, points with negative weight, and therefore imaginary radii. Secondly, real circles that do not touch the skin curve, e.g.,  $p_1$  in Figure 12. We can extend and shorten the segment between  $p_0$  and  $p_1$ , by moving  $p_1$  over a carefully chosen part of the affine hull, without changing the skin curve.

In the next lemma we will define where  $p_1$  may lie on the affine hull such that the convex hull of  $p$  and  $p_1$  remains the same.

**Lemma 5** *Let  $p_0$  and  $p_1$  be two weighted points that form a part of the envelope of a 1-flat  $h$  in the skin curve. Assume that  $p_0$  touches to  $h$  and  $p_1$  does not. Let  $f$  be the focus of  $p_0$  and  $p_1$ .*

*We can parameterize the affine hull of  $p_0$  and  $p_1$  using the parameter  $\gamma$ :*

$$q_\gamma = f + \gamma \frac{p_0 - f}{\|z_{p_0} - z_f\|}.$$

*Then  $h \subset \text{skn}^s\{p_0, q_\gamma\}$  for  $\gamma \in (-\sqrt{-w_f}, \sqrt{-w_f/t})$ .*

*Remark:* Before starting our proof, notice that  $w_f < 0$ , because if  $w_f \geq 0$  every point on the convex hull would contribute to the skin curve and no point  $p_1$  that does not contribute can be chosen.

*Proof.* Our proof consists of computing the two boundaries. The left boundary depends on the first point on the affine hull that generates a part of the other sheet. This is the point with zero weight. The last circle on the convex

hull that contributes to  $h$  forms the other boundary. All points in between do not change  $h$  and can be added or removed.

Let the point  $q_{\gamma_1}$  with zero weight be the first point on the affine hull of  $p_0$  and  $p_1$  that contributes to the skin curve on the other sheet. This point depends on the orthogonal focus  $g$  of  $p_0$  and  $p_1$ . Remember that  $g$  has the same location as  $f$  and is orthogonal to  $f$ , therefore  $w_g = -w_f$ .

Each point on the affine hull through  $p_0$  and  $p_1$  is orthogonal to  $g$ . Using this fact we can define the first point that generates a part of the other sheet. Using orthogonality and the distance function (Formula (3)) we have:

$$\begin{aligned} \|z_g - z_{q_{\gamma_1}}\|^2 - w_g - w_{q_{\gamma_1}} &= 0 \\ \|z_g - z_{q_{\gamma_1}}\| &= \sqrt{-w_g} \\ \gamma_2 &= -\sqrt{-w_g} \end{aligned}$$

Because  $w_g = -w_f$ , this corresponds to the left boundary.

For the other boundary, we will find the value of  $\gamma$  for the last weighted point  $q_{\gamma_2}$  in the affine hull of  $p_0$  and  $p_1$  that contributes to  $h$

Notice that this point  $q_{\gamma_2}$  contributes to the skin curve, in the point  $x$  where it touches the shrunken focus  $g$  of the orthogonal flat. Because the centers of  $g$  and  $q_{\gamma_2}$  both lie on  $F' = \{z_{p_0} | p_0 \in F\}$  and they only intersect in one point,  $x$  also has to lie on  $F'$ . From Equation (4) we know that the weight of  $q_{\gamma_2}$  is  $\frac{s}{t}w_g$  and we have

$$\begin{aligned} \|f - q_{\gamma_2}\| &= \|z_f - x\| + \|x - z_{q_{\gamma_2}}\| \\ &= \sqrt{w_g t} + \sqrt{w_{q_{\gamma_2}} s} \\ &= \sqrt{\frac{t^2}{t} w_g} + \sqrt{\frac{s^2}{t} w_g} \\ &= (s + t) \sqrt{w_g / t} \\ &= \sqrt{w_g / t}. \end{aligned}$$

This corresponds to the right boundary of the segment of values for  $\gamma$  where points do not contribute.

Using these two boundaries we get a set of values for  $\gamma$  for which a point  $q_{\gamma_2}$  does not contribute to the skin curve. Therefore these points can be added and removed without changing  $h$  as long as at least one point remains in the input set.

□

## 3.2 Reconstruction of skin curves

In this section we will specify the appropriate value for the shrinking factor. Using this value, we will then reconstruct the set of weighted points.

The reconstruction of the input set will be done in three steps. First, we will reconstruct all points that generate parts of a 0-flat. After that, we will generate a reconstruction complex for which each vertex inside the skin curve has to be added to the input set. Finally, we need to add some points to the input set that generate the hyperbolic caps, as defined in Lemma 5.

### 3.2.1 The shrinking factor

The shrinking factor  $s$  can be derived both from the mixed complex and from the hyperbolic patches. First, we will derive the shrinking factor from the mixed complex. Assume that a part of the mixed complex consists of a hyperbolic part that is connected to two parts of different arcs of circles, with their centers inside the skin curve. Let  $p$  and  $q$  be the centers defined by the two arcs. It follows from the definition of the mixed complex that  $\|p - p_s\| / \|p_s - \text{foc}(\{p, q\})\| = t/s$ , where  $t = 1 - s$ , and  $p_s$  is the intersection point of the edge of the mixed complex and the line through the centers of  $p$  and  $q$ . Similar reasoning can be done for other combinations of mixed cells.

The disadvantage of the previous method is that not every combination of mixed cells has to occur. The second method does not have this problem, it derives the shrinking factor from one hyperbolic part. If there are no such parts in the skin curve, there is no need to shrink the circles.

Assume that there exists a hyperbolic patch in the skin curve. Then we can rotate and translate this patch so, that the convex hull forms a part of the x-axis and the focus lies in the origin. The hyperbolic patch is then of the form  $-c_x x^2 + c_y y^2 + c = 0$ , where  $c_x$ ,  $c_y$  and  $c$  are fixed values with  $c_x \cdot c_y > 0$ . Using Equation (9), we can conclude that  $c_s \cdot c_x = 1/t$ ,  $c_s \cdot c_y = 1/s$  and  $c_s \cdot c$  is the weight of the focus, with  $c_s$  a constant scaling factor. We know that  $s + t = 1$ , and can derive  $c_s$  with the formula  $c_s = (c_x + c_y)/(c_x c_y)$ . This leads us to the formula for the shrinking factor

$$s = \frac{c_x}{c_x + c_y}.$$

### 3.2.2 Selecting points

The reconstruction of the input set  $P$  is by far the most difficult part of reconstructing a skin curve. Reconstruction is so difficult because points can contribute to the skin curve in so many ways. E.g., some points generate a part of a 0-flat. Reconstruction for these points is easy. Other points do not touch the skin curve, but do contribute to the skin curve by filling voids, or they generate points on the convex hull with other points in the input set that do contribute to the skin curve. Because all these points have to be found, and they are not defined uniquely, reconstruction of the input set is difficult.

In order to find all these points we divide the reconstruction of the input set in 3 parts. The first and the last part are straight forward. The first part reconstructs all points that generate a part of a 0-flat. After that, we will reconstruct all the points that generate other flats. This will be done using a function that specifies the maximal weight for every point, that when added, does not change the skin curve. The last part consists of adding points to the input set that generate some hyperbolic caps that are not reconstructed yet. Each step in the algorithm will be discussed in a separate paragraph.

**Generating 0-flats** It is easy to reconstruct all points that generate a part of a 0-flat. Given a part of the 0-flat, which is a circular arc with its center inside the skin curve, we can reconstruct the point in the input set it is generated from. Its center corresponds to the center of the arc and its weight is  $r_0^2/s$ , where  $r_0$  is the radius of the arc and  $s$  the shrinking factor.

**Generating 1- and 2-flats** 1- and 2-flats are generated from more than one point in the input set and they do not define these points uniquely. In Section 3.1 we computed the maximal weight for a point, so that it does not contribute to a certain part of a flat. We will extend this notion of contribution from one weighted point to a set of weighted points, and compute the maximal weight for a point, so that, after adding it to the input set, the skin curve is still the same.

Points that generate parts of a 0-flat have already been reconstructed. They also form the endpoints of two 1-flats. Therefore we can omit the parts of a 0-flat during the construction of function that generates the maximal weight.

In Section 3.1 we generated sets of points that can be added to the input set without changing a particular part of a flat in the skin curve. For a 2-flat this is the upward closure of the orthogonal flat of an orthogonal focus. For a 1-flat it is the intersection of the upward closure of two orthogonal flats of two orthogonal focuses. Let  $\phi_h$  be the function that returns the set of orthogonal focuses that generate the flats for a 1- or 2-flat  $h$ . The orthogonal focuses of all 1- and 2-flats form the set  $\Phi$ :

$$\Phi = \{g | g \in \phi_h, h \text{ a 1- or 2-flat in the skin curve}\}.$$

Every point in  $g \in \Phi$  makes sure that the skin curve in some part will not change after adding points that are (further than) orthogonal to  $g$ . Let  $\psi_g(x)$  define the maximal weight for a point  $x$ , such that the weighted point  $(x, \psi_g(x))$  is the largest weighted with center  $x$  that does not change  $g$ . Mathematically, we can write:

$$\psi_g(x) = (w_p | z_p = x \wedge p \perp g).$$

For the whole skin curve we can define a similar function  $\Psi$  that returns the point with maximal weight that does not change any hyperbolic patch,

$$\Psi(x) = \min\{\psi_g(x) | g \in \Phi\}.$$

Notice that  $\Psi$  defines the maximal weight for which a point can be added to the input set without changing the skin curve. The only restriction on the points that are added is that they have to lie inside the skin curve. Without this restriction it is possible to add points that do not change the skin curve, but do add new components.

The set  $P_0 \cup \{(x, \Psi(x)) | x \text{ lies inside the skin curve}\}$  is an infinitely large input set that reconstructs the skin curve, except for some hyperbolic caps. In the last part of the reconstruction we will add points, so that these caps are also generated. For now, we assume that they do not occur in the input set.

Most of the points in the input set defined above can be omitted, because they lie on the convex hull of other points. In order to reduce the number of points in the input set, we will generate a *reconstruction complex*.

The *reconstruction cell* of an orthogonal focus  $g \in \Phi$  consists of all weighted points  $p$  for which  $g$  is the first point that is no longer further than orthogonal if  $w_p$  is increased:

$$R_g = \{(x, \psi_g(x)) | \psi_g(x) = \Psi(x)\}$$

Each  $R_g$  is a part of the orthogonal flat of  $g$ . Combining several reconstruction cells, we can define edges and vertices in the reconstruction diagram. For a subset  $G \in \Phi$  we have:

$$\rho_G = \bigcap_{g \in G} R_g$$

which we will call a reconstruction  $\ell$ -cell ( $\ell = \dim \rho_G = 3 - |G|$ ), if it is not empty. In the non degenerated case, every 0-cell is a point, which is the intersection of three adjacent cells. An 1-cell is the line segment connecting two adjacent cell and a 2-cell is a nonempty reconstruction cell.

Notice that each point on a 1-cell or edge is orthogonal to at least two patches in the skin curve and each 0-cell or vertex is orthogonal to at least three patches. We already noticed that each reconstruction cell is part of an orthogonal flat. Each edge is the intersection of two reconstruction cells and is therefore also part of an orthogonal 1-flat. The endpoints of an edge correspond to the vertices. Adding these vertices to the input set would therefore be enough to reconstruct all points in the reconstruction complex.

Now we only need to find the vertices of the reconstruction complex. Notice that there is a finite number of vertices if and only if the skin curve consists of a finite number of patches. This can be seen because each patch generates at most 2 orthogonal points corresponding two faces in the reconstruction complex. The number of faces is thus also finite and with a finite number of faces it is not possible to construct an infinite number of vertices.

In order to be able to find these vertices, we need to manipulate the definition of the reconstruction cell. Let us re-examine this definition:

$$\begin{aligned} R_g &= \{(x, \psi_g(x)) | \psi_g(x) = \Psi(x)\} \\ &= \{(x, \psi_g(x)) | \psi_g(x) = \min\{\psi_{g'}(x) | g' \in \Phi\}\} \\ &= \{(x, \psi_g(x)) | \psi_g(x) \leq \psi_{g'}(x) \text{ for all } g' \in \Phi\} \\ &= \{(x, \psi_g(x)) | \pi_g(x) \leq \pi_{g'}(x) \text{ for all } g' \in \Phi\}. \end{aligned}$$

This is exactly the definition of a Voronoi cell for a point  $g \in \Phi$  as defined in Equation (11). Computing the Voronoi diagram of  $\Phi$  will lead to the requested diagram. After that it is easy to select the vertices that lie within the reconstruction complex and add them to the input set.

**Hyperbolic caps** The hyperbolic caps mentioned earlier are generated by points that do not contribute to the skin curve. In Lemma 5 we concluded that these points may lie on a certain segment of the affine hull. Let  $p$  be the point corresponding to the 0-flat to which the cap is attached and let  $q$  be the last point that touches the hyperbolic cap. This is the case for which the line segment  $z_p, z_q$  has minimal length. Every other point that generates this skin curve, also contains  $q$  in the convex hull with  $p$ . Because we are reconstructing a skin curve, we know that it is possible to add a point that generates this cap and we can add  $q$  to the input set.

### 3.2.3 The algorithm

The previous remarks lead to the following algorithm:

1.  $P = \emptyset$
2. calculate the shrinking factor  $s$
3. for each part  $h$  of a 0-flat do
4.      $P = P \cup \{p\}$ ,  $p$  the weighted point corresponding to the arc  $h$
5. initialize  $\mathcal{V}$  to the empty Voronoi diagram
6. compute  $\Phi$
7. for each  $g \in \Phi$  do
8.     add  $g$  to  $\mathcal{V}$
9. for each vertex  $v \in \mathcal{V}$  do
10.    if ( $v$  lies inside the skin curve) then
11.      $P = P \cup \{v\}$
12. for each hyperbolic cap  $h$  not generated by the skin curve of  $P$  do
13.    add the first point that generates  $h$  to  $P$
14. return  $P$  and  $s$

Let us analyze this algorithm. Line 1 and 5 are initialization. Line 2 is described in Section 3.2.1. Line 3 and 4 reconstruct all points that generate parts of 0-flats. Lines 6 – 11 generate the reconstruction diagram and then extract the vertices as described in the previous section, paragraph *Generating 1- and 2-flats*. Lines 12 and 13 generate the last hyperbolic caps discussed in the last paragraph of the previous section.

## 4 Morphing skin curves

In this section we will have a look at three methods of changing skin curves continuously. For the first method, we will introduce a parameter  $\alpha$  that can grow and shrink all points simultaneously. The morph consists of an increment or decrement of alpha.

The second method deforms one skin curve into another. Therefore we interpolate between two input sets and generate the skin curve of the new set.

Finally, we will combine these two methods and find a method to minimize the number of topological changes during the morph.

### 4.1 Growing alpha

This method is previously described in [Ede99b] and [CEF98]. For a parameter  $\alpha^2 \in \mathbb{R}$  we define  $p_\alpha = (z_p, w_p + \alpha^2)$ . For  $\alpha^2 < 0$  all points are shrunken and for  $\alpha^2 > 0$  they are grown. Let  $P_\alpha = \{p_\alpha | p \in P\}$ .

The Voronoi complex and the Delaunay triangulation do not vary with  $\alpha$ . Equation (11) says that a Voronoi cell of a point  $p_\alpha \in P_\alpha$  consists of all points that lie closer to  $p_\alpha$  than to any other point in  $P_\alpha$ . We can rewrite this, by expanding the weight function to  $\{x | \pi_p(x) - \alpha^2 \leq \pi_q(x) - \alpha^2 \text{ for all } q_\alpha \in P_\alpha\}$ , which is the same as Equation (11). The Voronoi cells are the basis for the Voronoi diagram and when they do not change, the whole diagram is the same. Because of the duality, the Delaunay triangulation does not change either.

This method of morphing is computed faster than the methods described in the next two section because the underlying data structure is the same for every image and only the different images have to be rendered.

Notice that the topology of the skin curve can change. This happens when a focus becomes a point. For all different cases see Table 2.

### 4.2 Cross section

Another way of morphing is proposed in [CEF98] and [CEFL98]. Its advantage is that each point can be manipulated individually. The main advantage is that the increase in weight is not necessarily the same for all points and that it's also possible to change the centers. This gives us a lot of freedom to change the first skin curve into the second. Therefore we will use a linear

morphing schedule. Assume that we have two base input sets  $P_0$  and  $P_1$ . An intermediate set  $P_t$  is defined for  $t \in [0, 1]$  as

$$\begin{aligned} P_t &= (1-t) \cdot P_0 + t \cdot P_1 \\ &= \{(1-t) \cdot p_0 + t \cdot p_1 \mid p_0 \in P_0 \wedge p_1 \in P_1\} \end{aligned} \quad (19)$$

Notice that for  $t = 0$  we have the first base input set en for  $t = 1$  the second. For a fixed shrink-value of  $s$ , define the skin curve corresponding to the set  $P_t$  as  $\Phi_t = \text{skn}^s P_t$ .

The cardinality of  $P_t$  is  $|P_0| \cdot |P_1|$ . At first sight, this may seem excessive, but in practice a lot of points have empty Voronoi sets. These points do not contribute to the skin curve and algorithms exist that only spend time on non redundant points. See e.g. [CEF98].

### 4.3 Combining two methods

The most complicated method is also described in [CEFL98]. It combines both previous methods and defines the space  $\Omega = [0, 1] \times \mathbb{R}^2$  with parameters  $t$  and  $\alpha^2$ . Any continuous curve through this plane, starting in  $(0, 0)$  and ending in  $(1, 0)$  forms a continuous morph between both skin curves. Notice that if  $P_0 = P_1$ , we have the morphing method of Section 4.1. The line through  $(0, 0)$  and  $(1, 0)$  does not change  $\alpha^2$  and is therefore the same as the method described in Section 4.2.

The great advantage of this method is that we can define a function  $f_X : [0, 1] \rightarrow \mathbb{R}^2$  that, given a Delaunay cell  $\delta_X$  gives the value for  $\alpha^2$ , such that the focus has zero weight. If  $\alpha^2 \geq f$  then the focus has a positive weight and the components are connected. On the other hand, if  $\alpha^2 < f$  they aren't.

Every  $f_X$  is a function of at most degree two, see [CEFL98, sec. 4] (NGHK: het is me niet gelukt om dit na te rekenen, zie ook het mathematica-

$\ell$	Change of the $s$ -body	Change of the skin curve
0	A new component appears	Void changes into a point and then into a circle
1	A bridge appears	Two sheets change into one sheet
2	A hole disappears	Circle changes into a point and then disappears.

Table 2: Change in topology when  $\alpha^2$  is increased and the focus of a Delaunay  $\ell$ -cell becomes real.

bestand). This splits up  $\Omega$  in several regions of skin curves with the same topology. This can be derived from the definition of  $f$ : all focuses have the same sign and thus the connections between the points in the input set are same. It is possible that two regions represent the same topology. Assume that there are two skin curves in both regions, then they have the same topology, but in order to transform one into the other, the topology has to change at least twice.

We will have a closer look at the dual graph  $G$  of the regions that decompose  $\Omega$ . We can derive several attributes (also called *signatures*) from both graphs. First of all, for every region we can compute the number of components the skin curve consists of. Another property is the number of holes. We will use these two properties to make an evaluation function for the morph. A morph is better than another, if there are less changes in topology during the morph. Using this function, we can find a path through the regions that minimizes this evaluation function. This is, in principle, the same as finding a shortest path in  $G$ , where the edges have weight equal to the number of topological changes.

## 5 Summary

The main goal of this article is to approximate an arbitrary curve with a skin curve, so that we could use the advantages of skin curves in morphing for any curve.

As a first step towards approximation we reconstructed the input parameters, the shrinking factor and the set of weighted points, from a skin curve. The base of this algorithm is the mixed complex, which can be extended to three dimensions for skin surfaces. It is beyond the scope of this Masters thesis, but it seems straight forward to extend this algorithm to reconstruct skin surfaces.

This algorithm is not suitable for approximation of an arbitrary curve. To accomplish this, we are currently working on another approach using medial axes.

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## A List of definitions

$p, q, f, g$	A weighed point consists of a pair $(z_p, w_p)$ of location and weight.
$P, Q$	A set of weighted points
$w_p$	Weight of a point is a real number. The radius of the corresponding circle is equal to its root. Negative weights correspond to imaginary circles
$z_p$	An element of $\mathbb{R}^d$ , specifying the center of the weighted point $p$
$\Pi(p)$	Bijection $\Pi(p) = (\xi_0, \dots, \xi_n,  z_p ^2 - w_p)$ with $z_p = (\xi_0, \dots, \xi_n)$ . In this vector space usual addition and multiplication with a constant hold. I.e., $p+q = \Pi^{-1}(\Pi(p) + \Pi(q))$ and $\gamma \cdot p = \Pi^{-1}(\gamma \cdot \Pi(p))$ .
$\pi_p(x)$	The distance between a weighted point $p$ and a point $x$ $\pi_p(x) = \ z_p - x\ ^2 - w_p$
$\pi_{p,q}$	The distance between two weighted points $p$ and $q$ $\pi_{p,q} = \ z_p - z_q\ ^2 - w_p - w_q$
$p \perp q$	$p$ and $q$ are orthogonal iff $\pi_{p,q} = 0$ and further than orthogonal iff $\pi_{p,q} > 0$
$\text{aff}(P)$	The affine hull of $P = \{p_0, p_1, \dots, p_n\}$ . $\text{aff}(P) = \{\sum_{i=0}^n \gamma_i \cdot p_i \mid \sum_{i=0}^n \gamma_i = 1\}$
$\text{conv}(P)$	The convex hull of $P = \{p_0, p_1, \dots, p_n\}$ is a subset of the affine hull. $\text{conv}(P) = \{\sum_{i=0}^n \gamma_i \cdot p_i \mid \sum_{i=0}^n \gamma_i = 1 \wedge \gamma_i \geq 0 \text{ for all } i\}$
A.i.	A set of points of points $P$ is affinely independent if for all $p \in P$ holds that $p \notin \text{aff}(P - \{p\})$
$k$ -flat	An affine hull of $k$ affinely independent points
$\text{foc}(P)$	The focus of a set $P$ is the point in $\text{aff}(P)$ with equal distance to each $p \in P$ . This is also the point with minimum weight in $\text{aff}(P)$ .
Orthogonal focus	The orthogonal focus of a set of weighted points $P$ is the point in the orthogonal flat with same center as $\text{foc}(P)$ .
$\text{ucl}(p)$	The upward closure of a weighted point $p$ . It is the set of all points with the same center as $p$ and smaller or equal weight.

$\text{bdy}^s P$	The $s$ -body. $\text{bdy}^s P = \bigcup \text{ucl}(\text{conv}(P)^s)$
$\text{skn}^s P$	The skin curve is the boundary of the $s$ -body $\text{skn}^s P = \text{bd } \text{bdy}^s P$
$V_p$	The Voronoi cell of $p$ contains all points that lie closer to $p$ than to any other point in the set of weighted points.
$\nu_{P'}$	A Voronoi $\ell$ -cell ( $\ell = \dim \nu_{P'} = 3 -  X $ ) is the non empty intersection of $V_p$ for each $p \in P'$ .
Voronoi diagram	The set of all Voronoi 0-, 1- and 2-cells
$\delta_{P'}$	A Delaunay $\ell$ -cell corresponds to the Voronoi $3 - \ell$ -cell. It is the $\text{conv}(P')$ if $\nu_{P'} \neq \emptyset$ and else it is $\emptyset$ .
Delaunay triangulation	The set of all Delaunay 0-, 1- and 2-cells
$\mu_X^s$	The mixed cell is an interpolated cell between the Delaunay triangulation and the Voronoi diagram. It contains a part of the skin curve that can be represented with a quadratic formula $\mu_X^s = s \cdot \nu_X + t \cdot \delta_X$
Mixed complex	The linear interpolated complex between the Delaunay triangulation and the Voronoi diagram. It consists of all mixed cells
Reconstruction complex	The Voronoi diagram of $P_{hi}$ . This diagram is used for reconstruction of weighed points in the input set.
$R_g$	The reconstruction cell of $g$ consist of all points with largest weight that do not change the skin curve and when the weight is increased, $g$ is the first point that is no longer further than orthogonal. $R_g = \{(x, \psi_g(x))   \psi_g(x) = \Psi(x)\}$
$\phi_h$	The set of orthogonal focuses that define the orthogonal flat for which points can be added to the input set without changing $h$ .
$\Phi$	All focusses defined by $\phi_h$ for any $h$ in the skin curve.
$\psi_g(x)$	The maximal weight a weighted point with center $x$ may have without changing the part of the skin curve that generated $g \in \Phi$ .
$\Psi(x)$	The maximal weight a weighted point with center $x$ may have without changing the skin curve. $\Psi(x) = \min\{\psi_g(x)   g \in \Phi\}$