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Thresholds for Information Transmission on Trees

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Abstract

A broadcasting process describes the transmission of information along a rooted infinite tree (a graph with no loops). The root of the tree is in a certain state, which is then transmitted to each of its children, and so on along the tree. The information transmission is imperfect, so that at each step, there is a certain probability that the child vertex will receive something different from what was transferred from the parent vertex. It is assumed that the error probability is the same for each parent-child pair on the tree. This model has found application, among others, in physics and genetics.

The question is whether the state of the root affects the states of the set of vertices at faraway levels of the tree, even when the distance to the root grows to infinity. Using arguments inspired on the Galton-Watson process, results can be inferred for the threshold separating error probabilities for which the state of the root affects the state of the whole tree, from error probabilities for which the opposite happens.

Contents

1 Introduction

A well-known children's game is the "telephone" game. In this game, a finite number of children is arranged in a certain order, after which the first child transmits a word to the second, the second child transmits the same word to the third, and so on until the last child has received the word. Then the word that the last child received is compared to the word that the first child transmitted.

Imagine that the number of children is large. Since it is possible that a miscommunication occurs between each pair of children, it is intuitively clear that the probability that the last child will receive the same word as the first child transmitted will be small, and when the number of children grows to infinity, this probability of a correct transmission drops to almost zero. Of course, the children's language has only a finite number of words, so the probability of a correct transmission from begin to does not entirely drop to zero.

To make the game more interesting, we can tell the children to transmit the word not to one new child, but to several – say three. If we have a large number of children, can we tell, by inventorising the words that the last children received, which word was intended to be transmitted? What happens when the number of children goes to infinity?

Of course, in practice it's not easy to get an infinity of children together to play a game. In physics, this kind of model can be more applicable. Imagine, for example, that we have an enormous quantity of atoms which have either positive or negative spin, and that they are connected treewise; imagine further that the spin of an atom is only influenced by its neighbours'. Does the spin of the atoms very far away from the root of the tree reflect in any way the spin of the root?

Although atoms cannot be connected in the form of a tree, in statistical physics this model has found extensive application as an approximation whith which calculations are easier.

2 Broadcasting on a Tree

A broadcasting process is the transmission of information along a graph. We are concerned with the case that the graph is an infinite tree with a root and with regular ramifications, so that each vertex in the tree has exactly d children. We will denote this tree by T_d, and its root by ρ. The set of vertices of T_d is denoted V_d, and the set of edges of T_d is denoted E_d.

For the different adjacency relations between vertices in T_d, we use the following familiar terminology:

Definition 1. *We say that a vertex v is the parent of the vertex w if v is the first vertex next to w on the unique path from w to the root ρ. In that case, we say that w is a child of v. We call the set of vertices in the path from w to ρ the ancestors of w, and those vertices for which v is an ancestor we call them the descendants of v.*

Definition 2. Branching number: *A cutset Π of an infinite tree T is defined as a set of vertices of T such that any infinite self-avoiding path emanating from the root must pass through some vertex of Π. The branching number of an infinite tree T is defined as*

$$br(T) = \sup\{\lambda \geq 1 : \inf_{\Pi \text{ cutset}} \sum_{v \in \Pi} \lambda^{-|v|} > 0\}$$

where $|v|$ is the distance of v from the root.

It's not hard to see that the branching number of the d-ary tree is d. Indeed, consider the sequence of cutsets L_n. $\sum_{v \in L_n} \lambda^{-|v|} = d^n \lambda^{-|n|} = \left(\frac{d}{\lambda}\right)^{-n}$. Letting $n \to \infty$, this value remains bounded away from 0 if $\lambda \geq d$.

The information to be transmitted is a certain element from a finite set $\mathcal{A} = \{1, \ldots, k\}$ (the alphabet). To each vertex v in T_d we associate one character out of $\mathcal{A}$. We say that the vertex v is in state $i \in \mathcal{A}$ when i is the character associated to v, and we then write $\sigma_v = i$.

Information is transmitted from each vertex to its d children: the state of v influences the state of its children. For each pair i and j in the alphabet $\mathcal{A}$ (i and j may be equal), there is a probability $0 \leq m_{i,j} \leq 1$ that a certain vertex is in state j given that its parent is in state i. This property is a Markov property: given the state of a vertex v, the state of each child of v depends on the state of v, but not on the state of its further ancestors up to ρ. Formally, if v_n is a vertex at distance n from ρ, and $v_{n-1}, v_{n-2}, \ldots, v_1, \rho$ are the vertices on the path from v_n to ρ:

$$\mathbf{P}(\sigma_{v_n} = j \mid \sigma_{v_{n-1}} = i, \sigma_{v_{n-2}} = i_{n-2}, \ldots, \sigma_{v_1} = i_1, \sigma_\rho = i_\rho) =$$

$$\mathbf{P}(\sigma_{v_n} = j \mid \sigma_{v_{n-1}} = i) = m_{i,j}$$

for all possible states $j, i, i_{n-2}, \ldots, i_1, i_\rho$.

We can place the values $m_{i,j},\ i,j \in \mathcal{A}$ in a matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,k} \\ m_{2,1} & m_{2,2} & \dots & m_{2,k} \\ \vdots & \ddots & & \vdots \\ m_{k,1} & m_{k,2} & \dots & m_{k,k} \end{pmatrix}$$

The matrix $\mathbf{M}$ is an example of a channel:

Definition 3. Channel *: A channel is a stochastic matrix describing the conditional distribution* $\mathbf{P}(Y \mid X)$ *of the output variable* Y *given the input* X*.*

We remind the reader that a stochastic matrix is a square matrix with positive real entries such that the sum of every row is equal to 1. We require that this channel must be irreducible and aperiodic, as explained below. We refer to [RO00] for more information.

In a Markov chain, we say that two states i and j communicate if there is at least one number n such that $\mathbf{P}(M^n(i) = j) > 0$; that is, it must be possible that after n iterations, the process gets in state j starting from state i. Clearly, each state communicates with itself, since $\mathbf{P}(M^0(i) = i) = 1$. If there is at least one pair of states which do not communicate, the Markov chain is said to be reducible. We rule out this possibility by stating that the Markov chain must be irreducible.

We say that a state $i \in \mathcal{A}$ has period p if $\mathbf{P}(M^n(i) = i) = 0$ whenever n is not divisible by p, and p is the largest number with this property. Thus, it is only possible to return to state i after $p, 2p, 3p, \dots$ iterations of M. If a state has period 1, we call it aperiodic. We require that all states be aperiodic.

3 Solvability and census-solvability

The reconstruction problem is to find out whether the state of the root has any influence on the state of the vertices far away from the root, even when the distance between those and the root goes to infinity. A trivial example is when each vertex' state is transmitted with probability 1 to all of its children, that is, when no miscommunications occur. In that case, all vertices in the tree end up in exactly the same state as the root, and there is no problem in finding out the state of the root from the state of vertices, however far away they may be. It is noticed that in this example, the Markov chain is reducible, since $\mathbf{P}(M^n(i) = j) = 0$ for all $i \neq j$ and all n; our requirement that the Markov chain be irreducible is to rule out this kind of trivialities.

A similar reason exists for our requirement that the Markov chain be aperiodic. In the case that there is a state i with period p, the state of the root affects the state of all levels of the tree in the sense that, if a vertex v is at distance n of the root ρ, n is no multiple of p, and $\sigma_\rho = i$, it is impossible that $\sigma_v = i$.

We say that the reconstruction problem is solvable if σ_ρ affects the state of the faraway levels. We say that the reconstruction problem is census-solvable if σ_ρ affects the *number* of vertices in faraway levels which are in each state. A rigorous definition for both concepts follows.

3.1 Solvability

We will denote by L_n the set of vertices of T_d at level n, that is, $L_n = \{v \in T_d \mid d(v, \rho) = n\}$. Clearly, $|L_n| = d^n$. If $v_1, v_2, \ldots, v_{d^n}$ are the vertices in L_n, we write $\sigma_{L_n} = (\sigma_{v_1}, \sigma_{v_2}, \ldots, \sigma_{v_{d^n}}) \in \mathcal{A}^{d^n}$ for the state of L_n.

We denote by $\mathbf{P}_n^i$ the conditional distribution of σ_{L_n} given $\sigma_\rho = i$, that is, for some $S \in \mathcal{A}^{d^n}$:

$$\mathbf{P}_n^i(S) = \mathbf{P}(\sigma_{L_n} = S \mid \sigma_\rho = i)$$

Definition 4. *The reconstruction problem is called solvable if the conditional distribution of σ_{L_n} given $\sigma_\rho = i$ is different from the conditional distribution of σ_{L_n} given $\sigma_\rho = j$, for some $i \neq j \in \mathcal{A}$ and even for limit values of n. Thus, the reconstruction problem is solvable if*

$$\lim_{n\to\infty} |\mathbf{P}_n^i - \mathbf{P}_n^j| > 0$$

for some $i \neq j \in \mathcal{A}$, where $|\cdot|$ denotes the total variation norm.

3.2 Census-solvability

We write $c_{L_n}(i) = \#\{v \in L_n \mid \sigma_v = i\} \in \mathbb{Z}$, and $c_{L_n} = \big(c_{L_n}(1), c_{L_n}(2), \ldots, c_{L_n}(k)\big) \in \mathbb{Z}^k$. Thus, $c_{L_n(i)}$ is the number of vertices in L_n which are in state i. We denote by $\mathbf{P}_n^{(c),i}$ the conditional distribution of c_{L_n} given $\sigma_\rho = i$, that is, for some $K \in \mathbb{Z}^k$:

$$\mathbf{P}_n^{(c),i}(K) = \mathbf{P}(c_{L_n} = K \mid \sigma_\rho = i)$$

Definition 5. *The reconstruction problem is called census-solvable if the conditional distribution of c_{L_n} given $\sigma_\rho = i$ is different from the conditional distribution of c_{L_n} given $\sigma_\rho = j$, for some $i \neq j \in \mathcal{A}$ and even n goes to infinity. Thus, the reconstruction problem is solvable if*

$$\lim_{n\to\infty} |\mathbf{P}_n^{(c),i} - \mathbf{P}_n^{(c),j}| > 0$$

for some $i \neq j \in \mathcal{A}$.

It is trivial that a reconstruction problem that is solvable will also be census-solvable. If the state of the root affects the configuration of the states of faraway levels, it also affects the number of vertices in those levels that are in every state.

According to these definitions, (census-) solvability of the reconstruction problem means that the state of the root affects the state of faraway levels. It's not inmediately clear that the reciprocal holds: that, when the reconstruction problem is (census-) solvable, given the state of a faraway level, it's possible to reconstruct the state of the root with more than trivial probability. This will turn out to be the case.

4 Equivalent definitions for solvability

The definition for solvability as stated in Definition 4 refers to the fact that the state of the root influences the state of all vertices in the tree, no matter how far away.

Definition 6. (Mutual information) *The mutual information of two random variables X and Y is defined as*

$$I(X;Y) = \sum_{x,y} \mathbf{P}(X=x, Y=y) \log \frac{\mathbf{P}(X=x, Y=y)}{\mathbf{P}(X=x)\mathbf{P}(Y=y)}$$

If X and Y are independent, $\mathbf{P}(X=x, Y=y) = \mathbf{P}(X=x)\mathbf{P}(Y=y)$, so the mutual information is 0 in that case. If X and Y are equal, on the other hand, the mutual information is the entropy of X.

It is proven in [MO01] that solvability can be equivalently defined as follows: the reconstruction problem is solvable if $\lim_{n\to\infty} I(\sigma_\rho; \sigma_{L_n}) > 0$.

Solvability can also equivalently be defined as follows. We denote by $\Delta_n(\pi)$ the probability of correctly inferring σ_ρ from σ_{L_n} given that σ_ρ is distributed according to a distribution vector π. The reconstruction problem is solvable if there is a $\tilde{\pi}$ such that

$$\liminf_{n\to\infty} \Delta_n(\tilde{\pi}) > \max_i \tilde{\pi}_i$$

5 Known theorems

We will denote by $\lambda_2(\mathbf{M})$ the second largest eigenvalue of the matrix $\mathbf{M}$, in absolute value. The following two theorems are well-established facts about solvability and census-solvability. The first one, by Mossel and Peres, is a consequence of a theorem by Kesten and Stigum from 1966.

Theorem A. *We consider the broadcasting process on the d-ary tree for which the transition probabilities are according to the channel $\mathbf{M}$. For this broadcasting process, the reconstruction problem is census-solvable if $d\lambda_2^2(\mathbf{M}) > 1$ and is not census-solvable if $d\lambda_2^2(\mathbf{M}) < 1$.*

Theorem (Bleher, Ruiz, Zagrebnov 1995). *If the state of the root is broadcasted on the d-ary tree with the following channel:*

$$\mathbf{M} = \begin{pmatrix} 1-\delta & \delta \\ \delta & 1-\delta \end{pmatrix} \quad (1)$$

then the reconstruction problem is solvable if and only if $d\lambda_2^2(\mathbf{M}) = d(1-2\delta)^2 > 1$.

6 Mossel's theorems

Theorem A and the Bleher-Ruiz-Zagrebnov theorem establish that the threshold values for solvability and census-solvability of the reconstruction problem are the same for the case that the alphabet is binary and the channel is symmetric. It was believed that solvability and census-solvability would have the same threshold value for all broadcasting processes. It is shown in [MO01] that this is not the case. In Theorem 1, it will be stated that the reconstruction problem may be solvable but not census-solvable for a broadcasting process with binary alphabet and very asymmetric transition matrix. Theorem 2 will state that the reconstruction problem for the broadcasting process with q-ary alphabet may be solvable if δ is sufficiently small and q is sufficiently large. In Theorem 6, we will state a continuity result about the threshold curve for the general binary channel.

Theorem 1 ([MO01])**.** *Consider the possibly asymmetric binary channel*

$$\mathbf{M} = \begin{pmatrix} 1-\delta_1 & \delta_1 \\ 1-\delta_2 & \delta_2 \end{pmatrix} \tag{2}$$

(Note that $\lambda_2(\mathbf{M}) = \delta_2 - \delta_1$). If $d\lambda_2(\mathbf{M}) > 1$, then there exists a $\delta > 0$ such that if $\delta_1 < \delta$, then the reconstruction problem is solvable.

Theorem 1 states that the reconstruction problem is solvable when $\lambda_2(\mathbf{M})$ and d are large enough, and δ_1 is small enough. Note that δ_1 is the probability of an error in the transmission of the first element of $\mathcal{A}$.

Remark. *Theorem 1 is purely an existence theorem. It does not yield any information on the actual threshold values between solvability and non-solvability, because the range from which δ_1 can be chosen is dependent on $\lambda(\mathbf{M})$ in the statement and in the proof of the theorem.*

Remark. *The combination of the Theorem A and Theorem 1 implies that if d and $\lambda_2(\mathbf{M})$ are such that $d\lambda_2^2(\mathbf{M}) < 1$ and $d\lambda_2(\mathbf{M}) > 1$, then the reconstruction problem is solvable for $\delta_1 < \delta$, but not census-solvable.*

Theorem 2 ([MO01])**.** *Consider the symmetric channel on q symbols*

$$\mathbf{M} = \begin{pmatrix} 1-(q-1)\delta & \delta & \dots & \delta \\ \delta & 1-(q-1)\delta & \delta & \dots \\ \vdots & \dots & \ddots & \vdots \\ \delta & \dots & \delta & 1-(q-1)\delta \end{pmatrix} \tag{3}$$

(Note that $\lambda_2(\mathbf{M}) = 1 - q\delta$). If $d\lambda_2(\mathbf{M}) > 1$, then there exists a Q such that if $q > Q$, then the reconstruction problem is solvable.

Theorem 2 states that the reconstruction problem is solvable if $\lambda_2(\mathbf{M})$, d, and the number of elements in the alphabet are large enough.

The following two propositions state that if the second eigenvalue is smaller than $\frac{1}{d}$ (in absolute value), the reconstruction problem becomes unsolvable.

Proposition 3 ([MO01])**.** *Let $\mathbf{M}$ be of the form (2) and let d be an integer such that $0 \le d\lambda_2(\mathbf{M}) = d(\delta_2 - \delta_1)$. Then the reconstruction problem is unsolvable for the d-ary tree and $\mathbf{M}$.*

Proposition 4 ([MO01])**.** *Let $\mathbf{M}$ be of the form (3) and let d be an integer such that $0 \le d\lambda_2(\mathbf{M}) = d(1 - q\delta) \le 1$. Then the reconstruction problem is unsolvable for the d-ary tree and $\mathbf{M}$.*

Theorem 5 ([MP03])**.** *Let $\mathbf{M}$ be of the form (2) and let d be an integer such that $d\frac{(\delta_2-\delta_1)^2}{\min\{\delta_1+\delta_2, 2-\delta_1-\delta_2\}} \le 1$. Then the reconstruction problem is unsolvable for the d-ary tree and $\mathbf{M}$.*

The following theorem states that there is a continuous curve above the diagonal of the unit (δ_1, δ_2)-square such that the reconstruction problem is solvable for the channel $\begin{pmatrix} 1-\delta_1 & \delta_1 \\ 1-\delta_2 & \delta_2 \end{pmatrix}$ and the d-ary tree above that curve:

Theorem 6. *For every infinite tree T, there is a nondecreasing function $\varphi : [0,\ 1-\frac{1}{d}] \to [0,1]$ which is continuous outside $\{0\}$, such that the reconstruction problem is solvable for the channel $\begin{pmatrix} 1-\delta_1 & \delta_1 \\ 1-\delta_2 & \delta_2 \end{pmatrix}$ and T when $\delta_2 > \varphi(\delta_1)$, and the reconstruction problem is not solvable for this channel and T when $\delta_2 < \varphi(\delta_1)$.*

We'll consider the broadcasting process with channel

$$\mathbf{M} = \begin{pmatrix} 1-\delta_1 & \delta_1 \\ 1-\delta_2 & \delta_2 \end{pmatrix}$$

Based on the Theorem A, the Bleher-Ruiz-Zagrebnov theorem and Proposition 5, we can divide the unit square, in which δ_1 and δ_2 lie, in areas such that the reconstruction problem for this broadcasting process is either census-solvable, solvable or unsolvable when δ_1 and δ_2 lie in them. Recall that, for this transition matrix, $\lambda_2(\mathbf{M}) = \delta_2 - \delta_1$.

From the Theorem A, the reconstruction problem is census-solvable if $d(\delta_2 - \delta_1)^2 > 1 \implies$ either $\delta_2 - \delta_1 > \frac{1}{\sqrt{d}}$ or $\delta_2 - \delta_1 < -\frac{1}{\sqrt{d}}$. These two areas of the unit square are coloured in Figure 1.

Proposition 5 states that the reconstruction problem is unsolvable when $\delta_2 < 1 - \delta_1$ and $\delta_1 + \frac{1}{2d} - \frac{1}{2}\sqrt{\frac{1}{d^2} + \frac{8}{d}\delta_1}\delta_2 \leq \delta_1 + \frac{1}{2d} + \frac{1}{2}\sqrt{\frac{1}{d^2} + \frac{8}{d}\delta_1}$. When $\delta_2 \geq 1 - \delta_1$, Proposition 5 establishes that the reconstruction problem is unsolvable if $\delta_1 - \frac{1}{2d} - \sqrt{\frac{2}{d} + \frac{1}{d^2} - \frac{8}{d}\delta_1} \leq \delta_2 \leq \delta_1 - \frac{1}{2d} + \sqrt{\frac{2}{d} + \frac{1}{d^2} - \frac{8}{d}\delta_1}$. These threshold curves are also shown in Figure 1.

Theorem 1 states that, if $d\lambda_2(\mathbf{M}) = d(\delta_2 - \delta_1) > 1 \implies \delta_2 - \delta_1 > \frac{1}{d}$, then there is a $\delta > 0$ such that the reconstruction problem is solvable if δ_1 (the probability of an error in the transmission of state 1) is smaller than δ. Moreover, when we rearrange the alphabet we see that the channel

$$\tilde{\mathbf{M}} = \begin{pmatrix} \delta_2 & 1-\delta_2 \\ \delta_1 & 1-\delta_1 \end{pmatrix}$$

is equivalent to the original channel. It follows from Theorem 1 that there is a $\tilde{\delta} > 0$ such that the reconstruction problem is solvable if $\delta_2 - \delta_1 > \frac{1}{d}$ and $1 - \delta_2 < \tilde{\delta} \implies \delta_2 > 1 - \tilde{\delta}$. Unfortunately, the theorem does not give sufficient information in order to find an area in the δ_1, δ_2 unit square such that the reconstruction problem is solvable for those δ_1 and δ_2.

Finally, Theorem 6 states that there is a continuous nondecreasing curve in the δ_0, δ_1 unit square such that the reconstruction problem is solvable when (δ_1, δ_2) lies above that curve and unsolvable in the opposite case. Unfortunately, the theorem again provides no further information about that curve, so that it is not possible to plot it in Figure 1.

7 Proof of Theorem 1 and Theorem 2

In order to prove the stated theorems and propositions, we will make use of random-cluster arguments. To each edge e of T_d we assign a value $\tau(e) \in \{0,1\}$. An edge e is said to be open if $\tau(e) = 1$. A Bernoulli percolation[1] with parameter

[1] This process is sometimes called independent bond percolation.

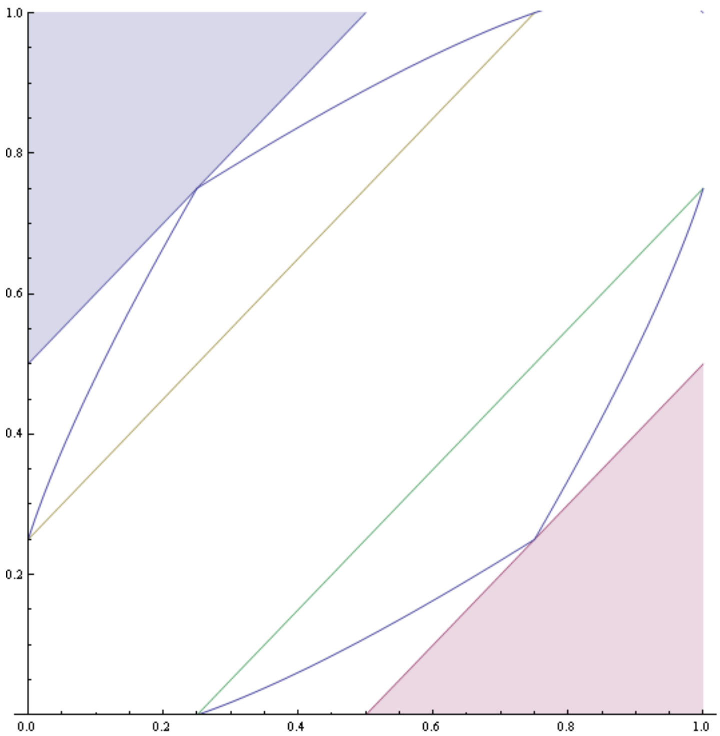

Figure 1: The reconstruction problem is census-solvable for δ_1 and δ_2 in the coloured areas. The problem is unsolvable for δ_1 and δ_2 in the area between the parabolic curves.

λ on T_d is a random process with state space $\{0,1\}^{E_d}$, with $\mathbf{P}(\tau(e) = 1) = \lambda$ independently for all $e \in E_d$; so each edge e can be open with probability λ and closed with probability $1 - \lambda$. The cluster (or component) $\mathcal{C}(v)$ of a vertex v is defined as the set of all vertices in T_d which are connected to v by a path of open edges.

We say that a subtree T' of T_d is an l-diluted b-regular subtree of T_d if all vertices at level il of T' have exactly b descendents at level $(i+1)l$, for all i.

Lemma 7. *Let T_d be the infinite d-ary tree rooted at ρ, and let $1/d < \lambda \leq 1$. There is a positive ε dependent on d and λ such that, for all $b \geq 1$, there exists an $l \geq 1$ such that, when a Bernoulli percolation is performed with a parameter $\lambda' \geq \lambda$ on T_d, then*

$$\mathbf{P}(\rho \textit{ is the root of an open } l\textit{-diluted } b\textit{-regular tree}) \geq \varepsilon \tag{4}$$

For the proof of Lemma 7, we need Lemma 8.

Lemma 8. *Let T_d be the d-ary tree, and let $1/d < \lambda \leq 1$. There is an $\varepsilon > 0$ such that, for all b, there is an l such that, when we perform Bernoulli percolation with parameter λ,*

$$\mathbf{P}(|\mathcal{C}(\rho) \cap L_l| \geq b) \geq \varepsilon \tag{5}$$

Lemma 8 says that, in case the percolation parameter λ is greater than $1/d$, there is a minimum positive value ε such that any number b of children of ρ is attained in a certain level L_l of the cluster of ρ, with a probability at least ε.

Proof. Let $Z_l = |\mathcal{C}(\rho) \cap L_l| \in \mathbf{N}_0$. We claim that $\mathbb{E}(Z_{l+1} \mid Z_l = z) = (d\lambda)^l z$, for all l. The claim clearly holds when $z = 0$. In the case that $z > 0$, take a vertex v in $\mathcal{C}(\rho) \cap L_l$. v has d children in T_d, and each of these pertains to the cluster of ρ with probability λ; hence, the number of children of v that pertain to the cluster of ρ is $Bin(d, \lambda)$-distributed, and its expectation is $d\lambda$. We sum up the expected number of children for all the z vertices in $\mathcal{C}(\rho) \cap L_l$ and find that the claim holds.

Next, we define $W_l = (d\lambda)^{-1} Z_l$. It is easy to check that, for $w \in \mathbf{N}$, $\mathbb{E}(W_{l+1} \mid W_l = w) = w$, so W_l is a positive martingale, which converges almost surely to some W.

It is proven in [Athreya] that $\mathbf{P}(W \neq 0) > 0$. Hence, there are $\varepsilon > 0, \varepsilon_1 > 0$ such that $\mathbf{P}(W \geq \varepsilon_1) > \varepsilon \implies \mathbf{P}(\lim_{l\to\infty}(d\lambda)^{-l} Z_l \geq \varepsilon_1) \geq \varepsilon \implies \lim_{l\to\infty} \mathbf{P}(Z_l \geq \varepsilon_1 (d\lambda)^l) \geq \varepsilon$. Finally, since $\varepsilon_1 (d\lambda)^l \to \infty$, there is a value K such that for $l > K$, $\varepsilon_1 (d\lambda)^l > b$. □

Proof of Lemma 7. For increasing values of λ', the probability of the left-hand side in 4 increases; hence it suffices to provide the proof for λ. Take ε such that for all $\tilde{b}$ there is an $\tilde{l}$ such that ρ has $\tilde{b}$ children or more in level $\tilde{l}$ of a λ-diluted subtree of T_d with probability at least ε. Lemma 8 assures that such an ε exists.

Take B such that $\mathbf{P}(Bin(\varepsilon/2, B) > b) > 1/2$, take l such that $\mathbf{P}(|\mathcal{C}(\rho) \cap L_l| \geq B) \geq \varepsilon$. By definition of ε, such an l can be found.

Denote by A_r the event that ρ is the root of rl levels of an l-diluted b-regular tree. Let $p_r = \mathbf{P}(A_r)$. Clearly, $p_0 = 1$, and

$$p_{r+1} = \mathbf{P}\left(|\mathcal{C}(\rho) \cap L_l| \geq B\right) \mathbf{P}\left(A_{r+1} \Big|\, |\mathcal{C}(\rho) \cap L_l| \geq B\right) +$$
$$\mathbf{P}\left(|\mathcal{C}(\rho) \cap L_l| < B\right) \mathbf{P}\left(A_{r+1} \Big|\, |\mathcal{C}(\rho) \cap L_l| < B\right) \implies$$
$$p_{r+1} \geq \mathbf{P}\left(|\mathcal{C}(\rho) \cap L_l| \geq B\right) \mathbf{P}\left(A_{r+1} \Big|\, |\mathcal{C}(\rho) \cap L_l| \geq B\right)$$
$$\geq \varepsilon \mathbf{P}(Bin(p_r, B) \geq b)$$

ε is the probability that ρ has at least B children in level l of the cluster of ρ. Each of these children is at the root of rl levels of an l-diluted b-regular tree with probability p_r. By definition of B, and given the induction hypothesis $p_r \geq \varepsilon/2$, also $p_{r+1} \geq \varepsilon/2$. □

The next lemma provides information for the case that λ is close to 1. It states that we can fix any probability $(1 - \varepsilon)$ smaller than 1, and then there is a λ (close to 1), so that when we perform a Bernoulli percolation with this parameter λ or a larger parameter, with probability $1 - \varepsilon$ or more, the cluster of ρ contains nearly all vertices of T.

Lemma 9. *Let T_d be the infinite d-ary tree rooted at ρ, and take $l \geq 1$ and $\varepsilon > 0$. There exists a $\lambda \leq 1$ such that, if one performs Bernoulli percolation with parameter $\lambda' \geq \lambda$ on T_d, then*

$$\mathbf{P}(\rho \textit{ is the root of an open } l\textit{-diluted } (d^l - 1)\textit{-regular tree}) \geq 1 - \varepsilon$$

Proof. By monotonicity, it's sufficient to provide the proof for λ. Define

$$f(p) = \mathbf{P}(Bin(p, d^l) \geq d^l - 1) = p^{d^l} + d^l(1-p)p^{d^l-1}$$

Then $f(1) = 1$ and $f'(1) = 0$. Therefore, there is $1 > p^* > 1 - \varepsilon$ such that $f(p^*) > p^*$. We now take $\lambda < 1$ such that

$$\mathbf{P}(|\mathcal{C}(\rho) \cap L_l| = d^l) \geq \frac{p^*}{f(p^*)}$$

We write p_r for the probability that ρ is at the root of an l-diluted (d^l-1)-regular tree of rl levels. Then, $p_0 = 1 \geq p^*$, and

$$p_{r+1} \geq \mathbf{P}(|\mathcal{C}(\rho) \cap L_l| = d^l) f(p_r) \geq \frac{p^*}{f(p^*)} f(p_r) \geq p^*$$

In this expression, $\mathbf{P}(|\mathcal{C}(\rho) \cap L_l| = d^l)$ is the probability that all the T_d-descendants of ρ in level l belong to the cluster of ρ, and $f(p_r)$ is the probability that each of those is at the root of rl levels of an l-diluted (d^l-1)-regular open tree. The induction hypothesis is that $p_r \geq p^*$, so that $f(p_r) \geq f(p^*)$, which generates the last inequality. Hence, it is proven that $p_r \geq p^* \geq 1 - \varepsilon$. □

We need a finer form of percolation to suit for dependence. A progressive percolation with parameter λ we define it as follows.

Definition 7. *A progressive percolation with parameter λ is the random process wich assigns either the value 0 or the value 1 to the edges in T_d as follows. If v is the parent of w, we assign to the edge $\{v, w\}$ the value 0 if v is not in the cluster of ρ. If v is in the cluster of ρ, on the other hand, we assign to $\{v, w\}$ the value 1 with probability λ, and the value 0 with probability $1 - \lambda$, independently. We say that an edge e is open if e has got assigned the value 1, and closed if it has got assigned the value 0.*

We remark that the progressive percolation as defined is the process which, given that the cluster of ρ has reached as far as a vertex v, extends the cluster of ρ to any child of v with probability λ.

Corollary 10. *Let T_d be the infinite d-ary tree rooted at ρ, and let $\lambda \leq 1$ be such that $d\lambda > 1$. Then there exists an $\varepsilon > 0$ with the property that for all $b \geq 1$ there is an $l \geq 1$ such that, when we perform a progressive percolation with parameter $\lambda' \geq \lambda$ on T_d,*

$$\mathbf{P}(\rho \textit{ is the root of an open } l\textit{-diluted } b\textit{-regular tree}) \geq \varepsilon$$

Corollary 11. *Let T_d be the infinite d-ary tree rooted at ρ, and take $l \geq 1$ and $\varepsilon > 0$. There exists a $\lambda \leq 1$ such that, if one performs progressive percolation with parameter $\lambda' \geq \lambda$ on T_d, then*

$$\mathbf{P}(\rho \textit{ is the root of an open } l\textit{-diluted } (d^l-1)\textit{-regular tree}) \geq 1 - \varepsilon$$

Proof of Corollary 10 and Corollary 11. The proofs of Lemma 8, Lemma 7 and Lemma 9 hold for progressive percolation as well as for Bernoulli percolation. □

The last lemma establishes that there can't be an l-diluted r-regular subtree of T_d and an l-diluted s-regular subtree of T_d with $r+s>d^l$ such that they have no vertices in common at level $l, 2l, 3l, \ldots$. They can't exist because they don't fit in the tree T_d.

Lemma 12. *Let r and s be such that $r+s>d^l$, and $n \geq 0$. Let T_d be the d-ary tree rooted at ρ. Let $T'=(V',E')$ be an l-diluted r-regular tree rooted at ρ, and let $T''=(V'',E'')$ be an l-diluted s-regular tree rooted at ρ. Then, for any n, it is impossible that $T' \cup T'' \subset T_d$ with $V' \cap V'' \cap L_{ln}=\emptyset$.*

Proof. The proof is by induction on n. The claim holds for $n=0$. Let $v \in V' \cap V'' \cap L_{ln}$. Then v has r descendants in $V' \cap L_{l(n+1)}$ and s descendants in $V'' \cap L_{l(n+1)}$. Since v has only d^l descendents in $L_{l(n+1)}$ and $d^l<r+s$, some of the descendants of v in $V' \cap L_{l(n+1)}$ must also belong to $T'' \cap L_{l(n+1)}$. □

With the help of Corollaries 10 to 11 and of Lemma 12, we will proceed to prove Theorem 1. The key argument of the proof is that, in the case that the root of T_d is in state $+$, there is a positive probability (that we will denote by ε) that from the root emanates an l-diluted binary subtree T' (which by definition extends to infinity), with the property that on level rl all the vertices of T' are in state $+$, for all r. On the other hand, if the root happens to be in state $-$, with probability $1-\varepsilon/2$ the tree T_d is almost fully in state $-$, so that there is no room in T_d for the subtree T' as described.

Proof of Theorem 1. We will denote the states of the chain by $-$ and $+$. As Corollary 10 states, when we perform a progressive percolation with parameter λ on the d-ary tree T_d rooted at ρ, and this λ is larger than $1/d$, then there is a positive probability ε that the cluster of ρ contains an l-diluted binary tree that extends all the way along T_d, up to infinity. On the other hand, Corollary 11 states that, for this given ε, there is a $\tilde{\lambda}$ such that, when we perform a progressive percolation with parameter at least $1-\tilde{\lambda}$, there is a (d^l-1)-regular subtree in the cluster of ρ (which again extends to infinity) with probability larger than $1-\varepsilon/2$.

If v is the parent vertex of w:

$$\mathbf{P}(\sigma_w=+ \mid \sigma_v=+)=\delta_2$$

$$\mathbf{P}(\sigma_w=- \mid \sigma_v=-)=1-\delta_1$$

We assign the value 1 to those edges of T_d which join two vertices which are in state $+$; the other edges get assigned the value 0. This is equivalent to performing a progressive percolation with parameter δ_2 on T_d. In the case that $\sigma_\rho=1$ and that $\delta_2 \geq \lambda$, by Corollary 10, with probability at least ε it happens that ρ is the root of an l-diluted binary tree, all of whose vertices are in state $+$, that extends to infinity along T_d.

We denote by A_{rl} the event that there exists some l-diluted binary tree T', all whose rl-level vertices are in state $+$. In the case that all the vertices in the cluster of ρ are in state $+$ and the cluster contains an l-diluted binary tree, clearly A_{rl} is realised; hence the probability of the latter, under the condition that ρ is in state $+$, is at least the probability of the former, under the same condition:

$$\mathbf{P}^+_{rl}(A_{rl}) \geq \mathbf{P}(\rho \text{ root of an } l\text{-diluted binary tree fully in state } + \mid \sigma_\rho=+) \geq \varepsilon \tag{6}$$

We can also prove that the probability of the realisation of A_{rl} is at most equal to $\varepsilon/2$ in the case that $\sigma_\rho = -$. To that end, we now assign the value 1 to those edges of T_d between two vertices in state $-$. This is equivalent to performing a progressive percolation with parameter $1-\delta_1$ on T_d. Corollary 11 implies that, in the case that $\sigma_\rho = -$, there is a $\delta > 0$ such that if $1-\delta \leq 1-\delta_1 < 1$, with probability at least $1-\varepsilon/2$ it happens that ρ is the root of an l-diluted (d^l-1)-regular tree fully in state $-$ that extends to infinity along T_d; here ε is taken the same as before. The realisation of the mentioned subtree makes A_{rl} impossible by Lemma 12, so indeed

$$\mathbf{P}^-_{rl}(A_{rl}) \leq \varepsilon/2 \tag{7}$$

Now, if $\delta_1 < \delta$ and $\delta_2 = \delta_1 + \lambda \geq 1/d$, then the following inequality holds for all r:

$$|\mathbf{P}^+_{rl}(A_{rl}) - \mathbf{P}^-_{rl}(A_{rl})| \geq \varepsilon/2 \tag{8}$$

It follows that the reconstruction problem is solvable when $\delta_1 \leq \delta$ and $\lambda_2(\mathbf{M}) = \delta_2 - \delta_1 > 1/d$, as we wanted to prove. □

Proof of Theorem 2. The proof of Theorem 2 is similar to that of Theorem 1. We fix $i \neq j \in \mathcal{A}$. Then, if v is the parent vertex of w:

$$\mathbf{P}(\sigma_w = i \mid \sigma_v = i) = 1-(q-1)\delta$$

$$\mathbf{P}(\sigma_w \neq i \mid \sigma_v = i') = 1-\delta, \text{ for all } i' \neq i$$

We assign the value 1 to those edges of T_d which join two vertices in state i; the other edges get assigned the value 0. This is equivalent to performing a progressive percolation with parameter $1-(q-1)\delta$ on T_d. When $1-(q-1)\delta > 1/d$, Corollary 10 implies that, in the case that $\sigma_\rho = i$, with probability at least ε it happens that there is an l-diluted binary subtree of T_d completely in state i.

We denote by A_{rl} the event that there is an l-diluted binary tree T' rooted at ρ such that all the vertices of T' in level rl are in state i. In the case that all the vertices in the cluster of ρ are in state i and the cluster of ρ contains any l-diluted binary tree, clearly A_{rl} happens; hence the probability of the latter, under the condition that $\sigma_\rho = i$, is at least the probability of the former, under the same condition:

$$\mathbf{P}^i_{rl}(A_{rl}) \geq \mathbf{P}(\rho \text{ root of an } l\text{-dil. binary tree fully in state } i \mid \sigma_\rho = i) \geq \varepsilon \tag{9}$$

In the case that $\sigma_\rho = j$, the probability of A_{rl} is at most $\varepsilon/2$. To show that, we now assign the value 1 to all edges of T_d between vertices which are not in state i. This is equivalent to performing a progressive percolation with parameter $1-\delta$ on T_d. When $\sigma_\rho = j$ and $\delta > 0$ is small enough, Corollary 11 implies that, with probability at least $1-\varepsilon/2$, ρ is at the root of an l-diluted (d^l-1)-regular subtree of T_d. The realisation of this subtree makes the realisation of A_{rl} impossible by Lemma 12, which shows indeed, for all r:

$$\mathbf{P}^j_{rl}(A_{rl}) \leq \varepsilon/2 \tag{10}$$

We take $\lambda = 1-q\delta \implies \delta = \frac{1-\lambda}{q}$. Let Q be such that if $q \geq Q$, then equation (10) holds for $\delta = \frac{1-\lambda}{q}$. Clearly, $1-(q-1)\delta > 1-q\delta = \lambda$, so equation (9) also holds.

It follows that the reconstruction problem is solvable for δ small enough and q large enough. □

8 Proof of Proposition 3 and Proposition 4

For the proofs of Proposition 3 and Proposition 4, we will use Markov chains whose matrices have the form

$$\mathbf{M}_{i,j} = \lambda \mathbf{N}_{i,j} + (1-\lambda)\nu_j \tag{11}$$

where $\mathbf{N} \in \mathbb{R}^{k\times k}$ is some transition matrix, $\nu \in \mathbb{R}^k$ is a distribution vector, and $0 \le \lambda \le 1$. The proof of Propositions 3 and 4 follows from the following two prepositions.

Proposition 13. *Suppose that* $\mathbf{M}$ *has the form (11). Then the reconstruction problem is unsolvable for* M *when* $d\lambda \le 1$.

Proposition 14. *All binary channels of the form (2) are of the form (11) with* $\lambda = \lambda_2(\mathbf{M})$. *All symmetric channels of the form (3) with* $\lambda = 1 - q\delta \le 0$ *are of the form (11) with* $\lambda = \lambda_2(\mathbf{M}) = 1 - q\delta$.

Proof of Proposition 13. The matrix $\mathbf{M}$ satisfies (11); hence, we can write the random function M as $M = XN + (1-X)Y$, where N is a random function such that $\mathbf{P}(N(i) = j) = \mathbf{N}_{i,j}$, Y is a random variable such that $\mathbf{P}(Y = j) = \nu_j$, X is a variable on $\{0,1\}$ such that $\mathbf{P}(X = 1) = \lambda$, and all these variables are independent.

To see this equality, consider:

$$\begin{gathered}\mathbf{P}(XN(i) + (1-X)Y = j) = \\ \mathbf{P}\Big(XN(i) + (1-X)Y = j \mid X = 1\Big)\mathbf{P}(X=1) + \\ \mathbf{P}\Big(XN(i) + (1-X)Y = j \mid X = 0\Big)\mathbf{P}(X=0) = \\ \lambda \mathbf{P}\Big(N(i) = j\Big) + (1-\lambda)\mathbf{P}\Big(Y = j\Big) = \\ \lambda \mathbf{N}_{i,j} + (1-\lambda)\nu_j = \mathbf{M}_{i,j}\end{gathered}$$

As a consequence of this, the broadcasting process on T_d with channel $\mathbf{M}$ can equivalently be constructed as follows:

1. For each vertex v, let N_v and Y_v be independent copies of N and Y.
2. Perform a percolation with parameter λ on T_d.
3. Fix $\sigma_\rho = i$.
4. Denote $\Gamma = (\tau(e))_{e\in E_d}$. Here as before, $\tau(e) \in \{0,1\}$ is the value of the edge e; Γ is the (random) state of E_d, the set of edges of T_d.
5. If v is the parent vertex of w, and σ_v is already assigned, we assign σ_w as follows. If the edge $\{v,w\}$ is open, then we set $\sigma_w = N_v(\sigma_v)$; otherwise, we set $\sigma_w = Y_w$.

In the broadcasting process as described, the state of each vertex with probability λ depends on its parent's state, and with probability $1-\lambda$ doesn't. Clearly, the state of level L_n can be affected by the state of ρ only in the case that a path exists from ρ to at least one vertex in L_n, on which path each vertex' state depends on its parent's; i.e., only in the case that the cluster of ρ extends to L_n. However, for each vertex v in $\mathcal{C}(\rho)$, the number of children that also belong to $\mathcal{C}(\rho)$ is a random number with expectation $d\lambda$. This Galton-Watson process is finite with probability 1, so for $n \to \infty$, the probability that $\mathcal{C}(\rho)$ extends to level L_n goes to 0. For that reason, the state of the root (a.s.) has no effect on the state of L_n when L_n goes to infinity, and the reconstruction problem is unsolvable. □

Proof of Proposition 14.

- For a symmetric channel of the form (3) with $\lambda = 1 - q\delta \leq 0$, we take $\mathbf{N} = \mathbf{I}$ the identity matrix, ν the uniform distribution and $\lambda = \lambda_2(\mathbf{M}) = 1 - q\delta$:

$$\mathbf{M} = \begin{pmatrix} 1-(q-1)\delta & \delta & \dots & \delta \\ \delta & 1-(q-1)\delta & \delta & \dots \\ \vdots & \dots & \ddots & \vdots \\ \delta & \dots & \delta & 1-(q-1)\delta \end{pmatrix} =$$

$$= (1-q\delta)\begin{pmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & 0 & 1 \end{pmatrix} + q\delta \begin{pmatrix} \frac{1}{q} & \frac{1}{q} & \dots & \frac{1}{q} \\ \frac{1}{q} & \dots & \ddots & \frac{1}{q} \\ \vdots & \ddots & & \vdots \\ \frac{1}{q} & \dots & \dots & \frac{1}{q} \end{pmatrix}$$

- For the general binary channel

$$\mathbf{M} = \begin{pmatrix} 1-a & a \\ 1-b & b \end{pmatrix} = (b-a)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-b+a)\begin{pmatrix} \frac{1-b}{1-b+a} & \frac{a}{1-b+a} \\ \frac{1-b}{1-b+a} & \frac{a}{1-b+a} \end{pmatrix}$$

$$= (a-b)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (1-a+b)\begin{pmatrix} \frac{1-a}{1-a+b} & \frac{b}{1-a+b} \\ \frac{1-a}{1-a+b} & \frac{b}{1-a+b} \end{pmatrix}$$

□

9 Topology of the area of solvability for general trees

In this section, we will prove that the area above the diagonal of the unit square in which the parameters δ_1 and δ_2 can lie in order to make the reconstruction problem solvable, does not misbehave: this area is connected; there are no isolated points in it; and its border curve is continuous and non-decreasing. We prove this by a geometric interpretation of the following probabilistic proposition.

Proposition 15. *Let $M^{(1)}$ and $M^{(2)}$ be channels such that*

$$\mathbf{M}^{(1)}_{i,j} = \lambda_1 \mathbf{N}_{i,j} + (1-\lambda_1)\nu_j$$

$$\mathbf{M}^{(2)}_{i,j} = \lambda_2 \mathbf{N}_{i,j} + (1-\lambda_1)\nu_j$$

with equal channels $\mathbf{N}$ and ν and with ν such that all rows of ν are equal, and $0 \leq \lambda_1 < \lambda_2$. If the reconstruction problem is solvable for $\mathbf{M}^{(1)}$ and the infinite tree T, then the reconstruction problem is also solvable for $\mathbf{M}^{(2)}$ and T.

Proof. Take λ_3 such that $\lambda_1 = \lambda_2\lambda_3$. Then: $\mathbf{M}^{(1)} = \lambda_2\lambda_3\mathbf{N} + (1-\lambda_2\lambda_3)\nu = \lambda_2\lambda_3\frac{1}{\lambda_2}[\mathbf{M}^{(2)} - (1-\lambda_2)] + (1-\lambda_2\lambda_3)\nu = \lambda_3\mathbf{M}^{(2)} + (1-\lambda_3)\nu$.

As in the proof of Proposition 13, we see that the broadcasting on T_d with the channel $\mathbf{M}^{(1)}$ is equivalent to first performing a Bernoulli percolation with parameter λ_3 on T. On the cluster of the root, we broadcast according to the channel $\mathbf{M}^{(2)}$; outside this cluster, the states of the vertices are assigned at random with distribution according to any row of ν (recall that all rows of ν are equal).

If we denote by σ_v the state which the vertex v gets assigned by the broadcasting with channel $\mathbf{M}^{(1)}$ and by τ_v the state which v gets assigned by the broadcasting with channel $\mathbf{M}^{(2)}$, we find that $\sigma_v = \tau_v$ if v is in the cluster of ρ, and σ_v is independent of σ_ρ otherwise.

It follows that, if the reconstruction problem is not solvable for the channel $\mathbf{M}^{(2)}$ and the d-ary tree, the reconstruction problem also is not solvable for the channel $\mathbf{M}^{(1)}$ and T_d, as was to be proven. □

Remark. *It does not play a role in the proof of Lemma 15 whether the tree is regular, hence the result holds for every infinite tree.*

Proposition 15 implies that, for each infinite tree T, there is a continuous nondecreasing curve above the diagonal of the unit square such that, when the point (δ_1, δ_2) lies above this curve, the reconstruction problem is solvable for the channel $\begin{pmatrix} 1-\delta_1 & \delta_1 \\ 1-\delta_2 & \delta_2 \end{pmatrix}$ and T. This is the statement of Theorem 6. In order to prove the theorem, we need two further lemmas.

We will denote $\mathbf{M}(a,b) = \begin{pmatrix} 1-a & a \\ 1-b & b \end{pmatrix}$, and we will denominate (a,b) the coordinates of $\mathbf{M}(a,b)$.

Lemma 16. *When $a < 1 - 1/br(T)$, the reconstruction problem is solvable for $\mathbf{M}(a,1)$ and the T. Similarly, when $b > 1/br(T)$, the reconstruction problem is solvable for $\mathbf{M}(0,b)$ and T.*

Proof. When ρ is in state 1 and the state is broadcasted according to $\mathbf{M}(a,1)$, the entire tree gets in state 1. On the other hand, if a vertex v is in state 0, each child of v is in state 0 with probability $1-a$. If we define an edge to be open when it joins two vertices in state 0, this is equivalent to performing a percolation with parameter $1-a$ on T. It is proven in [PE99] that this percolation has a positive probability of survival if $1-a > 1/\mathrm{br}(T)$, and this is the probability that at every level at least one vertex will be in state 0.

In the same manner, when $b > 1/d$ and $a = 0$ and we broadcast on T_d with the channel $\mathbf{M}(a,b)$, the entire tree gets in state 0 if the root is in state 0. On the other hand, if a vertex v is in state 1, each child of v is in state 1 with probability b; again by the result in [PE99], if $b > 1/\mathrm{br}(T)$, there is a positive probability that at least one vertex in every level will be in state 1. □

Lemma 17. *Let $\mathbf{M}(a,b) = \lambda\mathbf{M}(\alpha,\beta) + (1-\lambda)\nu$, with $\nu = \begin{pmatrix} 1-\gamma & g \\ 1-\gamma & \gamma \end{pmatrix}$ and $0 \le \lambda \le 1$. If for some real number t, $\mathbf{M}(\tilde{a},\tilde{b}) = (\lambda+t)\mathbf{M}(\alpha,\beta) + (1-\lambda-t)\nu$, then the vector $(\tilde{a}-a, \tilde{b}-b)$ is a scalar multiple of the vector $(\alpha-a, \beta-b)$.*

Proof.

$$a = \lambda\alpha + (1-\lambda)\gamma \implies \alpha - a = \alpha - \lambda\alpha - (1-\lambda)\gamma = (1-\lambda)(\alpha-\gamma)$$

$$b = \lambda\beta + (1-\lambda)\gamma \implies \beta - b = \beta - \lambda\beta - (1-\lambda)\gamma = (1-\lambda)(\beta-\gamma)$$

$$\tilde{a} = (\lambda+t)\alpha + (1-\lambda-t)\gamma = a + t(\alpha-\gamma)$$

$$\tilde{b} = (\lambda+t)\beta + (1-\lambda-t)\gamma = b + t(\beta-\gamma)$$

$$\big(\tilde{a}-a, \tilde{b}-b\big) = t\big(\alpha-\gamma, \beta-\gamma\big) = \frac{t}{1-\lambda}\big(\alpha-a, \beta-b\big)$$

□

Remark. *We see that, for $\lambda \le \tilde{\lambda} \le 1$, the coordinates of the channels $\mathbf{N} = \tilde{\lambda}\mathbf{M}(\alpha,\beta) + (1-\tilde{\lambda})\nu$ lie on the line segment between the points $\big(a,b\big)$ and $\big(\alpha,\beta\big)$. We remark that, given the fact that the reconstruction problem is solvable for the the tree T and $\mathbf{M}(a,b)$, the proposition implies that for all channels whose coordinates lie on the mentioned line segment, the reconstruction problem is solvable on T.*

Proof of Theorem 6. Fix an infinite tree T. For simplicity, during this proof, we will say "the channel $\mathbf{M}$ is solvable" to indicate that the reconstruction problem is solvable for $\mathbf{M}$ and the tree T, and "the channel $\mathbf{M}$ is unsolvable" to indicate the oposite.

First, we prove that for $0 \le t < 1/\mathrm{br}(T)$ there is a number $\varphi(t) \ge t$ such that $\mathbf{M}(t,b)$ is solvable when $b > \varphi(t)$ and $\mathbf{M}(t,b)$ is unsolvable when $b < \varphi(t)$.

Lemma 16 states that $\varphi(0) = 1/\mathrm{br}(T)$. For $t \ne 0$, we define $R(t) = \{\xi \in [t,1] \mid \mathbf{M}(t,\xi) \text{ is solvable}\}$. It follows from Lemma 16 that $R(t)$ is not the empty set if $0 \le t < 1 - \frac{1}{\mathrm{br}(T)}$. It follows from the completeness of $\mathbb{R}$ that $R(t)$ has an infimum. This infimum we will denote $\varphi(t) = \inf(R(t))$.

By definition of $\varphi(t)$, when $b < \varphi(t)$, $\mathbf{M}(t,b)$ is unsolvable. In order to see that $\mathbf{M}(t,b)$ is solvable when $b > \varphi(t)$, we decompose $\mathbf{M}(t,b)$ as follows:

$$\mathbf{M}(t,b) = \begin{pmatrix} 1-t & t \\ 1-b & b \end{pmatrix} = \frac{b-t}{1-t}\begin{pmatrix} 1-t & t \\ 0 & 1 \end{pmatrix} + \left(1 - \frac{b-t}{1-t}\right)\begin{pmatrix} 1-t & t \\ 1-t & t \end{pmatrix}$$

The line segment between $\big(t,b\big)$ and $\big(t,1\big)$ is vertical. When we take $b \downarrow \varphi(t)$, the channel $\mathbf{M}(t,b)$ is solvable by definition of $\varphi(t)$. By the remark following Lemma 17, all channels with coordinates on the mentioned vertical line are solvable. This proves that indeed the channel $\mathbf{M}(t,b)$ is solvable when $b > \varphi(t)$.

We remark that it is unknown whether the limiting case $\mathbf{M}(t,\varphi(t))$ itself is solvable.

Secondly, we proceed to prove that $\varphi : \ [0,\ 1 - \frac{1}{\mathrm{br}(T)}] \to [0,1]$ is nondecreasing, or equivalently, that when $\mathbf{M}(a,b)$ is solvable, also $\mathbf{M}(\tilde{a},b)$ is solvable whenever $\tilde{a} \le a$.

We need the following decomposition of $\mathbf{M}(a,b)$:

$$\mathbf{M}(a,b) = (1-a)\begin{pmatrix} 1 & 0 \\ \frac{b-1}{a-1} & \frac{a-b}{a-1} \end{pmatrix} + a\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

The remark following Lemma 17 implies that, in the case that $\mathbf{M}(a,b)$ is solvable, all the channels whose coordinates lie on the segment between $\big(a,b\big)$ and

$\left(0, \frac{a-b}{a-1}\right)$ are also solvable. The number $\frac{a-b}{a-1}$ is smaller than b (for all relevant values of a and b), so this line segment has positive slope. In combination with the first part of the proof, we conclude that all channels whose coordinates lie on this line segment or above it are solvable.

Now consider $\mathbf{M}(\tau, b)$ with $b > \varphi(\tau)$. This channel is solvable by the first part of the proof. We decompose $\mathbf{M}(\tau, b)$ as follows:

$$\mathbf{M}(\tau, b) = (1-\tau)\begin{pmatrix} 1 & 0 \\ \frac{1-b}{1-\tau} & \frac{b-\tau}{1-\tau} \end{pmatrix} + \tau \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{12}$$

Letting $b \downarrow \varphi(\tau)$, we find that the channels whose coordinates lie above the line segment between $\left(0, \frac{\tau-\varphi(\tau)}{\tau-1}\right)$ and $\left(\tau, \varphi(\tau)\right)$ are solvable, and specifically $\mathbf{M}(\tilde{a}, \varphi(\tau))$ is solvable whenever $\tilde{a} < \tau$. It follows that $\varphi(\tilde{a}) \leq \varphi(\tau)$ whenever $\tilde{a} < \tau$.

Finally, we prove that φ is continuous. Concretely, we prove that φ is right-continuous in every point of its domain excepting 0.

We require the following decomposition of $\mathbf{M}(a, b)$:

$$\mathbf{M}(a, b) = \begin{pmatrix} 1-a & a \\ 1-b & b \end{pmatrix} = b \begin{pmatrix} \frac{b-a}{b} & \frac{a}{b} \\ 0 & 1 \end{pmatrix} + (1-b)\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Again, if $\mathbf{M}(a, b)$ is solvable, so are all the channels whose coordinates lie on or above the line segment joining $\left(a, b\right)$ and $\left(\frac{a}{b}, 1\right)$. Taking $a = \tau$ and $b \downarrow \varphi(\tau)$, we find that all channels with coordinates above the line segment between $\left(\tau, \varphi(\tau)\right)$ and $\left(\tau/\varphi(\tau), 1\right)$ are solvable. For all $0 < \tau \leq 1$, if $0 < \varphi(\tau) < 1$, this segment has slope $\frac{1-\varphi(\tau)}{\tau/\varphi(\tau)-\tau} = \frac{\varphi(t)}{\tau}$.

In combination with the fact that φ is non-decreasing, it follows that if $\tilde{\tau} > \tau$, $\varphi(\tau) \leq \varphi(\tilde{\tau}) \leq \varphi(\tau) + (\tilde{\tau} - \tau)\frac{\varphi(\tau)}{\tau}$. This proves that φ is continuous outside 0. □

10 Miscellania

Sketch of the proof of Theorem 5. It is shown that, when $i, j \in \mathcal{A}$ are such that $\mathbf{M}_{i,l} = \mathbf{M}_{j,l}$ for all $l \in \mathcal{A}$, $l \neq i, j$, and there exist $0 \leq \varepsilon \leq 1$, $0 \leq \alpha$, $0 \leq \beta$ and $0 \leq \gamma$ such that the following holds:

$$\begin{pmatrix} \mathbf{M}_{i,i} & \mathbf{M}_{i,j} \\ \mathbf{M}_{j,i} & \mathbf{M}_{j,j} \end{pmatrix} = \alpha \begin{pmatrix} 1-\varepsilon & \varepsilon \\ 1-\varepsilon & \varepsilon \end{pmatrix} + \begin{pmatrix} \beta & \gamma \\ \beta & \gamma \end{pmatrix} \tag{13}$$

then $\lim_{n\to\infty} |\mathbf{P}_n^i - \mathbf{P}_n^j| = 0$. It is shown that (13) holds for the general binary channel (2) and for the general q-state Potts channel (3).

1. It is shown that without loss of generality we may take $\beta = \gamma = 0$.
2. The vertices which are not in state either i or j are percolated away. On the remaining cluster of ρ, we have a broadcasting with channel $\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}$.
3. A convenient upper bound is found for the mutual information between the state of the root and the state of the leaves when the channel is binary symmetric.

4. It is shown that for almost every realisation of the percolation in step 2, the upper bound for the mutual information between the state of the root and the state of cutsets far away in the tree goes to zero.

□

The following interesting theorem is proven in [MP03]:

Theorem 18. *Let $d > 1$. There is a channel $\mathbf{M}$ such that the configuration at the leaves of the 2-level d-ary tree is independent of the state of the root; nevertheless, there is a D such that the reconstruction problem is solvable for $\mathbf{M}$ and the B-ary tree.*

The proof of this theorem is constructive. A channel is found on the state space consisting of the polynomials of degree at most d over a field with at least $d+2$ elements. The choice of the constant coefficient of the child polynomial depends stochastically on the parent; the other coefficients are chosen uniformly from the field. It turns out that this channel does the job.

The following theorem extends the Bleher-Ruiz-Zagrebnov theorem to general trees, and is stated and proven in [PE99]:

Theorem 19. *The reconstruction problem for the channel $\begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}$ and a tree T is solvable when $1-2\varepsilon > (br(T))^{-1/2}$, and unsolvable when $1-2\varepsilon < (br(T))^{-1/2}$*

This result is proven by considering the tree as an electrical network, with a certain resistance associated to each edge.

In Figure 1, it can be seen that the results about solvability refer to the area of the unit (δ_1, δ_2)-square that lies above the diagonal. Intuitively, it could be expected that these results should be mirrored in the diagonal. However, consider the channel $\begin{pmatrix} 1-a & a \\ 1 & 0 \end{pmatrix}$ whose coordinates lie on the ground line. On the d-ary tree, if $b > 1/d$, the percolation argument applied in Lemma 16 does not apply.

If the root is in state 0, each of its children are in state 0 with probability a; if $a > 1/d$, this survives up to infinity with positive probability. Unfortunately, if the root is in state 1, all its children will be in state 0, so we may probably find that faraway levels are (a.s.) in the same state whether the root is in state 0 or 1. This shows that the percolation arguments, which can be applied above the diagonal, cannot be applied here.

The decomposition techniques from Theorem 6 can probably be applied below the diagonal of the unit square to show that there is the graph of a continuous function separating the area of solvability from the area of non-solvability

Of course, there is a critical value a_c such that, if $a < a_c$, the reconstruction problem is not solvable on T_d and the channel $\begin{pmatrix} 1-a & a \\ 0 & 1 \end{pmatrix}$. In [MA03], it is shown that, when this channel is parametrised by w as follows: $\begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 1 & 0 \end{pmatrix}$, the critical value w_c is at least $\frac{\log d - \log(\log d)}{d}$. Thus, if a_c is the critical value for a, $a_c = \frac{w_c}{1+w_c} \geq \frac{\log d - \log\log d}{d + \log d - \log\log d}$.

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