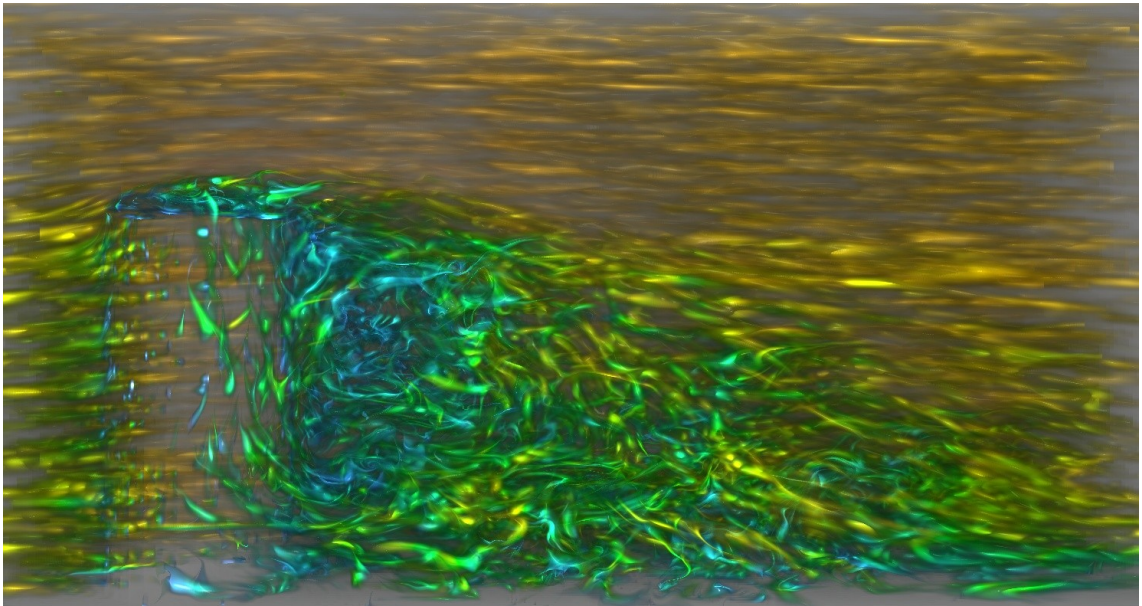




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Filtering in Large Eddy Simulation



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1 Introduction

We are all familiar with the concept of flow. Every day we feel the wind that blows and we see the water that moves. Observing the movement of fluids, one can immediately see that this is a very complex process. Whirls created by obstacles move fast and unpredictable and forces work in every direction causing the fluid to change its form constantly. For decades, mathematicians try to unfold the secrets behind the turbulent motion of fluids. Also nowadays, there is a wide range of applications involving flow of fluids. Think about aërodynamics, building dams and oil platforms. Solving these problems begins with the Navier-Stokes equations that form the base of this field of research. At first glance, just a set of equations containing mass and impulse balances combined with forces from the environment. However, in practice it is impossible to solve these equations exact and even with today's computers, solving by numerical simulation will ask too much calculating capacity. Every day engineers deal with problems too complex to solve with a computer. To still make an accurate simulation of turbulence, mathematicians came up with methods to simplify problems involving turbulent flow. By simulating only the larger structures in the flow and filtering out the smaller structures, computers are able to yield acceptable solutions. This form of simulating flow of fluids is called Large Eddy Simulation (LES). However, there is not one best way to perform LES. There are a lot of different features that can be adapted dependent to the problem that has to be solved. Think about the approximation used in the numerical simulation, the way the smaller structures are filtered out or the way these structures are being modelled. In this thesis, the focus will lie on the different ways to filter the solution. After a technical introduction to turbulent flow and LES in common, filtering will be discussed in more detail and different kinds of filters will be compared. These filters are first analyzed on a uniform grid. Later on, I choose a non-uniform grid to study the consequences for a filter.

2 Introduction to turbulent flow

2.1 Basic equations

In all problems due to flow of fluids, there is one set of equations which is the starting point of all solutions. These are the Navier-Stokes equations (NSE). One of the most common form in which these equations can be written is:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \frac{p}{\rho} + \nu \Delta \mathbf{u} + \mathbf{F} \quad (1)$$

Here, \mathbf{u} is the velocity-vector which depends on space and time. In practical problems, \mathbf{u} is in so in components: $\mathbf{u} = \mathbf{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$. The velocity is the quantity that has to be solved from the NSE. For the initial conditions we just assume

It is possible to make the incompressible NSE dimensionless by defining a characteristic length L and charactersitic velocity U . Now, define dimensionless variables $\tilde{x}, \tilde{y}, \tilde{z}$ in the following way:

$$(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{1}{L}(x, y, z), \tilde{t} = \frac{U}{L}t, \tilde{\mathbf{u}} = \frac{\mathbf{u}}{U}, \tilde{p} = \frac{p}{\rho U^2} \quad (2)$$

Substituting this into (1) gives us the dimensionless NSE (where $\tilde{\mathbf{u}}$ is renamed \mathbf{u}):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} \quad (3)$$

where $Re = \frac{\rho UL}{\mu} = \frac{UL}{\nu}$ is the Reynolds number. The Reynolds number is a dimensionless parameter that measures the ratio between the friction and convection term. To illustrate this, assume we have a stationary flow ($\frac{\partial \mathbf{u}}{\partial t} = 0$) without pressure ($\nabla p = 0$). Now, (3) becomes:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{Re} \Delta \mathbf{u} \quad (4)$$

The left-hand side of this equation is called convective term, which is a non-linear term. The right-hand side is the term caused by friction due to viscosity forces in the fluid. The Reynolds number Re gives the ratio between these two terms. For very small Re , the viscous forces are relatively large with respect to the non-linear term. Because of the high viscous forces, the flow will stay very stable. The flow lines lie perfectly in layers next to each other and are time-independent, this is called laminar flow. However, for very large Re the opposite happens. The non-linear term becomes very large with respect to the viscous forces and this will cause instability in the flow. Small perturbations in the flow can grow to larger perturbations and the flow loses its perfect laminar pattern. The flow lines will show a lot of small disturbances and the flow becomes time-dependent. This phenomenon is called turbulence. A turbulent flow contains 'eddies', in practice eddies are the whirls that can be seen if a fluid is disturbed by an obstacle. The reason for these eddies can be found by observing the non-linear term more in detail.

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \left(\begin{pmatrix} u \\ v \\ w \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \right) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} + \dots + w \frac{\partial u}{\partial z} \quad (5)$$

Now only focus on one of the contributing terms, for example $u \frac{\partial u}{\partial x}$ and assume that there is a sinusoidal disturbance in the flow with frequency k . The contributing term now becomes:

$$u \frac{\partial u}{\partial x} \sim k \sin(kx) \cos(kx) = \frac{1}{2} \sin(2kx) \quad (6)$$

The non-linear term causes an eddy with frequency k to become an eddy with twice its frequency, i.e. a wavelength twice as small. So the energy of the bigger eddies is being transported by the non-linear term to eddies of smaller size. This is called the energy-cascade. At the same time, the viscous term also changes:

$$\Delta \mathbf{u} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad (7)$$

Again, observe one of the contributing term with the same sinusoidal disturbance:

$$\frac{\partial^2 u}{\partial x^2} \sim -k^2 \sin(kx) \quad (8)$$

As k grows because of the non-linear term, the amplitude of the viscous term grows with k^2 . Because the Reynolds number is very high the viscous term is very small, but its

amplitude grows until it can compensate the convective term. So because of the non-linear term, smaller eddies will form until k reaches a value where viscous term can compensate the eddies. Hence, no smaller eddies can be formed and the energycascade stops.

In 1941, Kolmogorov stated that there is a balance between the production of small eddies and destruction due to viscous dissipation. He introduced a length- and time-scale based on the viscosity and energy dissipation $\epsilon \equiv \frac{dk}{dt}$, where k is the turbulent kinetic energy per mass unit:

$$l \sim \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \quad \tau \sim \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}} \quad (9)$$

Kolmogorov stated that these scales are the length- and time-scale of the smallest eddies possible. By observing the powers in the scales, we see that the smallest length-scale is proportional with $\nu^{\frac{3}{4}}$ and the smallest time-scale with $\nu^{\frac{1}{2}}$. In terms of the Reynolds number $Re = \frac{UL}{\nu} = \frac{l}{\tau\nu} = \frac{l^2}{\tau\nu}$, the smallest length-scale is proportional with $Re^{-\frac{3}{4}}$. In other words, the smallest length-scale is $Re^{\frac{3}{4}}$ times smaller than the biggest length-scale. If we want calculate the turbulent motion of all length scales by using a numerical method, the width of the grid has to be very small to perform calculations on the smallest length scale. In practice, we want to simulate problems in three dimensions so the grid has to be three-dimensional. In that case, the amount of grid points is proportional with $(Re^{\frac{3}{4}})^3 = Re^{\frac{9}{4}}$ including the time scale $Re^{\frac{1}{2}}$, the total amount of work for a computer simulation is proportional with $Re^{\frac{11}{4}} \approx Re^3$. Hence, if a Reynolds number becomes 10 times bigger, the amount of work becomes 1000 times bigger. Even with nowadays supercomputers, there is not enough capacity to solve problems involving turbulence. Despite this problem, we still want to solve the secrets behind turbulent flow. Therefore, we need to find methods to simplify our calculations and to minimize the amount of work necessary to get accurate results.

3 Large Eddy Simulation

Suppose somebody wants to perform a DNS for a problem involving turbulent flow. By using a fine grid he hopes to simulate even the smallest eddies to retrieve an accurate result. However, the available capacity of the computer is not enough to make this huge amount of calculations. Therefore, a coarser grid should be used to limit the work. With this coarser grid, only large eddies can be solved. However, in the previous section we saw that there is an interaction between larger and smaller eddies because of the energy cascade. As a consequence, solving larger eddies without taking into account the influence of the small would yield a wrong result. Lacking the capacity to solve smaller eddies, we have to model the smaller, unresolved motions in order to simulate the larger, resolved motions properly. This is the reason for introducing Large Eddy Simulation (LES). In the following section, there will be more mathematical examples of how LES is applied to make the idea more touchable.

3.1 The main idea of LES

First, we introduce a new notation for the NSE from (1). The velocity vector \mathbf{u} will be written per component i where we sum over every direction $j = 1, 2, 3$. To simplify the

notation the summation sign $\sum_{j=1}^3$ will be left out of the equations. Now, the assumption for incompressibility from (??) becomes:

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (10)$$

Here, x_i is the i -th space direction. For the non-linear term this means:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{\partial}{\partial x_j}(u_i u_j) = u_i \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} \quad (11)$$

and for the viscosity term:

$$\nu \Delta \mathbf{u} = \frac{\partial u_i}{\partial x_j \partial x_j} \quad (12)$$

With help of (10) the NSE of (1) can be written as:

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_j)}{\partial x_j} + \frac{\partial \Pi}{\partial x_i} = \nu \frac{\partial(2\nu S_{ij})}{\partial x_j} \quad (13)$$

where $S_{ij} = (\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})/2$ is strain-rate tensor and $\Pi = p/\rho$.

The idea of LES is that we find a \bar{u} of the solution u to connect the larger, resolved structures to the smaller, unresolved structures. Therefore we have to define an operation $u \rightarrow \bar{u}$. This operation is called the filtering operation which is one of the basic features of LES. The idea of this operation is that we 'filter' u in such a way that we get a \bar{u} which is based on only the larger resolved structures and the model where the small unresolved structures are taken into account. The filtered NSE with \bar{u} as a variable can now be solved. Examining \bar{u}_i in more detail, we first have to write $u_i = \bar{u}_i + u'_i$. Here, u'_i can be seen as the residue of u_i . For this residue, we assume $\overline{u'_i} = 0$. This assumption implies that filtering a second time does not change the result: $\overline{\bar{u}_i} = \bar{u}_i$. From these, it follows that $\overline{\bar{u}_i u'_i} = \bar{u}_i \overline{u'_i} = 0$. The filter operator also has a few useful properties:

$$\text{Linearity: } \overline{u_i + u_j} = \bar{u}_i + \bar{u}_j \quad (14)$$

$$\text{Commutates with differentiation in space and time: } \frac{\partial \bar{u}}{\partial x} = \overline{\frac{\partial u}{\partial x}}$$

The difficult part of the NSE is the non-linear term which contains $u_i \bar{u}_j$. With the aid of the previous properties we can this work this term out:

$$\overline{u_i \bar{u}_j} = \overline{(\bar{u}_i + u'_i)(\bar{u}_j + u'_j)} = \overline{\bar{u}_i \bar{u}_j + \bar{u}_i u'_j + \bar{u}_j u'_i + u'_i u'_j} = \bar{u}_i \bar{u}_j + \overline{u'_i u'_j} \quad (15)$$

The previous calculation shows that $u_i \bar{u}_j \neq \bar{u}_i \bar{u}_j$. This seems a bit strange because, at first glance, the term $\overline{u'_i u'_j}$ seems zero. Because u'_i and u'_j are very small, their product should be neglectable small. However, in general this is not true. For example, if a filter filters out high frequencies of a signal, these high frequencies are represented by u'_i and u'_j . The product of high frequent signals, however, can yield a low frequent signal which should not be filtered out. This is an example of a case where $\overline{u'_i u'_j} \neq 0$. Now, if a filter is applied to (13) a new set of equations is created:

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial(\bar{u}_i \bar{u}_j)}{\partial x_j} + \frac{\partial \bar{\Pi}}{\partial x_i} = \frac{\partial(\nu 2 \bar{S}_{ij})}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} \quad (16)$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (17)$$

where $\tau_{ij} = u_i \bar{u}_j - \bar{u}_i \bar{u}_j$

This set of equations only governs the large, resolved modes. The term τ_{ij} represents the impact of the unresolved modes, which has to be modelled. This feature of LES is called Subgrid-scale modelling (SGS modelling). Here, we will not focus on different SGS models but more on the filter operator.

A very common example of a filter is the volume-balance-approach of Schumann (1975). On a three dimensional grid, the average is defined as the integral of the solution over a certain volume V divided by the volume:

$$\bar{u} = \frac{1}{|V|} \int_V u(x) dx \quad (18)$$

This is an example of how the operation $u \rightarrow \bar{u}$ can be defined. By performing the operation, a part of the solution is 'filtered' out. There are many other ways to define a filter for a certain problem. An other filter was defined by Leonard(1974):

$$\bar{u}(x) = \int_{-\infty}^{\infty} G(x-s)u(s)ds \quad (19)$$

where $G(x)$ is called the transfer function with property $\int G(x)dx = 1$. This kind of integral is a convolution product, which maps a continuous function $u(x)$ to an other continuous function.

3.2 Filtering in frequency space

Using convolution filters gives access to more useful properties. To see this, we are going to observe our problem not in the physical space, as we are used to, but in Fourier space. The Fourier space, or also called frequency- or spectral space can be reached by applying the Fourier transform:

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x)e^{-ikx} dx \quad (20)$$

Here, x is the variable in physical space and k the variable in Fourier space and i is the standard imaginary unit. In Fourier space a convolution becomes a product, so equation (19) then reads:

$$\hat{\bar{u}}(k) = \hat{G}(k)\hat{u}(k) \quad (21)$$

It is now easily seen that, if a filter can be written as a convolution, taking the derivative and filtering commute. Because in Fourier space, differentiation is equivalent with multiplication by ik which yields the commutative property: $\frac{\partial \bar{u}}{\partial x} = \bar{\frac{\partial u}{\partial x}}$.

Besides this property, there are other reasons to consider filtering in Fourier space instead of physical space. If we compare (19) and (21), we see that (21) is a far more simpler form to write the filter operator. This will be very advantageous when analysing the filter in more detail. In (21) we can see that $\hat{G}(k)$ is a kind of weighth function for $\hat{u}(k)$. The way the transfer function is defined determines for all k in what proportion $\hat{u}(k)$ will be transferred to the filtered version $\hat{\bar{u}}(k)$. This way of observing filtering matches more

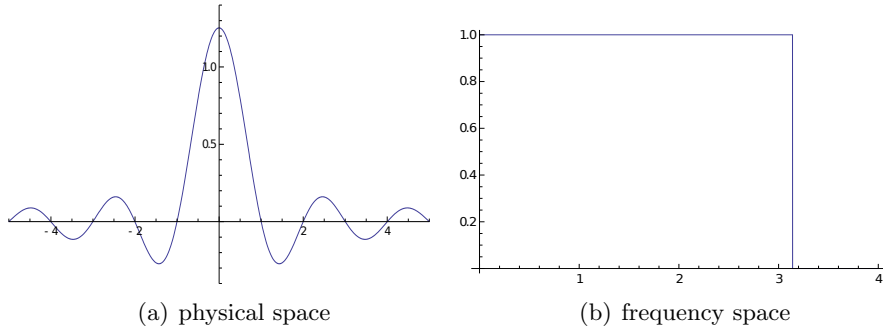


Figure 1: The sharp cut-off filter in both the physical and frequency space

with our intuition. Returning to the concept of LES, we want to filter out eddies of small scales. In Fourier space, small scales correspond with Fourier modes of high frequencies (k is high). So, a proper filter for application in LES should contain a transfer function $G(\hat{k})$ that has low values for higher k in order to eliminate these high frequencies, i.e. the eddies of smaller scales. In figure 1 an example of a filter is given. Here, the filter is represented in the physical and the frequency space. The right-hand side plot gives an idea about how lower frequencies are conserved and higher frequencies are filtered out of the solution. In the physical space (left plot), it is not so easy to see how filtering is performed. For this reason, filters are often observed in the frequency space. In the next section, this way of filtering called band-pass filtering will be discussed in more detail.

4 Discrete filtering

The previous section introduced the concept of filtering and the fact that this operation can be performed in many ways. Filtering can also be done in a discrete way. First, if one wants to observe a discrete filter, the involving area has to be discretized. This is done by creating a mesh which contains grid points ϕ_i and a mesh size h between this grid points. In a discrete filter F , the filtered value at the grid point $\bar{\phi}_i$ is determined by not only the grid point ϕ_i but also the $2N$ surrounding grid points. This is done by assigning weights to each of these grid points.

$$\bar{\phi}_i = F\phi_i = \sum_{l=-N}^N a_l \phi_{i+l} = a_{-N}\phi_{i-N} + \dots + a_0\phi_i + \dots + a_N\phi_{i+N} \quad (22)$$

Here, the weights are represented by the coefficients a_{-N}, \dots, a_N with the extra condition that $\sum_{l=-N}^N a_l = 1$. The way these coefficients are chosen specifies the filter. Observe that the previous equations are in one dimension and that physical problems like turbulent flow are often in three dimensions. Extending the discrete filter to three dimensions can be done by making a linear combination or a composition of three one-dimensional filters:

$$F^3 = \frac{1}{3} \sum_{i=1}^3 F^i \text{ or } F^3 = \frac{1}{3} \prod_{i=1}^3 F^i \quad (23)$$

where F^i is the filter in the i th space direction. For the sake of simplicity, only one-dimensional filters are being observed for now.

4.1 Discrete band-pass filtering

This section is dedicated to a special class of discrete filters, the discrete band-pass filter. This kind of filter is commonly used in filtering signals. A signal can be seen as a superposition of multiple signals with different frequencies. The idea of a band-pass filter is to 'cut-off' the signals with certain frequencies and to conserve the signals with frequencies which lie within a certain band. The same principle can be applied in solutions of the NSE. However, the solution $u(x)$ of the NSE is in the physical state space. To apply the band-pass filter the solution must be in the frequency space, this can be done by performing a Fourier transform. After this, the filter can be specified by making some demands. The purpose of filtering in LES is to conserve the larger resolved structures, that correspond with lower frequencies in frequency space, and to filter out the smaller structures, i.e. the higher frequencies. So the band of the filter corresponds to an interval $[0, k_c]$, where k_c is the cut-off frequency. Frequencies higher than k_c will be filtered out of the solution. In physical space, this frequency corresponds to the mesh size h , which is the smallest size where structures can be resolved.

A band-pass filter also possesses a transfer function $G(k)$ which describes in what way the frequencies are transferred by the filter. In the previous section, (21) stated that in the frequency space, filtering the solution is equivalent with multiplication with the transfer function. For example, the transfer function of an ideal band-pass filter with cut-off frequency k_c is the unit step function on the interval $[0, k_c]$:

$$G_{ideal}(k) = \begin{cases} 1, & \text{if } k \in [0, k_c] \\ 0, & \text{elsewhere} \end{cases} \quad (24)$$

Here, all the frequencies that not belong to the interval $[0, k_c]$ are perfectly filtered out (multiplying by 0), the rest is perfectly conserved (multiplying by 1). However, in practice, it is not possible to create this ideal filter in physical space. But it is important to use a filter with a $G(k)$ that approximates G_{ideal} .

Take a 3-point discrete filter in the form as in (22):

$$\bar{\phi}_n = \alpha\phi_{n-1} + (1 - 2\alpha)\phi_n + \alpha\phi_{n+1} \quad (25)$$

It is possible to calculate the transfer function of this filter by writing ϕ in Fourier space. This can be done by using the transformation $\phi(x) = e^{ikx}$. The mesh consists of grid points x_n with distance h between neighbours, so $x_{n-1} = (n-1)h, x_n = nh$ and $x_{n+1} = (n+1)h$. Now, ϕ_n can be written in Fourier space:

$$\begin{aligned} \phi_n &= e^{iknh} \\ \phi_{n-1} &= e^{ik(n-1)h} = e^{iknh} e^{-ikh} = \phi_n e^{-ikh} \\ \phi_{n+1} &= e^{ik(n+1)h} = e^{iknh} e^{ikh} = \phi_n e^{ikh} \end{aligned} \quad (26)$$

This can be plugged into the filter (25):

$$\begin{aligned}
\bar{\phi}_n &= \alpha\phi_{n-1} + (1 - 2\alpha)\phi_n + \alpha\phi_{n+1} = \alpha\phi_n e^{-ikh} + (1 - 2\alpha)\phi_n + \alpha\phi_n e^{ikh} \\
&= \phi_n(1 - 2\alpha + \alpha e^{-ikh} + \alpha e^{ikh}) = \phi_n(1 - \alpha(2 - 2\cos(kh))) \\
&= \phi_n(1 - 4\alpha\sin^2(\frac{kh}{2}))
\end{aligned} \tag{27}$$

It is now easily seen that the transfer function of this filter is $G(k) = 1 - 4\alpha\sin^2(\frac{kh}{2})$. For specifying α , a first restriction can be made. In filtering, solutions are being adapted but do not change sign, i.e $G(k) \geq 0$ in the entire frequency space. This yields a restriction for α : $0 \leq \alpha \leq \frac{1}{2}$. Plotting $G(k)$ together with $G_{ideal}(k)$ gives information about the quality of the filter. The goal is to minimize the difference between these two functions. This can be done by defining a norm for the function $G(k) - G_{ideal}(k)$ and then minimizing this norm over α . If this method provides an α for which the norm is minimal, then this α can be used to make the filter that is the best approximation of the ideal filter.

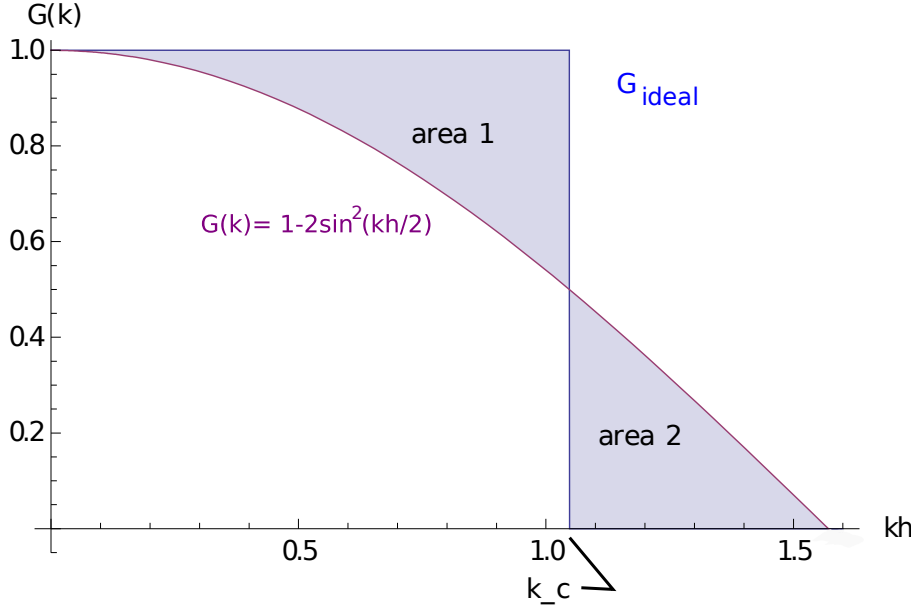


Figure 2: Example of $G(k)$ with $\alpha = 0.5$

A common norm for a function f is the 2-norm:

$$\|f\|_2 = \int |f|^2 dk \tag{28}$$

In this case $f = G(k) - G_{ideal}(k)$ and the integral will consist of two parts denoted as area 1 and 2 in figure 5. Area 1 is the part of low frequencies that should not be filtered. Area 2 are the higher frequencies that should be filtered. The aim is to minimize these areas over α by calculating these areas by integration using the norm of (28).

$$\begin{aligned}
&\int_0^{k_c} (1 - (1 - 4\alpha\sin^2(\frac{k}{2})))^2 dk + \int_{k_c}^{\frac{\pi}{2}} (1 - 4\alpha\sin^2(\frac{k}{2}))^2 dk \\
&= \alpha^2(3\pi - 8) + \alpha(4k_c - 2\pi + 4 - 4\sin(k_c)) + (\frac{\pi}{2} - k_c)
\end{aligned} \tag{29}$$

To improve the readability of the text only the result is given. For a detailed calculation the reader is referred to the appendix.

The last line provides the value which has to be minimized over α for a certain k_c , this can be called the value function $V(\alpha)$.

$$V(\alpha) = \alpha^2(3\pi - 8) + \alpha(4k_c - 2\pi + 4 - 4\sin(k_c)) + \left(\frac{\pi}{2} - k_c\right) \quad (30)$$

This line is plotted in the left of figure 3 for $k_c = \frac{5\pi}{12}$. Minimizing this function can be done by differentiating with respect to α and solving $\frac{\partial V(\alpha)}{\partial \alpha} = 0$.

$$\frac{\partial V(\alpha)}{\partial \alpha} = \alpha(6\pi - 16) + 4k_c - 2\pi + 4 - 4\sin(k_c) \implies \alpha_{opt}(k_c) = \frac{4k_c - 2\pi + 4 - 4\sin(k_c)}{16 - 6\pi} \quad (31)$$

Here α is also subject to the restriction that $0 \leq \alpha \leq \frac{1}{2}$, so now α_{opt} can be specified for all k_c . However, for practical purposes, only for $\frac{\pi}{4} \leq k_c \leq \frac{\pi}{2}$ is the relevant interval for the cut-off frequency. In figure 3 α_{opt} is plotted with respect to the k_c of our interest. At $k_c = K$, $\alpha_{opt}(k_c)$ from (31) intersects with $\alpha_{opt} = \frac{1}{2}$. So, as mentioned in Figure 3, for all $k_c < K$ the optimal value for α is $\alpha_{opt} = \frac{1}{2}$ (because of the restriction noted earlier). For all $k_c > K$, α_{opt} is determined by (31).

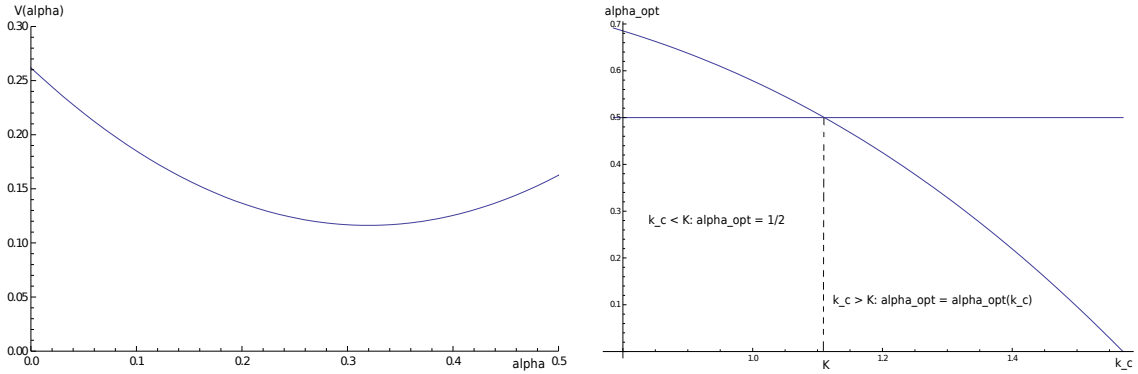


Figure 3: Left: $V(\alpha)$, right: α_{opt} versus k_c

For example, take $k_c = \frac{5\pi}{12}$. From (31) it follows that $\alpha_{opt} \approx 0.320$ and $V(\alpha_{opt}) \approx 0.116$. This value is the distance of $G(k)$ with respect to G_{ideal} using the norm of (28). It is also possible to calculate this for more elaborate filters. The question is, does the transfer function of a more elaborate filter yield a better approximation of the ideal filter? The previous 3-point filter can be compared with the 5-point filter:

$$\bar{\phi}_n = \beta\phi_{n-2} + \alpha\phi_{n-1} + (1 - 2\alpha - 2\beta)\phi_n + \alpha\phi_{n+1} + \beta\phi_{n+2} \quad (32)$$

In the same way as a 3-point filter, the transfer function can be found by substituting $\phi_n = e^{iknh}$.

$$\begin{aligned} \bar{\phi}_n &= \beta\phi_n e^{-2ikh} + \alpha\phi_n e^{-ikh} + (1 - 2\alpha)\phi_n + \alpha\phi_n e^{ikh} + \beta\phi_n e^{2ikh} \\ &= \phi_n(1 - 2(\alpha + \beta) + \alpha(e^{-ikh} + e^{ikh}) + \beta(e^{-2ikh} + e^{2ikh})) \\ &= \phi_n(1 - \alpha(2 - 2\cos(kh)) - \beta(2 - 2\cos(2kh))) \\ &= \phi_n(1 - 4\alpha\sin^2(\frac{kh}{2}) - 4\beta\sin^2(kh)) \end{aligned} \quad (33)$$

Again, the integral of (28) has to be calculated for the $G(k)$ of (33) by splitting this in two parts. This will yield a new value function for the 5-point filter. Again, the details of the calculation can be found in the appendix. For now, only the result is relevant for the text.

$$\begin{aligned}
V(\alpha, \beta) &= \int_0^{k_c} (1 - (1 - 4\alpha \sin^2(\frac{k}{2}) - 4\beta^2 \sin^2(k)))^2 dk + \int_{k_c}^{\frac{\pi}{2}} (1 - 4\alpha \sin^2(\frac{k}{2}) - 4\beta^2 \sin^2(k))^2 dk \\
&= \alpha^2(3\pi - 8) + 3\pi\beta^2 + \alpha\beta(4\pi + \frac{4}{3} - 8) + \alpha(4 - 2\pi) - 2\pi\beta \\
&\quad + k_c(4\alpha + 4\beta) - 4\alpha \sin(k_c) - 2\beta \sin(2k_c) + (\frac{\pi}{2} - k_c)
\end{aligned} \tag{34}$$

Now, the minimization has to be made with respect to the two unspecified α and β . The restriction $G(k) \geq 0$ is still present, this implies a first restriction on α and β : $\alpha + 2\beta \leq \frac{1}{2}$. It is also important that the coefficients a_{-2}, \dots, a_2 are positive, so two more restrictions emerge from this assumption: $1 - 2\alpha \geq 0$ and $1 - 2\beta \geq 0$. All together, the pair (α, β) is restricted to the square $0 \leq \alpha \leq \frac{1}{2}, 0 \leq \beta \leq \frac{1}{4}$.

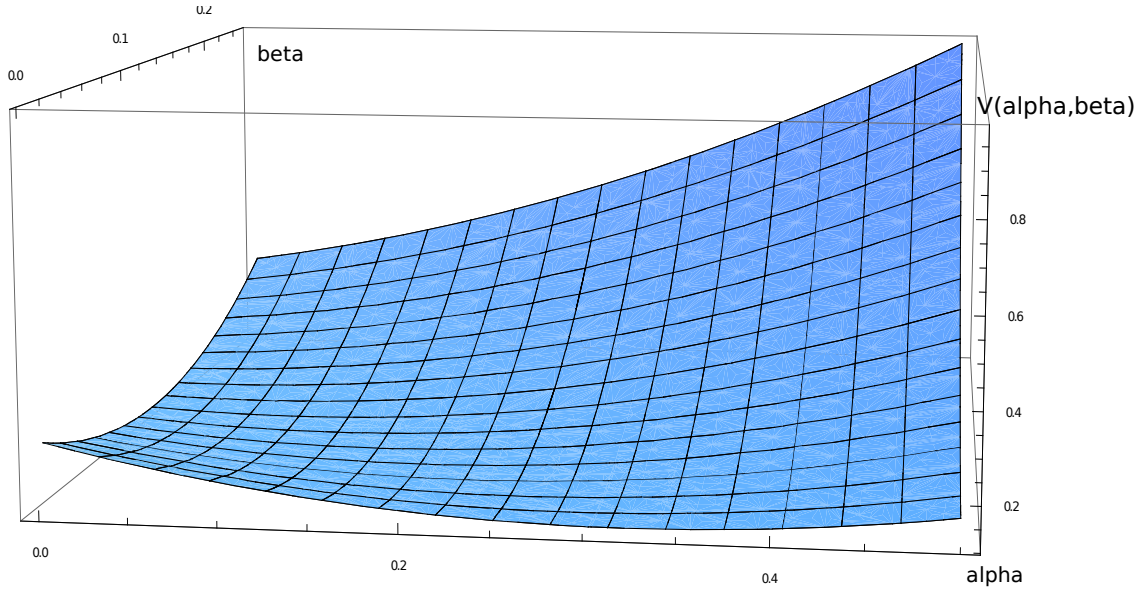


Figure 4: The surface $V(\alpha, \beta)$ on the square $0 \leq \alpha \leq \frac{1}{2}, 0 \leq \beta \leq \frac{1}{4}$

With again $k_c = \frac{5\pi}{12}$ and the help of Mathematica the minimum of (46) can be found on the square. The plot of the surface $V(\alpha, \beta)$ can be found in figure 4.1. For the code used in Mathematica to find the minimum on this surface I refer to the appendix. The result is $\alpha_{opt} \approx 0.269$ and $\beta_{opt} \approx 0.024$ and this gives the minimal value $V(\alpha_{opt}, \beta_{opt}) \approx 0.114$. This value is very close to the $V(\alpha_{opt})$ found for the 3-point filter. Also, β_{opt} is quite close to zero which is equivalent to the 3-point filter. Hence, a 5-point filter gives a better approximation of the ideal filter, however, it will cost more than the 3-point while the improvement of the result is quite small.

An other type of upgrading the 3-point filter is applying a second 3-point filter on the filtered value. This can be done in the following way:

$$\gamma\phi_{n-1}^- + (1 - 2\gamma)\phi_n^- + \gamma\phi_{n+1}^- = \alpha\phi_{n-1} + (1 - 2\alpha)\phi_n + \alpha\phi_{n+1} \tag{35}$$

Here, ϕ_{n-1}^- and ϕ_{n+1}^- are used, which are partially determined by ϕ_{n-2} and ϕ_{n+2} . So from a certain point of view, this filter is the same as a 5-point filter, but written in a more implicit form. Again, the question is: does this filter gives a better approximation of the ideal filter? Therefore, by substituting $\phi_n = e^{iknh}$ in (35), the transfer function must be determined:

$$\bar{\phi}_n(1 - 4\gamma \sin^2(\frac{kh}{2})) = \phi_n(1 - 4\alpha \sin^2(\frac{kh}{2})) \quad (36)$$

In both the left- and right-hand side, functions emerge which are familiar from the 3-point filter. Using $\bar{\phi}_n = \phi_n G(k)$ and dividing both sides by ϕ_n , the new transfer function can be determined by taking the function on the left-hand side to the right:

$$G(k) = \frac{1 - 4\alpha \sin^2(\frac{kh}{2})}{1 - 4\gamma \sin^2(\frac{kh}{2})} \quad (37)$$

Taking a better look at this $G(k)$ provides some restrictions on α and γ . First of all, $G(k) \geq 0$ for all k but this demand gives no restriction for α nor γ . Because of the fraction, the numerator and the denominator could both be negative, which gives a lot of freedom for α and γ . However, the coefficients in the filter should be positive, i.e. $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \gamma \leq \frac{1}{2}$. Also, the denominator should be unequal to zero for $k \in [0, \frac{\pi}{2}]$ because $G(k)$ should be finite. With $0 \leq \gamma \leq \frac{1}{2}$, the only possibility for the denominator to become zero is if $\gamma = \frac{1}{2}$ and $k = \frac{\pi}{2}$. Therefore, γ is further restricted to $0 \leq \gamma < \frac{1}{2}$. With this $G(k)$, the value function becomes a function of α and γ :

$$V(\alpha, \gamma) = \int_0^{k_c} (1 - (\frac{1 - 4\alpha \sin^2(\frac{kh}{2})}{1 - 4\gamma \sin^2(\frac{kh}{2})})^2) dk + \int_{k_c}^{\frac{\pi}{2}} (\frac{1 - 4\alpha \sin^2(\frac{kh}{2})}{1 - 4\gamma \sin^2(\frac{kh}{2})})^2 dk \quad (38)$$

As we saw in the previous filters, solving these integrals by hand takes a lot of time and effort. The goal is not to be able to solve these integrals exact, but to generate the value which indicates if this filter is a better approximation of the ideal filter than the previous cases. Therefore, it is sufficient to approximate this value by using numerical integration. In Matlab, a routine using the Simpson rules for integration, helps to solve to the integrals. In the appendix the details of the implementation can be found. With again $k_c = \frac{5\pi}{12}$, the minimum is found at $\alpha_{opt} = 0.5$ and $\gamma_{opt} = 0.375$. In this optimal point, the value function reads $V(\alpha_{opt}, \gamma_{opt}) = 0.0692$. We must take into account that this value is an approximation of the exact minimum. In the implementation the integral is solved at a finite amount of grid points on the (α, γ) -square, so the exact $(\alpha_{opt}, \gamma_{opt})$ cannot be found nor can we find the minimum at this point. Also, the routine for integration contains an error which will influence the minimum value found. Despite the numerical error, we can conclude that that the minimum is smaller than the value found for a 3- or 5-point filter. Where the 5-point filter is not a big improvement, the implicit version gives a beter result. Although the only difference between these two upgrades of the 3-point filter is the form, the implicit version gives a better approximation of the ideal filter than the explicit.

4.2 Filtering on a non-uniform grid

All the research done in the previous section was based on a uniform grid. However, in numerical simulation of physical problems it is not particularly necessary to use a uniform

grid. Moreover, using a non-uniform grid is advantageous in some ways. The most simple form of a non-uniform grid is the following:

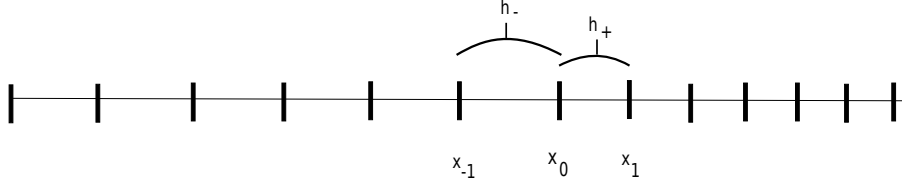


Figure 5: A non-uniform grid with two grid width's, the bigger h_- and the smaller h_+

Here, for all points x_i with $i \leq 0$ the width between x_i and x_{i-1} is h_- and for $i > 0$ this distance will be defined h_+ . This is nothing more than using two uniform grids and connecting them in the point x_0 . The advantage of using such a grid is that the user can specify where calculations should be more accurate. For example, imagine a simulation of the flow of a fluid that behaves laminar before a point x_0 and shows turbulent motion after this point, because of an obstacle at this point. Because the laminar motion has no activities at small scales, it is not necessary to make calculations on this scales, i.e. to use a fine grid before x_0 . This will save computer capacity which can be used for the simulation of the turbulent flow, which needs a finer grid to make the calculations at smaller scales.

In this section, the same filters as in the previous section will be applied on the non-uniform grid defined above. The purpose is to find out how a filter and its transfer function will be affected by changing the grid. It will be important to focus on what is happening at the transition point x_0 . The quotient $\frac{h_-}{h_+}$, which indicates how coarse the transition between the two parts of the grid is, will also influence the behaviour of the filter at x_0 .

The filter we are going to apply is the implicit filter of (35). The transfer function could be calculated by substituting $\phi(x) = e^{ikx}$, but now we cannot assume that $x_n = nh$, reminding that the grid is not uniform at all points. At all points, except for x_0 this would be a sufficient method, so let's take a better look at x_0 . Substiting $\phi_n(x) = e^{ikx_n}$ and $e^{ikx_n} = G(k)e^{ikx_n}$ at this transition point, the filter looks like:

$$G(k)(\gamma e^{ikx_{-1}} + (1 - 2\gamma)e^{ikx_0} + \gamma e^{ikx_1}) = \alpha e^{ikx_-} + (1 - 2\alpha)e^{ikx_0} + \alpha e^{ikx_1} \quad (39)$$

Dividing both sides by e^{ikx_0} gives us the transfer function $G(k)$:

$$G(k)(\gamma e^{ik(x_{-1}-x_0)} + (1 - 2\gamma) + \gamma e^{ik(x_1-x_0)}) = \alpha e^{-ik(x_{-1}-x_0)} + (1 - 2\alpha) + \alpha e^{ik(x_1-x_0)}$$

$$G(k)(\gamma e^{-ikh_-} + (1 - 2\gamma) + \gamma e^{ikh_+}) = \alpha e^{-ikh_-} + (1 - 2\alpha) + \alpha e^{ikh_+}$$

$$\implies G(k) = \frac{\alpha e^{-ikh_-} + (1-2\alpha) + \alpha e^{ikh_+}}{\gamma e^{-ikh_-} + (1-2\gamma) + \gamma e^{ikh_+}} \quad (40)$$

The $G(k)$ found in the latter calculation is of the same form as the transfer function of the implicit filter on a uniform grid. Again, we have a quotient of complex powers of e with parameters α and γ . But, the complex powers of e are not the same anymore

because we now have h_- and h_+ . On a uniform grid, only h determined the powers of e and application of trigonometric identities was possible to express $G(k)$ in real functions. For a non-uniform grid apparently, this is not the case and $G(k)$ stays a complex function.

Like before, the transfer function approximates an ideal transfer function $G_{ideal}(k)$. By minimizing over α and γ , we should be able to find a best approximating $G(k)$. This can be done by using the standard 2-norm for functions defined in (28). Although we now have a complex transfer function, we still can apply this norm. Here, $|f|$ is the modulus of the complex function which transforms our complex functions into a real expression. Hence, the following integral has to be solved:

$$V(\alpha, \gamma) = \int_0^{k_c} \left| 1 - \frac{\alpha e^{-ikh_-} + (1 - 2\alpha) + \alpha e^{ikh_+}}{\gamma e^{-ikh_-} + (1 - 2\gamma) + \gamma e^{ikh_+}} \right|^2 dk + \int_{k_c}^{\frac{\pi}{2}} \left| \frac{\alpha e^{-ikh_-} + (1 - 2) + \alpha e^{ikh_+}}{\gamma e^{-ikh_-} + (1 - 2\gamma) + \gamma e^{ikh_+}} \right|^2 dk \quad (41)$$

The value function $V(\alpha, \gamma)$ has to be minimized over α and γ providing a transfer function that is the best approximation of G_{ideal} . An other important feature for the analysis is the quotient $\frac{h_-}{h_+}$. We will assume that h_- belongs to the coarser part of the grid and h_+ to the finer part, i.e. $\frac{h_-}{h_+} > 1$. So, in the analysis, not only the (α, γ) -minimization, but it is also important to investigate what happens if we use different values for $\frac{h_-}{h_+}$.

First, solving the integrals of (41) by hand will be far too complicated, so again numerical integration will do the dirty work. For a list of values of $\frac{h_-}{h_+}$ in the interval $[1, 2]$ with a step of 0.05, we compute the minimum value of $V(\alpha, \gamma)$. At $\frac{h_-}{h_+} = 1$, we find $V(\alpha_{opt} = 0.5, \gamma_{opt} = 0.375) = 0.0692$ which agrees with the result found for the same filter on a uniform grid. The minimum value over all $\frac{h_-}{h_+}$ however, is found at $\frac{h_-}{h_+} = 1.15$ where $V(\alpha_{opt} = 0.475, \gamma_{opt} = 0.375) = 0.0613$. You should expect that a filter works the most efficient at the simplest grid, but apparently this is not the case. However, for higher $\frac{h_-}{h_+}$ the values of $V(\alpha_{opt}, \gamma_{opt})$ become higher, so for more coarser transitions the effectivity of the filter indeed drops.

An other remarkable result of the analysis lies in the place of the minimum at different $\frac{h_-}{h_+}$. If we take a look at the square $0 \leq \alpha \leq \frac{1}{2}, 0 \leq \gamma < \frac{1}{2}$ in (α, γ) -plane, we see that for $\frac{h_-}{h_+}$ close to 1, the minima lie close to $(\alpha, \gamma) = (0.5, 0.375)$. When $\frac{h_-}{h_+}$ is raised, the minima shift in a straight line over the square towards $(\alpha, \gamma) = (0.05, 0)$. If we raise $(\alpha, \gamma) = (0.5, 0.375)$ further to 5, the minimum takes a quick shift back to the neighbourhood of $(\alpha, \gamma) = (0.5, 0.375)$ where it begun. Now the process starts all over again and will repeat as $\frac{h_-}{h_+}$ is raised.

The last feature we want to investigate is how the optimal transfer function $G_{opt}(k)$ looks like in a plot with together with G_{ideal} . The goal is to see if we really get a function that approximates G_{ideal} and if the form of this function suits the idea of band-pass filtering. For several values of $\frac{h_-}{h_+}$, $G_{opt}(k)$ is plotted together with G_{ideal} in figure 6. Before plotting, one should remind that there are now two planes to represent the transfer function in. Because of the two different grid width's h_- and h_+ , $G(k)$ can be plotted against kh_- or kh_+ . First, we take h_- as the 'standard' grid width, so in all following figures $G(k)$ will be plotted against kh_- . But, in the figure is specified where k_c is located in the kh_- - and kh_+ -plane.

Observing the plot where $\frac{h_-}{h_+} = 1$, we see that the form of G_{opt} is what we expected.

For lower values of k , G_{opt} is close to 1, and when k reaches the cut-off frequency $k_c h_-$, G_{opt} drops to zero. If $\frac{h_-}{h_+}$ is raised to 1.7, the form partially stays the same. For lower k , G_{opt} still approximates G_{ideal} but at the end its line went up. This should not happen, because the frequencies higher than $k_c h_+$ should be filtered out. The line keeps going up if we raise $\frac{h_-}{h_+}$ to 3 and we even see that near $k = \frac{\pi}{2}$ is monotonically increasing. Raising $\frac{h_-}{h_+}$ further to 10 shows that G_{opt} becomes a wave. In fact, reminding that G contains complex powers of e , this result is not very strange. However, for discrete band-pass filtering this transfer function is not very appropriate. However, if we take $k_c h_+$ as the cutoff-frequency, the transfer function does suit the idea of filtering. It is just a matter of choosing whether to look at h_+ or h_- . Of course, we want a transfer function that suits both grid-width's so we can conclude that only for values of $\frac{h_-}{h_+}$ close to 1, a filter as in (35) will be appropriate.

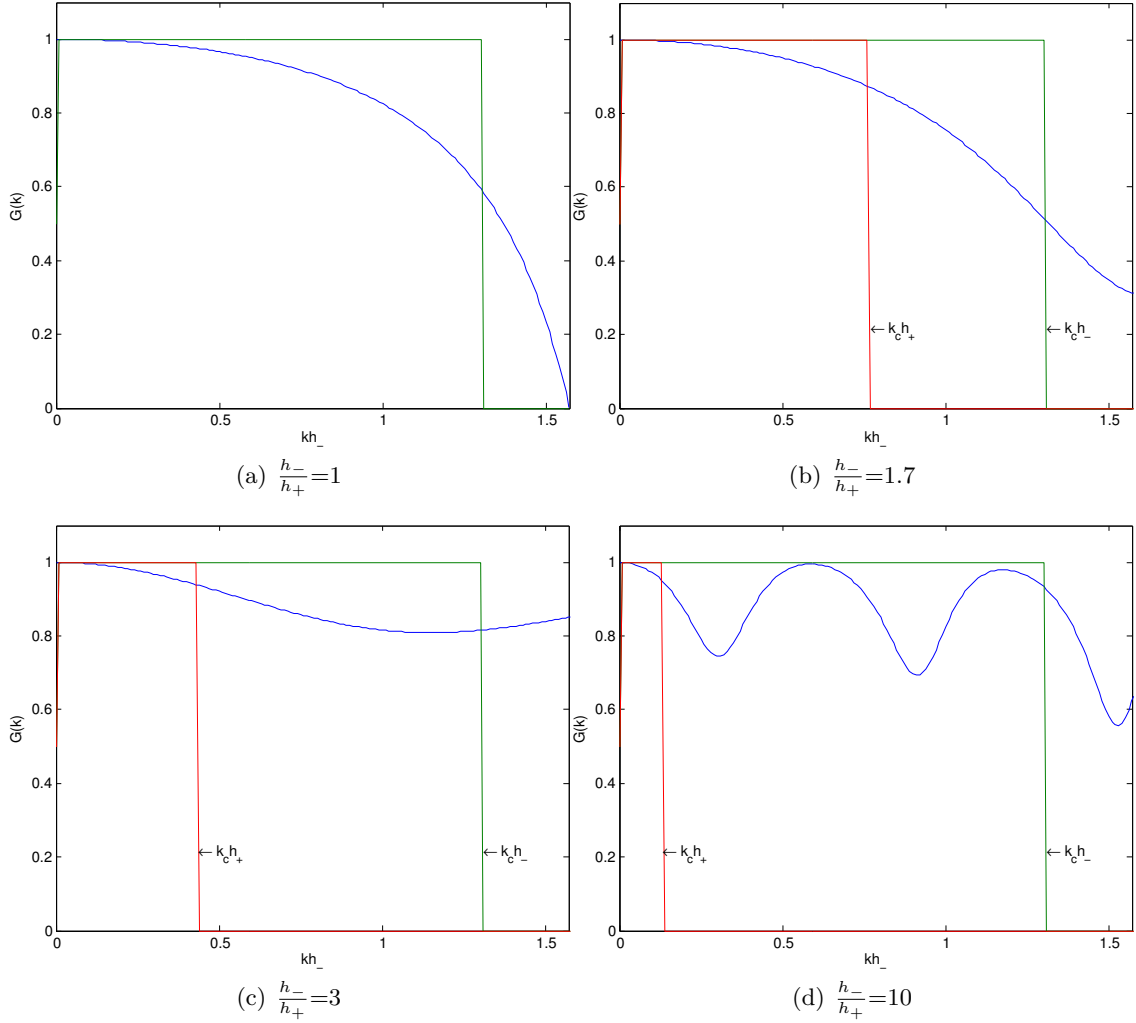


Figure 6: the blue line $G(k)$ and the green line $G_{ideal}(k)$ plotted for a few values of $\frac{h_-}{h_+} \geq 1$

Let us investigate non-uniform grids further by assuming $\frac{h_-}{h_+} < 1$ and plotting the transfer functions for certain values of $\frac{h_-}{h_+}$. The result are the plots showed in figure 7.

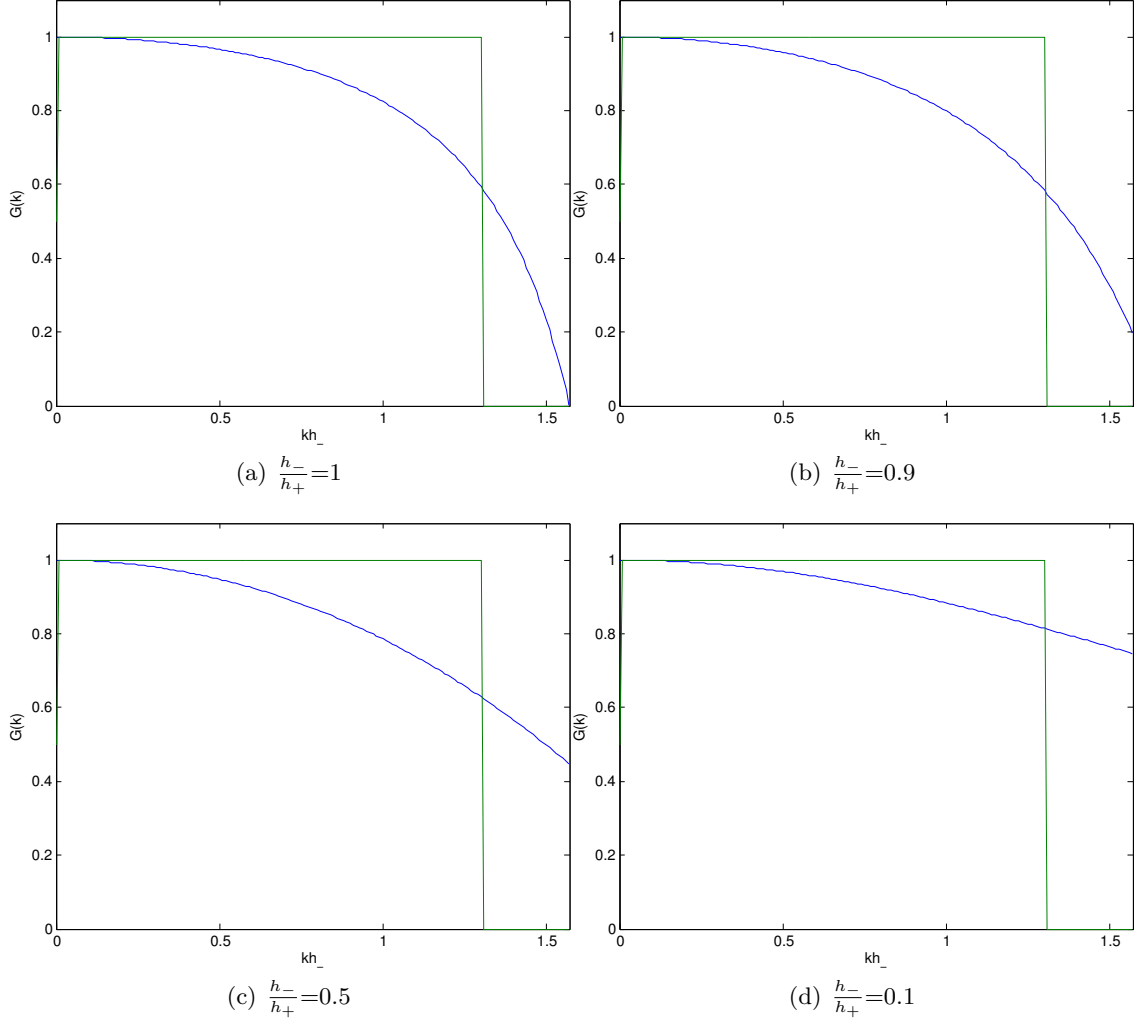


Figure 7: the blue line $G(k)$ and the green line $G_{ideal}(k)$ plotted for a few values of $\frac{h_-}{h_+} \leq 1$

As $\frac{h_-}{h_+}$ drops, the line of $G(k)$ goes up at $k = \frac{\pi}{2}$. As $\frac{h_-}{h_+}$ reaches 0.1, we see that the transfer function does not suit the ideal case. Still, lower frequencies than the cut-off are conserved, but after the cut-off we do not see the line going down towards zero. This is the same we saw in the previous case where $\frac{h_-}{h_+}$ was greater than 1. The difference between $\frac{h_-}{h_+} > 1$ and $\frac{h_-}{h_+} < 1$ is that in the first case the transfer function gets wave-like behaviour unlike the second case. This can be explained by going back to the expression of $G(k)$ in (40). The terms e^{ikh_+} and e^{ikh_-} are the terms that can cause the wave-like behaviour of the transfer function. If for instance h_- becomes bigger, we get waves with smaller wavelengths. This is also observed in figure 6. If e^{ikh_-} becomes smaller, the opposite happens. The wavelength becomes larger and cannot be observed in our interval of interest. Hence, the wave-like behaviour is also present for $\frac{h_-}{h_+} < 1$ but cannot be observed in the interval $k \in [0, \frac{\pi}{2}]$. As $\frac{h_-}{h_+}$ leaves 1, the transfer function gets more and more different than the ideal case. The wave-like behaviour of the transfer function does not suit the idea of discrete cut-off filtering.

To require a transfer function that suits a grid width h_- and h_+ , it is necessary that h_- and h_+ are not very distant from each other. This conclusion was found earlier when $G(k)$ was compared with $G_{ideal}(k)$. Here, we concluded that as $\frac{h_-}{h_+}$ leaves 1, i.e. h_- and h_+ get distant from each other, the transfer function gets more and more different than the ideal case. Intuitive, it is not very strange that a filter becomes less effective as we make our grid more complicated.

5 Conclusion

In this thesis, the aim was to understand the basic features in LES and then to discuss the concept of filtering in more detail. Different filters are compared by calculating their distance to an ideal filter. Finally, one filter is applied on a non-uniform grid to observe the consequences for the effectivity of the used filter in comparison to the uniform case. First, to understand the basics of LES, an introduction to the study of turbulent flow is necessary. The Navier-Stokes equations, turbulence, energy cascade and eddies are concepts that cannot be absent in such an introduction. Making the step to LES, more new concepts emerge like the distinction between larger and smaller eddies, physical and frequency space and of course filtering. Then focussing on filters, concepts like transfer function, cut-off frequency and discrete filters form the fundamentals of the research. The more practical research begins with discrete filters. Three filters are compared to the ideal filter: the 3-point filter, 5-point and the implicit 3-point filter. Finding the transfer functions belonging to these filters was the first step. With knowledge of calculus, Matlab and Mathematica it is possible to analyse and compare the transfer functions with the ideal filter. The result was that a 5-point filter is closer to an ideal filter than the 3-point version. This is not strange because the 5-point is an upgrade of the 3-point. However, the difference was quite small and one could ask if the improvement of the filter can compensate the costs. The implicit 3-point filter, which has the same degrees of freedom as the 5-point filter, does give a remarkable better result than the 5-point filter. Clearly, the implicit 3-point filter can be classified as the most effective of the three filters investigated. Therefore, this filter is used in the remaining part of the reasearch. After having assumed a uniform grid, now the filter is applied on a non-uniform grid. This non-uniform grid was first specified in more detail and next the new transfer function could be calculated. Doing the same research as earlier, the result was that applying a filter to a non-uniform grid will decrease the effectivity of the filter. Moreover, if the difference between the two grid width's becomes larger, the effectivity drops. Intuitive, we could expect that making the grid more complex will be disadvantageous for the filter to do its work.

6 Appendix

A Calculations of equations in the text

A.1 Equation 29

$$\begin{aligned}
& \int_0^{k_c} (1 - (1 - 4\alpha \sin^2(\frac{k}{2})))^2 dk + \int_{k_c}^{\frac{\pi}{2}} (1 - 4\alpha \sin^2(\frac{k}{2}))^2 dk \\
&= \int_0^{k_c} 16\alpha^2 \sin^4(\frac{k}{2}) dk + \int_{k_c}^{\frac{\pi}{2}} 1 - 8\alpha \sin^2(\frac{k}{2}) + 16\alpha^2 \sin^4(\frac{k}{2}) dk \\
&= 4\alpha^2 \int_0^{k_c} \frac{3}{2} - 2\cos(k) + \frac{1}{2}\cos(2k) dk + \int_{k_c}^{\frac{\pi}{2}} 1 - 4\alpha + 4\alpha^2 \cos(k) + 4\alpha^2(\frac{3}{2} - 2\cos(k) + \frac{1}{2}\cos(2k)) dk \\
&= 4\alpha^2 [\frac{3}{2}k - 2\sin(k) + \frac{1}{4}\sin(2k)]_0^{k_c} + [(1 - 4\alpha + 6\alpha^2)k + (4\alpha - 8\alpha^2)\sin(k) + \alpha^2 \sin(2k)]_{k_c}^{\frac{\pi}{2}} \\
&= 4\alpha^2(\frac{3}{2}k_c - 2\sin(k_c)) + \frac{1}{4}\sin(2k_c) + (1 - 4\alpha + 6\alpha^2)(\frac{\pi}{2} - k_c) + (4\alpha - 8\alpha^2)\sin(k_c) + \alpha^2 \sin(k_c) \\
&= \alpha^2(3\pi - 8) + \alpha(4k_c - 2\pi + 4 - 4\sin(k_c)) + (\frac{\pi}{2} - k_c)
\end{aligned} \tag{42}$$

A.2 Equation 34

Calculate:

$$\int_0^{k_c} (1 - (1 - 4\alpha \sin^2(\frac{k}{2}) - 4\beta^2 \sin^2(k)))^2 dk + \int_{k_c}^{\frac{\pi}{2}} (1 - 4\alpha \sin^2(\frac{k}{2}) - 4\beta^2 \sin^2(k))^2 dk \tag{43}$$

First area:

$$\begin{aligned}
& \int_0^{k_c} (1 - (1 - 4\alpha \sin^2(\frac{k}{2}) - 4\beta^2 \sin^2(k)))^2 dk \\
&= \int_0^{k_c} 16\alpha^2 \sin^4(\frac{k}{2}) + 16\beta^2 \sin^4(k) + 32\alpha\beta \sin^2(\frac{k}{2}) \sin^2(k) dk \\
&= [4\alpha^2(\frac{3}{2}k - 2\sin(k) + \frac{1}{4}\sin(2k)) + 4\beta^2(\frac{3}{2}k - \sin(2k) + \frac{1}{8}\sin(4k)) \\
&\quad + 4\alpha\beta(\sin(k) - \frac{1}{3}\sin(3k) + 2k - 2\sin(k) - \sin(2k))]_0^{k_c} \\
&= \alpha^2(6k_c - 8\sin(k_c) + \sin(2k_c)) + \beta^2(6k_c - 4\sin(2k_c) + \frac{1}{2}\sin(4k_c)) \\
&\quad + \alpha\beta(8k_c - 4\sin(k_c) - 4\sin(2k_c) - \frac{4}{3}\sin(3k_c))
\end{aligned} \tag{44}$$

Second area:

$$\begin{aligned}
& \int_{k_c}^{\frac{\pi}{2}} (1 - 4\alpha \sin^2(\frac{k}{2}) - 4\beta^2 \sin^2(k))^2 dk \\
&= \int_{k_c}^{\frac{\pi}{2}} 1 + 16\alpha^2 \sin^4(\frac{k}{2}) + 16\beta^2 \sin^4(k) - 8\alpha \sin^2(\frac{k}{2}) - 8\beta \sin^2(k) + 32\alpha\beta \sin^2(\frac{k}{2}) \sin^2(k) dk \\
&= [k + 4\alpha^2(\frac{3}{2}k - 2\sin(k) + \frac{1}{4}\sin(2k)) + 4\beta^2(\frac{3}{2}k - \sin(2k) + \frac{1}{8}\sin(4k)) \\
&\quad - 4\alpha(k + \sin(k)) - 4\beta(k + \frac{1}{2}\sin(2k)) + \alpha\beta(8k + 4\sin(k) - \frac{4}{3}\sin(3k) - 8\sin(k) - 4\sin(2k))]_{k_c}^{\frac{\pi}{2}} \\
&= (\frac{\pi}{2} - k_c) + 4\alpha^2(\frac{3\pi}{4} - \frac{3}{2}k_c - 2 + 2\sin(k_c) - \frac{1}{4}\sin(2k_c)) + 4\beta^2(\frac{3\pi}{4} - \frac{3}{2}k_c + \sin(2k_c) - \frac{1}{8}\sin(4k_c)) \\
&\quad - 4\alpha(\frac{\pi}{2} - k_c) + 4\alpha - 4\alpha\sin(k_c) - 4\beta(\frac{\pi}{2} - k_c) - 2\beta\sin(2k_c) \\
&\quad + \alpha\beta(8(\frac{\pi}{2} - k_c) + 4(1 - \sin(k_c)) + \frac{4}{3}(1 + \sin(3k_c)) - 8(1 - \sin(k_c)) + 4\sin(2k_c)) \\
&= \alpha^2(3\pi - 6k_c - 8 + 8\sin(k_c) - \sin(2k_c)) + \beta^2(3\pi - 6k_c + 4\sin(2k_c) - \frac{1}{2}\sin(4k_c)) \\
&\quad + \alpha\beta(4\pi - 8k_c + 4 - 4\sin(k_c) + \frac{4}{3} + \frac{4}{3}\sin(3k_c) - 8 + 8\sin(k_c) + 4\sin(2k_c)) \\
&\quad + \alpha(4k_c - 2\pi + 4 - 4\sin(k_c)) + \beta(4k_c - 2\pi - 2\sin(2k_c)) + (\frac{\pi}{2} - k_c)
\end{aligned} \tag{45}$$

In total, the value function becomes (44) + (45):

$$\begin{aligned}
V(\alpha, \beta) &= \alpha^2(6k_c - 6k_c - 8\sin(k_c) + \sin(2k_c)) + 3\pi - 8 + 8\sin(k_c) - \sin(2k_c) \\
&\quad + \beta^2(6k_c - 6k_c - 4\sin(2k_c) + \frac{1}{2}\sin(4k_c)) + 3\pi + 4\sin(2k_c) - \frac{1}{2}\sin(4k_c) \\
&\quad + \alpha\beta(8k_c - 4\sin(k_c) - 4\sin(2k_c) - \frac{4}{3}\sin(3k_c) + 4\pi - 8k_c + 4 - 4\sin(k_c) \\
&\quad + \frac{4}{3} + \frac{4}{3}\sin(3k_c) - 8 + 8\sin(k_c) + 4\sin(2k_c)) \\
&\quad + \alpha(4k_c - 2\pi + 4 - 4\sin(k_c)) + \beta(4k_c - 2\pi - 2\sin(2k_c)) + (\frac{\pi}{2} - k_c) \\
&= \alpha^2(3\pi - 8) + 3\pi\beta^2 + \alpha\beta(4\pi + \frac{4}{3} - 8) + \alpha(4 - 2\pi) - 2\pi\beta \\
&\quad + k_c(4\alpha + 4\beta) - 4\alpha\sin(k_c) - 2\beta\sin(2k_c) + (\frac{\pi}{2} - k_c)
\end{aligned} \tag{46}$$

B Mathematica code

The minimum of the value function in equation 34 is found using the following code in Mathematica:

```
In:= FindMinimum[
  alpha^2*(3*Pi - 8) + 3*Pi*beta^2 + alpha*beta*(4*Pi + 4/3 - 8)
  + alpha (4 - 2*Pi) - 2*Pi*beta + (5*Pi/12)*(4*alpha + 4*beta)
  - 4*alpha*Sin[5*Pi/12] - 2*beta*Sin[2*5*Pi/12] + (Pi/2 - 5*Pi/12),
  {{alpha, 0}, {beta, 0}}
```

```
Out:= {0.114248, {alpha -> 0.269339, beta -> 0.0243071}}
\
```

C Matlab programmes

```
function [] = plottransfer(m,n,refinement)
% Provides plots of the transfer function, "m" and "n" is the size
% of a matrix called "int", this matrix' entries are the numerical solution
% of the integral, the "refinement" is the quotient  $h_{-}/h_{+}$ 

cutoff = 5*pi/12;
index = simpson2(m,n,refinement , cutoff);
gammaopt = (1/(2*n))*(index(1,1)-1);
alphaopt = (1/(2*m))*(index(2,1)-1);

%specify cut-off frequency and retrieve the index of the smallest
%(in absolute sense) element of the matrix "int", from the index of this
%element the optimal alpha "alphaopt" and gamma "gammaopt" can be
%calculated

x = [0:0.01:20];
y1 = real((alphaopt*exp(-1i*x)+(1-2*alphaopt)+alphaopt*exp(refinement*1i*x))
  ./(gammaopt*exp(-1i*x)+(1-2*gammaopt)+gammaopt*exp(refinement*1i*x)));
y2 = (heaviside(x)-heaviside(x-cutoff));
y3 = (heaviside(x)-heaviside(x-cutoff/refinement));

% The transfer function "y1" is specified for optimal alpha and gamma
% "y2" is the ideal transfer function for  $h_{-}$ 
% "y3" is the ideal transfer function for  $h_{+}$ 

figure;
plot(x,y1,x,y2,x,y3)
axis([0 pi/2 0 1.1])
xlabel('k $h_{-}$ ')
ylabel('G(k)')

text(cutoff,0.2,'\leftarrow k_{c}h_{-}', 'HorizontalAlignment', 'left');
text(cutoff/refinement,0.2,'\leftarrow k_{c}h_{+}', 'HorizontalAlignment',
```

```

'left ');

% Specify axes and names for the plot

end

function [index] = simpson2(m,n,refinement , cutoff)

% Calculates the index of the smallest element of matrix "int"

for s = 1:(m+1)

for t = 1:n

alpha = (s-1)/(2*m);
gamma = (t-1)/(2*n);

% For each element of the matrix "int" an alpha and gamma are specified
% such that alpha and gamma are between 0 and 0.5, the matrix "int" can be
% seen as the square on which the point (alpha ,gamma) can lie.

[int1 , int2] = checkfunctions(alpha ,gamma,refinement , cutoff);

% checkfunctions returns the two values of the two parts of the integral

int(s,t) = int1 + int2;
% the matrix "int" is formed by filling the entries with solutions of the
% integral

end
end

[~,plaats] = min(min(int));
index(1)=plaats;
[~,plaats] = min(min(int '));
index(2)=plaats;
index = index ';

% The place of smallest entry of the matrix "int" is determined and is the
% output that is returned to the first function

end

function [int1 , int2] = checkfunctions(alpha ,gamma,refinement , cutoff)

% Calculates the integral by using the quad routine , a standard integrator

```

*% in Matlab using Simpson quadrature formula's, "f" and "g" are function
% handles defining the two functions in the integral*

```
f = @(k) (abs(1-(alpha*exp(-1i*k)+(1-2*alpha)+alpha*exp(refinement*1i*k))  
./ (gamma*exp(-1i*k)+(1-2*gamma)+gamma*exp(refinement*1i*k))))).^2;  
g = @(k) (abs((alpha*exp(-1i*k)+(1-2*alpha)+alpha*exp(refinement*1i*k))  
./ (gamma*exp(-1i*k)+(1-2*gamma)+gamma*exp(refinement*1i*k))))).^2;  
int1 = quad(f,0,cutoff);  
int2 = quad(g,cutoff,pi/2);
```

end

References

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