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# The opening between subspaces of a Hilbert Space

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# THE OPENING BETWEEN SUBSPACES OF A HILBERT SPACE

Abstract. Suppose one has two closed linear subspaces. Is their sum also closed? It turns out that it is not necessarily closed. This will be proved in this thesis. In this bachelorthesis, we develop a general way to determine whether $\mathfrak{M}+\mathfrak{N}$ is closed. It turns out that the notion of the opening between subspaces is important. The content of this thesis builds further on work done by F.Deutsch, [3], and J.-Ph. Labrousse, [14].

## 1. Introduction

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Is $\mathfrak{M}+\mathfrak{N}$ a closed linear subspace? It is clear that this is a linear subspace but it turns out that it is not necessarily closed. We will illustrate it by giving an example in which $\mathfrak{M}+\mathfrak{N}$ is not closed. As a consequence one might wonder under which circumstances $\mathfrak{M}+\mathfrak{N}$ is closed. In this bachelorthesis, we develop a general way to determine whether $\mathfrak{M}+\mathfrak{N}$ is closed. An important notion in the theory is that of the opening between two subspaces. The main result is that if the angle between $\mathfrak{M}$ and $\mathfrak{N}$ is less than 1, then $\mathfrak{M}+\mathfrak{N}$ is closed. After that, we will formulate this result in term of the gap between $\mathfrak{M}$ and $\mathfrak{N}$.

## 2. Some preliminaries

This section contains a number of results that are improtant for the theory developed in this Bachelor thesis. The theory about parallel and orthogonal projections is very important. For a treatment of the concepts and theorems involved, we refer the reader to appendix A and appendix B. Unless stated otherwise, $\mathfrak{M}$ and $\mathfrak{N}$ denote closed subspaces of a Hilbert space $\mathfrak{H}$.

2.1. **Some generalities.** The identities treated in this subsection will be used frequently in later sections.

**Lemma 2.1.** *Let $\mathfrak{M}$,$\mathfrak{N}$ be closed linear subspaces of $\mathfrak{H}$ then*

$$(\mathfrak{M}+\mathfrak{N})^{\perp} = \mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp}, \ \mathrm{clos}\,(\mathfrak{M}+\mathfrak{N}) = (\mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp})^{\perp}.$$

*Proof.* Assume that $x \in (\mathfrak{M}+\mathfrak{N})^{\perp}$ and $y \in \mathfrak{M}+\mathfrak{N}$, then $y$ can be written as $y = m+n$, with $m \in \mathfrak{M}, n \in \mathfrak{N}$. It follows that

$$0 = \langle x, y\rangle = \langle x, m+n\rangle = \langle x, m\rangle + \langle x, n\rangle = 0,$$

hence $x \in \mathfrak{M}^{\perp}$ and $x \in \mathfrak{N}^{\perp}$ so $(\mathfrak{M}+\mathfrak{N})^{\perp} \subset \mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp}$. Conversely, assume that $x \in \mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp}$. Then for all $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$,

$$\langle x, m\rangle + \langle x, n\rangle = \langle x, m+n\rangle = 0,$$

so $x \perp m+n$ which shows that $x \in (\mathfrak{M}+\mathfrak{N})^{\perp}$. So $(\mathfrak{M}+\mathfrak{N})^{\perp} = \mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp}$. The second identity follows easily from the first identity since

$$(\mathfrak{M}+\mathfrak{N})^{\perp\perp} = \mathrm{clos}\,(\mathfrak{M}+\mathfrak{N}) = (\mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp})^{\perp},$$

using the fact that $\mathfrak{K}^{\perp\perp} = \mathrm{clos}\,\mathfrak{K}$ for an arbitrary subset $\mathfrak{K}$ of $\mathfrak{H}$. □

From this lemma it is also clear that

$$(\mathfrak{M}^\perp + \mathfrak{N}^\perp)^\perp = \mathfrak{M} \cap \mathfrak{N}$$

and

$$\operatorname{clos}(\mathfrak{M}^\perp + \mathfrak{N}^\perp) = (\mathfrak{M} \cap \mathfrak{N})^\perp.$$

**Lemma 2.2.** *The following statements are equivalent:*

(1) $\mathfrak{M} + \mathfrak{N}$ *is closed;*
(2) $\mathfrak{M} + \mathfrak{N} = (\mathfrak{M}^\perp \cap \mathfrak{N}^\perp)^\perp$.

*Proof.* (1) $\Rightarrow$ (2) if $\mathfrak{M} + \mathfrak{N}$ is closed, then by lemma 2.1

$$\mathfrak{M} + \mathfrak{N} = \operatorname{clos}(\mathfrak{M} + \mathfrak{N}) = (\mathfrak{M}^\perp \cap \mathfrak{N}^\perp)^\perp.$$

(2) $\Rightarrow$ (1) since $(\mathfrak{M}^\perp \cap \mathfrak{N}^\perp)^\perp$ is an orthogonal complement of a set it is closed, so $\mathfrak{M} + \mathfrak{N}$ is closed. □

2.2. **Overlapping spaces.** Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. The *overlap* of $\mathfrak{M}$ and $\mathfrak{N}$ is defined by $\mathfrak{M} \cap \mathfrak{N}$. Note that the overlap is a closed linear subspace of $\mathfrak{H}$. We can write $\mathfrak{H}$ as $\mathfrak{H} = (\mathfrak{M} \cap \mathfrak{N})^\perp \oplus \mathfrak{M} \cap \mathfrak{N}$. A question which might arrise is how we can express $\mathfrak{M}$ and $\mathfrak{N}$ as a direct sum of a subset of $(\mathfrak{M} \cap \mathfrak{N})^\perp$ and a subset of $(\mathfrak{M} \cap \mathfrak{N})$.

**Lemma 2.3.** *The subspace* $\mathfrak{M}$ *can be written as*

$$\mathfrak{M} = \mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp \oplus (\mathfrak{M} \cap \mathfrak{N}).$$

*Proof.* Assume that $x \in \mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp \oplus (\mathfrak{M} \cap \mathfrak{N})$ then obviously $x \in \mathfrak{M}$. Conversely, assume that $y \in \mathfrak{M}$, because $\mathfrak{H} = (\mathfrak{M} \cap \mathfrak{N})^\perp \oplus (\mathfrak{M} \cap \mathfrak{N})$, $y$ can be written as $y = \alpha + \beta$ with $\alpha \in (\mathfrak{M} \cap \mathfrak{N})^\perp, \beta \in (\mathfrak{M} \cap \mathfrak{N})$. We have that $\beta, y \in \mathfrak{M}$ and since $\mathfrak{M}$ is a linear space we have that $\alpha \in \mathfrak{M}$. But $\alpha \in (\mathfrak{M} \cap \mathfrak{N})^\perp$ so $\alpha \in \mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp$, which shows that $y \in \mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp \oplus (\mathfrak{M} \cap \mathfrak{N})$. □

As a consequence of this lemma the subspace $\mathfrak{N}$ can be written as $\mathfrak{N} = \mathfrak{N} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp \oplus (\mathfrak{M} \cap \mathfrak{N})$. Now define the *'reduced' subspaces* $\mathfrak{M}_0$ and $\mathfrak{N}_0$ by

$$\mathfrak{M}_0 = \mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp, \quad \mathfrak{N}_0 = \mathfrak{N} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp. \tag{2.1}$$

Since $\mathfrak{M}$, $\mathfrak{N}$, $(\mathfrak{M} \cap \mathfrak{N})^\perp$ are closed linear spaces, the subspaces $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are closed linear subspaces. Furthermore, we see that

$$\mathfrak{M}_0 \cap \mathfrak{N}_0 = \{0\}. \tag{2.2}$$

**Lemma 2.4.** *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be closed linear subspaces of a Hilbert space* $\mathfrak{H}$*. Then*

$$\mathfrak{M}_0 + \mathfrak{N}_0 = (\mathfrak{M} + \mathfrak{N}) \cap (\mathfrak{M} \cap \mathfrak{N})^\perp \tag{2.3}$$

*and*

$$\mathfrak{M} + \mathfrak{N} = (\mathfrak{M}_0 + \mathfrak{N}_0) \oplus (\mathfrak{M} \cap \mathfrak{N}). \tag{2.4}$$

*In particular,* $\mathfrak{M} + \mathfrak{N}$ *is closed if and only if* $\mathfrak{M}_0 + \mathfrak{N}_0$ *is closed.*

*Proof.* In order to prove (2.3), let $x \in \mathfrak{M}_0 + \mathfrak{N}_0$, then $x$ can be written as $x = y + z$, where $y \in \mathfrak{M}_0$ and $z \in \mathfrak{N}_0$. Since $\mathfrak{M}_0 \subset \mathfrak{M}$ and $\mathfrak{N}_0 \subset \mathfrak{M}$, we see that $y + z = x \in (\mathfrak{M} + \mathfrak{N}) \cap (\mathfrak{M} \cap \mathfrak{N})^\perp$. In order to show the reversed inclusion let $u \in (\mathfrak{M} + \mathfrak{N}) \cap (\mathfrak{M} \cap \mathfrak{N})^\perp$. Then $u = x + y$ for some $x \in \mathfrak{M}$, $y \in \mathfrak{N}$, and $u \perp \mathfrak{M} \cap \mathfrak{N}$. Therefore $P_{\mathfrak{M} \cap \mathfrak{N}} u = 0$ and

$$u = u - P_{\mathfrak{M} \cap \mathfrak{N}} u = x - P_{\mathfrak{M} \cap \mathfrak{N}} x + y - P_{\mathfrak{M} \cap \mathfrak{N}} y \in \mathfrak{M}_0 + \mathfrak{N}_0,$$

since $I - P_{\mathfrak{M}\cap\mathfrak{N}}$ is the orthogonal projection onto $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$. Hence $\mathfrak{M}_0 + \mathfrak{N}_0 = (\mathfrak{M}+\mathfrak{N}) \cap (\mathfrak{M}\cap\mathfrak{N})^{\perp}$.

In order to prove (2.4), observe that if $x \in \mathfrak{M}_0 + \mathfrak{N}_0 \oplus (\mathfrak{M}\cap\mathfrak{N})$, then $x$ can be written as a sum of elements in $\mathfrak{M}$ and $\mathfrak{N}$ so $(\mathfrak{M}_0 + \mathfrak{N}_0) \oplus (\mathfrak{M}\cap\mathfrak{N}) \subset \mathfrak{M}+\mathfrak{N}$. In order to show the reversed inclusion let $u \in \mathfrak{M}+\mathfrak{N}$. Then $u = x+y$ for some $x \in \mathfrak{M}$, $y \in \mathfrak{N}$, and decompose both $x$ and $y$ with respect to $\mathfrak{H} = (\mathfrak{M}\cap\mathfrak{N})^{\perp} \oplus (\mathfrak{M}\cap\mathfrak{N})$:

$$x = x_0 + x_1, \quad y = y_0 + y_1, \quad x_0 \in \mathfrak{M}_0, \quad y \in \mathfrak{N}_0, \quad x_1, y_1 \in \mathfrak{M}\cap\mathfrak{N}.$$

Therefore $u = x_0 + y_0 + x_1 + y_1$ with $x_0 + y_0 \in \mathfrak{M}_0 + \mathfrak{N}_0$ and $x_1 + y_1 \in \mathfrak{M}\cap\mathfrak{N}$. Hence $\mathfrak{M}+\mathfrak{N} = (\mathfrak{M}_0 + \mathfrak{N}_0) \oplus (\mathfrak{M}\cap\mathfrak{N})$ has been shown.

The last statement follows from $\mathfrak{M}+\mathfrak{N} = (\mathfrak{M}_0 + \mathfrak{N}_0) \oplus (\mathfrak{M}\cap\mathfrak{N})$ since the summand $\mathfrak{M}\cap\mathfrak{N}$ in the orthogonal sum in the righthand side is closed. □

Notice that $\mathfrak{M}_0$, $\mathfrak{N}_0$, $\mathfrak{M}_0 + \mathfrak{N}_0$ are subspaces of $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$. Suppose that we consider this set as a Hilbert space on its own. This is allowed since $\mathfrak{M}\cap\mathfrak{N}$ is a closed linear subspace of $\mathfrak{H}$ (even a Hilbert space on its own, since any closed subset in a complete metric space is complete) so its orthogonal complement is a closed linear subspace and so a Hilbert space on its own. Then it turns out that we can write $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ as a sum of $\mathfrak{M}^{\perp}$ and $\mathfrak{M}_0$. This is the content of the next lemma.

**Lemma 2.5.** *Let $\mathfrak{M}$ and $\mathfrak{M}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}_0$ be as defined above. Then $\mathfrak{M}_0 \oplus \mathfrak{M}^{\perp} = (\mathfrak{M}\cap\mathfrak{N})^{\perp}$.*

*Proof.* Let $h \in \mathfrak{M}_0 \oplus M^{\perp}$, then we can write $h = x + y$, $x \in \mathfrak{M}_0$ and $y \in \mathfrak{M}^{\perp}$. It is clear that $x \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. Because $(\mathfrak{M}\cap\mathfrak{N})^{\perp} = \text{clos}\,(\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp})$, we see that $y \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$ and so $x + y \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. Conversely, assume that $h \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. Decompose $h$ as $h = h_0 + h_1$ with $h_0 \in \mathfrak{M}$ and $h_1 \in \mathfrak{M}^{\perp}$. Since $(\mathfrak{M}\cap\mathfrak{N})^{\perp} = \text{clos}\,(\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp})$, it follows that $h_1 \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. Because $\mathfrak{M}\cap\mathfrak{N}$ is a linear subspace of $\mathfrak{H}$ and $h, h_1 \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$ we must also have that $h_1 - h = h_0 \in (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. So $h_0 \in \mathfrak{M} \cap (\mathfrak{M}\cap\mathfrak{N})^{\perp} = \mathfrak{M}_0$, which shows that $h \in \mathfrak{M}_0 \oplus M^{\perp}$. So $\mathfrak{M}_0 \oplus M^{\perp} = (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. □

As a direct consequence of this lemma, $\mathfrak{N}_0 \oplus \mathfrak{N}^{\perp} = (\mathfrak{M}\cap\mathfrak{N})^{\perp}$. These decompositions for $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ show that

$$\mathfrak{M}^{\perp} = \mathfrak{M}_0^{\perp} + \{0\}, \mathfrak{N}^{\perp} = \mathfrak{N}_0^{\perp} + \{0\},$$

if we take the orthogonal complement in the Hilbert space $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$.

What is the point in defining the subspaces $\mathfrak{M}_0$ and $\mathfrak{N}_0$ and deriving the properties in the lemma‘s above? The notion of ’reduced’ subspaces turns out to be a very useful one. As an example of this, we give a prove for the result $\mathfrak{M}+\mathfrak{N}$ is closed if and only if $\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp}$ is closed in the next subsection. But before we can do that, we need the result in theorem 2.6.

**Theorem 2.6.** *The following statements are equivalent:*

(1) *$\mathfrak{M}+\mathfrak{N}$ is closed and $\mathfrak{M}\cap\mathfrak{N} = \{0\}$;*
(2) *there exists $c > 0$ such that*

$$c\,\|v\| \leq \|v + w\|\,. \tag{2.5}$$

*Proof.* (1) ⇒ (2) Because $\mathfrak{M}+\mathfrak{N}$ is a (by assumption) closed linear subspace of a Hilbert space, we can consider $\mathfrak{K} = \mathfrak{M}+\mathfrak{N}$ as a Hilbert space. The condition $\mathfrak{M}\cap\mathfrak{N} = \{0\}$ implies that the sum $\mathfrak{M}+\mathfrak{N}$ is direct. So, for each $u \in \mathfrak{K}$ there exist unique $v \in \mathfrak{M}$, $w \in \mathfrak{N}$ such that $u = v + w$. Now define the mapping $J : \mathfrak{H} \to \mathfrak{M}$ by $Ju = v$. This mapping is well-defined and its graph is closed. In order to see the closedness of the graph of $J$, assume that $u_n \to u$ and $Ju_n = v_n \to v$, then $u_n = v_n + w_n \to u$, which implies that $w_n = u_n - v_n \to w \in \mathfrak{N}$, which implies that $u = v + w$ and $Ju = v$. By the closed graph theorem, J is bounded, which means that

$$\|Ju\| \leq c' \|u\| \Leftrightarrow c \|Ju\| \leq \|u\|,$$

with $c = 1/c'$. This shows that

$$c \|v\| \leq \|v + w\|.$$

(2) $\Rightarrow$ (1), in order to see that $\mathfrak{M} + \mathfrak{N}$ is closed, let $v_n$ be a sequence in $\mathfrak{M}$ and $w_n$ a sequence in $\mathfrak{N}$ such that $v_n + w_n \to \phi$. Then $v_n + w_n$ is a Cauchy sequence and by (2.5) $v_n, w_n$ are Cauchy sequences. Since $\mathfrak{M}, \mathfrak{N}$ are closed, there is a $v \in \mathfrak{M}$ and a $w \in \mathfrak{N}$ such that $v_n \to v$ and $w_n \to w$. This implies that $v_n + w_n \to v + w = \phi \in \mathfrak{M} + \mathfrak{N}$. Therefore $\mathfrak{M} + \mathfrak{N}$ is closed. It remains to be shown that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. It is clear that $0 \in \mathfrak{M} \cap \mathfrak{N}$, so $\{0\} \subset \mathfrak{M} \cap \mathfrak{N}$. Conversely, assume that $v \in \mathfrak{M} \cap \mathfrak{N}$ then $-v \in \mathfrak{M} \cap \mathfrak{N}$ so $c\|v\| \leq 0$, but this can only be the case if $v = 0$, so $\mathfrak{M} \cap \mathfrak{N} \subset \{0\}$, which shows that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. ☐

**Remark 2.7.** The constant $c$ is given by

$$c = \|J\|,$$

the norm of the projection onto $\mathfrak{M}$ parallel to $\mathfrak{N}$.

### 2.3. Important theorem.

**Theorem 2.8.** *The following statements are equivalent:*

(1) $\mathfrak{M} + \mathfrak{N}$ *is closed;*
(2) $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ *is closed.*

*Proof.* By symmetry it is sufficient to proof (1) $\Rightarrow$ (2). The proof can be reduced to the case that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. Suppose that $\mathfrak{M} + \mathfrak{N}$ is closed and that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, then (by theorem 2.6) there exists a $c > 0$ such that

$$c \|v\| = \|v + w\|, \; v \in \mathfrak{M}, \; w \in \mathfrak{N}.$$

Note that if $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed, then $\{0\}^\perp = (\mathfrak{M} \cap \mathfrak{N})^\perp = \text{clos}\,(\mathfrak{M}^\perp + \mathfrak{N}^\perp) = \mathfrak{M}^\perp + \mathfrak{N}^\perp = \mathfrak{H}$. So we have to show that $\mathfrak{M}^\perp + \mathfrak{N}^\perp = \mathfrak{H}$. It is obvious that $\mathfrak{M}^\perp + \mathfrak{N}^\perp \subset \mathfrak{H}$, so we only need to proof the reversed inclusion. For each element $f \in \mathfrak{H}$, we can define the linear mappings $G, H : \mathfrak{M} + \mathfrak{N} \to \mathbb{C}$ by

$$G(u) = \langle v, f \rangle, \; H(u) = \langle w, f \rangle, \; u = v + w, \; v \in \mathfrak{M}, \; w \in \mathfrak{N}. \tag{2.6}$$

Extend the linear mappings $G, H$ trivially to all of $\mathfrak{H}$. Note that $|\langle v, f \rangle| \leq \|f\| \|\|v\|$ and since $\|f\|$ is constant, $G$ is bounded. Now replace $v$ by $w$ and we see that $H$ is bounded. Because $G$ and $H$ are bounded on all of $\mathfrak{H}$, we can use the Riesz representation theorem. By this theorem (see appendix C for a formulation of this theorem) there are unique elements $g, h \in \mathfrak{H}$ such that

$$G(u) = \langle u, g \rangle, \; H(u) = \langle u, f \rangle, \; u \in \mathfrak{H}. \tag{2.7}$$

If $u \in \mathfrak{N}$, then $G(u) = 0$ and if $u \in \mathfrak{M}$ then $H(u) = 0$, which shows that $g \in \mathfrak{N}^\perp$ and $h \in \mathfrak{M}^\perp$. Since $u = v + w$, with $v \in \mathfrak{M}, u \in \mathfrak{N}$, we have by equations (2.6) and (2.7) that

$$\langle u, f \rangle = \langle v, f \rangle + \langle w, f \rangle = \langle u, g \rangle + \langle u, h \rangle.$$

This can be rewritten as

$$\langle u, f\rangle = \langle u, f-g-h\rangle = 0.$$

This implies that $k = f-g-h \in (\mathfrak{M}+\mathfrak{N})^\perp = (\mathfrak{M}^\perp \cap \mathfrak{N}^\perp) \subset M^\perp$. So $f = k+g+h$, with $k \in \mathfrak{M}^\perp, g \in \mathfrak{N}^\perp, h \in \mathfrak{M}^\perp$, which shows that $f \in \mathfrak{M}^\perp + \mathfrak{N}^\perp$, so $\mathfrak{H} \subset \mathfrak{M}^\perp + \mathfrak{N}^\perp$. We now have shown that $\mathfrak{M}^\perp + \mathfrak{N}^\perp = \mathfrak{H}$. Suppose now that $\mathfrak{M} \cap \mathfrak{N} \neq \{0\}$, then $\mathfrak{K} := (\mathfrak{M} \cap \mathfrak{N})^\perp$ is a closed linear subspace of $\mathfrak{H}$ so this can be viewed as a Hilbert space on its own. Because $\mathfrak{M}^\perp + \mathfrak{N}^\perp = \mathfrak{M}_0^\perp + \mathfrak{N}_0^\perp$ and$(\mathfrak{M}_0 \cap \mathfrak{N}_0)^\perp = \mathfrak{K}$ if $\mathfrak{M}_0^\perp + \mathfrak{N}_0^\perp$ is closed, we can repeat the above argument with $\mathfrak{H}$ replaced by $\mathfrak{K}$ and $\mathfrak{M}, \mathfrak{N}$ replaced by respectively $\mathfrak{M}_0, \mathfrak{N}_0$. □

## 3. Motivation

It is a natural question to ask if $\mathfrak{M} + \mathfrak{N}$ is a closed linear subspace of $\mathfrak{H}$. It is obvious that $\mathfrak{M} + \mathfrak{N}$ is a linear subspace, but it turns out that $\mathfrak{M} + \mathfrak{N}$ is not necessarily closed. One goal of this section is to show this by constructing a counterexample. This is done in section 3.1. The counterexample is described by Stone, [15]. Another goal is to give examples of sufficient conditions for $\mathfrak{M} + \mathfrak{N}$ is closed. These examples can be found in section 3.2.

3.1. **Counterexample.** Let $\mathfrak{H}$ be a Hilbert space. Let $n \in \mathbb{N}$. Let $(\phi_n)_{n=1}^\infty$ be an orthonormal basis for $\mathfrak{H}$. Now define the closed subspace $\mathfrak{M}$ of $\mathfrak{H}$ by $\mathfrak{M} = \operatorname{clos\,span}(\phi_{2n})$. Let $a_n$ be a sequence of nonzero complex numbers with $\sum_{n=1}^\infty |a_n|^2 = 1$ and let $\theta_n$ be a sequence of real numbers in $(0, \frac{\pi}{2})$ such that

$$\sum_{n=1}^\infty \frac{|a_{2n-1}|^2}{\cos^2(\theta_n)} = \infty.$$

Now define the closed linear subspace $\mathfrak{N}$ of $\mathfrak{H}$ by

$$\mathfrak{N} = \operatorname{clos\,span}(\chi_n),$$

with

$$\chi_n = \cos(\theta_n)\phi_{2n-1} + \sin(\theta_n)\phi_{2n}.$$

A simple calculation shows that $\langle \chi_m, \chi_n\rangle = \delta_{mn}$. So $(\chi_n)_{n=1}^\infty$ is an orthonormal basis for $\mathfrak{N}$.

**Lemma 3.1.** *The subspaces $\mathfrak{M}$ and $\mathfrak{N}$,as defined above, have the following properties:*

(1) $\mathfrak{M} \cap \mathfrak{N} = \{0\}$;
(2) $\operatorname{clos\,span}(\mathfrak{M} + \mathfrak{N}) = \mathfrak{H}$;

*Proof.* To prove the first property, assume that $x \in \mathfrak{M} \cap \mathfrak{N}$. Then there are $c_n, d_n \in \ell^2$ such that

$$\sum_{n=1}^\infty c_n\phi_{2n} = \sum_{n=1}^\infty d_n\chi_n.$$

This can be rewritten as

$$\sum_{n=1}^\infty c_n\phi_{2n} - d_n(\cos(\theta_n)\phi_{2n-1} + \sin(\theta_n)\phi_{2n}) = 0,$$

which is equivalent to

$$\sum_{n=1}^{\infty}(c_n - d_n \sin(\theta_n))\phi_{2n} - d_n \cos(\theta_n)\phi_{2n-1} = 0.$$

Because the $\phi_{2n-1}$ are linearly independent we must have that $d_n = 0$ for all $n$. Furthermore, as a consequence of this we must have that $c_n = 0$. So $\mathfrak{M} \cap \mathfrak{N} \subset \{0\}$. It is clear that $\{0\} \subset \mathfrak{M} \cap \mathfrak{N}$, which proves (1).
For (2), observe that

$$\cos(\theta_n)\phi_{2n-1} + \sin(\theta_n)\phi_{2n} \in \mathfrak{N} \subset \mathfrak{M} + \mathfrak{N},\ \phi_{2n} \in \mathfrak{M} \subset \mathfrak{M} + \mathfrak{N},$$

so $\phi_{2n-1} \in \mathfrak{M}+\mathfrak{N}$, which shows that $\phi_n \in \mathfrak{M} + \mathfrak{N}$ for all $n$ and so $\operatorname{span}(\phi_n) \subset \mathfrak{M}+\mathfrak{N}$ so $\mathfrak{H} \subset \operatorname{clos} \mathfrak{M} + \mathfrak{N}$. It is easy to see that $\operatorname{clos} \mathfrak{M} + \mathfrak{N} \subset \mathfrak{H}$ because $\mathfrak{M} + \mathfrak{N}$ consists of sums of elemenst in $\mathfrak{H}$, so $\operatorname{clos} \operatorname{span}(\mathfrak{M} + \mathfrak{N}) = \mathfrak{H}$. □

**Lemma 3.2.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be the closed subspaces as defined above. The linear subspace $\mathfrak{M} + \mathfrak{N}$ is not closed.*

*Proof.* Assume to the contrary that $\mathfrak{M} + \mathfrak{N}$ is closed. Because of property (2) of lemma (3.1), this means that $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$. Define $f = \sum_{n=1}^{\infty} a_n \phi_n$, with $a_n$ as defined above. Then $f$ can be written as $f = g + h$, with $g \in \mathfrak{M}, h \in \mathfrak{N}$. Now observe that

$$a_{2n-1} = \langle f, \phi_{2n-1}\rangle \ = \langle g, \phi_{2n-1}\rangle + \langle h, \phi_{2n-1}\rangle = \langle h, \phi_{2n-1}\rangle\,,$$

since $g \in \mathfrak{M}$. Because $h \in \mathfrak{N}$, $h = \sum_{n=1}^{\infty} c_k \chi_k$ with $c_k = \langle h, \chi_k\rangle$. Now,

$$\begin{aligned}\langle h, \phi_{2n-1}\rangle &= \left\langle \sum_{k=1}^{\infty} c_k \chi_k, \phi_{2n-1} \right\rangle \\ &= \sum_{k=1}^{\infty} c_k \left\langle \cos(\theta_k)\phi_{2k-1} + \sin(\theta_k)\phi_{2k}, \phi_{2n-1}\right\rangle \\ &= c_n \cos(\theta_n).\end{aligned}$$

This calculation shows that

$$\langle h, \chi_n\rangle = c_n = \frac{a_{2n-1}}{\cos(\theta_n)} \Rightarrow \sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} \frac{|a_{2n-1}|^2}{\cos^2(\theta_n)} = \infty.$$

But this is in contradiction with the fact that $\|h\|^2 = \sum_{n=1}^{\infty} |c_n|^2$. So $\mathfrak{M} + \mathfrak{N}$ is not closed. □

3.2. **Examples of conditions for $\mathfrak{M} + \mathfrak{N}$ closed.** As a consequence of the counterexample in last subsection, one might wonder under which circumstances the sum $\mathfrak{M} + \mathfrak{N}$ is closed. In section 2.2 we saw that $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp}$ is closed. Using this theorem we have to determine wether $\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp}$ is closed instead of determining whether $\mathfrak{M} + \mathfrak{N}$ is closed. In order to do this, we have to produce a proof for the particular case we are investigating.

However, are there other (easier to use) criteria which whe can use to make the decision? We have seen theorem 2.6 which says that we have to find a $c > 0$ such that

$$c\,\|v\| = \|v + w\|\,,\ v \in \mathfrak{M},\ w \in \mathfrak{N}.$$

A drawback of this criterion is that it also implies that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. We can't use this theorem if $\mathfrak{M} \cap \mathfrak{N} \neq \{0\}$. The goal of this section is to show a few more examples of sufficient conditions for $\mathfrak{M} + \mathfrak{N}$ is closed.

**Lemma 3.3.** *If $\mathfrak{M} \perp \mathfrak{N}$, then $\mathfrak{M} + \mathfrak{N}$ is a closed direct sum*

*Proof.* (1), since $\mathfrak{M} \perp \mathfrak{N}$ implies that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ it follows that the sum is direct. To prove that $\mathfrak{M} + \mathfrak{N}$ is closed, let $f_n$ is a sequence in $\mathfrak{M}$ and $g_n$ is a sequence in $\mathfrak{N}$ and assume $f_n + g_n$ converges to $f + g \in \mathfrak{H}$. Then $f_n + g_n$ is a Cauchy sequence and by Pythagoras

$$\|f_n + g_n - (f_m + g_m)\|^2 = \|f_n - f_m\|^2 + \|g_n - g_m\|^2 .$$

Hence, $(f_n)$ is a Cauchy sequence in $\mathfrak{M}$ and $(g_n)$ is a Cauchy sequence in $\mathfrak{N}$. Since $\mathfrak{M}, \mathfrak{N}$ are both closed, there is a $f \in \mathfrak{M}$ to which $(f_n)$ converges and there is a $g \in \mathfrak{N}$ to which $(g_n)$ converges. So $f_n + g_n$ converges to $f + g \in \mathfrak{M} + \mathfrak{N}$ so this set contains all its limit points, which shows that $\mathfrak{M} + \mathfrak{N}$ is closed. □

This is a criterion with an obvious drawback: it is not very general. We can't say anything with this criterion if $\mathfrak{M}$ and $\mathfrak{N}$ are not perpendicular.

**Lemma 3.4.** *If $\dim \mathfrak{N} < \infty$, then $\mathfrak{M} + \mathfrak{N}$ is closed.*

*Proof.* Let $P_{\mathfrak{M}}$ be an orthogonal projection on $\mathfrak{M}$. Then

$$\mathfrak{M} + \mathfrak{N} = \mathfrak{M} \oplus (I - P_{\mathfrak{M}})\mathfrak{N}. \tag{3.1}$$

In order to see the equality in (3.1), first assume that $x \in \mathfrak{M} + \mathfrak{N}$, then $x = m + n$, where $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. Decompose $n$ as $n = h_0 + h_1$, with $h_0 \in \mathfrak{M}$ and $h_1 \in \mathfrak{M}^{\perp}$. Then

$$x = m + n = m + h_0 + h_1.$$

Notice that $m + h_0 \in \mathfrak{M}$ and $h_1 = (I - P_{\mathfrak{M}})n$, which shows that $\mathfrak{M} + \mathfrak{N} \subset \mathfrak{M} \oplus (I - P_{\mathfrak{M}})\mathfrak{N}$. Now, for the reversed inclusion, assume that $x \in \mathfrak{M} \oplus (I - P_{\mathfrak{M}})\mathfrak{N}$, then $x = m + h_2$, with $m \in \mathfrak{M}$ and $h_2 \in (I - P_{\mathfrak{M}})\mathfrak{N}$. This implies that there exists a $n_1 \in \mathfrak{N}$ such that $(I - P_{\mathfrak{M}})n_1 = h_2$ and $n_1 = h_1 + h_2$, where $h_1 \in \mathfrak{M}$. So

$$x = m + h_2 = m + h_2 + h_1 - h_1 = m - h_1 + h_1 + h_2 = m - h_1 + n_1.$$

Notice that $m - h_1 \in \mathfrak{M}$ and $n_1 \in \mathfrak{N}$. So $x \in \mathfrak{M} + \mathfrak{N}$. This shows that $\mathfrak{M} \oplus (I - P_{\mathfrak{M}})\mathfrak{N} \subset \mathfrak{M} + \mathfrak{N}$. So $\mathfrak{M} + \mathfrak{N} = \mathfrak{M} \oplus (I - P_{\mathfrak{M}})\mathfrak{N}$. Because $\mathfrak{N}$ is finite dimensional, there is a basis for $V$ with a finite number of elements. Assume that the set

$$V = \{v_1, v_2, ..., v_n\}$$

is a basis for $\mathfrak{N}$. An arbitrary $n \in \mathfrak{N}$ can be written as

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n,$$

where $c_1, c_2, ..., c_n$ are scalars in the field over which $\mathfrak{H}$ is a vector space. Because $I - P_{\mathfrak{M}}$ is a linear operator, it follows that

$$(I - P_{\mathfrak{M}})n = c_1(I - P_{\mathfrak{M}})v_1 + c_2(I - P_{\mathfrak{M}})v_2 + ... + c_n(I - P_{\mathfrak{M}})v_n.$$

Hence $I - P_{\mathfrak{M}}\mathfrak{N}$ is finite dimensional and closed. Notice that $(I - P_{\mathfrak{M}})\mathfrak{N} \subset \mathfrak{M}^{\perp}$, which implies that $\mathfrak{M} \perp (I - P_{\mathfrak{M}})\mathfrak{N}$ and hence by the previous lemma $\mathfrak{M} \oplus (I - P_{\mathfrak{M}})\mathfrak{N}$ is closed. So $\mathfrak{M} + \mathfrak{N}$ is closed. □

This result is not very general since we need that $\mathfrak{M}$(or $\mathfrak{N}$) is finite dimensional. We also want to be able to make some statements about the sum of two infinite dimensional subspaces of a Hilbert space.

## 4. The opening between subspaces

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}_0$ and $\mathfrak{N}_0$ be as defined in section 2.2. In the previous section we saw a number of sufficient conditions for $\mathfrak{M}+\mathfrak{N}$ is closed. All these conditions are not very general since we had assumed things about $\mathfrak{M}$ and $\mathfrak{N}$. This motivates us to look for a general criterion which can determine whether $\mathfrak{M}+\mathfrak{N}$ is closed or not, without having to assume anything about the subspaces $\mathfrak{M}$ and $\mathfrak{N}$, except that they are closed. In this section the notion of opening between subspaces will be introduced. As we will see in section 5, this enablus us to derive a criterion with the generality that we are looking for.

**Definition 4.1.** *The opening between* $\mathfrak{M}$ *and* $\mathfrak{N}$ *is defined by*

$$c(\mathfrak{M},\mathfrak{N}) := \sup\left\{|\langle x,y\rangle| : x\in\mathfrak{M}_0, \|x\|\le 1, y\in\mathfrak{N}_0, \|y\|\le 1\right\}.$$

**Definition 4.2.** *The minimal opening between* $\mathfrak{M}$ *and* $\mathfrak{N}$ *is defined by*

$$c_0(\mathfrak{M},\mathfrak{N}) := \sup\left\{|\langle x,y\rangle| : x\in\mathfrak{M}, \|x\|\le 1, y\in\mathfrak{N}, \|y\|\le 1\right\}.$$

**Remark 4.3.** Sometimes the notion of the angle $\alpha(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is introduced as

$$\cos(\alpha(\mathfrak{M},\mathfrak{N})) = c(\mathfrak{M},\mathfrak{N}),$$

see Friedrichs, [7] and the notion of minimum angle is introduced as

$$\cos(\alpha_0(\mathfrak{M},\mathfrak{N})) = c_0(\mathfrak{M},\mathfrak{N}),$$

see Dixmier, [4].

What properties do $c(\mathfrak{M},\mathfrak{N})$ and $c_0(\mathfrak{M},\mathfrak{N})$ have? Are they related to each other? If so, in what way are they related? In the next three lemma's we will answer these questions. Further we derive properties which will be used in the next section.

**Lemma 4.4.** *The openings* $c(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M},\mathfrak{N})$ *have the following properties:*

(1) $0\le c(\mathfrak{M},\mathfrak{N})\le c_0(\mathfrak{M},\mathfrak{N})\le 1$;
(2) $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{N},\mathfrak{M})$, $c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{N},\mathfrak{M})$;
(3) $c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}_0,\mathfrak{N}_0)$;
(4) *if* $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, *then* $c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N})$ *and* $\alpha(\mathfrak{M},\mathfrak{N}) = \alpha_0(\mathfrak{M},\mathfrak{N})$;
(5) *if* $\mathfrak{M}\cap\mathfrak{N}\neq\{0\}$, *then* $c_0(\mathfrak{M},\mathfrak{N}) = 1$ *and* $\alpha_0(\mathfrak{M},\mathfrak{N}) = 0$.

*Proof.* (1) It follows from the definition of $c(\mathfrak{M},\mathfrak{N})$ and $c_0(\mathfrak{M},\mathfrak{N})$ that

$$c(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M},\mathfrak{N})\ge 0.$$

Since the supremum in the definition of $c_0$ is taken over a larger set then the supremum in the definition of $c$, it follows that $c(\mathfrak{M},\mathfrak{N})\le c_0(\mathfrak{M},\mathfrak{N})$. In order to see that $c(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M},\mathfrak{N})\le 1$, note that $|\langle x,y\rangle|\le\|x\|\,\|y\|\le 1$ since $\|x\|, \|y\|\le 1$. For (2)

$$\begin{aligned} c_0(\mathfrak{M},\mathfrak{N}) &= \sup\left\{|\langle x,y\rangle| : x\in\mathfrak{M}, \|x\|\le 1, y\in\mathfrak{N}, \|y\|\le 1\right\} \\ &= \sup\left\{|\langle y,x\rangle| : x\in\mathfrak{M}, \|x\|\le 1, y\in\mathfrak{N}, \|y\|\le 1\right\} \\ &= c_0(\mathfrak{N},\mathfrak{M}). \end{aligned}$$

The proof of the symmetry of $c$ is similar.
(3) From the definition of $c_0$ and $c$,

$$c_0(\mathfrak{M}_0,\mathfrak{N}_0) = \sup\left\{|\langle x,y\rangle| : x\in\mathfrak{M}_0, \|x\|\le 1, y\in\mathfrak{N}_0, \|y\|\le 1\right\} = c(\mathfrak{M},\mathfrak{N}).$$

(4) If $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, then $(\mathfrak{M} \cap \mathfrak{N})^\perp = \mathfrak{H}$, so

$$\begin{aligned} c(\mathfrak{M}, \mathfrak{N}) &= \sup \{|\langle x, y\rangle| : x \in \mathfrak{M} \cap \mathfrak{H}, \|x\| \le 1, y \in \mathfrak{N} \cap \mathfrak{H}, \|y\| \le 1\} \\ &= \sup \{|\langle x, y\rangle| : x \in \mathfrak{M}, \|x\| \le 1, y \in \mathfrak{N}, \|y\| \le 1\} . \end{aligned}$$

It is obvious that $\alpha(\mathfrak{M}, \mathfrak{N}) = \alpha_0(\mathfrak{M}, \mathfrak{N})$.
(5) Suppose that $\mathfrak{M} \cap \mathfrak{N} \ne 0$ then there is $h \in \mathfrak{M} \cap \mathfrak{N}$ with $\|h\| = 1$, then this element is in $\mathfrak{M}$ and in $\mathfrak{N}$ so that $c_0(\mathfrak{M}, \mathfrak{N}) = 1$. It is obvious that $\alpha_0(\mathfrak{M}, \mathfrak{N}) = 0$. ☐

**Lemma 4.5.** *The openings $c(\mathfrak{M}, \mathfrak{N}), c_0(\mathfrak{M}, \mathfrak{N})$ have the following properties:*

(1) $c(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}, \mathfrak{N}_0) = c_0(\mathfrak{M}_0, \mathfrak{N})$;
(2) $|\langle x, y\rangle| \le c_0(\mathfrak{M}, \mathfrak{N}) \|x\| \|y\|$ *for all* $x \in \mathfrak{M}$, $y \in \mathfrak{N}$;
(3) $|\langle x, y\rangle| \le c(\mathfrak{M}, \mathfrak{N}) \|x\| \|y\|$ *for all* $x \in \mathfrak{M}$, $y \in \mathfrak{N}$ *and at least one of* $x$ *and* $y$ *is in* $(\mathfrak{M} \cap \mathfrak{N})^\perp$;
(4) $c_0(\mathfrak{M}, \mathfrak{N}) = \|P_\mathfrak{M} P_\mathfrak{N}\| = \|P_\mathfrak{M} P_\mathfrak{N} P_\mathfrak{M}\|^{\frac{1}{2}}$;
(5) $c(\mathfrak{M}, \mathfrak{N}) = \|P_\mathfrak{M} P_\mathfrak{N} - P_{\mathfrak{M}\cap\mathfrak{N}}\| = \left\|P_\mathfrak{M} P_\mathfrak{N} P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\right\| = \left\|P_\mathfrak{M} P_{(\mathfrak{M}\cap\mathfrak{N})^\perp} P_\mathfrak{N} P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\right\|$;
(6) $c_0(\mathfrak{M}, \mathfrak{N}) = 0 \Leftrightarrow \mathfrak{M} \perp \mathfrak{N}$ *(i.e.* $\mathfrak{M} \subset \mathfrak{N}^\perp$*)*;
(7) $c(\mathfrak{M}, \mathfrak{N}) = 0 \Leftrightarrow P_\mathfrak{M}$ *and* $P_\mathfrak{N}$ *commute.*

*Proof.* (1) By symmetry it is sufficient to prove the first equality:

$$\begin{aligned} c(\mathfrak{M}, \mathfrak{N}) &= c_0(\mathfrak{M}_0, \mathfrak{N}_0) \\ &= \sup\{|\langle x, y\rangle| : x \in \mathfrak{M}_0, y \in \mathfrak{N}_0, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{|\langle P_{\mathfrak{M}_0} x, P_{\mathfrak{N}_0} y\rangle| : x \in \mathfrak{H}, y \in \mathfrak{H}, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{\left|\left\langle P_{(\mathfrak{M}\cap\mathfrak{N})^\perp} P_\mathfrak{M} x, P_{(\mathfrak{M}\cap\mathfrak{N})^\perp} P_\mathfrak{N} y\right\rangle\right| : x \in \mathfrak{H}, y \in \mathfrak{H}, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{\left|\left\langle P_\mathfrak{M} x, P_{(\mathfrak{M}\cap\mathfrak{N})^\perp} P_\mathfrak{N} y\right\rangle\right| : x \in \mathfrak{H}, y \in \mathfrak{H}, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{\left|\left\langle P_\mathfrak{M} x, P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp} y\right\rangle\right| : x \in \mathfrak{H}, y \in \mathfrak{H}, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{|\langle x, y\rangle| : x \in \mathfrak{M}, y \in \mathfrak{N}_0, \|x\| \le 1, \|y\| \le 1\} \\ &= c_0(\mathfrak{M}, \mathfrak{N}_0). \end{aligned}$$

(2) Since $|\langle x, y\rangle| \le \|x\| \|y\|$ and $|\langle x, y\rangle| \le c_0(\mathfrak{M}, \mathfrak{N})$ it follows that $|\langle x, y\rangle| \le c_0(\mathfrak{M}, \mathfrak{N}) \|x\| \|y\|$ for all $x \in \mathfrak{M}$, $y \in \mathfrak{N}$.
(3) By parts (1) and (2) of this lemma $|\langle x, y\rangle| \le c_0(\mathfrak{M}, \mathfrak{N}) \|x\| \|y\| = c(\mathfrak{M}, \mathfrak{N}_0) \|x\| \|y\| = c(\mathfrak{M}_0, \mathfrak{N}) \|x\| \|y\|$, which shows that $|\langle x, y\rangle| \le c(\mathfrak{M}, \mathfrak{N}) \|x\| \|y\|$ for all $x \in \mathfrak{M}$, $y \in \mathfrak{N}$ and at least one of $x$ and $y$ is in $(\mathfrak{M} \cap \mathfrak{N})^\perp$.
(4) We have

$$\begin{aligned} c_0(\mathfrak{M}, \mathfrak{N}) &= \sup\{|\langle x, y\rangle| : x \in \mathfrak{M}, y \in \mathfrak{M}, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{|\langle P_\mathfrak{M} x, P_\mathfrak{N} y\rangle| : x, y \in \mathfrak{H}, \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{|\langle x, P_\mathfrak{M} P_\mathfrak{N} y\rangle| : x, y \in \mathfrak{H}, \|x\| \le 1, \|y\| \le 1\} \\ &= \|P_\mathfrak{M} P_\mathfrak{N}\| , \end{aligned}$$

which shows the first equality. Take $P = P_\mathfrak{M} P_\mathfrak{N}$. The identity $\|P^* P\| = \|P\|^2$ is valid in this situation, so we have that

$$c_0(\mathfrak{M}, \mathfrak{N}) = \|P\| = \|P^* P\|^{1/2} = \|P_\mathfrak{M} P_\mathfrak{N} P_\mathfrak{M}\|^{1/2} ,$$

which shows the second equality and hence $c_0(\mathfrak{M},\mathfrak{N}) = \|P_{\mathfrak{M}}P_{\mathfrak{N}}\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}}P_{\mathfrak{M}}\|^{\frac{1}{2}}$.
(5) By lemma 4.3 and part (4) of this lemma, we have that

$$c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}_0,\mathfrak{N}_0) = \|P_{\mathfrak{M}_0}P_{\mathfrak{N}_0}\|.$$

From this we see that

$$\|P_{\mathfrak{M}_0}P_{\mathfrak{N}_0}\| = \left\|P_{\mathfrak{M}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\right\|.$$

From this we deduce that

$$\left\|P_{\mathfrak{M}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\right\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}} - P_{\mathfrak{M}}P_{\mathfrak{N}}P_{\mathfrak{M}\cap\mathfrak{N}}\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}} - P_{\mathfrak{M}\cap\mathfrak{N}}\|.$$

(6) $c_0(\mathfrak{M},\mathfrak{N}) = 0 \Leftrightarrow \|P_{\mathfrak{M}}P_{\mathfrak{N}}\| = 0 \Leftrightarrow P_{\mathfrak{M}}P_{\mathfrak{N}} = 0 \Leftrightarrow \mathfrak{M} \perp \mathfrak{N}$.
(7) $c(\mathfrak{M},\mathfrak{N}) = 0 \Leftrightarrow \|P_{\mathfrak{M}}P_{\mathfrak{N}} - P_{\mathfrak{M}\cap\mathfrak{N}}\| = 0 \Leftrightarrow P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}\cap\mathfrak{N}} \Leftrightarrow P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{N}}P_{\mathfrak{M}}$.
□

**Lemma 4.6.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. If $c_0(\mathfrak{M},\mathfrak{N}) < 1$, then*

$$(1 - c_0(\mathfrak{M},\mathfrak{N}))(\|x\|^2 + \|y\|^2) \le \|x+y\|^2, \quad x \in \mathfrak{M}, \quad y \in \mathfrak{N}. \tag{4.1}$$

*Proof.* The identity

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\,(x,y), \quad x,y \in \mathfrak{H},$$

leads to the following inequalities

$$\begin{aligned} \|x\|^2 + \|y\|^2 &\le \|x+y\|^2 + 2|(x,y)| \\ &\le \|x+y\|^2 + 2c_0\|x\|\|y\| \\ &\le \|x+y\|^2 + c_0(\|x\|^2 + \|y\|^2), \quad x \in \mathfrak{M}, \quad y \in \mathfrak{N}, \end{aligned} \tag{4.2}$$

where $c_0 = c_0(\mathfrak{M},\mathfrak{N})$. Since $c_0 < 1$, (4.2) gives (4.1). □

## 5. Necessary and sufficient conditions in terms of openings

In the previous section we defined the notions of opening en minimal opening and derived a number of properties which they posses. In this section we will explain how these notions can be used to determine whether $\mathfrak{M}+\mathfrak{N}$ is closed if $\mathfrak{M}$ and $\mathfrak{N}$ are closed linear subspaces of a Hilbert space $\mathfrak{H}$.

**Proposition 5.1.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*
- (i) $c_0(\mathfrak{M},\mathfrak{N}) < 1$;
- (ii) $\mathfrak{M}\cap\mathfrak{N} = \{0\}$ *and* $\mathfrak{M}+\mathfrak{N}$ *is closed.*

*Proof.* (i) ⇒ (ii) First it is shown that $\mathfrak{M}\cap\mathfrak{N} = \{0\}$. Assume that $\mathfrak{M}\cap\mathfrak{N} \ne \{0\}$, then by lemma 4.4, $c(\mathfrak{M},\mathfrak{N}) = 1$. But this is in contradiction with the assumption that $c(\mathfrak{M},\mathfrak{N}) < 1$, so $\mathfrak{M}\cap\mathfrak{N} = \{0\}$. In order to see that $\mathfrak{M}+\mathfrak{N}$ is closed, let $u_n$ be a sequence in $\mathfrak{M}+\mathfrak{N}$ converging to $u \in \mathfrak{H}$, so that

$$u_n = x_n + y_n, \quad x_n \in \mathfrak{M}, \quad y_n \in \mathfrak{N}.$$

It follows from (4.1) that

$$(1 - c_0(\mathfrak{M},\mathfrak{N}))(\|x_n\|^2 + \|y_n\|^2) \le \|u_n\|^2.$$

Since $u_n$ is a Cauchy sequence, also $x_n$ and $y_n$ are Cauchy sequences. Hence there exist elements $x \in \mathfrak{M}$ and $y \in \mathfrak{N}$, so that $x_n \to x$ in $\mathfrak{M}$ and $y_n \to y$ in $\mathfrak{N}$. Hence, $u = x + y \in \mathfrak{M} + \mathfrak{N}$. Thus $\mathfrak{M} + \mathfrak{N}$ is closed.

(ii) $\Rightarrow$ (i) Assume that $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ and that $\mathfrak{M} + \mathfrak{N}$ is closed. Hence, there exists $\rho > 0$ such that

$$\rho\|x\| \leq \|x + y\|, \quad x \in \mathfrak{M},\ y \in \mathfrak{N}.$$

Now suppose that $c_0(\mathfrak{M}, \mathfrak{N}) = 1$. Then there exist sequences $x_n \in \mathfrak{M}$ and $y_n \in \mathfrak{N}$, such that

$$(x_n, y_n) \to 1, \quad \|x_n\| = \|y_n\| = 1.$$

Hence, it follows that

$$\rho^2 \leq \|x_n - y_n\|^2 = \|x_n\|^2 - 2\mathrm{Re}\,(x_n, y_n) + \|y_n\|^2 = 2(1 - \mathrm{Re}\,(x_n, y_n)) \to 0,$$

which leads to a contradiction. Thus it follows that $c_0(\mathfrak{M}, \mathfrak{N}) < 1$. □

**Proposition 5.2.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $c(\mathfrak{M}, \mathfrak{N}) < 1$*;*
(ii) $\mathfrak{M} + \mathfrak{N}$ *is closed.*

*Proof.* The condition $c(\mathfrak{M}, \mathfrak{N}) < 1$ is equivalent to $c_0(\mathfrak{M}_0, \mathfrak{N}_0) < 1$, where $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are as defined in section 2.2. This is equivalent to $\mathfrak{M}_0 + \mathfrak{N}_0$ is closed. According to lemma 2.4, $\mathfrak{M}_0 + \mathfrak{N}_0$ is closed if and only if $\mathfrak{M} + \mathfrak{N}$ is closed. □

We have managed to derive a criterion which has the generality that we were looking for. In order to decide whether $\mathfrak{M} + \mathfrak{N}$ is closed, we just have to calculate the opening or the minimal opening between the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ and check that it is less than 1.

**Example 5.3.** In section 3.1 we discussed an example in which $\mathfrak{M}$ and $\mathfrak{N}$ is closed, but $\mathfrak{M} + \mathfrak{N}$ is not. With the theory of this section, we could have shown that $\mathfrak{M} + \mathfrak{N}$ is not closed in another way. Recall that every element in $\mathfrak{M}$ can be written als a linear combination of $\phi_{2n}$ and every element in $\mathfrak{N}$ can be written as a linear combination of $\chi_n$, where $\chi_n = \cos(\theta_n)\phi_{2n-1} + \sin(\theta_n)\phi_{2n}$. Let's calculate $c_0(\mathfrak{M}, \mathfrak{N})$. For notational convenience, define

$$A := \{x \in \mathfrak{M}, y \in \mathfrak{N}, \|x\|, \|y\| \leq 1\}$$

By definition of $c_0(\mathfrak{M}, \mathfrak{N})$,

$$\begin{aligned} c_0(\mathfrak{M}, \mathfrak{N}) &= \sup_{x,y \in A} |\langle x, y\rangle| \\ &= \sup \left| \left\langle \sum_{n=1}^{\infty} c_n \phi_{2n}, \sum_{k=1}^{\infty} d_k \chi_k \right\rangle \right| \\ &= \sup \sum_{n=1}^{\infty} |\langle c_n \phi_{2n}, d_n \sin(\theta_n)\phi_{2n}\rangle| \\ &= \sup \sin(\theta_n) \end{aligned}$$

Assume that $\sup_{n\in\mathbb{N}} \sin(\theta_n) = 1$, then

$$\sup_{n\in\mathbb{N}} \frac{1}{\cos^2(\theta_n)} = \sup_{n\in\mathbb{N}} \frac{1}{1 - \sin^2(\theta_n)} = \infty.$$

So we can construct a sequence $a_n$ $(n \in \mathbb{N})$ in $\ell^2$ such that

$$\sum_{n=1}^{\infty} \frac{|a_{2n-1}|^2}{\cos^2(\theta_n)} = \infty.$$

This defines an element $f \in \mathfrak{H}$ which is not in $\mathfrak{M} + \mathfrak{N}$.

In section 2.2, we proved that $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed. We can determine whether $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed by using the opening between $\mathfrak{M}^\perp$ and $\mathfrak{N}^\perp$. A logical question to ask at this point is: can we relate $c(\mathfrak{M}, \mathfrak{N})$ to $c(\mathfrak{M}^\perp, \mathfrak{N}^\perp)$? It turns out that $c(\mathfrak{M}, \mathfrak{N}) = c(\mathfrak{M}^\perp, \mathfrak{N}^\perp)$, see corollary 5.6. Theorem 5.5 is used in the proof of corollary 5.6.

**Example 5.4.** It is not in general the case that $c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp)$. For instance, if $\mathfrak{M} + \mathfrak{N}$ is closed and $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. Then it does not necessarily follow that $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$. Take for example $\mathfrak{M}^\perp = \mathfrak{N}^\perp = \mathfrak{H}$, then $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, but $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \mathfrak{H}$.

The following theorem concerns a relationship between the conorm and the opening. See appendix D for the definition and some properties of the conorm.

**Theorem 5.5.** *Let $c(\mathfrak{M}, \mathfrak{N})$ be the opening between the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ and let $\gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}})$ be the conorm of the operator $(I - P_{\mathfrak{N}})P_{\mathfrak{M}}$. There is the following relation between $c$ and $\gamma$:*

$$c(\mathfrak{M}, \mathfrak{N})^2 + \gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = 1.$$

*Proof.* By definition of the conorm,

$$\gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = \inf \left\{ \frac{\|(I - P_{\mathfrak{N}})P_{\mathfrak{M}}h\|^2}{\|h\|^2} : h \in \ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^\perp \right\}.$$

First we show that

$$\ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^\perp = \mathfrak{M}_0. \tag{5.1}$$

In order to do that we show that $\ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}}) = (\mathfrak{M} \cap \mathfrak{N}) \oplus \mathfrak{M}^\perp$. Assume that $h \in \ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}})$. We can decompose $h$ as $h = x + y$ with $x \in \mathfrak{M}$, $y \in \mathfrak{M}^\perp$. Then

$$\begin{aligned} 0 &= (I - P_{\mathfrak{N}})P_{\mathfrak{M}}h \\ &= (I - P_{\mathfrak{N}})P_{\mathfrak{M}}(x + y) \\ &= (I - P_{\mathfrak{N}})P_{\mathfrak{M}}x + (I - P_{\mathfrak{N}})P_{\mathfrak{M}}y \\ &= (I - P_{\mathfrak{N}})P_{\mathfrak{M}}x \\ &= (I - P_{\mathfrak{N}})x, \end{aligned}$$

so $x \in \mathfrak{N}$. So $x \in \mathfrak{M} \cap \mathfrak{N}$. This shows that $h \in \mathfrak{M} \cap \mathfrak{N} \oplus \mathfrak{M}^\perp$. Conversely, assume that $h \in (\mathfrak{M} \cap \mathfrak{N}) \oplus \mathfrak{M}^\perp$, then $h$ can be written as $h = x + y$ with $x \in \mathfrak{M} \cap \mathfrak{N}$, $y \in \mathfrak{M}^\perp$. Then

$$(I - P_n)P_{\mathfrak{M}}h = (I - P_n)P_{\mathfrak{M}}x + (I - P_n)P_{\mathfrak{M}}y = 0,$$

which shows that $(\mathfrak{M} \cap \mathfrak{N}) \oplus \mathfrak{M}^\perp \subset \ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}})$. So $\ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}}) = (\mathfrak{M} \cap \mathfrak{N}) \oplus \mathfrak{M}^\perp$. From this follows that $\ker\, ((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^\perp = \mathfrak{M}_0$ Secondly, we show that

$$\frac{\|(I - P_{\mathfrak{M}})P_{\mathfrak{N}}\|^2}{\|h\|^2} = \frac{\|(P_{\mathfrak{M}}h\|^2}{\|h\|^2} - \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2}. \tag{5.2}$$

Notice that

$$\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}\|^2 = \langle (I-P_{\mathfrak{N}})P_{\mathfrak{M}}, (I-P_{\mathfrak{N}})P_{\mathfrak{M}}\rangle = \langle P_{\mathfrak{M}}, P_{\mathfrak{M}}\rangle - \langle P_{\mathfrak{M}}h, P_{\mathfrak{N}}P_{\mathfrak{M}}h\rangle\,,$$

which shows that

$$\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}\|^2 = \|P_{\mathfrak{M}}\|^2 - \|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2\,.$$

From this it is easy to see that $\frac{\|(I-P_{\mathfrak{M}})P_{\mathfrak{N}}\|^2}{\|h\|^2} = \frac{\|(P_{\mathfrak{M}}h\|^2}{\|h\|^2} - \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2}$.

By using previous lemma‘s and (5.1) and (5.2), we obtain

$$\begin{aligned}
\gamma((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2 &= \inf\left\{\frac{\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h\|^2}{\|h\|^2} : h\in\mathfrak{M}_0\right\}\\
&= \inf\left\{\frac{\|P_{\mathfrak{M}}h\|^2}{\|h\|^2} - \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2} : h\in\mathfrak{M}_0\right\}\\
&= 1-\sup\left\{\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2} : h\in\mathfrak{M}_0\right\}\\
&= 1-\sup\left\{\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}_0}k\|^2}{\|k\|^2} : k\in\mathfrak{H}\right\}\\
&= 1-\|P_{\mathfrak{N}}P_{\mathfrak{M}_0}\|^2\\
&= 1-c_0(\mathfrak{N},\mathfrak{M}_0)^2\\
&= 1-c_0(\mathfrak{M}_0,\mathfrak{N})^2\\
&= 1-c(\mathfrak{M},\mathfrak{N})^2,
\end{aligned} \tag{5.3}$$

which shows that $c(\mathfrak{M},\mathfrak{N})^2+\gamma((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2=1$. □

**Corollary 5.6.** $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$.

*Proof.* By theorem 5.5, $c(\mathfrak{N},\mathfrak{M})^2+\gamma((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = 1$ and $c(\mathfrak{M}^\perp,\mathfrak{M}^\perp)^2 + \gamma((P_{\mathfrak{N}})(I-P_{\mathfrak{M}}))^2 = 1$. Since $\gamma(A)=\gamma(A^*)$, we have that $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{N},\mathfrak{M}) = c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$. So $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$. □

**Example 5.7.** Theorem 2.8 can be proved using the theory of the last two sections. Suppose that $\mathfrak{M}+\mathfrak{N}$ is closed, this is equivalent to $c(\mathfrak{M},\mathfrak{N})<1$ if and only if $c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)<1$, which is equivalent to $\mathfrak{M}^\perp+\mathfrak{N}^\perp$ is closed.

## 6. Neccessary and sufficient conditions in terms of the gap

In the previous section we derived neccessary and sufficient conditions for $\mathfrak{M}+\mathfrak{N}$ is closed in terms of openings. In this section we will derive relations between openings and operatornorms.

Define the *gap between* $\mathfrak{M}$, $\mathfrak{N}$ by

$$g(\mathfrak{M},\mathfrak{N}) = \|P_{\mathfrak{M}}-P_{\mathfrak{N}}\|\,,$$

where $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ are the orthogonal projections onto $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

In the following lemma, we state some properties of the gap.

**Lemma 6.1.** *The gap $g(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ has the following properties:*

(1) $g(\mathfrak{M},\mathfrak{N}) = g(\mathfrak{N},\mathfrak{M})$;
(2) $g(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = g(\mathfrak{M},\mathfrak{N})$;

(3) $g(\mathfrak{M},\mathfrak{N}) \leq 1$.

*Proof.* For (1)

$$g(\mathfrak{M},\mathfrak{N}) = \|P_{\mathfrak{M}} - P_{\mathfrak{N}}\| = \|-(P_{\mathfrak{N}} - P_{\mathfrak{M}})\| = \|P_{\mathfrak{N}} - P_{\mathfrak{M}}\| = g(\mathfrak{N},\mathfrak{M}).$$

For (2)

$$g(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp}) = \|(I - P_{\mathfrak{M}}) - (I - P_{\mathfrak{N}})\| = \|P_{\mathfrak{M}} - P_{\mathfrak{N}}\| = g(\mathfrak{M},\mathfrak{N})$$

For (3) Observe that

$$\begin{aligned}\|(P_{\mathfrak{M}} - P_{\mathfrak{N}})h\|^2 &= \|P_{\mathfrak{M}}(I - P_{\mathfrak{N}})h\|^2 + \|(I - P_{\mathfrak{M}})P_{\mathfrak{N}}h\|^2 \\ &\leq \|(I - P_{\mathfrak{N}})h\|^2 + \|P_{\mathfrak{N}}\|^2 \\ &= \|h\|^2,\end{aligned}$$

which implies that

$$\sup_{h\in\mathfrak{H}} \frac{\|(P_{\mathfrak{M}} - P_{\mathfrak{N}})h\|^2}{\|h\|^2} \leq 1,$$

so $g(\mathfrak{M},\mathfrak{N}) \leq 1$ □

The following propositions show relations between

$$g(\mathfrak{M},\mathfrak{N}), \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp}))$$

and between

$$c(\mathfrak{M},\mathfrak{N}), \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp})).$$

These relations enable us to express the theory of the previous section in terms of the gap between two subspaces.

**Proposition 6.2.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear spaces of a Hilbert space $\mathfrak{H}$. Then*

$$\max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp})) = g(\mathfrak{M},\mathfrak{N}^{\perp}). \tag{6.1}$$

*In particular, if $c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp})$, then $c_0(\mathfrak{M},\mathfrak{N}) = g(\mathfrak{M},\mathfrak{N}^{\perp})$.*

*Proof.* First we show that $\max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp})) \leq g(\mathfrak{M},\mathfrak{N}^{\perp})$. Observe that

$$\begin{aligned}c_0(\mathfrak{M},\mathfrak{N}) &= c_0(\mathfrak{N},\mathfrak{M}) \\ &= \|P_{\mathfrak{M}}P_{\mathfrak{N}}\| \\ &= \|P_{\mathfrak{M}}(I - P_{\mathfrak{N}^{\perp}})\| \\ &= \|(P_{\mathfrak{M}} - P_{\mathfrak{N}^{\perp}})P_{\mathfrak{M}}\| \\ &\leq \|P_{\mathfrak{M}} - P_{\mathfrak{N}^{\perp}}\| \\ &= g(\mathfrak{M},\mathfrak{N}^{\perp})\end{aligned}$$

So $c_0(\mathfrak{M},\mathfrak{N}) \leq g(\mathfrak{M},\mathfrak{N}^{\perp})$. As a consequence of this and lemma 6.1, also

$$c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp}) \leq g(\mathfrak{M},\mathfrak{N}^{\perp}).$$

So

$$\max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp})) \leq g(\mathfrak{M},\mathfrak{N}^{\perp}).$$

Secondly we show that $g(\mathfrak{M},\mathfrak{N}) \le \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp))$. Let $u \in \mathfrak{H}$, then

$$\begin{aligned}\|(P_\mathfrak{M} - P_{\mathfrak{N}^\perp})u\|^2 &= \|(I - P_{\mathfrak{N}^\perp})P_\mathfrak{M}u - P_{\mathfrak{N}^\perp}(I - P_\mathfrak{M})u\|^2 \\ &= \|P_\mathfrak{N}P_\mathfrak{M}u\|^2 + \|P_{\mathfrak{N}^\perp}P_{\mathfrak{M}^\perp}u\|^2 \\ &\le \|P_\mathfrak{N}P_\mathfrak{M}\|^2\,\|P_\mathfrak{M}u\|^2 + \|P_{\mathfrak{N}^\perp}P_{\mathfrak{M}^\perp}\|^2\,\|P_{\mathfrak{M}^\perp}u\|^2 \\ &= c_0(\mathfrak{M},\mathfrak{N})^2\,\|P_\mathfrak{M}u\|^2 + c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)^2\,\|P_{\mathfrak{M}^\perp}u\|^2 .\end{aligned}$$

It is sufficient to show that if $c_0(\mathfrak{M},\mathfrak{N}) \ge c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$, then $g(\mathfrak{M},\mathfrak{N}^\perp) \le c_0(\mathfrak{M},\mathfrak{N})$. The proof for the case that $c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp) \ge c_0(\mathfrak{M},\mathfrak{N})$ is completely analogous.

$$\begin{aligned}\|(P_\mathfrak{M} - P_{\mathfrak{N}^\perp})u\|^2 &\le c_0(\mathfrak{M},\mathfrak{N})^2(\|P_\mathfrak{M}u\|^2 + \|P_{\mathfrak{M}^\perp}u\|^2). \\ &= c_0(\mathfrak{M},\mathfrak{N})^2(\|x\|^2 + \|y\|^2), \\ &= c_0(\mathfrak{M},\mathfrak{N})^2 \cdot \|u\|^2\end{aligned}$$

where $u = x + y$, $x \in \mathfrak{M}$ and $y \in \mathfrak{M}^\perp$. This equation shows that

$$\frac{\|(P_\mathfrak{M} - P_{\mathfrak{N}^\perp})u\|^2}{\|u\|^2} \le c_0(\mathfrak{M},\mathfrak{N})^2,$$

so

$$g(\mathfrak{M},\mathfrak{N}^\perp) \le c_0(\mathfrak{M},\mathfrak{N}).$$

From this, we see that $g(\mathfrak{M},\mathfrak{N}^\perp) \le \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp))$. If $c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$, then it is clear from (6.1) that $c_0(\mathfrak{M},\mathfrak{N}) = g(\mathfrak{M},\mathfrak{N}^\perp)$. ☐

**Corollary 6.3.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear spaces in $\mathfrak{H}$. Then*

$$\begin{aligned}c(\mathfrak{M},\mathfrak{N}) &\le \min(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)) \\ &\le \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)) = g(\mathfrak{M},\mathfrak{N}^\perp).\end{aligned} \tag{6.2}$$

*Moreover, if $\mathfrak{M}\cap\mathfrak{N} = \{0\}$ and $M^\perp \cap \mathfrak{N}^\perp = \{0\}$, then*

$$c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = g(\mathfrak{M},\mathfrak{N}^\perp). \tag{6.3}$$

*Proof.* From the previous lemma, we have that

$$\max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)) = g(\mathfrak{M},\mathfrak{N}^\perp).$$

It is obvious that

$$\min(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)) \le \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)).$$

In order to see that $c(\mathfrak{M},\mathfrak{N}) \le \min(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp))$, remember that we showed in section 4 that $c(\mathfrak{M},\mathfrak{N}) \le c_0(\mathfrak{M},\mathfrak{N})$. Since $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$ and $c(\mathfrak{M}^\perp,\mathfrak{N}^\perp) \le c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$, it follows that $c(\mathfrak{M},\mathfrak{N}) \le c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$ so

$$c(\mathfrak{M},\mathfrak{N}) \le \min(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)).$$

This shows the chain of inequalities in (6.2).

The conditions $\mathfrak{M}\cap\mathfrak{N} = \{0\}$ and $\mathfrak{M}^\perp\cap\mathfrak{N}^\perp = \{0\}$ imply that $c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N})$ and $c(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$ (lemma 4.4). By corollary 5.6, $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)$, so

$$c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp).$$

By the previous lemma, $\max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)) = g(\mathfrak{M},\mathfrak{N}^\perp)$, so the chain of equalities in (6.3) has been shown. ☐

We have established that $g(\mathfrak{M}, \mathfrak{N}^\perp) = \max(c_0(\mathfrak{M}, \mathfrak{N}), c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp))$. If $g(\mathfrak{M}, \mathfrak{N}^\perp) < 1$ then $c_0(\mathfrak{M}, \mathfrak{N}) < 1$. Because $c(\mathfrak{M}, \mathfrak{N}) \le c_0(\mathfrak{M}, \mathfrak{N})$, we have that $\mathfrak{M} + \mathfrak{N}$ is closed if and only if $g(\mathfrak{M}, \mathfrak{N}^\perp) < 1$.

**Proposition 6.4.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(1) $g(\mathfrak{M}, \mathfrak{N}^\perp) < 1$;
(2) $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}, \mathfrak{M} \cap \mathfrak{N} = \{0\}$.

*If either of these equivalent conditions hold, then the chain of equalities in* (6.3) *is satisfied.*

*Proof.* (1) $\Rightarrow$ (2), proposition (6.2) implies that $c_0(\mathfrak{M}, \mathfrak{N}) < 1$, $c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp) < 1$ and hence $c(\mathfrak{M}, \mathfrak{N}) = c(\mathfrak{M}^\perp, \mathfrak{N}^\perp) < 1$.The condition $c_0(\mathfrak{M}, \mathfrak{N}) < 1$ implies that $\mathfrak{M} + \mathfrak{N}$ is closed and $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. The condition $c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp) < 1$ shows that $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed and that $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$, so $\operatorname{clos}(\mathfrak{M} + \mathfrak{N}) = \mathfrak{H}$. Hence $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$. (2) $\Rightarrow$ (1), if $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$ and $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, then it follows that $c_0(\mathfrak{M}, \mathfrak{N}) < 1$. Moreover $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed and $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$, so that $c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp) < 1$. Since $c_0(\mathfrak{M}, \mathfrak{N}) < 1$ and $c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp) < 1$, it follows that $g(\mathfrak{M}, \mathfrak{N}^\perp) < 1$. □

The following result comes from Ljance, [11].

**Lemma 6.5.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ such that*

$$\mathfrak{H} = \mathfrak{M} + \mathfrak{N},\ \mathfrak{M} \cap \mathfrak{N} = \{0\}.$$

*Then the gap $g(\mathfrak{M}, \mathfrak{N}^\perp)$ is given by*

$$g(\mathfrak{M}, \mathfrak{N}^\perp) = \sqrt{1 - \frac{1}{\|P\|^2}},$$

*where $P$ is the projection onto $\mathfrak{M}$, parallel to $\mathfrak{N}$.*

*Proof.* The condition $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ implies that $(\mathfrak{M} \cap \mathfrak{N})^\perp = \mathfrak{H}$ so $\mathfrak{M}_0 = \mathfrak{M}$. Furthermore, since $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$, it follows that $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$. According to corollary 6.3 we have the chain of equalities

$$g(\mathfrak{M}, \mathfrak{N}^\perp) = c(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp).$$

Using this and theorem 5.5, we see that

$$\begin{aligned} c(\mathfrak{M}, \mathfrak{N})^2 &= g(\mathfrak{M}, \mathfrak{N}^\perp)^2 \\ &= 1 - \gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^2, \end{aligned}$$

where $\gamma((I-P_{\mathfrak{N}})P_{\mathfrak{M}})$ denotes the conorm of the operator $(I-P_{\mathfrak{N}})P_{\mathfrak{M}}$. So $g(\mathfrak{M}, \mathfrak{N}^\perp) = \sqrt{1 - \gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}})^2}$. It remains to be shown that $\gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}}) = \|P\|$, where $P$ is the projection onto $\mathfrak{M}$ parallel to $\mathfrak{N}$. First note that

$$\begin{aligned} \gamma((I - P_{\mathfrak{N}})P_{\mathfrak{M}}) &= \inf \left\{ \frac{\|(I - P_{\mathfrak{N}})h\|}{\|h\|} : h \in \mathfrak{M} \right\} \\ &= \left( \sup \left\{ \frac{\|h\|}{\|(I - P_{\mathfrak{N}})h\|} : h \in \mathfrak{M} \right\} \right)^{-1}. \end{aligned}$$

Note that the mapping

$$(I - P_{\mathfrak{N}})h = h - P_{\mathfrak{N}}h \to h,\ h \in \mathfrak{M}$$

is defined on a subset of $\mathfrak{M} + \mathfrak{N}$ (this is because $h \in \mathfrak{M}$ and $P_{\mathfrak{N}}h \in \mathfrak{N}$) and maps onto $\mathfrak{M}$. Extend this mapping to all of $\mathfrak{H}$ as follows

$$h - P_{\mathfrak{N}}h + k \to h,\ h \in \mathfrak{M},\ k \in \mathfrak{N}.$$

This extenstion is exactly the projction onto $\mathfrak{M}$ parallel to $\mathfrak{N}$. By Pythagoras it is the case that

$$\|h - P_{\mathfrak{N}}h + k\|^2 = \|h - P_{\mathfrak{N}}h\|^2 + \|k\|^2,\ h \in \mathfrak{M},\ k \in \mathfrak{N}.$$

So

$$\|P\| = \sup\left\{ \frac{\|h\|}{\sqrt{\|h - P_{\mathfrak{N}}h\|^2 + \|k\|^2}} : h \in \mathfrak{M}, k \in \mathfrak{N} \right\}.$$

This shows that $g(\mathfrak{M}, \mathfrak{N}^{\perp}) = \sqrt{1 - \frac{1}{\|P\|^2}}$. □

## 7. Method of Alternating Projections

In the previous six sections, we discussed the closedness of the sum of two closed linear subspaces of a Hilbert space. We derived a necessary and sufficient condition in terms of the opening between two subspaces. As we will see in this section, the notion of the opening between two subspaces can be used to derive an approximation for the rate of convergence and approximation of error bounds of the method of alternating projections. The goal of this section is to give a short illustration of how openings can be used.

Let $\mathfrak{M}_1, \mathfrak{M}_2, ..., \mathfrak{M}_n$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}$ be the intersection of $\mathfrak{M}_1, \mathfrak{M}_2, ..., \mathfrak{M}_n$, that is,

$$\mathfrak{M} = \cap_{i=1}^{n} \mathfrak{M}_i,$$

so that $\mathfrak{M}$ is a closed linear subspace of $\mathfrak{H}$. The goal of the method of alternating projections is to approximate the best approximation of $x \in \mathfrak{H}$ in $\mathfrak{M}$. The way in which this is done is by cycling through the individual subspaces $\mathfrak{M}_i$. More precisely, set

$$x_0 = x,$$
$$x_1 = P_{\mathfrak{M}_1}x,\ x_2 = P_{\mathfrak{M}_2}x_1,\ \cdot\cdot,\ x_n = P_{\mathfrak{M}_n}x_{n-1},$$
$$x_{n+1} = P_{\mathfrak{M}_1}x_n,\ x_{n+2} = P_{\mathfrak{M}_2}x_{n+1},\ \cdots$$

It has been shown by Halperin and von Neumann that $x_n \to P_{\mathfrak{M}}x$ as $n \to \infty$. If we consider the subsequence $x_{nk}$ of $x_n$, then, as a consequence of this

$$\lim_{n\to\infty} \|(P_{\mathfrak{M}_k}...P_{\mathfrak{M}_2}P_{\mathfrak{M}_1})^n x - P_{\mathfrak{M}}x\| = 0.$$

So, $(P_{\mathfrak{M}_k}...P_{\mathfrak{M}_2}P_{\mathfrak{M}_1})^n x$ converges to $x$. But what can be said about the rate of convergence? According to Deutsch, [3] the convergence can be arbitrarily slow. However, we can find an upper bound for the rate of convergence. The rate of convergence is governed by the norm of the operator $(P_{\mathfrak{M}_n} \cdots P_{\mathfrak{M}_2}P_{\mathfrak{M}_1})^k - P_{\mathfrak{M}}$. Now we have that

$$\|(P_{\mathfrak{M}_k} \cdot ... \cdot P_{\mathfrak{M}_2}P_{\mathfrak{M}_1})^n x - P_{\mathfrak{M}}x\| \leq E_k(n)\, \|x\|, \tag{7.1}$$

for all $x \in \mathfrak{H}$ where

$$E_k(n) := \|(P_{\mathfrak{M}_k} \cdot ... \cdot P_{\mathfrak{M}_2}P_{\mathfrak{M}_1})^n - P_{\mathfrak{M}_n}\|$$

is the smallest constant (independent of $x$) which works in (7.1).

**Lemma 7.1.** *For $E_k(n)$ the following equalities hold:*

$$\begin{aligned} E_k(n) :&= \|(P_{\mathfrak{M}_k} ... P_{\mathfrak{M}_2} P_{\mathfrak{M}_1})^n - P_{\mathfrak{M}}\| \\ &= \|(P_{\mathfrak{M}_k} ... P_{\mathfrak{M}_2} P_{\mathfrak{M}_1}(P_{\mathfrak{M}^\perp})^n\| \\ &= \|Q_k ... Q_2 Q_1)^n\| . \end{aligned}$$

*Proof.* Since $\mathfrak{M} \subset \mathfrak{M}_i$, one of the lemma's about orthogonal projections implies that $P_{\mathfrak{M}}$ commutes with $\mathrm{P}_{\mathfrak{M}_i}$,$\mathrm{P}_{\mathfrak{M}_i} P_{\mathfrak{M}} = P_{\mathfrak{M}}$ and $P_{\mathfrak{M}_i} P_{\mathfrak{M}^\perp} = P_{\mathfrak{M}_i \cap \mathfrak{M}^\perp}$. Since $P_{\mathfrak{M}^\perp}$ is idempotent we have that

$$\begin{aligned} \|(P_{\mathfrak{M}_k} ... P_{\mathfrak{M}_2} P_{\mathfrak{M}_1})^n - P_{\mathfrak{M}}\| &= \|(P_{\mathfrak{M}_k} ... P_{\mathfrak{M}_2} P_{\mathfrak{M}_1})^n P_{\mathfrak{M}}\| \\ &= \|(P_{\mathfrak{M}_k} ... P_{\mathfrak{M}_2} P_{\mathfrak{M}_1})^n P_{\mathfrak{M}^\perp}\| \\ &= \|(P_{\mathfrak{M}_k} ... P_{\mathfrak{M}_2} P_{\mathfrak{M}_1} P_{\mathfrak{M}^\perp})^n\| \\ &= \|(Q_k ... Q_2 Q_1)^n .\| \end{aligned}$$

□

**Lemma 7.2.** *Assume that $k = 2$, then an upper bound for $E_2(n)$ is given by*

$$E_2(n) \leq c(\mathfrak{M}_1, \mathfrak{M}_2)^n.$$

*Proof.* By the previous lemma we have that

$$E_2(n) = \|(P_{\mathfrak{M}_2} P_{\mathfrak{M}_1} P_{\mathfrak{M}^\perp})^n\| .$$

Notice that

$$\|(P_{\mathfrak{M}_2} P_{\mathfrak{M}_1} P_{\mathfrak{M}^\perp})^n\| \leq \|P_{\mathfrak{M}_2} P_{\mathfrak{M}_1}(P_{\mathfrak{M}^\perp})\|^n = c(\mathfrak{M}_1, \mathfrak{M}_2)^n.$$

So

$$E_2(n) \leq c(\mathfrak{M}_1, \mathfrak{M}_2)^n.$$

□

This example shows that we can express $E_n(k)$ in terms of the opening between subspaces. In the following theorem we state and prove a much better expression for $E_2(n)$ in terms of the opening between subspaces.

**Theorem 7.3.** *Assume that $k = 2$, then an upper bound for $E_2(n)$ is*

$$E_2(n) = c(\mathfrak{M}_1, \mathfrak{M}_2)^{2n-1}, n = 1, 2, ...$$

*Proof.* By lemma 7.1, we have that $E_2(n) = \|(Q_2 Q_1)^n\|$, where $Q_i = P_{\mathfrak{M}_i \cap (\mathfrak{M}_1 \cap \mathfrak{M}_2)^\perp}$ $(i = 1, 2)$. Because

$$[(Q_2 Q_1)^n]^* = [(Q_2 Q_1)^*]^n = (Q_1 Q_2)^n,$$

we have

$$\begin{aligned} \|(Q_2 Q_1)^n\|^2 &= \|(Q_2 Q_1)^n [(Q_2 Q_1)^n]^* \|] \\ &= \|(Q_2 Q_1)^n (Q_1 Q_2)^n\| \\ &= \left\|(Q_2 Q_1 Q_2)^{2n-1}\right\| . \end{aligned} \tag{7.2}$$

Because $Q_2 Q_1 Q_2$ is selfadjoint, hence normal, we have that

$$\left\|(Q_2 Q_1 Q_2)^{2n-1}\right\| = \|(Q_2 Q_1 Q_2)\|^{2n-1} .$$

Playing a little further with these projections yields:

$$\|Q_2Q_1Q_2\| = \|Q_2Q_1Q_1Q_2\| = \|(Q_2Q_1)(Q_2Q_1)^*\| = \|Q_2Q_1\|^2 . \tag{7.3}$$

Combination of (7.2) and (7.3) yields

$$\|Q_2Q_1)^n\|^2 = \|(Q_2Q_1Q_2)\|^{2n-1} = \|Q_2Q_1\|^{2n-1} .$$

This implies that

$$E_2(n) = \|(Q_2Q_1)^n\| = \|Q_2Q_1\|^{2n-1} .$$

Now notice that

$$\|Q_2Q_1\| = \|P_{\mathfrak{M}_2}P_{\mathfrak{M}^\perp}P_1P_\perp\| = \|P_{\mathfrak{M}_2}P_{\mathfrak{M}_1}P_{\mathfrak{M}^\perp}\| = c(\mathfrak{M}_1, \mathfrak{M}_2).$$

So we have shown that $E_2(n) = c(\mathfrak{M}_1, \mathfrak{M}_2)^{2n-1}$. □

Theorem 7.1 says something about the rate of convergence if $\mathfrak{M}$ is the intersection of two closed linear subspaces. An upper bound can also be found for intersections of any number $\geq 2$ of closed linear subspaces. In the following theorem we state the result.

**Theorem 7.4.** ***(Smith-Solomon-Wagner,*** [16]***)*** *Let $k \geq 2$ and*

$$c_i = c(\mathfrak{M}_i, \cap_{j=i+1}^n \mathfrak{M}_j),\ i = 1, 2, ..., n-1.$$

*Then*

$$E_k(n) \leq c^n$$

*where*

$$c = [1 - \prod_{i=1}^{n-1}(i - c_i^2)]^{1/2}$$

## Appendix A. projections

A linear operator $P \in \mathbf{B}(\mathfrak{H})$ is called a *projection* if $P^2 = P$.

**Lemma A.1.** *Let $P \in \mathbf{B}(\mathfrak{H})$ be a projection, then*

(1) *$I - P$ is a projection;*
(2) *$\operatorname{ran} P = \ker\,(I - P)$;*
(3) *$\ker\, P = \operatorname{ran}\,(I - P)$.*

*Proof.* (1) By definition $P^2 = P$, so $(I-P)^2 = I - P - P + P = I - P$, which shows that $(I - P)$ is a projection.
(2) Assume that $h \in \operatorname{ran} P$ then there exists a $x \in \mathfrak{H}$ such that $h = Px$, then we have $(I - P)h = Px - Px = 0$ and so $h \in \ker\,(I - P)$. So $\operatorname{ran} P \subset \ker\,(I - P)$. Conversely assume that $h \in \ker\,(I - P)$, then $(I - P)h = 0$, so $h = Ph$ and hence $h \in \operatorname{ran} P$. So $\ker\,(I - P) \subset \operatorname{ran} P$. So $\operatorname{ran} P = \ker\,(I - P)$.
(3) This follows from (2) by replacing $P$ by $(I - P)$. □

**Corollary A.2.** *Let $P \in \boldsymbol{B}(\mathfrak{H})$ be a projection, then $\mathfrak{H} = \operatorname{ran} P + \ker\, P$ and this sum is direct.*

*Proof.* Because $\operatorname{ran} P, \ker\, P \subset \mathfrak{H}$ also $\operatorname{ran} P + \ker\, P \subset \mathfrak{H}$. To proof the reversed inclusion, assume that $f \in H$. Then $f$ can be written as

$$f = Pf + (I - P)f,\ Pf \in \operatorname{ran} P,\ (I - P)f \in \operatorname{ran}\,(I - P),$$

because $(I - P)f = f - Pf$. By lemma A.1, $\operatorname{ran}\,(I - P) = \ker\,(P)$, so $f \in \operatorname{ran} P + \ker\, P$. So we have shown that $\mathfrak{H} = \operatorname{ran} P + \ker\, P$. In order to show that the sum is direct, assume that $f \in \operatorname{ran} P \cap \ker P$, then there is $h \in \mathfrak{H}$ such that $f = Ph$, and $Pf = P^2 h = Ph = 0$. So $\operatorname{ran} P \cap \ker P = \{0\}$, which proves that $\operatorname{ran} P + \ker P$ is a direct sum. □

**Lemma A.3.** *If $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$, then there is a projection $P : \mathfrak{H} \to \mathfrak{H}$ with $\operatorname{ran} P = \mathfrak{M}$ and $\ker P = \mathfrak{N}$.*

*Proof.* Assume that $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$. Then for all $x \in \mathfrak{H}$ we have the decomposition $y + z$, with $y \in \mathfrak{M}$ and $z \in \mathfrak{N}$. If we define an operator $P$ with $Px = y$, then we have the desired projection. □

## Appendix B. orthogonal projections

Given an arbitrary point in a Hilbert space $\mathfrak{H}$ and a linear subspace $\mathfrak{M}$ of $\mathfrak{H}$. Can we find a point in $\mathfrak{M}$ which is the closest point near $x$? The answer to this question is the content of the first theorem in this appendix.

**Theorem B.1.** *Let $\mathfrak{M}$ be a closed linear subspace of a Hilbert space $\mathfrak{H}$. Then*

(1) *For each $x \in \mathfrak{H}$ there is a unique closest point $y \in \mathfrak{M}$ such that*

$$\|x - y\| = \min_{z \in \mathfrak{M}} \|x - z\| \,; \tag{B.1}$$

(2) *The point $y \in \mathfrak{M}$ closest to $x \in \mathfrak{H}$ is the unique element of $\mathfrak{M}$ with the property that $x - y \in \mathfrak{M}^{\perp}$.*

*Proof.* Let $d$ be the distance of $x$ from $\mathfrak{M}$. Then

$$d = \inf\{\|x - z\| \,|z \in \mathfrak{M}\}.$$

From this definition follows that there is a sequence of elements $y_n \in \mathfrak{M}$ such that

$$\lim_{n \to \infty} \|x - y_n\| = d,$$

so for all $\epsilon > 0$, there is an integer $N$ sucht that

$$\|x - y_n\| \le d + \epsilon,$$

when $n \ge N$. Using the parallelogram rule, we have

$$\|y_m - y_n\|^2 + \|2x - y_m - y_n\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 .$$

Because $(y_m + y_n)/2 \in \mathfrak{M}$, we see from the definition of $d$ that

$$\|x - (y_m + y_n)/2\| \ge d.$$

Combine the definition of $d$ with the above equations for all $m, n \ge N$ in the following way:

$$\begin{aligned} \|y_m - y_n\|^2 &= 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|2x - y_m - y_n\|^2 \\ &\le 4(d + \epsilon)^2 - 4d^2 \\ &\le 4\epsilon(2d + \epsilon) \end{aligned}$$

Since we can choose $\epsilon$ as small as we like, $y_n - y_m \to 0$, so $y_n$ is a Cauchy sequence. Since $\mathfrak{H}$ is complete, there is a $y$ to which $y_n$ converges and because $\mathfrak{M}$ is closed, we have $y \in \mathfrak{M}$. Because the norm is continuous, we have shown that $\|x - y\| = \lim_{n\to\infty} \|x - y_n\| = d$. In order to show the uniqueness of this $y$, suppose that $y$

and $y'$ both minimize the distance to $x$. We have to show that $\|y - y'\| = 0$. By the parallellogram identity we see that

$$2\|x - y\|^2 + 2\|x - y'\|^2 = \|2x - y - y'\|^2 + \|y - y'\|^2 .$$

Rewriting this equation and using that $\|x - y\| = \|x - y'\| = d$, we see that

$$\|y - y'\|^2 = 4d^2 - 4\|x - (y + y')/2\|^2 \leq 0,$$

so $\|y - y'\| = 0$ so $y = y'$. This proves part 1 of this theorem.
Is the vector $x - y$ orthogonal to $\mathfrak{M}$ for all $y \in \mathfrak{M}$? Because $y$ minimizes the distance to $x$, we have for every $\lambda \in \mathbb{C}$ and $z \in \mathfrak{M}$ that

$$\|x - y\|^2 \leq \|x - y + \lambda z\|^2 .$$

Expanding the righthand side, we obtain

$$2\mathrm{Re}\left(\lambda \langle x - y, z\rangle\right) \leq |\lambda|^2 \|z\|^2 .$$

Suppose that $\langle x - y, z\rangle = |\langle x - y, z\rangle| e^{i\phi}$. If we choose $\lambda = \epsilon e^{-i\phi}$, where $\epsilon > 0$ and divide by $\epsilon$, we get

$$2|\langle x - y, z\rangle| \leq \epsilon \|z\|^2 .$$

Taking the limit as $\epsilon \to 0^+$, we find that $\langle x - y, z\rangle = 0$, so $(x - y) \in \mathfrak{M}^\perp$. Finally it needs to be shown that $y$ is the only element in $\mathfrak{M}$ such that $x - y \in \mathfrak{M}^\perp$. Suppose to the contrary that there is a $y' \neq y$ with the property that also $x - y' \in \mathfrak{M}^\perp$. For any $z \in \mathfrak{M}$ we have

$$\langle z, y - y'\rangle = \langle z, x - y'\rangle - \langle z, x - y\rangle = 0$$

In particular, we may take $z = y - y'$, and therefore we must have $y = y'$. □

As a consequence of this theorem, each element $h$ in a Hilbert space $\mathfrak{H}$ can be uniquely written as $h = x + y$ where $x \in \mathfrak{M}$ and $y \in \mathfrak{M}^\perp$. An *orthogonal projection* onto a closed subspace $\mathfrak{M}$ of $\mathfrak{H}$ is the mapping $P_{\mathfrak{M}} : \mathfrak{H} \to \mathfrak{M}$ which associates with each $x \in \mathfrak{H}$ its unique nearest point(in the sense of theorem $B$.1) in $\mathfrak{M}$.

In some literature an orthogonal projection is defined as an operator which is self-adjoint and idempotent. This is the definition which we will used most in this appendix. In the next lemma, we show that these definitions are equivalent.

**Lemma B.2.** *Let $P_{\mathfrak{M}}$ be a bounded linear operator on a Hilbert space $\mathfrak{H}$. Then $P_M$ is an orthogonal projection if and only if $P_{\mathfrak{M}}^2 = P_{\mathfrak{M}}$ and $P_{\mathfrak{M}}^* = P_{\mathfrak{M}}$.*

*Proof.* Suppose that $P_{\mathfrak{M}}$. Because $\mathfrak{M}$ is closed, $x \in \mathfrak{H}$ can be written as $x = h_1 + h_2$ with $h_1 \in \mathfrak{M}$ and $h_2 \in \mathfrak{M}^\perp$. so $P_{\mathfrak{M}} = h_1 \in \mathfrak{M}$ and so $P_{\mathfrak{M}}^2 x = P_{\mathfrak{M}} h_1 = h_1$. Suppose now that $y \in \mathfrak{H}$, then $y$ can be written as $x = k_1 + k_2$ with $k_1 \in \mathfrak{M}$ and $k_2 \in \mathfrak{M}^\perp$. Then

$$\begin{aligned}\langle P_{\mathfrak{M}} x, y\rangle = \langle h_1, y\rangle &= \langle h_1, k_1 + k_2\rangle \\ &= \langle h_1, k_1\rangle = \langle h_1 + h_2, k_1\rangle \\ &= \langle x, P_{\mathfrak{M}} y\rangle .\end{aligned}$$

But also $\langle P_{\mathfrak{M}} x, y\rangle = \langle x, P_{\mathfrak{M}}^* y\rangle$, so $P_{\mathfrak{M}}^* = P_{\mathfrak{M}}$. Now, assume that $P_{\mathfrak{M}}^2 = P_{\mathfrak{M}}$ and $P_{\mathfrak{M}}^* = P_{\mathfrak{M}}$. Let $y_n$ be a sequence defined by $y_n = P_{\mathfrak{M}} x_n$ and $y_n \to z$. Notice that $P_{\mathfrak{M}} y_n = P_{\mathfrak{M}}^2 x_n = P_{\mathfrak{M}} x_n = y_n$. So $\lim_{n\to\infty} y_n = \lim_{n\to\infty} P_{\mathfrak{M}} y_n = P_{\mathfrak{M}}(z)$. So $z \in \mathfrak{M}$ and so $\mathfrak{M}$ is closed. Let $y \in \mathfrak{M}$ then

$$\langle x - P_{\mathfrak{M}} x, y\rangle = \left\langle P_{\mathfrak{M}} - P_{\mathfrak{M}}^2, P_{\mathfrak{M}} y\right\rangle = \langle 0, P_{\mathfrak{M}} y\rangle = 0.$$

□

B.1. **Properties.** In this subsection we state a number of useful properties of orthogonal projections.

**Proposition B.3.** *If $P$ is a nonzero orthogonal projection, then $\|P\| = 1$.*

*Proof.* Let $x \in \mathfrak{H}$ and $Px \neq 0$, then we have

$$\|Px\| = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \leq \|x\|\,,$$

so $\|P\| \leq 1$. If $P \neq 0$ than there exists an $x \in \mathfrak{H}$, with $Px \neq 0$ and $\|P(P(x))\| = \|Px\|$ so that $\|P\| \geq 1$. □

**Lemma B.4.** *Suppose that $P_{\mathfrak{M}}$ is an orthogonal projection, then $I - P_{\mathfrak{M}}$ is an orthogonal projection and $P_{M^\perp} = I - P_{\mathfrak{M}}$.*

*Proof.* Suppose that $P_{\mathfrak{M}}$ is an orthogonal projection, then $(I - P_{\mathfrak{M}})^2 = I - 2P_{\mathfrak{M}} + P_{\mathfrak{M}} = I - P_{\mathfrak{M}}$. Furthermore $(I - P_{\mathfrak{M}})^* = I - P_{\mathfrak{M}}^* = I - P_{\mathfrak{M}}$. So $I - P_{\mathfrak{M}}$ is an orthogonal projection. Let $x \in \mathfrak{H}$. Because $P_{\mathfrak{M}}$ is an orthogonal projection, $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}$ and $x$ can be uniquely written as $x = m + m_\perp$, with $m \in \mathfrak{M}, m_\perp \in \mathfrak{M}^\perp$. So $x = P_{\mathfrak{M}}x + P_{\mathfrak{M}^\perp}x$ and $P_{\mathfrak{M}^\perp}x = Ix - P_{\mathfrak{M}}x$. So $P_{M^\perp} = I - P_{\mathfrak{M}}$. □

The following theorem can be found in [1].

**Theorem B.5.** *If $P$ denotes an orthogonal projection, then $P \in$ **B(H)** and $\operatorname{ran} P$ is closed.*

*Proof.* Assume that $P$ is an orthogonal projection, then $P$ is bounded since

$$\|Ph\| = \langle Ph, h\rangle \leq \|Ph\|\,\|h\|\,,$$

so $\|Ph\| \leq \|h\|$. Define $\mathfrak{M}$

$$\mathfrak{M} = \{g \in \mathfrak{H}\,|Pg = g\}\,.$$

Since $P$ is a linear operator, it is easy to see that $\mathfrak{M}$ is a linear subspace of $\mathfrak{H}$. The subspace $\mathfrak{M}$ is also closed. In order to see this, let $g_n$ be a sequence in $\mathfrak{M}$ which converges to $x$ as $n \to \infty$. Now

$$Pg - g_n = P(g - g_n),$$

since $g_n \in \mathfrak{M}$. Taking limits as $n \to \infty$ yields

$$\|P(g - g)\| \leq \|P\|\,\|g - g\| = \|g - g\| = 0,$$

so $g \in \mathfrak{M}$. In order to show that $P_{\mathfrak{M}} = P$, let $h \in \mathfrak{H}$ be arbitrary. Then $g = Ph \in \mathfrak{M}$. For an arbitrary $g' \in \mathfrak{M}$ it needs to be shown that $\langle Ph, g'\rangle = \langle P_{\mathfrak{M}}h, g'\rangle$. Notice that

$$\langle Ph, g'\rangle = \langle h, P'\rangle = \langle h, g'\rangle$$

and

$$\langle P_{\mathfrak{M}}h, g'\rangle = \langle h, P_{\mathfrak{M}}g'\rangle = \langle h, g'\rangle\,,$$

which concludes the proof. □

**Lemma B.6.** *The following statements are equivalent:*

(1) *$P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ commute;*
(2) *$P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}\cap\mathfrak{N}}$ is an orthogonal projection;*

*Proof.* To prove (1) $\Rightarrow$ (2), assume that $P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{N}}P_{\mathfrak{M}}$. Then

$$(P_{\mathfrak{N}}P_{\mathfrak{M}})^2 = P_{\mathfrak{N}}P_{\mathfrak{M}}P_{\mathfrak{N}}P_{\mathfrak{M}} = P_{\mathfrak{M}}^2 P_{\mathfrak{N}}^2 = P_{\mathfrak{M}}P_{\mathfrak{N}},$$

which shows that $P$ is an orthogonal projection. To show that $P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}\cap\mathfrak{N}}$, let $x \in \mathfrak{M} \cap \mathfrak{N}$, then

$$P_{\mathfrak{M}}P_{\mathfrak{N}}x = x,$$

while for $y \in \mathfrak{M}^\perp + \mathfrak{N}^\perp$, we have that $P_{\mathfrak{M}}P_{\mathfrak{N}}y = 0$ since

$$P_{\mathfrak{M}}P_{\mathfrak{N}}(\mathfrak{M}^\perp + \mathfrak{N}^\perp) = P_{\mathfrak{M}}P_{\mathfrak{N}}(\mathfrak{M}^\perp) = P_{\mathfrak{N}}P_{\mathfrak{M}}(\mathfrak{M}^\perp) = \{0\}\,.$$

By continuity there follows that $P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}\cap\mathfrak{N}}$. (2) $\Rightarrow$ (1), $P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}\cap\mathfrak{N}} = P_{\mathfrak{N}\cap\mathfrak{M}} = P_{\mathfrak{N}}P_{\mathfrak{M}}$. □

**Corollary B.7.** *The following statements are equivalent:*

(1) $P_{\mathfrak{M}}P_{\mathfrak{N}} = 0$;
(2) $P_{\mathfrak{N}}P_{\mathfrak{M}} = 0$;
(3) $\mathfrak{M} \perp \mathfrak{N}$.

*Proof.* To prove (1) $\Leftrightarrow$ (2) it is sufficient to show (1) $\Rightarrow$ (2). The assumption $P_{\mathfrak{M}}P_{\mathfrak{N}} = 0$ implies that $P_{\mathfrak{M}}P_{\mathfrak{N}}$ is self-adoint and hence $P_{\mathfrak{N}}P_{\mathfrak{M}} = 0$. In order to proof (2) $\Rightarrow$ (3), observe that for all $f \in \mathfrak{M}$ and for all $g \in \mathfrak{N}$

$$\langle f, g\rangle = \langle P_{\mathfrak{M}}f, P_{\mathfrak{N}}g\rangle = \langle P_{\mathfrak{N}}P_{\mathfrak{M}}f, g\rangle = 0$$

so $\mathfrak{M} \perp \mathfrak{N}$. (3) $\Rightarrow$ (2): for all $h, k \in \mathfrak{H}$, it follows that

$$\langle P_{\mathfrak{N}}P_{\mathfrak{M}}h, k\rangle = \langle P_{\mathfrak{M}}h, P_{\mathfrak{N}}k\rangle = 0,$$

because $\mathfrak{M} \perp \mathfrak{N}$. Hence $P_{\mathfrak{N}}P_{\mathfrak{M}} = 0$ □

**Corollary B.8.** *The following statements are equivalent:*

(1) $P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}}$;
(2) $P_{\mathfrak{N}}P_{\mathfrak{M}} = P_{\mathfrak{M}}$;
(3) $\mathfrak{M} \subset \mathfrak{N}$.

*Proof.* To prove (1) $\Leftrightarrow$ (2), it is sufficient to proof (1) $\Rightarrow$ (2). Note that $P_{\mathfrak{M}}P_{\mathfrak{N}} = P_{\mathfrak{M}}$ is selfadjoint and hence $P_{\mathfrak{N}}P_{\mathfrak{M}} = P_{\mathfrak{M}}$. To prove (2) $\Rightarrow$ (3), let $x \in \mathfrak{M}$, then $P_{\mathfrak{N}}P_{\mathfrak{M}}x = P_{\mathfrak{M}}x = x$, so that $x \in \mathfrak{N}$. To prove (3) $\Rightarrow$ (2), assume that $f \in \mathfrak{M} \subset \mathfrak{N}$, then $P_{\mathfrak{M}}f = f$, so that $P_{\mathfrak{N}}P_{\mathfrak{M}}f = P_{\mathfrak{N}}f = f$. We have shown that $P_{\mathfrak{N}}P_{\mathfrak{M}} = P_{\mathfrak{M}}$ for all $f \in \mathfrak{M}$. It is also clear that for all $g \in \mathfrak{M}^\perp, P_{\mathfrak{N}}P_{\mathfrak{M}}g = 0 = P_{\mathfrak{M}}g$. Because $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$, we have shown that for all $h \in \mathfrak{H}$ that $P_{\mathfrak{N}}P_{\mathfrak{M}} = P_{\mathfrak{M}}$. □

**Lemma B.9.** *The following statements are equivalent.*

(1) $P_{\mathfrak{M}}$ *and* $P_{\mathfrak{N}}$ *commute;*
(2) $P_{\mathfrak{M}^\perp}$ *and* $P_{\mathfrak{N}}$ *commute;*
(3) $P_{\mathfrak{M}}$ *and* $P_{\mathfrak{N}^\perp}$ *commute;*
(4) $P_{\mathfrak{M}^\perp}$ *and* $P_{\mathfrak{N}^\perp}$ *commute.*

*Proof.* It suffices to proof the equivalence of (1) and (3). This equivalence can easily be seen from

$$P_{\mathfrak{M}}P_{\mathfrak{N}^\perp} - P_{\mathfrak{N}^\perp}P_{\mathfrak{M}} = P_{\mathfrak{M}}(I - P_{\mathfrak{N}}) - (I - P_{\mathfrak{N}})P_{\mathfrak{M}} = -P_{\mathfrak{M}}P_{\mathfrak{N}} + P_{\mathfrak{N}}P_{\mathfrak{M}} = 0,$$

so $P_{\mathfrak{M}}, P_{\mathfrak{N}}$ commute. □

**Corollary B.10.** *The following statements are equivalent:*

(1) $P_{\mathfrak{M}}$ *and* $P_{\mathfrak{N}}$ *commute;*

(2) $P_{\overline{\mathfrak{M}+\mathfrak{N}}} = P_{\mathfrak{M}} + P_{\mathfrak{N}} - P_{\mathfrak{M}} P_{\mathfrak{N}}$

*Proof.* (1) ⇒ (2), notice that

$$P_{\overline{\mathfrak{M}+\mathfrak{N}}} = I - P_{\mathfrak{M}^\perp \cap \mathfrak{N}^\perp} = I - (I - P_{\mathfrak{M}})(I - P_{\mathfrak{N}}) = P_{\mathfrak{M}} + P_{\mathfrak{N}} - P_{\mathfrak{M}} P_{\mathfrak{N}}.$$

(2) ⇒ (1), notice that

$$I - P_{\mathfrak{M}^\perp \cap \mathfrak{N}^\perp} = P_{\mathfrak{M}^\perp}) + P_{\mathfrak{N}^\perp}) + (I - P_{\mathfrak{M}})(I - P_{\mathfrak{N}}) = I - P_{\mathfrak{M}^\perp} P_{\mathfrak{N}^\perp}.$$

This equation is equivalent to

$$P_{\mathfrak{M}^\perp \cap \mathfrak{N}^\perp} = P_{\mathfrak{M}^\perp} P_{\mathfrak{N}^\perp}.$$

But this implies with the aid of a previous lemma that $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ commute. □

**Lemma B.11.** *The following statements are equivalent:*

(1) $P_{\mathfrak{M}} + P_{\mathfrak{N}}$ *is an orthogonal projection;*
(2) $P_{\mathfrak{M}} P_{\mathfrak{N}} = 0$.

*In this case* $P_{\mathfrak{M}} + P_{\mathfrak{N}} = P_{\mathfrak{M} \oplus \mathfrak{N}}$.

*Proof.* (1) ⇒ (2), denote the projection $P_{\mathfrak{M}} + P_{\mathfrak{N}}$ by $Q$. Then we have the identity

$$\langle Qf, f \rangle = \langle P_{\mathfrak{M}} f, f \rangle + \langle P_{\mathfrak{N}} f, f \rangle,$$

which implies that

$$0 \leq \|P_{\mathfrak{M}} f\|^2 + \|P_{\mathfrak{N}} f\|^2 = \|Qf\|^2 \leq \|f\|^2.$$

By choosing $f = P_{\mathfrak{N}} h$ we obtain

$$\|P_{\mathfrak{M}} P_{\mathfrak{N}} h\|^2 \leq 0$$

and hence $P_{\mathfrak{M}} P_{\mathfrak{N}} = 0$. In order to show that $P_{\mathfrak{M}} + P_{\mathfrak{N}} = P_{\mathfrak{M} \oplus \mathfrak{N}}$, simply notice that $P_{\mathfrak{M}} P_{\mathfrak{N}} = 0$ implies that $\mathfrak{M} \perp \mathfrak{N}$ and hence $\mathfrak{M} + \mathfrak{N}$ is a direct sum, and that $P_{\mathfrak{M}} P_{\mathfrak{N}} = 0$ implies that $P_{\mathfrak{M}}, P_{\mathfrak{N}}$ commute and hence by the previous lemma $P_{\mathfrak{M}} + P_{\mathfrak{N}} = P_{\mathfrak{M}+\mathfrak{N}}$. So $P_{\mathfrak{M}} + P_{\mathfrak{N}} = P_{\mathfrak{M} \oplus \mathfrak{N}}$. (2) ⇒ (1), because $P_{\mathfrak{M}} P_{\mathfrak{N}} = 0$, $P_{\mathfrak{M}} P_{\mathfrak{N}}$ is selfadjoint. The identity

$$(P_{\mathfrak{M}} + P_{\mathfrak{N}})^2 = P_{\mathfrak{M}} + P_{\mathfrak{M}} P_{\mathfrak{N}} + P_{\mathfrak{N}} P_{\mathfrak{M}} + P_{\mathfrak{N}} = P_{\mathfrak{M}} + P_{\mathfrak{N}},$$

because $P_{\mathfrak{M}} P_{\mathfrak{N}}$ is selfadjoint. □

**Corollary B.12.** *The following statements are equivalent:*

(1) $\mathfrak{M} \subset \mathfrak{N}$*;*
(2) $P_{\mathfrak{N}} - P_{\mathfrak{M}}$ *is an orthogonal projection*

*Proof.* (1) ⇒ (2), it is clear that $P_{\mathfrak{N}} - P_{\mathfrak{M}}$ is selfadjoint. So it remains to be shown that $P_{\mathfrak{N}} - P_{\mathfrak{M}}$ is idempotent:

$$(P_{\mathfrak{N}} - P_{\mathfrak{M}})^2 = (P_{\mathfrak{N}} - P_{\mathfrak{M}})(P_{\mathfrak{N}} - P_{\mathfrak{M}}) = P_{\mathfrak{N}} - P_{\mathfrak{N}} P_{\mathfrak{M}} - P_{\mathfrak{M}} P_{\mathfrak{N}} + P_{\mathfrak{M}} = P_{\mathfrak{N}} - P_{\mathfrak{M}},$$

by corollary B.8. (2) ⇒ (1), assume that $P_{\mathfrak{N}} - P_{\mathfrak{M}} = P_x$ is an orthogonal projection, then the orthogonal projection $P_{\mathfrak{N}}$ can be written as $P_{\mathfrak{M}} + P_x$, which implies that $x \perp \mathfrak{M}$. Hence $\mathfrak{N} = \mathfrak{M} \oplus x$ and so $\mathfrak{M} \subset \mathfrak{N}$. □

**Lemma B.13.** *The following statements are equivalent:*

(1) $\mathfrak{M} \subset \mathfrak{N}$*;*
(2) $P_{\mathfrak{M}} \leq P_{\mathfrak{N}}$*;*
(3) $\|P_{\mathfrak{M}} h\| \leq \|P_{\mathfrak{N}} h\|$.

*Proof.* (1) $\Rightarrow$ (2), since $\mathfrak{M} \subset \mathfrak{N}$, $P_{\mathfrak{N}} - P_{\mathfrak{M}}$ is an orthogonal projection. Because $\|P_{\mathfrak{N}} - P_{\mathfrak{M}}\| \geq 0$ we have by the equivalence of parts 2 and 3 of this lemma that $0 \leq P_{\mathfrak{N}} - P_{\mathfrak{M}}$ which shows that $P_{\mathfrak{M}} \leq P_{\mathfrak{N}}$. (2) $\Rightarrow$ (1), suppose that, for $f \in \mathfrak{H}$, $P_{\mathfrak{N}}h = 0$ then $P_{\mathfrak{M}}h = 0$ because $P_{\mathfrak{M}} \leq P_{\mathfrak{N}}$. So we have that $\mathfrak{N}^{\perp} \subset \mathfrak{M}^{\perp}$. Taking orthogonal complements on both sides and noting that $\mathfrak{M}$ and $\mathfrak{N}$ are closed, we have that $\mathfrak{M} \subset \mathfrak{N}$.
For (2) $\Leftrightarrow$ (3), Assume that $\|P_{\mathfrak{M}}h\| \leq \|P_{\mathfrak{N}}h\|$, then

$$\|P_{\mathfrak{M}}h\| \leq \|P_{\mathfrak{N}}h\| \Leftrightarrow \|P_{\mathfrak{M}}h\|^2 \leq \|P_{\mathfrak{N}}h^2\| \Leftrightarrow \langle P_{\mathfrak{M}}h, h\rangle \leq \langle P_{\mathfrak{N}}h, h\rangle,$$

which is equivalent to

$$\langle (P_{\mathfrak{M}} - P_{\mathfrak{N}})h, h\rangle \leq 0,$$

so

$$P_{\mathfrak{M}} - P_{\mathfrak{N}} \leq 0 \Leftrightarrow P_{\mathfrak{M}} \leq P_{\mathfrak{N}}.$$

□

**Lemma B.14.** *The following statements are equivalent:*

(1) $P_{\mathfrak{M}}$ *and* $P_{\mathfrak{N}}$ *commute;*
(2) $\mathfrak{M} = (\mathfrak{M} \cap \mathfrak{N}) \oplus (\mathfrak{M} \cap \mathfrak{N}^{\perp})$.

*Proof.* Assume that $\mathfrak{M} = (\mathfrak{M} \cap \mathfrak{N}) \oplus (\mathfrak{M} \cap \mathfrak{N}^{\perp})$. This is equivalent to $P_{\mathfrak{M}\cap\mathfrak{N}} P_{\mathfrak{M}\cap\mathfrak{N}^{\perp}} = 0$ if and only if

$$P_{\mathfrak{M}} = P_{(\mathfrak{M}\cap\mathfrak{N})\oplus(\mathfrak{M}\cap\mathfrak{N}^{\perp})} = P_{\mathfrak{M}\cap\mathfrak{N}} + P_{\mathfrak{M}\cap\mathfrak{N}^{\perp}}$$

if and only if

$$P_{\mathfrak{N}} P_{\mathfrak{M}} = P_{\mathfrak{N}}(P_{\mathfrak{M}\cap\mathfrak{N}} + P_{\mathfrak{M}\cap\mathfrak{N}^{\perp}}) = P_{\mathfrak{N}} P_{\mathfrak{M}\cap\mathfrak{N}} = P_{\mathfrak{M}\cap\mathfrak{N}}$$

if and only if $P_{\mathfrak{M}}, P_{\mathfrak{N}}$ commute, which shows the equivalence. □

## Appendix C. Three background results

In this appendix, we mention three background results which we used at some occasions in the text. First we mention a lemma about the operator norm, secondly we mention the closed graph theorem and after that we mention the Riesz representation theorem.

**Lemma C.1.** *Let $A$ be a bounded linear operator. Then*

$$\|A\| = \sup\{|\langle Ax, y\rangle| : x, y \in \mathfrak{H}, \|x\|, \|y\| \leq 1\} \tag{C.1}$$

*Proof.* For notational convenience, define

$$X := \{x \in \mathfrak{H}, \|x\| = 1, y \in \mathfrak{H}, \|y\| = 1\}$$

Because

$$\begin{aligned}\sup_{x,y\in X}\{|\langle Ax, y\rangle|\} &\geq \sup\{\left|\left\langle Ax, \frac{Ax}{\|Ax\|}\right\rangle\right| : x \in \mathfrak{H}, \|x\| = 1\} \\ &= \sup\{\|Ax\| : \|x\| = 1\} \\ &= \|A\|,\end{aligned}$$

we have that $\|A\| \leq \sup\{|\langle Ax, y\rangle| : x, y \in \mathfrak{H}, \|x\|, \|y\| \leq 1\}$. Furthermore,

$$\begin{aligned}\sup|\langle Ax, y\rangle| &\leq \|Ax\| \cdot \|y\| \\ &\leq \|A\| \cdot \|x\| \cdot \|y\| \\ &\leq \|A\|.\end{aligned}$$

So $\sup\{|\langle Ax, y\rangle| : x, y \in \mathfrak{H}, \|x\|, \|y\| \leq 1\} \leq \|A\|$. So $\|A\| = \sup\{|\langle Ax, y\rangle| : x, y \in \mathfrak{H}, \|x\|, \|y\| \leq 1\}$ □

**Theorem C.2** (**Closed Graph theorem**). *A linear mapping from of a Banach space $\mathfrak{M}$ to a Banach space $\mathfrak{N}$, defined on all of $\mathfrak{M}$, is bounded if and only if its graph is a closed subset of $\mathfrak{M} \times \mathfrak{N}$.*

*Proof.* A proof can be found in [2]. □

**Theorem C.3** (**Riesz representation theorem**). *Let $\mathfrak{H}$ be a Hilbert space and let $F$ be a continuous linear functional on $\mathfrak{H}$. There exists a unique $y \in \mathfrak{H}$ such that*

$$F(x) = \langle x, y\rangle$$

*for all $x \in \mathfrak{H}$. Moreover $\|y\| = \|F\|$.*

*Proof.* A proof can be found in [5]. □

## Appendix D. Conorm and its properties

In section 5, we derive a relation between the angle and the conorm of an operator. In this section, we give a definition of the conorm of a linear operator and mention two important results.

**Definition D.1.** *Let $A$ be a linear operator in $\boldsymbol{B}(\mathfrak{H})$, then the conorm of $A$ is given by*

$$\gamma(A) := \inf\left\{ \frac{\|Af\|}{\|f\|} : f \in ker(A)^{\perp}, f \neq 0 \right\}.$$

From the Closed Graph Theorem (Theorem 3.1) follows that $\operatorname{ran} A$ is closed if and only if the graph of the inverse of $A$ restricted to $(\ker\,(A)))^{\perp}$ is bounded.

**Proposition D.2.** *Let $A$ be a linear operator from a Hilbert space $\mathfrak{H}$ to an Hilbert space $\mathfrak{K}$ Then the following statements are equivalent:*

(1) $\gamma(A) > 0$;
(2) $\operatorname{ran}(A)$ *is closed.*

*Proof.* As noticed earlier, $\operatorname{ran} A$ is closed if and only if the graph of the inverse of $A$ restricted to $(\ker\,(A)^{\perp})$ is bounded. This means that there exists a $m > 0$ such that

$$\|f\| \leq m \left\|f^{'}\right\|,$$

$f \in \operatorname{ran} A^{-1}$ and $f^{'} = Af \in \operatorname{ran}(A)$. This implies that

$$\frac{\|Af\|}{\|f\|} \geq \frac{1}{m}.$$

Since $m > 0$ we must have that $\gamma(A) > 0$. □

**Lemma D.3.** *Let $A$ be a linear operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and $A^*$ the adjoint operator of $A$. Then*

$$\gamma(A) = \gamma(A^*) \tag{D.1}$$

*Proof.* A proof can be found in [8] □

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