Massive states of bosonic strings

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August 2008

Abstract

Some basic properties of the bosonic string theory are discussed. First the classical string theory is discussed and two different methods to quantise, the covariant approach and the light-cone gauge approach. Using the covariant approach one can get the whole amount of information about which states are possible at different massive levels, but only by tough calculations. The light-cone gauge approach is much easier to perform but does not give the whole amount of information. It is tried to find a new method which, only by comparing the dimensions of the representations, gives the same amount of information as the covariant approach, without doing the tough calculations. A method is found which succeeds for the lowest five massive levels but it is not proved that it will work in general. More investigation and more proofs are needed, but it can give some tools to analyse the massive states of bosonic strings.

Contents

1 Introduction

String theory, in being an attempt to unify gravity and all the other forces in nature, as well as all the particles, in just one conceptual structure is a very dominant and exciting branch of the contemporary theoretical physics. It is a very promising candidate for a unified theory. String theory is a quantum theory of gravity.

There is not just one string theory. Two broad subdivisions can be made. One between string theories which consider only closed strings and theories which consider both closed and open strings. Open strings do have free endpoints and closed string do not have endpoints at all because they are closed.

The second broad subdivision is between bosonic string theories and superstring theories. The vibrational modes of bosonic strings represent only the bosonic particles. For describing the fermionic particles as well one needs superstringtheory.

This thesis will be only about the open bosonic strings. Though fermionic particles can't be described, the most important concepts of string theory can be explained with the bosonic theory. The bosonic string theory is much simpler and thus the concepts can be made clear much easier. Open strings can be closed to form closed strings and if one understand the open strings, the closed string will not give much new phenomena to understand.

Vibrational modes can represent massive as well as massless particles. Most of the time physicists are working with string theory they do not give any attention to the massive states. They are only interested in the massless states which can be interpreted as well known particles. But because the massive states follow from the same method as the massless states it is interesting to know what kind of massive states one can expect.

First the classical theory of strings will be described. Because string theory needs to be a quantum theory to be a candidate for a unified theory, it has to be quantised. The quantisation of this theory can be done in two different ways. Both of these approaches does have their own (dis)advantages. In trying to find the ideal method for the interested but lazy physicist, these two approaches will be combined. There will be made an attempt to short-cut the tough calculation of the covariant approach by finding a simpler method.

2 Classical String Theory

In string theory everything is described in terms of strings and the vibrations on these strings. All strings are the same, apart from the distinction between the open en closed strings. On these strings are certain vibrational modes possible which determines what kind of particle is represented by the string. Depending on at which level the string is only certain vibrational modes are possible. To determine what kind of modes are possible the equations of motions have to be derived.

2.1 The Action

First of all it is convenient to introduce the action from which we can derive the equations of motions. This action is proportional to the proper area of the world sheet of the string. The world sheet is the surface spanned by the string when it is moving through spacetime.

Before looking at the action the coordinates have to be introduced. X^μ are the coordinates of the embedding space-time and $\sigma^\mu = (\tau, \sigma)$ are the coordinates on the string. Here will be made use of the convention for denoting the derivatives of the space-time coordinates:

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \;,\; X^{\mu'} \equiv \frac{\partial X^\mu}{\partial \sigma}$$

The action is then given by:

$$\begin{aligned} S = &- T \int dA \\ = &- T \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \\ = &- T \int d^2\sigma \sqrt{-\gamma} \end{aligned} \tag{1}$$

This form is called the Nambu-Goto form of the action. Here $\gamma = \det(\gamma_{\alpha\beta})$, while $\gamma_{\alpha\beta}$ is the induced metric on the world-sheet:

$$\gamma_{\alpha\beta} \equiv \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta}$$

more explicitly, in matrix notation:

$$\gamma_{\alpha\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{bmatrix}$$

Let's introduce a new world-sheet metric, $h_{\alpha\beta}(\tau, \sigma)$. This $h_{\alpha\beta}$ is a dynamical variable, so it will give its own equations of motions. If this new metric is introduced it gives the Polyakov form of the string action:

$$\begin{aligned} S = &- \frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \\ = &- \frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \gamma_{\alpha\beta} \end{aligned} \tag{2}$$

It can be shown (see [4]) that this form is classically equivalent to the Nambu-Goto form of the action (1).

This action is invariant under reparametrisation. We can use this fact to simplify the form of our metric. We can set $h_{\alpha\beta} = \rho^2(\sigma)\eta_{\alpha\beta}$. Here $\eta_{\alpha\beta}$ is the two-dimensional Minkowski metric, given by:

$$\eta = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

With this restriction on our metric $h_{\alpha\beta}$, this metric is called conformally flat. The choice for this restriction is called the conformal gauge and in this gauge (see e.g. [4]):

$$\sqrt{-h}h^{\alpha\beta} = \eta^{\alpha\beta}$$

With this the action can be rewritten in the form:

$$S = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X \partial_\beta X \tag{3}$$

This is the Polyakov form of the action and from this action we can easily derive the equations of motion. By varying the action with respect to X^μ we find the equations of motion:

$$\Box X^\mu = 0$$

By doing this the boundary terms have to set to zero. This can be done in two different ways. First one can impose the so-called Neumann boundary condition:

$$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0$$

Another possibility is that it is imposed that the endpoints of the string are fixed. This can be done by imposing $\delta X_\mu(\tau, 0) = \delta X_\mu(\tau, \pi) = 0$. This implies the Dirichlet boundary conditions:

$$\partial_\tau X^\mu(\tau, 0) = \partial_\tau X^\mu(\tau, \pi) = 0$$

2.2 Constraints

Because our metric is a dynamical variable in our action, we have to get the equations of motion for the metric also. By the variation of our action with respect to the metric we get the equations of motion which actually are constraints on our equation of motions for our coordinates. These constraints are given by[1]:

$$T_{\alpha\beta} \equiv \gamma_{\alpha\beta} - \frac{1}{2}(h \cdot \gamma)h_{\alpha\beta} = 0 \tag{4}$$

Our action (3) is scale invariant. Due to this:

$$\eta^{\alpha\beta}T_{\alpha\beta} = 0$$

From this we have:

$$\begin{aligned} T_{00} &= T_{11} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(x')^2 \\ T_{01} &= T_{10} = \dot{x} \cdot x' = 0 \end{aligned}$$

Summarizing we have as our constraints:

$$(\dot{x} \pm x')^2 = 0$$

[1] For the derivation of these constraints see e.g. [1] or [3]

2.3 Solution

It is possible to solve the equation of motion of the string. A general solution is given by:

$$X^{\mu}(\tau,\sigma) = q^{\mu} + \frac{1}{\pi T}P^{\mu}\tau + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^{\mu} e^{-in\tau}\cos n\sigma \tag{5}$$

Here q^{μ} is the position of the center of mass of the string, P^{μ} the total momentum and ℓ just a parameter with dimension of length. From this solution follows for the constraints:

$$\dot{x} \pm x' = \ell \sum_{n} \alpha_n^{\mu} e^{in(\tau\pm\sigma)}$$

where we defined:

$$\alpha_0^{\mu} = \frac{1}{\pi T \ell}P^{\mu}$$

For convenience later on we can rewrite the constraints:

$$L_n \equiv \frac{1}{2}\sum_{m} \alpha_m^{\mu}\alpha_{n-m.\mu} = 0$$

This last calculation is not trivial but it can be shown[2] by using Fourier analysis that $L_n = 0$ indeed implies that $T_{\alpha\beta} = 0$.

3 Quantisation

Up to here only the classical theory of strings is considered. But a quantum theory is needed, so the results found so far have to be quantised. This quantisation can be done in two different ways, namely by the covariant or the light-cone gauge approach. Each way has its own advantages and disadvantages.

But let's first introduce the quantum operators and their commutators. No calculations will be performed, only the results will be given[3]. First of all we have the equal-time commutation relation between the coordinate and momenta

$$[X^{\mu}(\tau,\sigma), P_{\nu}(\tau,\sigma')] = i\delta^{\mu}_{\nu}\hbar\delta(\sigma-\sigma')$$

There exist also the oscillator operators: $\alpha_n \begin{cases} n<0\text{: creation operator} \\ n>0\text{: annihilation operator} \end{cases}$

Their commutation relations are given by

$$[\alpha_n^{\mu}, \alpha_{\nu m}] = n\delta_{m+n,0}\delta^{\mu}_{\nu}$$

[2] [3] says a little more about this calculation

[3] most of these calculations are written down in [4] or [2]

Note also that

$$\alpha^{\mu}_{-n} = \alpha^{\mu\dagger}_{n}$$

Furthermore, the constraint operators:

$$L_n = \frac{1}{2}\sum_m \alpha^{\mu}_{m}\alpha_{n-m,\mu}$$

$$L_0 = \frac{1}{2\pi\hbar T}P^2 + \sum_{n=1}^{\infty}\alpha^{\mu}_{-n}\alpha_{n\mu} = \frac{\alpha'}{\hbar}P^2 + N^{(\alpha)}$$

with $N^{(\alpha)}$ the level operator:

$$N^{(\alpha)} = \sum_{n=1}^{\infty}\alpha^{\mu}_{-n}\alpha_{\mu n}$$

and α' some constant:

$$\alpha' = \frac{1}{2\pi T}$$

The definition for L_0 is slightly different due to normal ordering problems[4] which would arise when the general definition of L_n for $n = 0$ would be used.
From this follows:

$$[L_n, \alpha^{\mu}_{m}] = -m\alpha^{\mu}_{n+m}$$

The last important commutation relation is between two constraint operators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{26}{12}m(m^2-1)\delta_{m+n,0}$$

The 26 in the last term actually is the dimension of the embedding space in which the string lives. This dimension follows from analysis of the quantised states. See e.g. [1] or [4]

3.1 Covariant Approach

In the covariant approach one imposes the constraints by demanding that for physical states holds:

$$L_n|\phi\rangle = 0 \text{ for } n > 0, \quad (L_0 - 1)|\phi\rangle = 0$$

Once again the slightly modificated constraint for L_0 is due to the normal ordering problems. The value of the constant in the constraint for L_0, which is just 1, is determined by the same analysis as the dimension of the embedding space.

[4] see e.g. [2]

To find the possible states for a certain level in the covariant approach, one has to do four things:

1. write down the most general form of the state
2. determine the constraints
3. determine the dimensions of the contributing states
4. determine the physical null states

The most general state of a level contains contributions consisting of the ground state $|0\rangle$ with creation operators which raise it to the appropriate level. States are just denoted with in the bra or ket the eigenvalue of N, i.e. its level.
The constraints are determined by imposing the conditions for the L_n-operators on the states. The dimension of the states can be calculated by analysing the indices of the state and their symmetries. For further explanation of this calculation see Appendix A.
Physical null states are states which are physical but do not contribute to any measurable property. This is because they have zero inner product with itself as well as with each other physical state. Let's take a look at some examples.

- N=0:

The general form of the level zero state is just $|0\rangle$. For $N = 0$ only the L_0 constraint gives a real constraint because the L_n for $n \geq 1$ automatically annihilates $|0\rangle$. This is always the case for L_n and $|N_0\rangle$ with $n \geq N_0$. This is due to the fact that a state can't have a negative eigenvalue of the N-operator[5].
The L_0 constraint always determines the mass of the state:

$$(L_0 - 1)|0\rangle = 0 \Rightarrow M^2 = -\frac{\hbar}{\alpha'}$$

So this state has a negative mass and thus a momentum satisfying $k^2 > 0$ what is inconsistent with special relativity. This state is called the tachyon and it can be avoided in superstring theory, but not in the bosonic theory.

- N=1:

The general form of the state at level one is:

$$|1\rangle = \xi^{\mu}\alpha_{-1,\mu}|0\rangle$$

Here ξ^{μ} is some polarisation vector. The constraints give:

$$\begin{aligned}(L_0 - 1)|1\rangle =& 0 \Rightarrow M^2 = k^2 = 0\\ L_1|1\rangle =& 0 \Rightarrow k \cdot \xi = 0\end{aligned}$$

[5]see e.g. [4]

For $L_{n\geq 2}$ no new constraints are given. This constraints can be satisfied by 25 different states. But there is one null state. This state is of the form:

$$|\psi_1\rangle = L_{-1}|\chi\rangle$$

This state is orthogonal to all physical states $|\psi_{Ph}\rangle$, by definition of the physical states:

$$\langle\psi_1|\psi_{Ph}\rangle = \langle\chi_1|L_1|\psi_{Ph}\rangle = 0$$

Furthermore it is physical itself:

$$\begin{aligned} L_n|\psi_1\rangle =&0 \;\leftrightarrow\; (2L_0 + L_{-1}L_1)|\chi_1\rangle = 0 \\ (L_0 - 1)|\psi_1\rangle =&0 \;\leftrightarrow\; L_{-1}L_0|\chi\rangle = 0 \end{aligned}$$

Due to this null state the dimension of the state at this level is lowered to 24. This state corresponds to the well known photon. For a more deliberate description of the correspondence between this representation and the massless vector field of the photon, see [1].

- N=2:

The most general form of a level two state is

$$|2\rangle = [\xi^{\alpha\beta}(\alpha_{-1})^2 + \xi^{\alpha}\alpha_{-2}]|0\rangle \tag{6}$$

where $\xi^{\alpha\beta}$ is a symmetric matrix. The total dimension of this state is

$$\frac{d(d+1)}{2} + d = \frac{d(d+3)}{2} = 377 \;\text{ for } d = 26$$

From the constraints we get:

$$\begin{aligned} (L_0 - 1)|2\rangle =&0 \Rightarrow M^2 = \frac{\hbar}{\alpha'} \\ L_1|2\rangle =&0 \Rightarrow \xi^{\mu} = -\sqrt{\frac{2\alpha'}{\hbar}}k_{\rho}\xi^{\mu\rho} \\ L_2|2\rangle =&0 \Rightarrow \frac{1}{2}\xi^{\rho\sigma}\eta_{\rho\sigma} = -\sqrt{\frac{2\alpha'}{\hbar}}k_{\mu}\xi^{\mu} \end{aligned}$$

So this is the first massive level and the $d+1$ constraints from L_1 and L_2 reduce the dimension of this level to $\frac{d(d+1)}{2} - 1$. Because these states are massive they have to correspond to irreducible $SO(d-1)$ states. This is indeed the case. We get the right number of states if we take a massive scalar (1), one massive vector $(d-1)$ and one massive symmetric traceless 2-tensor $(\frac{d(d-1)}{2} - 1)$.

But there can be found two different kinds of physical null states. These are of the form:

$$|\phi_1\rangle = \chi_\mu L_{-1}\alpha^\mu_{-1}|0\rangle$$
$$|\phi_2\rangle = (L_{-2} + \frac{3}{2}L^2_{-1})|0\rangle$$

These correspond to a massive scalar and a massive vector. These two states are physical null states and they do not contribute to any measurable property. Actually these states are pure gauge: if we add a null state to a physical state nothing measurable will be altered. So we are left over with only the massive traceless 2-tensor of dimension $\frac{d(d-1)}{2} - 1 = 324$ for $d = 26$. So this is the only possible state on the second level.

3.2 Light-cone gauge approach

In the light-cone gauge approach first the constraints of the equations of motion are solved. This can be done by simplifying the coordinate system which has been used for the worldsheet. If a coordinate transformation on our action (3) and our constraints (4) is applied, the new coordinates have to satisfy some conditions. These conditions[6] are given by:

$$\partial^2\tilde{\tau} = \partial^2\tilde{\sigma}$$

Here $\tilde{\tau}$ and $\tilde{\sigma}$ are the new coordinates on the string. These conditions are satisfied by the following coordinate transformation:

$$\tilde{\tau} = \frac{n \cdot X}{n \cdot P}\pi T$$
$$\tilde{\sigma} = \frac{\pi}{n \cdot P}\int_0^\sigma d\sigma' n \cdot P^\tau(\tau, \sigma')$$

Let's now introduce first the light-cone coordinates

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^1)$$

and then the light-cone gauge

$$n^\mu = (-1, 1, 0, \ldots, 0)$$

From this follows that

$$\tilde{X}^+ = \frac{P^+}{\pi T}\tilde{\tau} \tag{7}$$

[6]for derivation of this conditions see [1]

Let's drop the tilde everywhere. The constraints can be rewritten with this in the form:

$$\dot{X}^- \pm X^{-\prime} = \frac{\pi T}{2P^+}(\dot{X}^i \pm X^{i\prime})^2$$

Now is it possible to solve for α_n^-. We get:

$$\alpha_n^- = \frac{1}{2}\sqrt{\frac{\hbar}{2\alpha'}}\sum_k \alpha_k^i \alpha_{n-k}^i \tag{8}$$

with

$$\alpha_0^i = \sqrt{\frac{\hbar}{2\alpha'}}P^i$$

With these equations a complete solution is given of the equations of motions and the constraints. By comparing of (7) with (5) we conclude that there are no dynamics in de X^+-coordinate and from (8) it can be concluded that there are no independent vibrational modes in de X^--coordinate[7].

3.3 Quantisation of the Light-Cone gauge

For quantizing in the light-cone gauge approach we use the same operators as in the covariant approach but in the light-cone coordinates. Because there are no constraints which have to be imposed by hand and no physical null states the procedure in the light-cone gauge approach is much simpler. There are only two things to do:

1. write down the most general form of the state
2. determine the dimensions of the contributing states

The most general form of the state is of course the same as in the covariant approach. Determining the dimension of the states is even more simpler because there are no constraints to be taken into account. For the calculation of the dimensions of the different states see Appendix A.
Let's take a look again at the examples. Level zero still corresponds to the tachyon. At level one the general state is:

$$|1\rangle = \xi^\mu \alpha_{-1,\mu}|0\rangle$$

This corresponds in the light-cone gauge to 24 dimensional representation. For level two we have:

$$|2\rangle = [\xi^{\alpha\beta}(\alpha_{-1})^2 + \xi^\alpha \alpha_{-2}]|0\rangle$$

with dimension

$$\frac{(d-2)(d-1)}{2} + d - 2 = \frac{d(d-1)}{2} = 324 \text{ for } d = 26$$

[7] For a more deliberate derivation of this solution and the use and meaning of the light-cone gauge and coordinates see [4] or [1]

These are exactly the dimensions given by the covariant approach. But in the light-cone gauge approach the physical interpretation of the states is not obvious. You can get the right total dimension but you don't know how to interpret them. Only by comparing the two different approaches you get the full amount of information.

3.4 Comparison of the two approaches

So let's take a look at the (dis)advantages of the two different approaches:

Light-Cone Gauge:

- gives the right number of states (relative easy to get)
- the physical meaning of the representations is not clear

Covariant Approach:

- it is not easy to get the right number of states, due to the null-states
- physical meaning is clear, after a lot of work

In combining the advantages of the two you can try to get a strategy which gives you the same amount of information with less effort. In this approach there are four steps:

- write down general state
- determine dimensions of the states (both in the light-cone and in the covariant approach)
- determine total dimension of the light-cone gauge states
- find appropriate covariant representations which gives the same total dimension

Let's take a look again on the second level. The general form of this state is given by (6):

$$|2\rangle = [\xi^{\alpha\beta}(\alpha_{-1})^2 + \xi^{\alpha}\alpha_{-2}]|0\rangle$$

In table 1 the dimensions of the different representations are written down. Here we see that the total dimension of the light-cone gauge is 324 and that the only way to get 324 as the total of dimension of the covariant representations also, is to take the traceless 2-tensor with dimension 324. This means that the states represented by the other representations, the massless vector and scalar, have to be null states. So, without determining the physical null states explicitly, it is possible to find out which states are allowed in the second level. Just because the only way to get the right total dimension is allowing only the symmetric traceless 2-tensor.

4 Conjectures and their verifications

On the basis of the results found in the section before, a first conjecture was made:

1) By combining the two different approaches it is possible to get the same amount of information as in the covariant approach only by comparing the dimensions of the different representations. That means: The combination of covariant representations which combines to the same amount as the total dimension of the representations in the light-cone gauge is unique and therefore has to be the combination of physical states at a certain level.

We can check this conjecture for some of the lower massive levels. In table 1 are the results given for the first three massive levels.

Level	States	Representation	Covariant	LCG
$N=2$	$(\alpha_{-1})^2$	$\square\square$	**324**	300
	α_{-2}	$\square$	25	24
	scalar	$\bullet$	1	
			324	324
$N=3$	$(\alpha_{-1})^3$	$\square\square\square$	**2900**	2600
	$\alpha_{-2}\alpha_{-1}$	$\square\square$	324	300
		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	**300**	276
	α_{-3}	$\square$	25	24
	scalar	$\bullet$	1	
			3200	3200
$N=4$	$(\alpha_{-1})^4$	$\square\square\square\square$	20150	17550
	$\alpha_{-2}(\alpha_{-1})^2$	$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$	5175	4600
		$\square\square\square$	2900	2600
	$(\alpha_{-2})^2$	$\square\square$	324	300
	$\alpha_{-3}\alpha_{-1}$	$\square\square$	324	300
		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	300	276
	α_{-4}	$\square$	25	24
	scalar	$\bullet$	1	
				25650

Table 1. Results of counting on the dimensions of representations of the two different approaches, the covariant and the light-cone gauge (LCG) approach. The bold numbers in the covariant column sum op to the same total amount as the whole LCG column. For the meaning of de Young-tableaus in the representation collumn and the calculation of the dimensions see Appendix A. The results for level 5 and 6 are given in Appendix B.

Here we can see that for the first two levels the approach succeeds in finding a unique set of states. For level $N=4$ however, there is no unique solution due to two facts. First there are two ways to sum op the dimensions of the covariant representations to the total amount of 25650, namely:

$$\begin{array}{ccccccc} \square\square\square\square & + & \begin{array}{l}\square\square\\ \square\end{array} & + & \square\square & + & \bullet \\ 20150 & & 5175 & & 324 & & 1 \end{array} \tag{9}$$

or

$$\begin{array}{ccccccc} \square\square\square\square & + & \begin{array}{l}\square\square\\ \square\end{array} & + & \begin{array}{c}\square\\ \square\end{array} & + & \square \\ 20150 & & 5175 & & 300 & & 25 \end{array} \tag{10}$$

Apart from that, in the first possibility a $\square\square$ representation is used but there are two of them. So its is not possible to determine which one would be the null state and which one the one contributing to the physical states at this level. Or maybe is the physical state even a combination of these two states. Both these two problems take the edge of our conjecture. So it proves not to be possible to get the same amount of information just by counting the dimensions of the representations of the two different approaches.

Two more conjectures were made:

2) It is possible to decompose the contributing physical states of the covariant representation in irreducible traceless $SO(24)$-representations got by decomposing the light-cone gauge representations.

3) This decomposability is a criterion on the basis of which it is possible to make distinction between two possible combinations of states which both were found with the method described in conjecture **1)**

Take for example level four. First let's decompose the light-cone gauge representations in irreducible traceless $SO(24)$-representations:

$$\underset{17550}{\square\square\square\square} = \underset{17250}{\square\square\square\square} + \underset{299}{\square\square} + \underset{1}{\bullet}$$
$$\underset{4600}{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} = \underset{4576}{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} + \underset{24}{\square}$$
$$\underset{2600}{\square\square\square} = \underset{2576}{\square\square\square} + \underset{24}{\square}$$
$$\underset{300}{\square\square} = \underset{299}{\square\square} + \underset{1}{\bullet}$$
$$\underset{300}{\square\square} = \underset{299}{\square\square} + \underset{1}{\bullet}$$
$$\underset{276}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \underset{276}{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$
$$\underset{24}{\square} = \underset{24}{\square}$$
$$\underset{1}{\bullet} = \underset{1}{\bullet}$$

So, these representations have been made irreducible by taking apart all the traces they contained. Now it can be tried to decompose the found combination of representations of the covariant approach in the same irreducible traceless $SO(24)$-representations. This turns out to be possible. Let's take the first combination (9):

$$\underset{20150}{\square\square\square\square} = \underset{17250}{\square\square\square\square} + \underset{2576}{\square\square\square} + \underset{299}{\square\square} + \underset{24}{\square} + \underset{1}{\bullet}$$
$$\underset{5175}{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} = \underset{4575}{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} + \underset{276}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \underset{24}{\square} + \underset{299}{\square\square} + \underset{1}{\bullet}$$
$$\underset{324}{\square\square} = \underset{299}{\square\square} + \underset{24}{\square} + \underset{1}{\bullet}$$
$$\underset{1}{\bullet} = \underset{1}{\bullet}$$

Here exactly the same $SO(24)$ representations have been used as above. But if we take a look at the second combinations (10), and compare it with the analysis of the first combination, it seems not to be possible to decompose this combination in the same way. Only the two smallest representations of the second combination differ from that of the first combination, but by trying to decompose this combination in the same way, would give us:

$$\underset{300}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \underset{299}{\square\square} + \underset{1}{\bullet}$$
$$\underset{25}{\square} = \underset{24}{\square} + \underset{1}{\bullet}$$

And this seems not to be correct, because $\square\square$ is not likely to be decomposed in $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. Trying to do it in an other way, would not work either, because then there have to be used other representations than that which are given by the decomposition of the covariant representations:

$$\begin{array}{ccccc} \begin{smallmatrix}\square\\\square\end{smallmatrix} & = & \begin{smallmatrix}\square\\\square\end{smallmatrix} & + & \square \\ 300 & & 276 & & 24 \\ \square & = & \square & + & \bullet \\ 25 & & 24 & & 1 \end{array}$$

In this case there are, for example, two $\begin{smallmatrix}\square\\\square\end{smallmatrix}$'s used but in the decomposition of the covariant representations there was only one of this representation.
Conjectures **2)** and **3)** can be checked for the other massive levels in the same way. By checking this for levels $N = 2$ until $N = 6$ both the conjectures are ratified. This means that for level $2, \ldots, 6$ it is proved to be possible, when the method of conjecture **1)** gives just one possible combination, to decompose the representations of this combination in irreducible traceless $SO(24)$ representations. When there is not found a unique combination (this is the case at level 4 and 6, in both cases there are two possible combinations, see appendix B) one of these combinations can be decomposed into the $SO(24)$-representations and the other can not. So it distinguishes the two found combinations.
So, for the first five massive states it is possible to either find just one unique combination or, if there are two different combinations, to make a distinction between them. This fact was the reason for another conjecture:

4) The physical states are given by the combination of states of which the representations can be decomposed into irreducible traceless $SO(24)$ representations.

This conjecture is checked and confirmed up to level 4 for which the physical states were know from [2]. For higher levels it is neither checked nor can be given a proof of this statement here.
If this conjecture is true some statements can be made about which states will contribute at a certain level. If conjecture **4)** is true they seem to have be true also, but they can be true also if conjecture **4)** is not. So they can be stated as independent conjectures:

5) At level $N = n$ states of the form $(\alpha_{-1})^n$ as well as states of the form $(\alpha_{-1})^{n-2}\alpha_{-2}$ have to be always among the physical states.

6) In general: states with representations which have 'new features' have to be among the physical states.

The 'new features' of the representations are features of representations which appear for the first time at the lists of the covariant or the $SO(24)$-representations at a certain level. The lower levels do not contain representations with this feature. One can think of features as the first representation with k indices (as in conjecture **5)**) or the first representations with m asymmetrical indices.

These conjectures are based on the fact that if conjecture **4)** is true it has to be the case that each irreducible traceless $SO(24)$-representations has to be used as a composite of the covariant representations. So, if there is in the list of the $SO(24)$-representations a representation of the form $\ydiagram{2,2}$ there has to be a representation in the covariant approach in which $\ydiagram{2,2}$ is 'contained'. If it is the first time that the $\ydiagram{2,2}$ appears in the list with the $SO(24)$-representations, the only representation in which it is contained in the covariant list is also of the form $\ydiagram{2,2}$. This is of course due to the fact that each other representations which contains it, would have more indices, like

$$\ydiagram{2,2,1} \text{ or } \ydiagram{3,2}.$$

These conjectures have not been proved but only confirmed for the first five massive levels. See also Appendix B.
It is been tried to perform the analysis on level seven also but because of some uncertainties about how to decompose certain representations, no conclusion are based on that analysis. Actually, this analysis was a lot of work and it is easy to make errors. It is not clear what has to be concluded from this analysis but it shows that also this approach is not without disadvantages. The analysis for higher levels become quite a lot of work.

5 Conclusion

It can be concluded that it is not possible to get the full amount of information about which states can exist at a certain level just by looking for a unique combination of covariant representations which dimension sums op to the same amount as the total dimension of the light-cone gauge representations. But, at least for the first five massive states, you can get either a unique combination or you can make a distinction between the two alternatives by looking at the decomposability of the representations in irreducible traceless $SO(24)$-representations. It is not proved that this is possible for all states nor it is proved that the combination with the decomposable states is the one with the physical states at a certain level. But if this would be true some more predictions can be made about which states do contribute to the physical states. These predictions follow from the fact that in that case you have to use all the $SO(24)$-representations, so you have to look for covariant representations in which they (and especially the one with features which no representation had on a lower level) are 'contained'. Again these statements are not proved but only ratified for the five lowest massive levels.
So, maybe it is possible to short-cut the tough calculation on the constraints and the physical null states by just looking at the dimensions of the representations of the states. But this is not sure yet because none of the necessary proofs can be given.

Apart from that, one can wonder if this new method is much easier to perform and more transparent than just performing the covariant approach with using the light-cone gauge approach to determine the number of possible states. Anyway, if one wants to analyse the massive states of bosonic strings it would be helpful to have two different approaches to get to the same answer. But before it can be used the conjectures stated above have to be proved.

Appendices

A Young-tableaus and the dimensions of the different representations

To calculate the dimensions of representations it is convenient to make use of Young-tableaus. Young-tableaus of states are generated in the following way:

- for each index the Young-tableau has one $\square$
- for symmetric indices the $\square$'s are placed on the same horizontal line
- for asymmetric indices the $\square$'s are placed beneath each other
- indices of the same kind of creation operators are always symmetric
- indices on different kind of creation operators can be symmetric as well as asymmetric
- the length of the column i, with $i < j$, must be bigger then or equal to the length of column j
- the length of the row i, with $i < j$, must be bigger then or equal to the length of row j

Let us take for example a level four state of the form

$$\xi^{\mu\nu\rho}\alpha_{\mu,-2}\alpha_{\nu,-1}\alpha_{\rho,-1}|0\rangle$$

This level give rise to the following representations:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array}$$

The dimensions of such representations is easily calculated using these tableaus. This can be done in the following way:

- put in each $\square$ a number due to the following rules:
  - put in the upper-left $\square$ the dimension in which you are working
  - for each $\square$ you go to the right you raise the number with one
  - for each $\square$ you go to beneath you lower the number with one
- multiply all these numbers and divide it by another number which you get by:
  - put in each $\square$ the sum of the $\square$'s which are to the right of it and the $\square$'s which are beneath it plus one (for itself)

  – multiply these numbers to get the number by which you have to divide

So for the representations of the level four state this gives:

$$\square\square\square \Rightarrow \frac{D(D+1)(D+2)}{3\cdot 2\cdot 1}$$

$$\begin{array}{ll}\square&\square\\\square&\end{array} \Rightarrow \frac{D(D+1)(D-1)}{3\cdot 1\cdot 1}$$

The bosonic stringtheory is using a 26-dimensional space-time as the embedding space. In the different approach for quantisations each index counts for a different amount. In the light-cone gauge approach each index counts for $d-2=24$. This is because of the fact that in two coordinates there are no (independent) osscilations.
In the covariant approach the constraints determine the dimensions of the states. It proves to be that all the states in the covariant approach can be represented by the irreducible representations of the $SO(25)$ group. For this reason the indices counts in the covariant approach for $d-1=25$. But the representations have to be made traceless by hand. This can be done by subtracting the dimension of the Young-tableaus of the state with two horizontal $\square$'s less. These correspond of course to two symmetric indices which are contracted. So, this gives:

$$\square\square\square \Rightarrow \begin{cases} \text{Light-cone: } \dfrac{(d-2)(d-1)d}{6} = 2600 \\ \text{Covariant: } \dfrac{(d-1)d(d+1)}{6} - (d-1) = 2900 \end{cases}$$

$$\begin{array}{ll}\square&\square\\\square&\end{array} \Rightarrow \begin{cases} \text{Light-cone: } \dfrac{(d-2)(d-1)(d-3)}{3} = 4600 \\ \text{Covariant: } \dfrac{(d-1)d(d-2)}{3} - (d-1) = 5175 \end{cases}$$

B Results for the massive levels 5 and 6

Level	States	Representation	Covariant	LCG	Decom-posable	Not-Dec.
$N = 5$	$(\alpha_{-1})^5$	□□□□□	115830	98280	115830	
	$\alpha_{-2}(\alpha-1)^3$	□□□□	20150	17550		
		□□□ □	52350	44850	52350	
	$\alpha_{-3}(\alpha_{-1})^2$	□□□	2900	2600		
		□□ □	5175	4600	5175	
	$\alpha_{-4}\alpha_{-1}$	□□	324	300		
	α_{-5}	□	25	24		
	scalar	•	1		1	
	$(\alpha_{-2})^2\alpha_{-1}$	□□□	2900	2600	2900	
		□□ □	5175	4600		
	$\alpha_{-3}\alpha_{-2}$	□□	324	300		
		□ □	300	276		
				176256	176256	
$N = 6$	$(\alpha_{-1})^6$	□□□□□□	573300	475020	573300	573300
	$\alpha_{-2}(\alpha_{-1})^4$	□□□□□	115830	98280		
		□□□□ □	387920	322920	387920	387920
	$\alpha_{-3}(\alpha-1)^3$	□□□□	20150	17550		
		□□□ □	52350	44850		52350
	$\alpha_{-4}(\alpha_{-1})^2$	□□□	2900	2600	2900	2900
		□□ □	5175	4600	5175	5175
	$\alpha_{-5}\alpha_{-1}$	□□	324	300	324	
		□ □	300	276		300
	α_{-6}	□	25	24	25	25
	scalar	•	1		1	
	$(\alpha_{-2})^2(\alpha_{-1})^2$	□□□□	20150	17550	20150	
		□□ □□	32175	27600	32175	
	$(\alpha_{-2})^3$	□□□	2900	2600		
	$\alpha_{-3}\alpha_{-2}\alpha_{-1}$	□□□	2900	2600		
		□□ □	5175	4600		
		□□ □	2300	2024	2300	2300
	$(\alpha_{-3})^2$	□□	324	300		
	$\alpha_{-4}\alpha_{-2}$	□□	324	300		
		□ □	300	276		
				1024270	1024270	1024270

In this table the results of the method described in conjecture **1)** are given for $N = 5, 6$. For level five there is one unique combination which gives the right total amount. For level six there are two different combinations possible. For one of these combinations all the states are decomposable in $SO(24)$-representations but for the other that's not possible. That this is not possible can easily be seen, because the second combination is not containing the $\boxplus$ representation or something which it contains (see conjecture **5)** and **6)** and the explanation about them).

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