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On moduli of hyperelliptic curves of genus two

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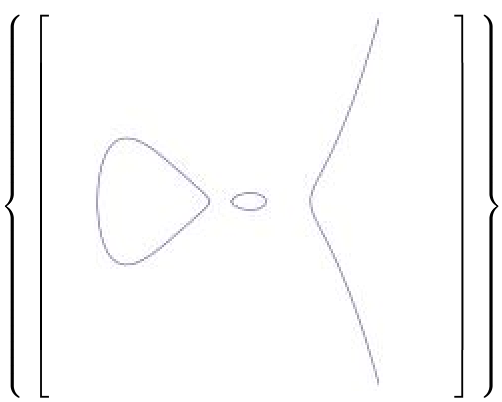

Hyperelliptic curves$_{/\mathbb{R}}$

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Abstract

Hyperelliptic curves of genus g are curves of genus $g \geq 2$ for which the canonical map to $\mathbb{P}^{g-1}$ is not an isomorphism onto its image. We state three equivalent definitions of a hyperelliptic curve of given genus, study some of its properties and determine one of the possible forms of its equation. This gives an explicit notion of isomorphic hyperelliptic curves of given genus. We obtain a description of the set, $\mathcal{M}_2$, consisting of isomorphism classes of hyperelliptic curves of genus two, parameterized in terms of invariants of the space of binary sextics. We prove that $\mathcal{M}_2$ gives rise to the coarse variety of moduli, M_2, of hyperelliptic curves of genus two. Finally, we state the so-called *Igusa invariants* in terms of our invariants and use our explicit description of M_2 to prove that it contains an unique singularity which we will describe in detail.

Contents

Chapter 1

Introduction

The moduli space of hyperelliptic curves

Hyperelliptic curves of genus g are curves of genus $g \geq 2$ for which the canonical map to $\mathbb{P}^{g-1}$ is not an isomorphism onto its image. Obtaining a mathematical description of the 'same' hyperelliptic curves of given genus suggests to classify all hyperelliptic curves of given genus. The classifying space is called the moduli space, say M, of hyperelliptic curves of given genus. That is, M is a space (topological, manifold, variety, scheme, stack) such that each point of M corresponds with an isomorphism class of hyperelliptic curves of given genus. Equivalently, M parameterizes all hyperelliptic curves of given genus. Notice, an irreducible, non-singular curve of genus g and a connected, compact Riemann surface of genus g can be considered as the 'same' objects. Hence, a moduli space of curves of genus g classifies all possible complex structures on a Riemann surface of genus g.

Problem statement, motivation and main results

We want to construct the moduli space of hyperelliptic curves of genus two. Therefore, we have to describe hyperelliptic curves of genus two in such a manner that we have a notion of isomorphic hyperelliptic curves of genus two. Once obtained such a description, we have to construct the set, say $\mathcal{M}_2$, of isomorphism classes of hyperelliptic curves of genus two. Then we have to put the structure of an algebraic space on $\mathcal{M}_2$ such that it gives rise to the moduli space, say M_2, of hyperelliptic curves of genus two. Finally, we study singularities of M_2.

Our goal is to work as elementary as possible. We will make the theory accessible to all graduate students within mathematics with a minimality of background knowledge of Algebraic Geometry and Riemann surfaces. This contrasts with the current, standard, literature on moduli spaces. Following our approach, the reader becomes familiar with the abstract notions, techniques and concepts which are standard in moduli theory. Furthermore, after studying our work the reader is prepared to study, and make contributions to, moduli theory.

We obtained a bijective correspondence $\mathcal{M}_2 \leftrightarrow \mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ where $\mathbb{Z}/5\mathbb{Z}$ acts on $\mathbb{A}^3(\mathbb{C})$ by $(x, y, z) \mapsto (\zeta x, \zeta^2 y, \zeta^3 z)$. Here ζ is a fifth root of unity. The set $\mathcal{M}_2$ gives rise to the coarse variety of moduli, M_2, of hyperelliptic curves of genus two which is isomorphic to $\operatorname{Spec} \mathbb{C}[y_1^5 y_5^{-1}, y_2^5 y_5^{-2}, y_3^5 y_5^{-3}]$ where the y_i's are independent variables of degree i. We proved that M_2 contains an unique singularity which corresponds to the isomorphism class of the hyperelliptic curve C of genus two given by $C : y^2 = x^6 - x$. We obtained this result using the failure of the *Implicit Function Theorem*. This approach is not usual but provides a lot of insight. Furthermore, we state the so-called *Igusa invariants* in terms of our invariants. Using these invariants, one can construct and study the moduli space of hyperelliptic curves of genus two in which the curves are taken over an arbitrary field. Even over fields of characteristic two.

Related work

There exists a rich literature about the variety of moduli of algebraic objects. Since the variety of moduli of general algebraic objects is very hard to construct directly, most of the literature is devoted to the variety of moduli of curves (of low genus). In general, we refer to D. Mumford, J. Fogarty and F. Kirwan [16] which gives criteria for the existence of varieties of moduli. Also the book of J. Harris and I. Morrison [1] is a good reference for general theory of curves and their moduli. Probably the most cited work in case of hyperelliptic curves of genus two over fields of arbitrary characteristic is the paper of Jun-Ichi Igusa [13].

Overview

While the distinction between theorems, lemmas, propositions and corollaries is purely subjective, they have a different system of numbering. Also definitions have a separate system of numbering. However, it will be clear from the context which results - and how these results - are related to each other.
We start introducing hyperelliptic curves of given genus in Chapter 2. In Section 2.1 we state three equivalent definitions of hyperelliptic curves of given genus and in Section 2.2 we deduce properties of them which are necessary but sufficient to construct the variety of moduli of hyperelliptic curves of genus two. In Chapter 3 we study sets of isomorphism classes of curves of genus $0, 1$ and 2.
In Chapter 4 we construct the coarse variety of moduli of hyperelliptic curves of genus two. In Section 4.1 we state a formal definition of moduli spaces. In addition we give some general properties of moduli spaces. In Section 4.2 we obtain the coarse variety of moduli of curves of genus $0, 1$ and 2.
In Chapter 5 we study the singularities of the variety of moduli of hyperelliptic curves of genus two.
In Chapter 8 (the Appendix) we give a brief introduction on the theory of schemes from the point of view of moduli of hyperelliptic curves of given genus. Furthermore, we introduce the concept of a stack in the case of elliptic curves and give an explicit description of the classical invariants of binary sextics.

Prerequisites

The prerequisites consist of a first course on Riemann surfaces and a first course on Algebraic Geometry. For the former we refer to Otto Forster [9] and for the latter we refer to Robin Hartshorne [11]. As our approach to the subject will be as elementary as possible we do not require knowledge of Category Theory. If necessary, we will provide background knowledge on the spot in the particular case of curves. Similarly, we provide some background knowledge of *Invariant Theory* on the spot. Throughout our thesis we use frequently, even without mentioning, the well known fact that a connected, compact Riemann surface of genus g can be considered as an irreducible, non-singular curve of genus g. Conversely, any irreducible, non-singular curve of genus g can be considered as a connected, compact Riemann surface of genus g.

Chapter 2

The objects of study

In this chapter we introduce hyperelliptic curves of given genus over the complex number field as in Hartshorne [11], but slightly more elementary. Having a definition of hyperelliptic curves we deduce their main properties which are necessary but sufficient to construct and study the variety of moduli of hyperelliptic curves of genus two.

2.1 Defining hyperelliptic curves

Unless otherwise stated, all our work will be done over the complex number field and the genus will be in $\mathbb{Z}_{\geq 2}$. By curve we mean an irreducible, non-singular curve over $\mathbb{C}$. During our thesis we write $\mathcal{M}(C) := \{f \mid f$ meromorphic function on $C\}$ and $\Omega(C) := H^0(C, \Omega_C) := \{\omega \mid \omega$ holomorphic 1-form on $C\}$ where C is a curve of genus g. While not using cohomology theory we emphasize that this, seemingly highbrow, notation directly implies that all our work can be done coordinate free. Since C can be considered as a connected, compact Riemann surface of genus g it follows that $\mathcal{M}(C)$ is, in our case, a $\mathbb{C}$-algebra and $H^0(C, \Omega_C)$ is, in our case, a $\mathbb{C}$-vector space. It follows from the *Theorem of Riemann-Roch* that $\dim_{\mathbb{C}} H^0(C, \Omega_C) = g$ (See Otto Forster [9] remark 17.10). This suggests that the genus of a curve C can be defined as the dimension of the space of holomorphic 1-forms on C which we will call the *geometric genus*. It follows from *Serre duality* that the dimension of $H^0(C, \Omega_C)$ equals the dimension of the first cohomology group of the structure sheaf, $\mathcal{O}_C$, on C, i.e., $H^0(C, \Omega_C) = H^1(C, \mathcal{O}_C)$. This means that we can define the genus of C to be the dimension of $H^1(C, \mathcal{O}_C)$ which we will call the *arithmetic genus*. Moreover, the genus of a Riemann surface can also be defined as the number of holes of the Riemann surface which we will call the *topological genus*. Since we work over the complex number field, all these definitions of the genus of a curve are equivalent.

Definition 1 (Canonical map). *Let C be a curve of genus $g \geq 2$ and let $(\omega_1, \ldots, \omega_g)$ be a basis for $H^0(C, \Omega_C)$. The map $\varphi_K : C \to \mathbb{P}^{g-1}(\mathbb{C})$ given by $p \mapsto [\omega_1(p) : \ldots : \omega_g(p)]$ is called the canonical map of C.*

Notice, a holomorphic 1-form ω on C at a point p of C with coordinate chart (U, ψ) can be written as $\omega(p) = f(\psi(p))d\psi$ where f is a holomorphic function on C. Hence, we have to consider the image of p under the canonical map as $[f_1(\psi(p)) : \ldots : f_g(\psi(p))]$ where the f_i's are holomorphic functions on C. Since the $\mathbb{C}$-vector space of holomorphic 1-forms on C is represented by the zeroth cohomology group of C over the sheaf Ω_C of holomorphic 1-forms it follows that it is coordinate free. Therefore, we will still write $[\omega_1(p) : \ldots : \omega_g(p)]$ instead of $[f_1(\psi(p)) : \ldots : f_g(\psi(p))]$. Furthermore, let $\mathrm{Div}(C) \ni D = \sum_{p \in C} m_p(p)$ where $m_p \in \mathbb{Z}$ for all $p \in C$, i.e., D is a *Weil-Divisor*. We define $\mathcal{L}(D)$ to be the set of meromorphic functions $f \in \mathcal{M}(C)^*$ such that, for a point q from the divisor D, if (i) $m_q < 0$ then f must have a zero of multiplicity $\geq |m_q|$ at q, i.e., $m_q < 0$ implies $\mathrm{ord}_q(f) \geq -m_q$ which implies that f must have a zero of order $\geq |m_q|$ at q. Similarly, if (ii) $m_q \geq 0$ then f can have a pole of multiplicity $\leq m_q$. Hence, $\mathcal{L}(D) = \{f \in \mathcal{C}^* \mid \operatorname{div} f + D \geq 0\} \cup \{0\}$. Furthermore, one proves that $\mathcal{L}(D)$, in our case, is a $\mathbb{C}$-vector space. Moreover, define $\Omega^1(D) := \{\omega \in \mathcal{M}^1(C) \quad \operatorname{div} \omega \geq D\}$ where $\mathcal{M}^1(C) := \{\omega \mid \omega \text{ meromorphic 1-form on } C\}$. That is, if $\omega \in \Omega^1(D)$ and if (i) $m_p < 0$, then ω can have a pole at p of multiplicity $\leq |m_p|$ and (ii) if $m_p > 0$, then ω must have a zero at p of multiplicity $\geq m_p$. Furthermore, if $D \geq 0$ then $\omega \in \Omega(D)$ means that ω is a holomorphic differential and has a zero at p of multiplicity $\geq m_p$ and $\Omega(D)$ is a linear subspace of $\Omega(C)$. We substitute, frequently, the notion of $\Omega(D)$ for $\Omega^1(D)$.

Lemma 1. *Let C be a curve of genus g. Then the canonical map of C is well-defined.*

Proof. The map φ_K is well-defined iff there exists an index $i \in \{1, \ldots, g\}$ such that $\omega_i(p) \neq 0$ for all points p on C where $(\omega_1, \ldots, \omega_g)$ is a basis of $\Omega(C)$. From the defintion of $\Omega(*)$ it follows that $\Omega(p) \subset \Omega(C)$. Furthermore, $\omega \in \Omega(C)$ implies that $\omega = \mathbb{C}\omega_1 \oplus \ldots \oplus \mathbb{C}\omega_g$. Then $\omega(p) = \mathbb{C}\omega_1(p) + \ldots \mathbb{C}\omega_g(p) = 0$; so $\Omega(p) \supset \Omega(C)$. Hence, $\Omega(p) = \Omega(C)$ and, hence, the *index of specialty* of p, $i(p)$, yields

$$\begin{aligned} i(p) &:= \dim_{\mathbb{C}} \Omega^1(p) \quad \text{(See O. Forster [9])} \\ &= \dim_{\mathbb{C}} \Omega(p) \\ &= \dim_{\mathbb{C}} \Omega(C) \\ &= g. \end{aligned}$$

From the *Theorem of Riemann-Roch* it follows that

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{L}(p) &= \deg p - g + 1 + i(p) \\ &= 1 - g + 1 + g \\ &= 2. \end{aligned}$$

Hence, $\mathcal{L}(p) = \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot f$ where f is a non-constant, meromorphic function such that $f^{-1}(\infty) = p$, i.e., $\deg f = 1$. Since C is a connected, compact Riemann surface and $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$ a compact Riemann surface, i.e., the Riemann sphere, it follows that the non-constant, meromorphic map $f : C \to \mathbb{P}^1$ is surjective (See O. Forster [9], Theorem 9). Hence, $C \cong \mathbb{P}^1$ and, hence, using the *Riemann-Hurwitz Formula*, $g = \text{genus}(C) = \text{genus}(\mathbb{P}^1) = 0$. Which is a contradiction because we supposed that $g \geq 2$. □

Theorem 1. *Let C be a curve of genus g and let the canonical map of C be not injective. Then there exists a holomorphic double covering map $C \to \mathbb{P}^1$.*

Proof. Since φ_K is not injective it follows that there exist points $p, q \in C$ such that $p \neq q$ and $\omega_i(p) = \lambda\omega_i(q)$ for all $i \in \{1, \ldots, g\}$ and for all $\lambda \in \mathbb{C}^*$. Define $\text{Div}(C) \ni D := p + q$ and let $\omega_1, \ldots, \omega_g$ be a basis for $\Omega(C)$. If $\omega \in \Omega(C)$, then $\omega = \lambda_1\omega_1 + \ldots + \lambda_g\omega_g$ where $\lambda_j \in \mathbb{C}$ for $j = 1, \ldots, g$. Therefore, $\omega(p) = 0$ iff $\lambda_1\omega_1(p) + \ldots \lambda_g\omega_g(p) = 0$ iff $\lambda_1(\lambda\omega_1(p)) + \ldots \lambda_g(\lambda\omega_g(p)) = 0$ iff $\omega(q) = 0$ which implies that $\Omega(p) = \Omega(p+q) = \Omega(D)$ (See also our proof of Lemma 1). Consider a linear combination $\lambda_1'\omega_1(p) + \ldots + \lambda_g'\omega_g(p) = 0$ in which we treat $\lambda_1', \ldots, \lambda_g'$ as being its variables. It than follows from Lemma 1 that this linear combination is linearly independent.Hence, there exist $(g-1)$ linear independent solutions $(\lambda_1', \ldots, \lambda_g') \in \mathbb{C}^g$ to the linear combination. Since $\Omega(p)$ is the $\mathbb{C}$-linear space which consists of elements as given as the linear combination it follows that $\Omega(p)$ consists of $(g-1)$ $\mathbb{C}$-linear independent elements. Hence, $i(p) = g - 1$. Since we computed that $\Omega(p) = \Omega(D)$ it follows that $g - 1 = i(p) = i(D)$ and, hence, from the *Theorem of Riemann-Roch* it follows that $\mathcal{L}(D) = \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot f$ where $f \in \mathcal{M}(C)$ non-constant. Considering C as a connected, compact Riemann surface of genus g implies that $f : C \to \mathbb{P}^1$ is a holomorphic map of degree 2. Notice, if $f^{-1}(\infty) = p$ (or q) we obtain $C \cong \mathbb{P}^1$ which contradicts the assumption $g \geq 2$. Hence, $f : C \to \mathbb{P}^1$ is a holomorphic double covering map. □

Theorem 1 suggests naturally the following definition.

Definition 2 (Hyperelliptic curve)**.** *Let C be a curve of genus g. We call C a hyperelliptic curve of genus g if the canonical map is not-injective and we call C a non-hyperelliptic curve of genus g if the canonical map is injective.*

However, the following definition of hyperelliptic curves of genus ≥ 2 is often used, e.g., by O. Forster [9] and R. Hartshorne [11].

Definition 3 (Hyperelliptic curve)**.** *Let C be a curve of genus g. The curve C is a hyperelliptic curve of genus g if there exists a holomorphic double covering map $C \to \mathbb{P}^1$ and C is a non-hyperelliptic curve of genus g if there does not exist a holomorphic double covering map $C \to \mathbb{P}^1$.*

Corollary 1. *Definition 2 and Definition 3 are equivalent.*

Proof. This is an immediate consequence of Theorem 1. □

As a consequence of Theorem 1, it is immediately clear that the following definition of a hyperelliptic curves of given genus is equivalent to Definition 2.

Definition 4 (Hyperelliptic curve)**.** *Let C be a curve of genus g. We call C a hyperelliptic curve of genus g if the canonical map is not an isomorphism onto its image and call C a non-hyperelliptic curve of genus g if the canonical map is an isomorphism onto its image, i.e., if the canonical map is an embedding.*

Throughout our thesis we will use the three obtained - equivalent - definitions of hyperelliptic and non-hyperelliptic curves interchangeably depending on which is the most suitable. Having a formal definition of hyperelliptic curves, objects for which we will study the problem of moduli, we deduce in the following section some of the major properties of hyperelliptic curves which are necessary but sufficient to construct and study the variety of moduli of hyperelliptic curves of genus two.

2.2 Properties of hyperelliptic curves

In this section we state some properties of hyperelliptic curves of given genus. Much more properties of them are stated by H.M. Farkas and I. Kra [8]. It is well-known that one has a group law on the rational points of an elliptic curve, say E. This is not so clear in the case of a hyperelliptic curve, say C, of genus g. Let $C(\mathbb{C})$ be the $\mathbb{C}$-rational points of C. One can define a group law on $C(\mathbb{C})$ using the theory of Jacobians, i.e., the group of divisors on C of degree 0 modulo rational equivalence. Since we work over the complex numbers it will turn out that the Jacobian variety is simply a complex torus $\mathbb{C}^g/\Lambda$ where $\Lambda \subset \mathbb{C}^g$ is a lattice. The idea is to transfer the abelian group law of $\mathbb{C}^g/\Lambda$ onto $C(\mathbb{C})$. For an extensive overview of this subject we refer to C. Birkenhake and H. Lange [3]. For us - to construct and study the moduli space of hyperelliptic curves of genus two - it is sufficient to know that a hyperelliptic curve is a double cover of the Riemann sphere. Moreover, we will prove that a hyperelliptic curve of genus g is a double cover of the Riemann sphere ramified at six mutually different points of the Riemann sphere.

Corollary 2. *Let C be a curve of genus 2. Then C is a hyperelliptic curve of genus 2.*

Proof. Suppose C is non-hyperelliptic. Then $\varphi_K : C \to \mathbb{P}^1$ is an isomorphism. Hence, genus(C) = genus$(\mathbb{P}^1)$ which is a contradiction. Therefore, C is a hyperelliptic curve of genus two. □

Theorem 2. *Let C be a curve of genus g. The curve C is a hyperelliptic curve of genus g iff $C:\ y^2 = \prod_{i=1}^{2g+2}(x-\mu_i)$ with x, y coordinate functions on $\mathbb{A}^2(\mathbb{C})$ and $\mu_i \neq \mu_j$ for all $i \neq j$.*

Proof. There exists a holomorphic double covering map $x : C \to \mathbb{P}^1$ such that $x^{-1}(\infty) = p + q$ where $p, q \in C$ and $p \neq q$. Let $\mathrm{Div}(C) \ni D := p + q$. Obviously, $\dim_{\mathbb{C}} \mathcal{L}(D) \geq 2$ and, hence, $1, x \in \mathcal{L}(D)$. The $\mathbb{C}$-vector space $\mathcal{L}((g+1)D)$ consists of meromorphic functions on C which may have a pole at p or q of order $\leq 2g+2$. Hence $1, x, \ldots, x^{g+1} \in \mathcal{L}((g+1)D)$ and are $\mathbb{C}$-linear independent. Since the *Theorem of Riemann Roch* implies that $\dim_{\mathbb{C}}(\mathcal{L}(D)) = g+3$ it follows that there exists a meromorphic function $y \in \mathcal{L}((g+1)D)$ with a pole at p or q of order $2g+1$ and $y \notin \mathbb{C}[X]$. The space $\mathcal{L}((2g+1)D)$ consists of meromorphic functions on C which may have a pole at p or q of order $\leq 4g+2$. Obviously, $1, x, \ldots, x^{2g+2}, y, xy, \ldots, x^g y, y^2 \in \mathcal{L}((2g+1)D)$. It follows from the *Theorem of Riemann*

Roch that $\dim_{\mathbb{C}}(\mathcal{L}((2g+1)D)) = 3g+3$. Hence, $1,\ldots,y^2$ are $\mathbb{C}$-linear dependent in $\mathcal{L}((2g+1)D)$. Normalization of the obtained linear combination gives $C:\ y^2 = \prod_{i=1}^{2g+2}(x-\mu_i)$. Now let C be given by $C:\ y^2 = \prod_{i=1}^{2g+2}(x-\mu_i)$ where x,y are coordinate functions on $\mathbb{A}^2(\mathbb{C})$ and $\mu_i \neq \mu_j$ for all $i \neq j$. It follows form O. Forster [9] example 8.10 that C is a connected, compact Riemann surface which admits a holomorphic double covering map $C \to \mathbb{P}^1$. Using the *Riemann Hurwitz formula* it follows that the genus of C equals g. Hence, C is a hyperelliptic curve of genus g. □

Let C be a hyperelliptic curve of genus g. By the *Riemann-Hurwitz Formula* (See O. Forster [9]) we have

$$\text{genus}(C) = \frac{\#\,\text{ramification points}}{2} + \deg(C \to \mathbb{P}^1)\cdot(\text{genus}(\mathbb{P}^1)-1)+1.$$

Remark 1. *This form of the Riemann-Hurwitz formula uses that we can consider the case in which the ramification points are simple. We refer to R. Hartshorne [11] or H.M. Farkas and I. Kra [8] for its general form.*

Since $\text{genus}(C) = g,\ \deg(C \to \mathbb{P}^1) = 2$ and $\text{genus}(\mathbb{P}^1) = 0$ it follows that

$$\#\,\text{ramification points} = 2g+2.$$

Hence, we may think of a hyperelliptic curve of genus g as a double cover of the Riemann sphere ramified at $2g+2$ distinct points. This or Theorem 2 suggest(s) that every hyperelliptic curve C of genus g can uniquely be associated to an unordered $(2g+2)$-tuple consisting of distinct points of $\mathbb{P}^1$ up to automorphism of $\mathbb{P}^1$. Conversely, an unordered $(2g+2)$-tuple consisting of distinct points of $\mathbb{P}^1$ can uniquely be associated to an expression of the form $y^2 = \prod_{i=1}^{2g+2}(x-\mu_i)$ up to scalar multiplication. It follows from Theorem 2 that such expressions are hyperelliptic curve of genus g. Hence, we proved the following proposition.

Proposition 1. *Every hyperelliptic curve C of genus g can be identified, in a natural manner, with an unordered $(2g+2)$-tuple $\mu = (\mu_1,\ldots,\mu_{2g+2}) \in \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 - \Delta$. Here Δ denotes the diagonal. In such a case we write C_μ instead of C.*

An automorphism M of $\mathbb{P}^1$ is a matrix $M \in \text{PGl}_2(\mathbb{C})$. For any triple (μ_1,μ_2,μ_3) consisting of distinct points of $\mathbb{P}^1$ there exists an unique automorphism of $\mathbb{P}^1$ which 'maps' (μ_1,μ_2,μ_3) onto $(0,1,\infty)$ (See A.F. Beardon [2] Theorem 4.1.1) . This observation, together with Theorem 2 and Proposition 1 proves the following corollary.

Corollary 3. *Let C be a curve of genus g. The curve C is a hyperelliptic curve of genus g iff $C:\ y^2 = x\cdot(x-1)\cdot(x-\mu_1)\cdot\ldots\cdot(x-\mu_{2g-1})$ where x,y are coordinate functions on $\mathbb{A}^2(\mathbb{C})$ and $\mu_i \in \mathbb{P}^1$ for $i=1,\ldots,2g-1$ such that $\mu_i \neq \mu_j$ for all $i \neq j$. Moreover, C can be identified with an unordered $(2g-1)$-tuple $\mu = (\mu_1,\ldots,\mu_{2g-1}) \in \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 - \Delta$, where Δ is the diagonal, such that $\mu_i \in \mathbb{C} - \{0,1\}$ for $i=1,\ldots,2g-1$.*

The form of a hyperelliptic curve C of genus g as given as in Corollary 3 is usually called the *Rosenhain normal* form of C. Furthermore, consider an unordered $(2g+2)$-tuple $(\mu_1,\ldots,\mu_{2g+2}) \in \mathbb{P}^{2g+2}(\mathbb{C})$ such that $\mu_i \neq \mu_j$ for all $i \neq j$ and let $M \in \text{PGl}_2$. Then $(\mu_1,\ldots,\mu_{2g+2})$ and the unordered $(2g+2)$-tuple $(M\mu_1,\ldots,M\mu_{2g+2})$ are equivalent unordered $(2g+2)$-tuples. That is, two unordered $(2g+2)$-tuples $\mu := (\mu_1,\ldots,\mu_{2g+2})$ and $\mu' := (\mu'_1,\ldots,\mu'_{2g+2})$ can be identified, which will be denoted by $\mu \sim \mu'$, iff there exists an automorphism M of $\mathbb{P}^1$ such that $(M\mu_1,\ldots,M\mu_{2g+2}) = (\mu'_1,\ldots,\mu'_g)$. The relation is well-defined and it is an equivalence relation since it is (i) reflexive since $\text{Id}_2 \in \text{PGl}_2$, it is (ii) symmetric since any matrix in PGl_2 is non-singular and it is (iii) transitive by standard matrix multiplication. This observation together with Corollary 1 implies that the following definition is well-defined.

Definition 5 (Equivalent hyperelliptic curves)**.** *Let C_μ and $C_{\mu'}$ be two hyperelliptic curves of genus g. We say that C_μ and $C_{\mu'}$ are equivalent hyperelliptic curves of genus g, which we denote by $C_\mu \sim C_{\mu'}$, iff $\mu \sim \mu'$.*

Proposition 2. *Let C_μ and $C_{\mu'}$ be two hyperelliptic curves of genus g such that $C_\mu \sim C_{\mu'}$. Then there exists an isomorphism $C_{\mu'} \to C_\mu$.*

Proof. Since $C_\mu \sim C_{\mu'}$ there exists a matrix $M \in \mathrm{PGl}_2(\mathbb{C})$ such that $(\mu_1, \ldots, \mu_{2g+2}) = (\mu'_1, \ldots, \mu'_{2g+2}) = (M\mu_1, \ldots, M\mu_{2g+2})$. Let $f(x,z) = \lambda_0 x^6 + \lambda_1 x^5 z + \ldots + \lambda_5 x z^5 + \lambda_6$. Notice that, $\lambda_0, \ldots \lambda_6 \in \mathbb{C}[\mu_1, \ldots, \mu_6]$ where $\mu_1, \ldots, \mu_6$ are the zeros of f. The matrix M acting by linear transformation on the coordinates (x,z). Explicitly,

$$\mathrm{PGL}_2 \ni M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x,y) \mapsto \begin{cases} x = ax' + bz' \\ z = cz' + dz' \end{cases}$$

Substituting these expressions of x and z into $f(x,z)$ and rewriting this expression yields

$$f(x,z) = \lambda'_0 (x')^6 + \lambda'_1 (x')^5 z' + \ldots + \lambda_5 x' (z')^5 + \lambda_6 (z')^6$$

(For details we refer to the beginning of Section 3.3). If $C_\mu = C : y^2 = f(x,1) = f(x)$ and if we let f' be the polynomial $f'(s) = \lambda'_0 s^6 + \lambda'_1 s^5 + \ldots + \lambda_5 s + \lambda'_6$, then the hyperelliptic curves C and $C_{\mu'} = C' : (y')^2 = f'(x)$ are isomorphic. More precise, the map

$$(x', y') \mapsto (x,y) = \left(\frac{ax'+b}{cx'+d}, \frac{y'}{(cx'+d)^{g+1}} \right)$$

induces an isomorphism

$$C' \xrightarrow{\cong} C$$

as

$$\left(\frac{y'}{(cx'+d)^{g+1}} \right)^2 = f\left(\frac{ax'+b}{cx'+d} \right).$$

□

Notice, we only proved that two equivalent, unordered $(2g+2)$-tuples μ and μ' both consisting of distinct points of $\mathbb{P}^1$ induces isomorphic hyperelliptic curves C_μ and $C_{\mu'}$ of genus g, respectively. Obviously the contrary is true. That is, $C_{\mu'} \cong C_\mu$ implies $\mu' \sim \mu$ since their polynomials f and f', respectively, are uniquely determined by their zeros up to multiplication by a constant. Furthermore, Proposition 2 suggests some terminology.

Definition 6 (Isomorphic hyperelliptic curves)**.** *We say that two hyperelliptic curves $C = C_\mu$ and $C' = C_{\mu'}$ of genus g are isomorphic, which we denote by $C \cong C'$, iff $\mu \sim \mu'$. The class of hyperelliptic curves of genus g isomorphic to an hyperelliptic curve C of genus g is called an isomorphism class of hyperelliptic curves and is denoted by $[C]$.*

Notice, Definition 6 states an explicit isomorphism between hyperelliptic curves of given genus. We also have a 'normal' notion of isomorphic hyperelliptic curves of given genus. It remains to prove that these two notions of isomorphic hyperelliptic curves of given genus are equivalent. Let C and C' be hyperelliptic curves of genus g and let $\varphi : C \to C'$ be a 'normal' isomorphism. The map $\iota : C \to C$ which interchanges both sheets of C is an automorphism on C (See Definition 8). Similarly, let $\iota' : C' \to C'$ interchange the sheets of C'. Then $\varphi \circ \iota \circ \varphi^{-1} = \iota'$ is an involution of C' where ι and ι' are unique. Since our explicit isomorphism of hyperelliptic curves of given genus is induced through automorphisms on the Riemann sphere, one proves that both isomorphic relations for hyperelliptic curves of given genus are equivalent. This will become clearer at the end where we consider automorphisms of (hyperelliptic) curves of given genus.
The set of isomorphism classes of hyperelliptic curves of given genus will be the study of the rest of our thesis. Indeed, we will construct it, put the structure of a normal, quasi-projective variety on it and study its singularities. From Theorem 2 it follows that for all $g \in \mathbb{Z}_{\geq 2}$ there exists a hyperelliptic curve of genus g. If we rewrite the Fermat Curve $C :\ x^n + y^n = 1$ in $\mathbb{A}^2(\mathbb{C})$ where $n \in \mathbb{N}$ like $y^n = x^n - 1$ and

take $n = 4$, it is well-known that its genus equals 3, i.e., genus$(C) = 2^{-1} \cdot ((n-1)(n-2))$. One can prove that the Fermat Curve C is a non-hyperelliptic curve of genus 3. In case $n = 5, 6, 7, \ldots$ one also proves, in a similar manner as in the case of $n = 4$, that the Fermat Curve is a non-hyperelliptic curve of genus $g = 6, 10, 15, \ldots$, respectively, i.e., the Fermat curve of genus $3, 6, 10, 15, \ldots$ is a non-hyperelliptic curve of genus $g = 3, 6, 10, 15, \ldots$, respectively, since it is a smooth, projective curves in $\mathbb{P}^2(\mathbb{C})$ (See R. Hartshorne [11]). Furthermore, H. Farkas and I. Kra [8] give an example of a non-hyperelliptic curve of any genus $g \in \mathbb{Z}_{\geq 4}$. This observation proves the following lemma.

Lemma 2. *For all $g \in \mathbb{Z}_{>2}$ there exists hyperelliptic curves of genus g and non-hyperelliptic curves of genus g.*

The disadvantage of the preceding discussion is that we do not know what it is means for two non-hyperelliptic curves of given genus to be isomorphic. However, intuitively it is clear that we may speak of isomorphic non-hyperelliptic curves. For what will follows, we will assume that such an intuition can be made mathematically rigorous. Our goal is to put structure on the set $\mathcal{H}_g$ of isomorphism classes of hyperelliptic curves of genus g. Generalizing this means that one would put structure on the set $\mathcal{M}_g$ of isomorphism classes of curves of genus g. As consequence of Lemma 2, if $g \geq 3$ one has to put the structure on the set $\mathcal{N}_g$ of isomorphism classes of non-hyperelliptic curves of genus g and glue $\mathcal{N}_g$ with $\mathcal{H}_g$ to obtain $\mathcal{M}_g$. In case of $g \geq 3$, the following discussion suggests to put structure on $\mathcal{N}_g$ rather than $\mathcal{H}_g$.
We defined that a curve C of genus g is a non-hyperelliptic curve of genus g if the canonical map $\varphi_K : C \to \mathbb{P}^{g-1}(\mathbb{C})$ is an embedding. According to R. Hartshorne [11] we call this the *canonical embedding*. According to R. Hartshorne [11] this suggest the following definition.

Definition 7 (Canonical curve)**.** *Let C be a non-hyperelliptic curve of genus g. The image $\varphi_K(C)$ of the canonical embedding $\varphi_K : C \to \mathbb{P}^{g-1}(\mathbb{C})$ is a curve of degree $2g-2$ which we call the canonical curve.*

This suggest the following proposition. We will prove it slightly more elementary as R. Hartshorne [11].

Proposition 3. *Let C be a non-hyperelliptic curve of genus three. Then $\varphi_K(C)$ is an algebraic quartic curve in $\mathbb{P}^2(\mathbb{C})$.*

Proof. It follows from the Definition of *the canonical map* and from the Definition of *non-hyperelliptic curves* that $\varphi_K(C) \subset \mathbb{P}^2(\mathbb{C})$. From Definition 7 it follows that $\deg \varphi_K(C) = 4$. Hence, $\varphi_K(C)$ is an algebraic quartic curve in $\mathbb{P}^2(\mathbb{C})$. □

Let $F(X_0, X_1, X_2)$ be a homogeneous polynomial of degree four and suppose that the equation $F(X_0, X_1, X_2) = 0$ corresponds with a canonical curve $\varphi_K(C)$. Obviously, there exists 14 monomials of degree four in the variables X_0, X_1 and X_2. The singularities of $\varphi_K(C)$ are the solutions (x_0, x_1, x_2) satisfying, simultaneously, the equations $\frac{\partial}{\partial X_i} F(X_0, X_1, X_2) = 0$ for $i = 0, 1, 2$. Therefore, the 14 coefficients corresponding to the 14 monomials of degree four in the variables X_0, X_1 and X_2 satisfies a non-trivial set of equations. Hence, these 14 coefficients induces a complex, projective variety $V \subset \mathbb{P}^{14}(\mathbb{C})$. Furthermore, a matrix which corresponds to a linear automorphism of $\mathbb{P}^2(\mathbb{C})$ is a (3×3)-square matrix which is invertible. That is, the equivalence classes of linear automorphisms of $\mathbb{P}^2(\mathbb{C})$ where the equivalence relation is given by $A \sim \lambda \cdot \mathrm{Id}_3\, A$ for $\lambda \in \mathbb{C} - \{0\}$ form an open subset U of $\mathbb{P}^8(\mathbb{C})$. Here A is an invertible, (3×3)-square matrix corresponding to a linear automorphism of $\mathbb{P}^2(\mathbb{C})$. Obviously, it forms an open subset of $\mathbb{P}^8(\mathbb{C})$ since $\dim \mathrm{Aut}(\mathbb{P}^2(\mathbb{C})) = 8$ and also the corresponding orbits are of dimension eight, i.e., any curve has finitely many automorphisms which implies that the the dimension of the orbits should be equal to eight. Combining that $X \subset \mathbb{P}^{14}(\mathbb{C})$ and $U \subset \mathbb{P}^8(\mathbb{C})$ implies that there exists $\infty^{14} - \infty^8 = \infty^6$ inequivalent quartic curves in $\mathbb{P}^2(\mathbb{C})$. This proves the following proposition.

Proposition 4. *The number of inequivalent quartic curves in $\mathbb{P}^2(\mathbb{C})$ equals ∞^6.*

From Proposition 3 it follows that all quartic curves in $\mathbb{P}^2(\mathbb{C})$ correspond with non-hyperelliptic curves of genus three. Combining this with Proposition 4 proves the following lemma.

Lemma 3. *There exists ∞^6 non-hyperelliptic curves of genus* 3.

We now determine the number of hyperelliptic curves of genus g. From Definition 6 it follows that two hyperelliptic curves C and C' of genus g are isomorphic iff their corresponding unordered $(2g{+}2)$-tuples μ and μ' of distinct points of $\mathbb{P}^1$, respectively, are equivalent. Obviously, if $\mu' = M\mu$ then $\mu' = \lambda \cdot M\mu$ for all $\lambda \in \mathbb{P}^1 - \{0\}$. Otherwise stated, scalar multiplication does not change the equivalence relation $\mu \sim \mu'$. Therefore, we can fix one of the elements of M. Using the *Riemann Hurwitz formula* gives $2g{+}2$ isomorphism classes of hyperelliptic curves of genus g. Since there exists an unique automorphism of $\mathbb{P}^1$ such that $(\mu_1, \mu_2, \mu_3) \mapsto (0, 1, \infty)$ where $\mu_1, \mu_2, \mu_3 \in \mathbb{P}^1$ distinct points (See A. F. Beardon [2]) it follows that there exists $2g{+}2{-}3 = 2g{-}1$ isomorphism classes of hyperelliptic curves of genus g. Later on we will see that this also suggests that the dimension of the variety of moduli of hyperelliptic curves of genus g equals $2g - 1$ and in particular that the dimension of the variety of moduli of hyperelliptic curves of genus two equals three. Anyway, we proved the following proposition.

Lemma 4. *There exists ∞^{2g-1} non-isomorphic hyperelliptic curves of genus g.*

Corollary 4. *There exists ∞^5 non-isomorphic hyperelliptic curves of genus three.*

Proof. Substitute $g = 3$ into the result of Lemma 4. □

If we compare the number of hyperelliptic curves of genus three with the number of non-hyperelliptic curves of genus three it is clear that almost all curves of genus three are non-hyperelliptic curves of genus three. In case of curves of genus two we found that all curves are hyperelliptic curves of genus two. That is, the number of hyperelliptic curves of genus g is negligible compared with the number of non-hyperelliptic curves of genus g if g becomes large, say $g \geq 3$. As our goal is to put structure on $\mathcal{H}_g$, these observations suggest that this is meaningful only in the case if the genus < 3 since otherwise all most all curves are non-hyperelliptic curves. As a consequence of Lemma 2 we obtain $\mathcal{H}_2 = \mathcal{M}_2$. Therefore, we will restrict ourselves mainly to the case of curves of genus two. A precise description of $\mathcal{M}_2$ and putting structure on it will be the central topic of what will follow. The last property of hyperelliptic curves we want to discuss is that of automorphisms on them. As it will turn out later on (See Chapter 5), hyperelliptic curves of given genus having 'extra automorphisms' disturb the structure of their moduli space.

Definition 8 (Automorphism)**.** *Let C be a curve of genus g. A map $f : C \to C$ which is holomorphic, bijective and for which the inverse f^{-1} is holomorphic is called an automorphism of C. We denote the set of all automorphisms of C by* $\mathrm{Aut}(C)$.

Proposition 5. *Let C be a curve of genus g. Then $\mathrm{Aut}(C)$ is a group and if C is a hyperelliptic curve of genus g, then $\mathrm{Aut}(C) \neq \{\mathrm{id}\}$.*

Proof. The first assertion is easily verified by a direct computation. From Theorem 2 it follows that $C : y^2 = \prod_{i=1}^{2g+2}(x - \mu_i)$, i.e., $C = \{(x, y) \in \mathbb{A}^2(\mathbb{C}) \mid y^2 - \prod_{i=1}^{2g+2}(x - \mu_i) = 0\}$. Obviously, the map $\iota : C \to C$ given by $(x, y) \mapsto (x, -y)$ is holomorphic and bijective such that its inverse is also holomorphic and leaving the $2g + 2$ points of ramification unchanged. Hence, $\mathrm{Aut}(C) \neq \{\mathrm{id}\}$. □

In standard literature the map $\iota : C \to C$ from the proof of Proposition 5 is called the *hyperelliptic involution*. Intuitively, the hyperelliptic involution interchanges the sheets of $C \to \varphi_K(C)$ in which C has to be considered as a Riemann surface. Clearly, the hyperelliptic involution has the additional property that $\iota^2 = \mathrm{Id}$ and ι is non-trivial.

Theorem 3. *Given the hyperelliptic curves of genus two $C_0 : y^2 = x^6 - 1, C_1 : y^2 = x^6 + a_1x^3 + b_1, C_2 : y^2 = x^5 + a_2x^3 + b_2, C_3 : y^2 = x^5 - x, C_4 : y^2 = x^6 + a_4x^4 + b_4x^2 + c_4$ and $C_5 : y^2 = x^5 - 1$, then $\mathrm{Aut}(C_0)/ < \iota > \cong D_6, \mathrm{Aut}(C_1)/ < \iota > \cong D_3, \mathrm{Aut}(C_2)/ < \iota > \cong D_2, \mathrm{Aut}(C_3)/ < \iota > \cong S_4, \mathrm{Aut}(C_4)/ < \iota > \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(C_5)/ < \iota > \cong \mathbb{Z}/5\mathbb{Z}$ where ι is the hyperelliptic involution. Here the a's, b's and c's are - sufficient - generally chosen.*

Proof. See Oskar Bolza [4]. □

Notice, since ι is an automorphism of every hyperelliptic curve and since ι is normal in the group $\mathrm{Aut}(C)$ where C is a hyperelliptic curve of given genus it follows that $\mathrm{Aut}\,/ < \iota >$ is well-defined. We will call this group the *reduced group of automorphisms* of C. So far we obtained the main properties of hyperelliptic curves of genus g. In the next chapter we start defining and studying hyperelliptic curves of genus g all at once.

Chapter 3

Sets of isomorphism classes of curves of given genus

Instead of studying each curve of given genus separately we want to construct and study the space which consists of isomorphism classes of curves of given genus, i.e., a space classifying all curves of given genus. A natural approach would be to first construct the set consisting of all isomorphism classes of curves of given genus and than put structure on it. We define $\mathcal{M}_g := \{[C] \mid C \text{ a curve of genus } g\}$ and $\mathcal{H}_g := \{[C] \mid C \text{ hyperelliptic curve of genus } g\}$. From Corollary 2 it follows that $\mathcal{M}_2 = \mathcal{H}_2$.

3.1 The case of curves of genus zero

In this short section we consider the set of isomorphism classes of curves of genus zero.

Lemma 5. *Let C be a curve of genus zero. Then C is isomorphic to $\mathbb{P}^1$.*

Proof. In particular $\mathrm{Div}(C) \ni D := p$ where $p \in C$. Then $0 \le i(D) \le \dim_{\mathbb{C}} H^0(C, \Omega_C) = g = 0$. Therefore, $i(D) = 0$. From the *Theorem of Riemann Roch* it than follows that $\mathcal{L}(D) \cong \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot f$ where $f : C \to \mathbb{P}^1$ such that $f^{-1}(\infty) = p$. Therefore, f is injective. Since C and $\mathbb{P}^1$ are compact f is also surjective. Hence, $C \cong \mathbb{P}^1$. □

Corollary 5. *Let $P \subset \mathbb{C}$ be a singleton subset. Then there exists a bijective correspondence $\mathcal{M}_0 \leftrightarrow P$.*

Proof. It follows from Lemma 5 that all curves of genus zero are isomorphic to $\mathbb{P}^1$. Hence, there exists a bijective map $\varphi : \mathcal{M}_0 \to Q$ where Q is a singleton subset of $\mathbb{C}$ since we suppose curves to be defined over $\mathbb{C}$. Since Q is a singleton subset there exists an unique bijective map $\psi : Q \to P$. Hence, $\psi \circ \varphi : \mathcal{M}_0 \to P$ is a bijective map. □

In the following section we consider sets of isomorphism classes of curves of genus 1, i.e., elliptic curves.

3.2 The case of elliptic curves

Let E be an elliptic curve over $\mathbb{C}$. Every elliptic curve E is isomorphic to an elliptic curve $E_\mu : \ y^2 = x(x-1)(x-\mu)$ with x, y coordinate functions on $\mathbb{A}^2(\mathbb{C})$ and $\mu \in \mathbb{C} - \{0, 1\}$ (See J.H. Silverman [18], Proposition 1.7). As was the case for hyperelliptic curves, E_μ can be identified with the unordered four tuple $(0, 1, \infty, \mu)$.

Remark 2. *Equivalently stated, an elliptic curve E is a double cover of the $\mathbb{P}^1$ ramified at $0, 1, \infty$ and μ where $\mu \in \mathbb{C} - \{0\}$. This may suggests that E is a hyperelliptic curve of genus one. By our conventions of Definition 1 this is not true. Moreover, it also follows from Definition 1 that it is 'meaningless' to speak of hyperelliptic curves of genus one (and zero) since the canonical map, maps onto $\mathbb{P}^{g-1}(\mathbb{C})$.*

Moreover, two different elliptic curves E_μ and $E_{\mu'}$ are isomorphic iff $(0, 1, \infty, \mu) \sim (0, 1, \infty, \mu')$. It follows from the cross-ratio (See A.F. Beardon [2]) that $E_\mu \cong E_{\mu'}$ iff $\mu' \in I := \{\mu, \mu^{-1}, 1-\mu, (1-\mu)^{-1}, \mu(1-\mu)^{-1}, (1-\mu)(\mu)^{-1}\}$. Finally we define a map $j : \mathbb{C} \to \mathbb{C}$ given by $j(z) = 256 \cdot (z^2 - z + 1)^3 (z^2(z-1))^{-2}$. This map is surjective and takes the same values for all $z \in I$ (See J.H. Silverman [18], Proposition 1.7).

Lemma 6. *The map $\psi : \mathcal{M}_1 \to \mathbb{A}^1(\mathbb{C})$ given by $E \mapsto (\mu \mapsto j(\mu))$ is a bijection.*

Proof. As $j(\mu) = j(\mu')$ iff $\mu, \mu' \in I$ iff $E_\mu \cong E_{\mu'}$ it follows that ψ is injective. Since there exists a point $\mu \in \mathbb{C}$ such that $E = E_\mu$ and j is surjective, it follows that ψ is surjective. Hence, there exists a bijective correspondence $\mathcal{M}_1 \leftrightarrow \mathbb{A}^1(\mathbb{C})$. □

Another way of looking at the bijective correspondence $\mathcal{M}_1 \leftrightarrow \mathbb{A}^1(\mathbb{C})$ is to consider it like: above any point on the affine line lies an isomorphism class of elliptic curves. Again we cannot say anything about whether or not $\mathcal{M}_1$ is a variety using that $\mathbb{A}^1(\mathbb{C})$ is a variety. A detailed description how to circumvent this 'problem' is delayed until the next chapter. In the following section we will construct $\mathcal{M}_2$ as parameterization in terms of invariants of the space of binary sextics.

3.3 The case of hyperelliptic curves of genus two

We construct $\mathcal{M}_2$ using *Invariant Theory*. For that reason we gradually introduce some terminology. Consider the homogeneous polynomial $f(x,z) = \lambda_0 x^6 + \lambda_1 x^5 z + \lambda_2 x^4 z^2 + \lambda_3 x^3 z^3 + \lambda_4 x^2 z^4 + \lambda_5 x z^5 + \lambda_6 z^6 \in \mathbb{C}[x,z]$. Then $f(x) := f(x,1)$ is the dehomogenized form of $f(x,z)$. We will call both $f(x)$ and $f(x,z)$ binary sextics and we will denote $\mathcal{B}_6$ for the set of these binary sextics, i.e., $\mathcal{B}_6 := \{f \in \mathbb{C}[x,z] \mid f(x,z) = \lambda_0 x^6 + \lambda_1 x^5 z + \ldots + \lambda_5 x z^5 + \lambda_6 z^6\}$. A linear transformation acts on $f \in \mathcal{B}_6$ by

$$\mathrm{Gl}_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x,z) \mapsto f'(x',z') := f(ax'+bz', cx'+dz') \text{ for all } f \in \mathcal{B}_6.$$

That is, a linear transformation gives rise to a map

$$\mathrm{Gl}_2 \to \mathrm{Gl}_7,\ \gamma \mapsto (f \mapsto \gamma f).$$

Notice, $f'(x',z')$ is itself a binary sextic. Applying a linear transformation on $f'(x',z')$ yields again a binary sextic $f''(x'',z'')$. A linear transformation on $\mathcal{B}_6$ is well-defined if it is invertible, if the linear transformed binary sextic is of the same order as the original binary sextic and if any two successively applied linear transformations can be replaced by a single linear transformation. Notice, the linear transformation acts on $f(x,z)$ by

$$\mathrm{Gl}_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x,z) \mapsto f(ax'+bz', cx'+dz')$$

which implies that we obtain a system of equations

$$\begin{cases} x = ax' + bz' \\ z = cx' + dz'. \end{cases}$$

We can solve this system of equations in terms of x', z' since the linear transformation is induced by a matrix of Gl_2, i.e., is invertible. Therefore, the action of a linear transformation on $f(x,z)$ is invertible. The other two properties are obviously clear. Formally, this proves the following lemma.

Lemma 7. *The action of a linear transformation $\gamma \in \mathrm{Gl}_2$ on a binary sextic $f \in \mathcal{B}_6$ given by $\gamma \mapsto (f \mapsto \gamma f)$ is well-defined.*

The obvious question arising from successively applying linear transformations on binary sextics is the following. Do there exist properties which simultaneously hold for all binary sextics obtained from successively applying linear transformation on them? This suggests the following terminology.

Definition 9 (Invariants)**.** *Let $f \in \mathcal{B}_6$. An invariant of f of degree k is a polynomial $F \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]$ such that $F(\gamma\lambda) = (\det\gamma)^{6k}F(\lambda)$ for all $\gamma \in \mathrm{Gl}_2$ where $\lambda := (\lambda_0, \ldots, \lambda_6)$. Here $k \in \mathbb{Z}$ is fixed. Furthermore, we denote the set of all invariants by $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Gl}_2}$.*

Notice, Sl_2 is a subgroup of Gl_2 with the property that all matrices in Sl_2 have determinant equal to one. Therefore, if we replace Gl_2 by Sl_2 in Definition 9 we obtain that $F \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]$ is an invariant of some binary sextic f iff $F(\delta\lambda) = F(\lambda)$ for all $\delta \in \mathrm{Sl}_2$. We will call these invariants *classical invariants* and denote $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ for the *set of classical invariants*. Moreover, the seemingly innocent conventions of Definition 9 will turn out to be probably the most important in our thesis since elements of $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ parameterize $\mathcal{M}_2$.

Proposition 6. *The classical set of invariants is a graded ring.*

Proof. First we prove that the classical set of invariants is a subring of $\mathbb{C}[\lambda_0, \ldots, \lambda_6]$. It is clear that $\mathbb{C} \subset \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$; so $1 \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$. If $F, G \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ then $(F - G)(\lambda) = F(\lambda) - G(\lambda) = F(\gamma\lambda) - G(\gamma\lambda) = (F - G)(\gamma\lambda)$ for all $\gamma \in \mathrm{Sl}_2$; so $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ is a subgroup. Finally, $(FG)(\lambda) = F(\lambda)G(\lambda) = F(\gamma\lambda)G(\gamma\lambda) = (FG)(\gamma\lambda)$ for all $\gamma \in \mathrm{Sl}_2$; so $FG \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$. We conclude that $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ is a subring of $\mathbb{C}[\lambda_0, \ldots, \lambda_6]$. It remains to prove that the classical ring of invariants is graded. Let $F \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$. Then F is homogeneous of degree, say d. This implies that $F = F_1 + \ldots + F_n$ where F_i is homogeneous of degree d. Hence, $F(\gamma\lambda) = F_1(\gamma\lambda) + \ldots + F_n(\gamma\lambda) = (\det\gamma)^{6d}F(\lambda) = F(\lambda) = F_1(\lambda) + \ldots + F_n(\lambda)$ for all $\gamma \in \mathrm{Sl}_2$. By induction on the number of monomials F_i it follows that $F_i(\gamma\lambda) = (\det\gamma)^{6d}F_i(\lambda) = F_i(\lambda)$ for $i = 1, \ldots, n$. Hence, $F_1, \ldots, F_n \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$. That is, each homogeneous component of the polynomial F is a classical invariant as well. Hence, $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ is a graded ring. □

From this moment, we will speak of the *ring of classical invariants* instead of the *set of classical invariants.* Since it will turn out that elements of the classical ring of invariants parameterize $\mathcal{M}_2$ we now start making this graded ring explicit. This amounts finding, suitable, generators for the classical ring of invariants. Therefore, consider $f \in \mathcal{B}_6$. By the *Fundamental Theorem of Algebra* we can write $f(x, z)$ as a homogeneous polynomial

$$g(x, z) = \lambda_0 \cdot (x - \mu_1 z) \cdot \ldots \cdot (x - \mu_6 z),$$

where $\mu_1, \ldots, \mu_6$ are the zeros of f in xz^{-1} where we require that $z \neq 0$. If we restrict ourselves to the case of hyperelliptic curves of given genus it follows from Theorem 2 it is than clear that we may require $z \neq 0$. Moreover, the quotients $\lambda_i\lambda_0^{-1}$ for $i = 0, \ldots, 6$ are polynomial functions of the zeros $\mu_1, \ldots, \mu_6$ of f. Since an invariant $F \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Gl}_2}$ can be written like $F = \lambda_0^p F\left(1, \lambda_1\lambda_0^{-1}, \ldots, \lambda_6\lambda_0^{-1}\right)$ it follows that we can write an invariant in terms of its zeros by multiplying with a suitable factor of λ_0.

Definition 10 (Discriminant)**.** *Let $f \in \mathcal{B}_6$ and let $\mu_1, \ldots, \mu_6$ be its zeros. The discriminant $D(\lambda)$ of f is defined as*

$$D(\lambda) = \lambda_0^{10} \prod_{i<j} (\mu_i - \mu_j)^2.$$

In the sequel, we use the shorthand notation $(ij)^2$ for $(\mu_i - \mu_j)^2$.

Lemma 8. *The discriminant $D(\lambda)$ of $f \in \mathcal{B}_6$ is an invariant of degree ten.*

Proof. From the preceding discussion it follows that we can write, with abuse of notation, $f(x,z) = \lambda_0(x-\mu_1 z)\cdot\ldots\cdot(x-\mu_6 z)$ where $\lambda_0 \in \mathbb{P}^1$ and $\mu_1,\ldots,\mu_6$ are the zeros of $f(x,z)$. Applying a linear transformation $\gamma \in \mathrm{Gl}_2$ on $f(x,z)$ yields

$$\begin{aligned}
\mathrm{Gl}_2 \ni \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x,z) &= \lambda_0 \prod_{i=1}^{6} ((ax+bz) - \mu_i(cx+dz)) \\
&= \lambda_0 \prod_{i=1}^{6} ((a-\mu_i\cdot c)x + (b-\mu_i\cdot d)z) \\
&= \lambda_0 \prod_{i=1}^{6} \left((a-\mu_i\cdot c)\left(x - \frac{d\mu_i - b}{-c\mu_i + a}\right) z\right) \\
&= f(a,c)\cdot\prod_{i=1}^{6}\left(x - \left(\frac{d\mu_i - b}{-c\mu_i + a}\right) z\right).
\end{aligned}$$

It is clear from the last expression that γ acts on μ_i by $\frac{d\mu_i - b}{-c\mu_i + a}$ for all $i = 1,\ldots,6$. Hence, γ acts on the difference $(\mu_i - \mu_j)^2$ by

$$(\mu_i-\mu_j)^2 \mapsto \frac{d\mu_i - b}{-c\mu_i + a} - \frac{d\mu_j - b}{-c\mu_j + a} = \left(\frac{\det(\gamma)}{(c\mu_i - a)(c\mu_j - a)}\right)^2 (\mu_i - \mu_j)^2$$

for all $i = 1,\ldots,6$. Substituting this expression into $D(\lambda)$ yields

$$\begin{aligned}
D(\lambda\gamma) &= f(a,c)^{10}\cdot\prod_{i<j}\left(\left(\frac{\det(\gamma)}{(c\mu_i - a)(c\mu_j - a)}\right)^2(\mu_i-\mu_j)^2\right) \\
&= f(a,c)^{10}\cdot(\det(\gamma))^{60}\cdot\left(\prod_{i<j}(c\mu_i - a)^{-2}(c\mu_j - a)^{-2}\right)\cdot D(\lambda) \\
&= f(a,c)^{10}\cdot(\det(\gamma))^{60}\cdot g(a,c)^{-10}\cdot D(\lambda) \\
&= \det(\gamma)^{60}\cdot D(\lambda).
\end{aligned}$$

Hence, $D(\lambda)$ is an invariant of f of degree ten. □

The following lemma is due to Jun-Ichi Igusa [13]. We will provide a proof in our conventions instead of using the *Fundamental Theorem of Symmetric functions* and mention that it is clear that these invariants only depend on their orbits.

Lemma 9. *Let $f \in \mathcal{B}_6$ and let $\mu_1,\ldots,\mu_6$ be its zeros. Then the expressions*

$$\begin{aligned}
A(\lambda) &= \lambda_0^2 \sum_{\textit{fifteen}} (12)^2(34)^2(56)^2, \\
B(\lambda) &= \lambda_0^4 \sum_{\textit{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2 \textit{ and} \\
C(\lambda) &= \lambda_0^6 \sum_{\textit{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(61)^2(14)^2(25)^2(36)^2
\end{aligned}$$

are invariants of f of degree two, four and six, respectively.

Notice, the subscript 'fifteen' in the expression of $A(\lambda)$ refers to the following. There are 15 possibilities to arrange the six zeros of f in three groups each consisting of two zeros of f. The subscript 'ten' in the expression of $B(\lambda)$ is due too the fact that there are ten possibilities to arrange the six zeros of f first in two groups each consisting of three of them and, secondly, arrange the two groups each consisting of three zeros into three groups of two such that each zero appears the same number of times (two times) in each complete arrangement. The subscript 'sixty' of $C(\lambda)$ is due too the fact that there are ten possibilities to arrange the six zeros of f as in the case of $B(\lambda)$ and there are six 'pairings' between these ten groups. Therefore, the subscript in $C(\lambda)$ needs to be $10 \cdot 6 = 60$. In the Appendix (Chapter 8) we state explicitly the invariants A, B, C and D because their full description will be useful to use computer software packages to manipulate them. Moreover, all properties of invariants given in our thesis can be proved by a direct computation using the full description of them.

Proof. With abuse of notation, write $f = f(x,z) = \lambda_0(x-\mu_1 z)\cdot\ldots\cdot(x-\mu_6 z)$ where $\lambda_0 \in \mathbb{P}^1$ and $\mu_1,\ldots,\mu_6$ are the zeros of $f(x,z)$. As in the proof of Lemma 8, it follows that $\gamma \in \mathrm{Gl}_2$ acts on $(\mu_i-\mu_j)^2$ by $\det(\gamma)^2((c\mu_i-a)(c\mu_j-a))^{-2}(\mu_i-\mu_j)^2$. Notice, by construction of $A(\lambda), B(\lambda)$ and $C(\lambda)$ it follows that every zero of $f(x,z)$ appears the same number of times in every term of the summation of these expressions. Hence, substitution of the result of the linear transformation γ acting on $(\mu_i-\mu_j)$ in $A(\lambda), B(\lambda)$ and $C(\lambda)$ yields $A(\gamma\lambda) = (\det(\gamma))^{12}A(\lambda), B(\gamma\lambda) = (\det(\gamma))^{24}B(\lambda)$ and $C(\gamma\lambda) = (\det(\gamma))^{36}C(\lambda)$. Hence, $A(\lambda), B(\lambda)$ and $C(\lambda)$ are invariants of $f(x,z)$ of degree two, four and six, respectively. □

In Jun-Ichi Igusa [14] one more invariant E of degree 15 is mentioned.

Lemma 10. *Let $f \in \mathcal{B}_6$ and let $\mu_1,\ldots,\mu_6$ be its zeros. Then the expression*

$$E(\lambda) = \lambda_0^{15} \prod_{\textit{fifteen}} \det\begin{pmatrix} 1 & \mu_1+\mu_2 & \mu_1\mu_2 \\ 1 & \mu_3+\mu_4 & \mu_3\mu_4 \\ 1 & \mu_5+\mu_6 & \mu_5\mu_6 \end{pmatrix}.$$

is an invariant of f of degree 15.

Notice, the subscript 'fifteen' refers to the same 'fifteen' as in the case of the invariant $A(\lambda)$ as given as before.

Proof. By a direct computation, substituting how a linear transformation $\gamma \in \mathrm{Gl}_2$ acts on μ_i for $i = 1,\ldots,6$ gives $E(\gamma\lambda) = (\det\gamma)^{90}E(\lambda)$. Hence, $E(\lambda)$ is an invariant of f of degree 15. □

Lemma 11. *Let $f \in \mathcal{B}_6$ and let $\mu_1,\ldots,\mu_6$ be its zeros. Then there exists an unique, irreducible relation $E^2 = F(A,B,C,D)$ where $F \in \mathbb{C}[A,\ldots,D]$ a homogeneous polynomial of degree* 30.

Proof. Using the explicit description of the invariants $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ and $E(\lambda)$ as given as in the Appendix (Chapter 8) it follows that

$$\begin{aligned} E^2 \quad = \quad & -\frac{4}{9}\cdot(B^2A^2C^3+C^5+B^3C^3)-\frac{1}{54}\cdot AB^7-\frac{4}{81}\cdot(A^3C^4+A^2B^5C)-\frac{13}{81}\cdot CB^6+\frac{1}{9}\cdot DB^5- \\ & \frac{1}{2}\cdot(D^3-CBD^2)-\frac{8}{243}\cdot A^3B^3C^2+\frac{1}{12}\cdot AB^2D^2-\frac{2}{3}\cdot(BAC^4-DC^2B^2-DAC^3)- \\ & \frac{14}{27}\cdot B^4AC^2+\frac{1}{3}\cdot AB^3DC+\frac{2}{9}\cdot A^2BDC^2. \end{aligned}$$

Since $\deg A = 2, \deg B = 4, \deg C = 6, \deg D = 10$ and $\deg E = 15$ it is easily verified that each term in the above expression is of degree 30. Using a computer software package it is easily shown that the right hand side of the above equality does not factor, i.e., the right hand side is irreducible. Obviously, the right hand side is uniquely determined up to scalar multiplication. Hence, there exists an unique, irreducible relation $E^2 = F(A,B,C,D)$ where $F \in \mathbb{C}[A,B,C,D]$ is homogeneous of degree 30 given by

$$\begin{aligned} F(A,B,C,D) \quad = \quad & -\frac{4}{9}\cdot(B^2A^2C^3+C^5+B^3C^3)-\frac{1}{54}\cdot AB^7-\frac{4}{81}\cdot(A^3C^4+A^2B^5C)-\frac{13}{81}\cdot CB^6+ \\ & \frac{1}{9}\cdot DB^5-\frac{1}{2}\cdot(D^3-CBD^2)-\frac{8}{243}\cdot A^3B^3C^2+\frac{1}{12}\cdot AB^2D^2-\frac{2}{3}\cdot(BAC^4- \\ & DC^2B^2-DAC^3)-\frac{14}{27}\cdot B^4AC^2+\frac{1}{3}\cdot AB^3DC+\frac{2}{9}\cdot A^2BDC^2 \end{aligned}$$

□

The following corollary is a direct consequence of Lemma 9, Lemma 11 and the Definition of *classical invariants.*

Corollary 6. *The invariants $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ and $E(\lambda)$ of binary sextics are classical invariants.*

Let us consider whether or not the classical invariants vanish. This simplifies the question whether or not these classical invariants are independent with respect to each other.

Proposition 7. *Let $f \in \mathcal{B}_6$ and let $\mu_1, \ldots, \mu_6$ be its zeros. The discriminant $D(\lambda)$ of f is non-vanishing iff all zeros of f differ from each other.*

Proof. First assume that $D(\lambda) = \lambda_0^{10} \prod_{i<j} (\mu_i - \mu_j)^2 \neq 0$. Then $(\mu_i - \mu_j) \neq 0$ for all pairs (i, j) such that $1 \leq i < j \leq 6$. Hence, $\mu_i \neq \mu_j$ for all $i, j \in \{1, \ldots, 6\}$ such that $i \neq j$. Conversely, assume $\mu_i \neq \mu_j$ for all $i, j \in \{1, \ldots, 6\}$ and $i \neq j$. Then $(\mu_i - \mu_j)^2 \neq 0$ for all $1 \leq i < j \leq 6$. Hence, $D(\lambda) \neq 0$. □

Otherwise stated, the discriminant of a binary sextic $f \in \mathcal{B}_6$ vanishes iff f has at least one zero of (algebraic) multiplicity > 1.

Proposition 8. *Let $f \in \mathcal{B}_6$ and let $\mu_1, \ldots, \mu_6$ be its zeros. Suppose f consists of a zero of algebraic multiplicity three. Then the (classical) invariants $B(\lambda), C(\lambda), D(\lambda)$ and $E(\lambda)$ of f vanish.*

Proof. From Proposition 7 it follows that $D(\lambda) = 0$. Consider the (classical) invariant $B(\lambda)$. Since it is obtained by taking pairs from the arrangement of $\mu_1, \ldots, \mu_6$ into two groups each consisting of three different elements and because every zero μ_i for $i = 1, \ldots, 6$ appears in every term of $B(\lambda)$ the same number of times it follows that each of these terms consist, without loss of generality, of a factor $(\mu_1 - \mu_2), (\mu_1 - \mu_3)$ or $(\mu_2 - \mu_3)$. It than follows that $B(\lambda) = 0$ if we pick $\mu_1 = \mu_2 = \mu_3$. Since every term of $C(\lambda)$ is a factor of a term of $B(\lambda)$ it follows that $C(\lambda) = 0$. An explicit calculation - using a computer software package - shows that, without loss of generality, if $\mu_1 = \mu_2 = \mu_3$ then $E(\lambda) = 0$. □

Proposition 9. *Let $f \in \mathcal{B}_6$ and let $\mu_1 \ldots, \mu_6$ be its zeros. Suppose f consists of a zero of algebraic multiplicity four. Then the (classical) invariant $A(\lambda)$ of f vanishes.*

Proof. Every term of $A(\lambda)$ is obtained by arranging $\mu_1, \ldots, \mu_6$ into three groups each consisting of two different zeros μ_i for $i = 1, \ldots, 6$. Assume, without loss of generality, that $\mu_1 = \ldots = \mu_4 = 0$ and $\mu_4 \neq \mu_5 \neq \mu_6$. Then the only non-zero factors are given by $f_{kl} := (\mu_k - \mu_l)$ where $k = 1, \ldots, 6$ and $l = 5, 6$ such that $k \neq l$. By construction of $A(\lambda)$ it follows that each μ_i for $i = 1, \ldots, 6$ appears the same number of times in each term of $A(\lambda)$. Hence, the factors f_{kl} do not form a term of $A(\lambda)$. Therefore, $A(\lambda) = 0$. □

The results of Proposition 7, 8 and 9 suggest the following. If we assume that f has of a zero of algebraic multiplicity 3 it follows that, in general, $B = C = D = E = 0$ and $A \neq 0$. Therefore, A is independent of B, C, D and E. If f consists of a zero of algebraic multiplicity 2 it follows that $D = 0$ and that $A, B, C, E \neq 0$. Therefore, D is independent of A, B, C and E. It follows from the definition of B and C that they are independent. Since the (classical) invariant E is of odd degree and the (classical) invariants A, B, C and D are of even degree it follows that E must be independent of A, B, C and D. These observations proves the following lemma.

Lemma 12. *Let $f \in \mathcal{B}_6$. Then the (classical) invariants A, B, C, D and E of f are independent with respect to each other.*

Since the (classical) invariants A, B, C, D and E are independent with respect to each other we claim that these invariants generate the ring of classical invariants. There are now several ways to proceed in proving this claim. One possibility is to apply the so-called *First Fundamental Theorem of* Sl_2 which can be found in almost any books on *Invariant Theory.* However, we use the so-called *Hilbert* *series.* From Proposition 6 it follows that the ring of classical invariants is graded. Obviously, the ring $\mathbb{C}[\lambda_0,\dots,\lambda_6]$ is graded. If we define $\mathbb{C}[\lambda_0,\dots,\lambda_6]_d \subset \mathbb{C}[\lambda_0,\dots,\lambda_6]$ to be the subspace of homogeneous polynomials of degree d we can write

$$\mathbb{C}[\lambda_0,\dots,\lambda_6] = \bigoplus_{d\geq 0}\mathbb{C}[\lambda_0,\dots,\lambda_6]_d \text{ and } \mathbb{C}[\lambda_0,\dots,\lambda_6]^{\mathrm{Sl}_2} = \bigoplus_{d\geq 0}\mathbb{C}[\lambda_0,\dots,\lambda_6]^{\mathrm{Sl}_2}\cap\mathbb{C}[\lambda_0,\dots,\lambda_6]_d.$$

Definition 11 (Hilbert series)**.** *Let G be a group acting on a graded ring S, denote by S^G the graded ring of invariants and let S_d where $d\in\mathbb{Z}_{\geq 0}$ be the subspace of homogeneous elements of S of degree d. The Hilbert series $P(t)$ of S^G is defined to be the formal power series*

$$P(t) := \sum_{d\geq 0}(\dim S^G \cap S_d)\cdot t^d \in \mathbb{Z}[[t]].$$

Obviously it makes sense to substitute $\mathbb{C}[\lambda_0,\dots,\lambda_6]$ for S and Sl_2 for G in Definition 11. In that case, the Hilbert series is a generating function for the dimensions of homogeneous components of the classical ring of invariants.

Theorem 4. *The dimension of $\mathbb{C}[\lambda_0,\dots,\lambda_e]^{\mathrm{Sl}_2}_d$ where $e,d\in\mathbb{Z}_{\geq 0}$ is given by*

$$\mathrm{Hil}(e,d) := \dim_{\mathbb{C}}\mathbb{C}[\lambda_0,\dots,\lambda_e]^{\mathrm{Sl}_2}_d = \begin{cases} \left[\frac{(1-s^{d+1})\cdot\ldots\cdot(1-s^{d+e})}{(1-s^2)\cdot\ldots\cdot(1-s^e)}\right]_{\frac{e\cdot d}{2}} & e\cdot d \textit{ even}\\ 0 & \textit{otherwise,}\end{cases}$$

Proof. For a proof we refer to I. Schur [17], Satz 2.22. □

Notice, in our case $e=6$ in Theorem 4. Therefore, the product $e\cdot d$ equals $6d$ which is even for all $d\in\mathbb{Z}_{\geq 0}$. Hence, in the case of the classical ring of invariants we have to compute

$$\mathrm{Hil}(6,d) = \left[\frac{(1-s^{d+1})(1-s^{d+2})(1-s^{d+3})(1-s^{d+4})(1-s^{d+5})(1-s^{d+6})}{(1-s^2)(1-s^3)(1-s^4)(1-s^5)(1-s^6)}\right]_{3d}.$$

Expanding the numerator of $\mathrm{Hil}(6,d)$ gives

$$\begin{aligned}\mathrm{Num}(\mathrm{Hil}(6,d)) \quad = \quad & 1-s^{1+d}-s^{2+d}-s^{3+d}-s^{4+d}-s^{5+d}-s^{6+d}+s^{3+2d}+s^{4+2d}+2s^{5+2d}+\\ & 2s^{6+2d}+3s^{7+2d}+2s^{8+2d}+2s^{9+2d}+s^{10+2d}+s^{11+2d}-s^{6+3d}-s^{7+3d}-\\ & 2s^{8+3d}-3s^{9+3d}-3s^{10+3d}-3s^{11+3d}-3s^{12+3d}-2s^{13+3d}-s^{14+3d}-s^{15+3d}\\ & +s^{10+4d}+s^{11+4d}+2s^{12+4d}+2s^{13+4d}+3s^{14+4d}+2s^{15+4d}+2s^{16+4d}+\\ & s^{17+4d}+s^{18+4d}-s^{15+5d}-s^{16+5d}-s^{17+5d}-s^{18+5d}-s^{19+5d}-s^{20+5d}+\\ & s^{21+6d}.\end{aligned}$$

In the expansion of $\mathrm{Num}(\mathrm{Hil}(6,d))$ we can forget about the terms of degree greater than $3\cdot d$. Therefore, $\mathrm{Num}(\mathrm{Hil}(6,d))$ reduces, with abuse of notation, to

$$\begin{aligned}\mathrm{Num}(\mathrm{Hil}(6,d)) \quad = \quad & 1-s^{1+d}-s^{2+d}-s^{3+d}-s^{4+d}-s^{5+d}-s^{6+d}+s^{3+2d}+s^{4+2d}+2s^{5+2d}+\\ & 2s^{6+2d}+3s^{7+2d}+2s^{8+2d}+2s^{9+2d}+s^{10+2d}+s^{11+2d}.\end{aligned}$$

Substitution of the reduced numerator of $\mathrm{Hil}(6,d)$ into $\mathrm{Hil}(6,d)$ and rearranging the terms by its degree gives

$$\begin{aligned}\mathrm{Hil}(6,d) \quad &= \quad \left[\frac{\mathrm{Num}(\mathrm{Hil}(6,d))}{(1-s^2)(1-s^3)(1-s^4)(1-s^5)(1-s^6)}\right]_{3d}\\ &= \quad \left[\frac{1}{(1-s^{\frac{2}{3}})\cdot\ldots\cdot(1-s^2)}\right]_d - \left[\frac{s-s^2-s^3-s^4-s^5-s^6}{(1-u)\cdot\ldots\cdot(1-u^3)}\right]_d\\ &\quad + \left[\frac{u^3+u^4+2u^5+2u^6+3u^7+2u^8+2u^9+u^{10}+u^{11}}{(1-u^2)\cdot\ldots\cdot(1-u^6)}\right]_d.\end{aligned}$$

One simplifies the expression between the squared brackets. Combining these expressions, considered as fractions, gives the Hilbert series.

Lemma 13. *The Hilbert series* $P^{(6)}(t)$ *of* $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ *is given by*

$$P^{(6)}(t) := \frac{(1-t^{30})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})(1-t^{15})}.$$

A worked example, in the case of binary forms of degree 5, can be found in I. Schur [17]. It is a well-known fact in *Invariant Theory* that - the denominator of - $P^{(6)}(t)$ implies that the classic ring of invariants consists of five generators of degree two, four, six, ten and 15, respectively. That is, the series expansion of $P^{(6)}(t)$ is given by

$$S^{(6)}(t) = 1 + t^2 + 2t^4 + 3t^6 + 4t^8 + 6t^{10} + 8t^{12} + 10t^{14} + t^{15} + 13t^{16} + t^{17} + 2t^{19} + \mathcal{O}(t)^{20} \in \mathbb{Z}[[t]].$$

Here the integral coefficient of t^n where $n \in \mathbb{N}$ in $S^{(6)}(t)$ is the number of linear independent invariants of degree n. Hence, from $S^{(6)}(t)$ it follows that there is one degree 2 invariant which we called A. There exist two invariants of degree four. That is, A^2 and a 'new' one which we called B. There exists three invariants of degree six. Namely, A^3, AB and a 'new' one which we called C. In case of the invariants of degree eight we obtain A^4, A^2B, AC and B^2. Therefore, we do not obtain 'new' invariants. Inductively, we find the invariants A, B, C, D and E. As a consequence, these invariants are independent with respect to each other. This was already observed in a direct manner by computing whether or not the invariants A, B, C, D and E vanish. Furthermore, $P^{(6)}(t)$ also implies that there exists a single relation between these five generators of degree 30 which we explicitly stated in Lemma 11. We will need one more result from *Invariant Theory*.

Lemma 14. *Let* $s_{g_1}, \ldots, s_{g_n}$ *be the generators of* S^G *and let* $\deg s_{g_i} = d_i$ *for* $1 \leq i \leq n$. *Then the Hilbert series* $P(t)$ *of* S^G *is given by*

$$P(t) = \frac{F(t)}{(1-t^{d_1}) \cdot \ldots \cdot (1-t^{d_n})}.$$

Here $F(t) \in \mathbb{Z}[t]$ *is a homogeneous polynomial.*

Proof. For a proof we refer to I. Schur [17]. □

From Lemma 8, 9 and 10 we obtain five invariants A, B, C, D and E of degree two, four, six, ten and 15, respectively. Moreover, $E^2 = F(A, B, C, D)$ where F is an irreducible, homogeneous polynomial of degree 30. Therefore, $E^2 \in \mathbb{C}[A, B, C, D]$. Obviously, $\mathbb{C}[A, B, C, D] \oplus E \cdot \mathbb{C}[A, B, C, D]$ is a subring of the classical ring of invariants since $A, B, C, D, E \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$. More precise, $\mathbb{C}[A, \ldots, D] \oplus E \cdot \mathbb{C}[A, \ldots, D]$ is a module of $\mathbb{C}[A, \ldots, E]$. Using the relation $E^2 = F(A, B, C, D)$ it directly follows from Lemma 14 that the Hilbert series of $\mathbb{C}[A, B, C, D] \oplus E \cdot \mathbb{C}[A, B, C, D]$ equals the Hilbert series $P^{(6)}(t)$ of the classical ring of invariants. Hence, the two rings $\mathbb{C}[A, B, C, D] \oplus E \cdot \mathbb{C}[A, B, C, D]$ and the classical ring of invariants coincide. This proves the following theorem.

Theorem 5. *The classical ring of invariants is given by*

$$\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2} = \mathbb{C}[A, B, C, D, E]/(E^2 - F(A, B, C, D)).$$

Obviously, it follows from Theorem 5 that the invariants A, B, C, D and E generate the ring of classical invariants. Given this explicit description of the ring of classical invariants of binary sextics we want to show how we obtain $\mathcal{M}_2$ from it. From Theorem 2 it follows that a hyperelliptic curve $\tilde{C}$ of genus 2 is of the form $\tilde{C} : y^2 = g(x, 1)$ where $g \in \mathcal{B}_6$ such that g consists of six different zeros contained in $\mathbb{P}^1$. In terms of classical invariants this implies that $D \neq 0$.
Consider two binary sextics $f, f' \in \mathcal{B}_6$. Let μ and μ' be unordered six tuples consisting of the zeros of f and f', respectively. In Section 2.2 we proved that f and f' are isomorphic iff $\mu' = M\mu$

for some $M \in \mathrm{PGl}_2$. Otherwise stated, f and f' are isomorphic iff f and f' are PGl_2 conjugated. Therefore, the equation $I(f) = 0$ for all $f \in \mathcal{B}_6$ and for all $I \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ is well-defined. Notice, $\mathrm{PGl}_2 = \mathrm{PSl}_2 = \mathrm{Sl}_2 / \{\pm \mathrm{Id}\}$ and that the latter group $\{\pm \mathrm{Id}\}$ acts trivial on any $f \in \mathcal{B}_6$. Hence, the classical ring of invariants is the coordinate ring of $\mathcal{B}_6/\sim$, i.e., $\mathbb{C}[\mathcal{B}_6/\sim] = \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$. Here we write $I(f)$ for a classical invariant I with respect to f and $\mathcal{B}_6/\sim$ for some subspace of $\mathcal{B}_6$ which we obtained by putting some equivalence relation $\sim$ on the space $\mathcal{B}_6$. Moreover, this equivalence relation is induced by the equivalence relation of unordered 6-tuples which consist of mutually distinct points of the Riemann sphere and which can be considered as the zeros of a binary sextic. This can be done quite similar as in the case of our explicit isomorphic relation of hyperelliptic curves of given genus. Notice, $\mathrm{PGl}_2 = \mathrm{PSl}_2 = \mathrm{Sl}_2 / \{\pm \mathrm{Id}\}$ gives a natural way of using $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ instead of $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Gl}_2}$ and vice versa. Two hyperelliptic curves $\tilde{C} : y^2 = f(x)$ and $\tilde{C}' : y^2 = f'(x)$ of genus 2 are isomorphic iff the corresponding sextics f and f' are PGl_2 conjugated. Therefore, if I is a classical invariant, then the equation $I(\tilde{C}) = 0$ makes sense via $I(f) = 0$ for all $f \in \mathcal{B}_6$. This implies that the classical invariants with the additional property $D \neq 0$ parameterize $\mathcal{M}_2$.
Since the classical ring of invariants is graded we have to compute the so-called projective spectrum, Proj, of the classical ring of invariants with the additional condition $D \neq 0$ to obtain $\mathcal{M}_2$ as parameterization in terms of (classical) invariants of $\mathcal{B}_6$. Intuitively, the variety (scheme) Proj is the union of varieties (schemes) Spec taken with respect to the grading of the ring under consideration. For the construction of the variety (scheme) Spec we refer to the Appendix (See Chapter 8). Here we are concerned with the construction of the variety (scheme) Proj, the so-called projective spectrum of a graded ring. We refer to R. Hartshorne [11], section 2.2 and to D. Eisenbud and J. Harris [7] for an extensive overview. Moreover, Harsthorne gives Proj in terms of a topological space. We need this description to prove Proposition 12 and to get an intuitive idea of Proj. The approach of Eisenbud and Harris is in terms of localizations. This enables us to obtain an affine description of Proj.
Consider a graded ring S with grading $S = \bigoplus_{d \geq 0} S_d$ where S_d consists of the elements of S homogeneous of degree d such that $S_d S_e \subset S_{d+e}$. We define the variety (scheme) $\mathrm{Proj}\, S$, *the projective spectrum of* S. The points of $\mathrm{Proj}\, S$ correspond to homogeneous prime ideals $\wp \subset S$ that do not contain the homogeneous ideal $S_+ := \bigoplus_{d>0} S_d$. We define a topology, the so-called Zariski topology, on $\mathrm{Proj}\, S$ by setting $Z(I) := \{\wp \mid \wp \text{ do not contain } S_+ \text{ and } \wp \supset I\}$ where $I \subset S$ a homogeneous ideal. In terms of localizations, let $d \in \mathbb{Z}_{>0}$ and let $\psi_s : S \to S_s$ be the localization of S with respect to $s \in S_d$ and $s \neq 0$.

Proposition 10. *The localization $S_s = S[r]/(rs - 1)$ is $\mathbb{Z}$-graded.*

Proof. We know that $S[r] = \bigoplus_{(e,f)} S_e \cdot r^f$. Since we want that $rs \in S[r]_0$ we decided that $S_e \cdot r^f \in S[r]_{e-df}$. Therefore, we obtain that $S[r]_n = \bigoplus_{e-fd=n} S_e r^f$; so $rs - 1 \in S[r]_0$. Therefore, the ideal $(rs-1)$ is homogeneous and the grading on $S[r]$ induces a grading on S_s such that $(S_s)_n = \{ts^{-m} \mid t \in S_{n+md}\}$. □

Proposition 10 suggests the following notation. We define $S_{(s)} := (S_s)_0 := \{ts^{-m} \mid t \in S_{md}\}$ and $D_+(s) := \mathrm{Spec}(S_{(s)})$. In what will follows we continue to use this notation.

Proposition 11. *If $e \in \mathbb{Z}_{>0}$ and $r \in S_e$, then $s^e r^{-d} \in S_{(r)}$ and $r^d s^{-e} \in S_{(s)}$.*

Proof. We defined $S_{(r)} := \{tr^{-m} \mid t \in S_{em}\}$. Since $s \in S_d$ it follows that $s^e \in S_{d \cdot e}$. In particular, if $m = d$ it follows that $s^e r^{-d} \in S_{(r)}$. Similarly, it follows that $r^d s^{-e} \in S_{(s)}$. □

Proposition 12. *There exists isomorphisms $(S_{(s)})_{r^d s^{-e}} \cong S_{(rs)} \cong (S_{(r)})_{s^e r^{-d}}$.*

Proof. From Proposition 11 it follows that $r^d s^{-e} \in S_{(r)}$. By definition of $(S(-))_*$ it than follows that $(S_{(s)})_{r^d s^{-e}} = (S_{(s)})_{(r)} = S_{(sr)} = S_{(rs)} = (S_{(r)})_{(s)} = (S_{(r)})_{s^e r^{-d}}$. Where we use $s^e r^{-d} \in S_{(s)}$ in the last equality. Hence, we obtain isomorphisms $(S_{(s)})_{r^d s^{-e}} \cong S_{(rs)} \cong (S_{(r)})_{s^e r^{-d}}$. □

From Proposition 12 it follows that $D_+(sr)$ is open in $D_+(r)$ and open in $D_+(s)$. The variety (scheme) obtained by 'glueing' all these $D_+(-)$'s is defined to be $\mathrm{Proj}\, S$. Notice, if $T \subset \cup_{d>0} S_d$ such that

$D_+(T)^C := Z(T) \subset \mathrm{Spec}(S)$, then for $s \in S$ it follows that $D_+(s)$ is a cover of $\mathrm{Proj}\, S$. Moreover, let $n \in \mathbb{Z}_{\geq 0}$ and let $d_0, \ldots, d_n \in \mathbb{N}$. Given $S := \mathbb{C}[x_0, \ldots, x_n]$ the grading for which S_d consists of a $\mathbb{C}$-basis of monomials $x_0^{m_0}, \ldots, x_n^{m_n}$ such that $\sum_i d_i m_i = d$. Then $\mathrm{Proj}\, S := \mathbb{P}(d_0, \ldots, d_n)$ over $\mathbb{C}$ which we will call *weighted projective space*. Notice, the weighted projective space $\mathbb{P}(d_0, \ldots, d_n)$ over $\mathbb{C}$ is covered by $D_+(x_i)$. The preceding discussion suggests the following formal definition.

Definition 12 (Weighted projective space)**.** *Let $d_0, \ldots, d_n \in \mathbb{Z}_{>0}$ and define $S = S(d_0, \ldots, d_n)$ to be the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ graded by $\deg x_i = d_i$. The weighted projective space $\mathbb{P}(d_0, \ldots, d_n)$ is defined as $\mathbb{P}(d_0, \ldots, d_n) = \mathrm{Proj}\, S$*

Just as ordinary projective space over $\mathbb{C}$ may be realized as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^*$ acting by scalar multiplication we can consider weighted projective space similarly. To be precise, let $x_0, \ldots, x_n$ coordinates on $\mathbb{A}^{n+1}(\mathbb{C})$ and let the group $\mathbb{C}^*$ act by $\eta(x_0, \ldots, x_n) = (\eta^{d_0} x_0, \ldots, \eta^{d_n} x_n)$. Then $\mathbb{P}(d_0, \ldots, d_n)$ is the quotient $(\mathbb{A}^{n+1}(\mathbb{C}) - \{0\})/\mathbb{C}^*$.

Example ($\mathbb{P}(1,2,3)$ as affine space). Define $S := \mathbb{C}[x_0, x_1, x_2]$ where $\deg x_0 = 1, \deg x_1 = 2$ and $\deg x_3 = 3$, i.e., $\mathrm{Proj}\, S = \mathbb{P}(1,2,3)$. Consider the localization $\mathbb{C}[x_0, x_0^{-1}, x_1, x_2]_0$ with respect to the multiplicative set $M_{x_0} = \{1, x_0, x_0^2, \ldots\}$. Since $\deg x_0 = 1$ we obtain $S_{(x_0)} := (S_{x_0})_0 := \{t x_0^{-m} \mid t \in S_m\}$. An easy computation yields Table 3.1 which consists of elements of $S_{(x_0)}$. From Table 3.1 it

m	t
1	$\frac{x_0}{x_0} = 1$
2	$\frac{x_1}{x_0^2}, \frac{x_0^2}{x_0^2} = 1$
3	$\frac{x_0 x_1}{x_0^3} = \frac{x_1}{x_0^2}, \frac{x_2}{x_0^3}, \frac{x_0^3}{x_0^3} = 1$

Table 3.1: *Elements of $S_{(x_0)}$.*

follows that $S_{(x_0)} = \mathbb{C}[x_1 x_0^{-2}, x_2 x_0^{-3}]$. Hence, $D_+(x_0) = \mathrm{Spec}\, \mathbb{C}[x_1 x_0^{-2}, x_2 x_0^{-3}] \cong \mathbb{A}^2(\mathbb{C})$. Now consider the localization $\mathbb{C}[x_0, x_1, x_1^{-1}, x_2]_0$ with respect to the multiplicative set $M_{x_1} = \{1, x_1, x_1^2, \ldots\}$. Since $\deg x_1 = 2$ we obtain $S_{(x_1)} := (S_{x_1})_0 := \{t x_1^{-m} \mid t \in S_{2m}\}$. A direct computation yields Table 3.2 which consists of element of $S_{(x_1)}$. From Table 3.2 it follows that $u := x_0^2 x_1^{-1}, v := x_0 x_2 x_1^{-2}$

m	t
1	$\frac{x_0^2}{x_1}, \frac{x_1}{x_1} = 1$
2	$\frac{x_0^4}{x_0^2} = \left(\frac{x_0^2}{x_1}\right)^2, \frac{x_0^2 x_1}{x_1^2} = \frac{x_0^2}{x_1}, \frac{x_0 x_2}{x_1^2}, \frac{x_1^2}{x_1^2} = 1$
3	$\frac{x_0 x^6}{x_1^3} = \left(\frac{x_0^2}{x_1}\right)^3, \frac{x_0^4 x_1}{x_1^3} = \left(\frac{x_0^2}{x_1}\right)^2, \frac{x_0^3 x_2}{x_1^3} = \frac{x_0^2}{x_1} \cdot \frac{x_0 x_2}{x_1^2}, \frac{x_0^2 x_1^2}{x_1^3} = \frac{x_0^2}{x_1}, \frac{x_0 x_1 x_2}{x_1^3} = \frac{x_0 x_2}{x_1^2}, \frac{x_1^3}{x_1^3} = 1, \frac{x_2^2}{x_1^3}$

Table 3.2: *Elements of $S_{(x_1)}$.*

and $w := x_2^2 x_1^{-3}$ are the generating elements. Moreover, it is easily seen that $u \cdot v = w^2$. Hence, $D_+(x_1) = \mathrm{Spec}\left(\mathbb{C}[u,v,w]/\left(uv - w^2\right)\right)$. From Algebraic Geometry we know that this prime spectrum corresponds with the 'standard' cone in $\mathbb{A}^3(\mathbb{C})$.

We are now in the position to construct $\mathcal{M}_2$ as parameterization in terms of invariants of $\mathcal{B}_6$ and give it an affine description.

Proposition 13. *There exists an isomorphism*

$$\mathrm{Proj}\left(\mathbb{C}[A,B,C,D,E]/\left(E^2 - F(A,B,C,D)\right)\right) \xrightarrow{\cong} \mathrm{Proj}\, \mathbb{C}[A,B,C,D].$$

Proof. The grading of $\mathbb{C}[A,B,C,D,E]$ in $\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D))$ is obtained by $\deg A = 2, \deg B = 4, \deg C = 6, \deg D = 10$ and $\deg E = 15$. Furthermore, F is an irreducible, homogeneous polynomial of degree 30. Consider $(\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)))^{\text{even}} \subset \mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D))$ consisting of elements of even degree. Then $(\mathbb{C}[A,B,C,D,E]/\,(E^2-F(A,B,C,D)))^{\text{even}}$ is generated by A,B,C and D since these (classical) invariants are of even degree and E is of odd degree such that $E^2 = F(A,B,C,D)$. Hence, $(\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)))^{\text{even}} \cong \mathbb{C}[A,B,C,D]$. Obviously, the $\mathbb{C}$-algebra $\mathbb{C}[A,B,C,D]$ is graded. This implies that $\operatorname{Proj}\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D))^{\text{even}} \cong \operatorname{Proj}\mathbb{C}[A,B,C,D]$.
Results to prove that $\operatorname{Proj}\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)) \cong \operatorname{Proj}(\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)))^{\text{even}}$. From the definition of weighted projective spaces it follows that $\operatorname{Proj}\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)) \subseteq \mathbb{P}(2,4,6,10,15)$. Therefore, a point $p \in \operatorname{Proj}\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D))$ is of the form $p=(a,b,c,d,e) \in \mathbb{P}(2,4,6,10,15)$. Since $\mathbb{P}(2,4,6,10,15) = \mathbb{A}^4 - \{0\}/\mathbb{C}^*$ where $\mathbb{C}^*$ acts by $\lambda(a,b,c,d,e) \mapsto (\lambda^2 a, \lambda^4 b, \lambda^6 c, \lambda^{10} d, \lambda^{15} e)$ it follows that $(a,b,c,d,e) = (a,b,c,d,-e)$. Here we take $\lambda = -1$. Hence, $\operatorname{Proj}\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)) \cong \operatorname{Proj}(\mathbb{C}[A,B,C,D,E]/(E^2-F(A,B,C,D)))^{\text{even}}$. □

Due to Alexander Grothendieck the result of Proposition 13 is not surprising as it is well-known that the ring of classical invariants is generated by classical invariants of even degree.

Lemma 15. *The projective prime spectrum of the classical ring of invariants is given by*

$$\operatorname{Proj}\left(\mathbb{C}[\lambda_0,\ldots,\lambda_6]^{\mathrm{Sl}_2}\right) = \mathbb{P}(2,4,6,10).$$

Proof. From Theorem 5 it follows that $\mathbb{C}[\lambda_0,\ldots,\lambda_6]^{\mathrm{Sl}_2} = \mathbb{C}[A,B,C,D,E]/\left(E^2-G(A,B,C,D)\right)$. Furthermore, $\deg A = 2, \deg B = 4, \deg C = 6$ and $\deg D = 10$. Hence, combining Proposition 13 with the definition of weighted projective space gives $\operatorname{Proj}\mathbb{C}[\lambda_0,\ldots,\lambda_6]^{\mathrm{Sl}_2} = \mathbb{P}(2,4,6,10)$. □

Lemma 16. *The weighted projective space* $\mathbb{P}(2,4,6,10)$ *is isomorphic to the weighted projective space* $\mathbb{P}(1,2,3,5)$.

Proof. The projective space $\mathbb{P}(2,4,6,10)$ is obtained as the quotient of $\mathbb{C}^4-\{0\}$ by $\mathbb{C}^*$ where $\eta \in \mathbb{C}^*$ acts by $(A,B,C,D) \mapsto (\eta^2 A, \zeta^4 B, \eta^6 C, \eta^{10} D)$. If we define $\eta' := \zeta^2 \in \mathbb{C}^*$, then $\eta' \in \mathbb{C}^*$ acts by $(A,B,C,D) \mapsto (\eta' A, (\zeta')^2 B, (\eta')^3 C, (\eta')^5 D)$. Obviously, this map is an isomorphisms. Hence, $\mathbb{P}(2,4,6,10) \cong \mathbb{P}(1,2,3,5)$. □

Corollary 7. *The projective prime spectrum of the classical rings of invariants is given by*

$$\operatorname{Proj}\left(\mathbb{C}[\lambda_0,\ldots,\lambda_6]^{\mathrm{Sl}_2}\right) = \mathbb{P}(1,2,3,5)$$

Proof. This immediately follows from Lemma 15 and Lemma 16. □

Theorem 6. *There exists an isomorphism* $\mathcal{B}_6 \xrightarrow{\cong} \mathbb{P}(1,2,3,5)$.

Proof. We already know that the ring of classical invariants $\mathbb{C}[\lambda_0,\ldots,\lambda_6]^{\mathrm{Sl}_2}$ is the function field on $\mathcal{B}_6$. Since the ring of classical invariants is graded it follows that $\mathcal{B}_6 \cong \operatorname{Proj}\left(\mathbb{C}[\lambda_0,\ldots,\lambda_6]^{\mathrm{Sl}_2}\right)$ which is equal to $\mathbb{P}(1,2,3,5)$ by Corollary 7. □

Corollary 8. *Define* $Z(D) := \{D=0\} \subset \mathbb{P}(1,2,3,5)$. *That is,* $Z(D)$ *is the closed subspace of binary sextics consisting of a root of algebraic multiplicity two. Then* $Z(D) \cong \mathbb{P}(1,2,3)$.

Proof. A binary sextic $f \in \mathcal{B}_6$ consists of a zero of algebraic multiplicity two iff $D(f)=0$. Here we write $D(f)$ for the invariant D of f. Therefore, the divisor $D=0$ corresponds to the set $Z(D)$. Hence, $Z(D) \cong \operatorname{Proj}(\mathbb{C}[A,B,C])$. By definition of weighted projective space this implies that $Z(D) = \mathbb{P}(1,2,3)$. □

Notice, in the foregoing example we showed that Aff($\mathbb{P}(1,2,3)$), by which we mean the affine description of $\mathbb{P}(1,2,3)$, corresponds to the 'standard' cone in $\mathbb{A}^3(\mathbb{C})$. The result of the following corollary will be used to prove that the variety of moduli of hyperelliptic curves of genus two is a so-called normal variety.

Corollary 9. *Define $\mathcal{B}_{6,2} \subset \mathcal{B}_6$ to be the set consisting of binary sextics which have two zeros of algebraic multiplicity two. Then $\mathcal{B}_{6,2} = Z(-B^3 + 9C^2 + A^2B^2 - 6ABC) \subset \mathbb{P}(1,2,3)$.*

Proof. Let $f \in \mathcal{B}_{6,2}$ and let $\mu_1, \ldots, \mu_6$ be its zeros. From Proposition 7 it follows that $D = D(f) = 0$. Here we write $I(f)$ for $I \in \mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ with respect to $f \in \mathcal{B}_6$. Without loss of generality, assume that $\mu_1 = \mu_2 = 0$ and $\mu_3 = \mu_4$. Using a computer software package it is easily verified that

$$\begin{aligned} A = A(f) &= 2\mu_4^2(4\mu_5^2\mu_6^2 - 4\mu_4\mu_5\mu_6(\mu_5+\mu_6) + \mu_4^2(3\mu_5^2 - 2\mu_5\mu_6 + 3\mu_6^2)), \\ B = B(f) &= 4\mu_4^4(\mu_4-\mu_5)^2\mu_5^2(\mu_4-\mu_6)^2\mu_6^2 \text{ and} \\ C = C(f) &= B(f)\cdot(\mu_5^2\mu_6^2 - \mu_4\mu_5\mu_6(\mu_5+\mu_6) + \mu_4^2(\mu_5^2 - \mu_5\mu_6 + \mu_6^2)). \end{aligned}$$

and $-B^3(f) + 9C^2(f) + A^2(f)B^2(f) - 6A(f)B(f)\,C(f) = 0$. Since $f \in \mathcal{B}_{6,2}$ is arbitrarily chosen it follows that $\mathcal{B}_{6,2} = Z(-B^3 + 9C^2 + A^2B^2 - 6ABC) \subset Z(D) \cong \mathbb{P}(1,2,3)$. □

As was observed earlier, any hyperelliptic curve of genus two can be associated in a natural manner to a binary sextic. In terms of binary sextics it follows from Theorem 2 that $\mathcal{M}_2$ is parameterized through the subspace of $\mathcal{B}_6$ consisting of binary sextics which have six different zeros. Let us be precise.

Corollary 10. *There exists a bijective correspondence $\mathcal{M}_2 \leftrightarrow \operatorname{Spec}\mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$.*

Proof. From Corollary 8 it follows that $Z(D) \cong \operatorname{Proj}\mathbb{C}[A,B,C]$. It follows from Theorem 2 that any hyperelliptic curve induces a binary sextic $f \in \mathcal{B}_6$ consisting of six distinct zeros. Hence, the complement $Z(D)^C$ of $Z(D)$ parameterizes $\mathcal{M}_2$. Obviously, $Z(D)^C$ is the Zariski open subset $D_+(D) = \operatorname{Spec}(\mathbb{C}[A,B,C,D]_D)_0 = \operatorname{Spec}(\{tD^{-m} \mid t \in \mathbb{C}[A,B,C,D]_{5m}\})$. An elementary - but long and tedious - computation similar as in the worked example $\mathbb{P}(1,2,3)$ proves that the elements A^5D^{-1}, B^5D^{-2} and C^5D^{-3} generate $(\mathbb{C}[A,B,C,D]_D)_0$. Hence, $Z(D)^C = D_+(D) = \operatorname{Spec}\mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$. Since $Z(D)^C$ parameterizes $\mathcal{M}_2$ it follows that there exists an bijective correspondence $\mathcal{M}_2 \leftrightarrow \mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$. □

In terms of Jun-Ichi Igusa [13], Corollary 10 states that $\mathcal{M}_2 \leftrightarrow \operatorname{Spec}\mathbb{C}[y_1^5{y_5}^{-1}, y_2^5{y_5}^{-2}, y_3^5{y_5}^{-3}]$ in a natural manner. Here the y_i's are independent variables of degree i. The difference between our result and the result of Jun-Ichi Igusa [13] is that we obtain $\mathcal{M}_2$ over $\mathbb{C}$ and Jun-Ichi Igusa obtains $\mathcal{M}_2$ over $\mathbb{Z}$. Moreover, Jun-Ichi Igusa [13] also considers the case when the hyperelliptic curves of genus two are taken over an arbitrary field and in particular over a field of characteristic two. However, our result is much more elementary and we describe $\mathcal{M}_2$ much more explicit compared with Jun-Ichi Igusa [13]. To further familiarize $\mathcal{M}_2$ we will determine $\operatorname{Spec}\mathbb{C}[y_1^5y_5^{-1}, y_2^5y_5^{-2}, y_3^5y_5^{-3}]$. The affine space $\mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ is the affine 3-space $\mathbb{A}^3(\mathbb{C})$ modulo the finite abelian group $\mathbb{Z}/5\mathbb{Z}$ acting by $(x,y,z) \mapsto (\zeta x, \zeta^2 y, \zeta^3 z)$ where ζ is a fifth-root of unity. That is, points which are identified with each other in the quotient $\mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ are those points for which (x,y,z) yields the same curves as $(\zeta x, \zeta^2 y, \zeta^3 z)$ where x,y,z are coordinate functions on $\mathbb{A}^3(\mathbb{C})$. Multiplying the (classical) invariants A, B, C and D with ζ does not disturb the structure of $\operatorname{Spec}\mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$. Hence, there exists a bijective correspondence $\operatorname{Spec}\mathbb{C}[y_1^5y_5^{-1}, y_2^5y_5^{-2}, y_3^5y_5^{-3}] \leftrightarrow \mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$. Together with Corollary 10 this proves the following proposition.

Proposition 14. *Their exists a bijective correspondence $\mathcal{M}_2 \leftrightarrow \mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$.*

Notice that multiplying the (classical) invariants A, B, C and D with ζ disturb the structure of $(\mathbb{C}[A,B,C,D,D^{-1}])_0$ unlike $\operatorname{Spec}\mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$. However, $\mathcal{M}_2 \leftrightarrow \mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ implies that the affine 3-space over the complex number field is a five sheeted cover of $\mathcal{M}_2$ and the only singularity of $\mathcal{M}_2$ corresponds to the point $(0,0,0)$. Moreover, the action of the finite abelian group

$\mathbb{Z}/5\mathbb{Z}$ on x^5 gives $\zeta^5 x^5 = x^5$ since ζ is a fifth root of unity. That is, x^5 is an 'invariant' under the action $\mathbb{Z}/5\mathbb{Z}$ on the coordinates (x, y, z). Likewise, one easily verifies that $y^5, z^5, x^3y, x^2y, xy^2, xz^3$ and yz are 'invariants' under the action of $\mathbb{Z}/5\mathbb{Z}$ on (x, y, z). In turn, this gives a map $\mathbb{A}^3(\mathbb{C}) \to \mathbb{A}^8(\mathbb{C})$ which factors through the quotient. This proves the following lemma.

Lemma 17. *There exists an embedding* $\mathcal{M}_2 \hookrightarrow \mathbb{A}^8(\mathbb{C})$.

Unlike standard projective space $\mathbb{P}(1, 1, 1, 1)$ the affine variety needs, in our case, ten generating functions (See Jun-Ichi Igusa [13], Theorem 6). In the next chapter we will put the structure of a normal, quasi-projective variety on $\mathcal{M}_2$. Formally, we prove that $\mathcal{M}_2$ induces to the coarse moduli space of hyperelliptic curves of genus two.

Remark 3. *We constructed* $\mathcal{M}_2$ *using Invariant Theory. We believe that this is a quite elementary way to obtain* $\mathcal{M}_2$*. However, it has some disadvantages. Among others, during the construction we missed an interpretation of objects we wanted to consider. However, other constructions like The Teichmüller approach or The Hodge theory approach to obtain* $\mathcal{M}_2$ *can be found in J. Harris and I. Morrison [10]. We like to thank Prof. dr. Eduard Looijenga from the University of Utrecht (The Netherlands) for giving an extensive overview of Teichmüller Theory. The set* $\mathcal{M}_2$ *may also be constructed using so-called Theta-functions. We like to thank Dr. Robin de Jong from the University of Leiden (The Netherlands) for giving an extensive - and useful - course on Jacobians and Theta functions. Some ideas of constructing* $\mathcal{M}_2$ *using Theta-functions can be found in H.M. Farkas and I. Kra [8].*

Chapter 4

The variety of moduli of curves of genus two

In Section 4.1 we state the problem of moduli and its solution. In Section 4.2 we apply this general theory to $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ to put structure on these sets. Our main interest is the case of $\mathcal{M}_2$.

4.1 Defining moduli spaces of curves

From Algebraic Geometry we know that a projective variety $X \subset \mathbb{P}^n(k)$ over a field k is defined as the common zero set of homogeneous polynomials $F_i(X_0, \ldots, X_n)$ which generate a prime ideal $\wp \subset k[X_0, \ldots, X_n]$. One of our degrees of freedom in defining such a variety is that we are interested in the prime ideal $\wp$ instead of the particular choice of homogeneous polynomials F_i generating $\wp$. We obtain a second degree of freedom if we define a variety X as the set of points which are solutions to equations $F_i = 0$ where the F_i's are homogeneous polynomials in any field in which these equations makes sense. That is, we do not pre-define a field in which we will look up for (non-trivial) solutions. These two degrees of freedom of a variety X makes it possible to describe X as a functor. Since we prerequisite only a first course on Algebraic Geometry and Riemann surfaces we make this idea precise right here on site. We end up with a definition of the normal, quasi-projective variety of moduli of algebraic objects. In fact, this definition is in line with the definition according to J. Harris and I. Morrison [10].

Definition 13 (S-valued points)**.** *Let $X \subset \mathbb{P}^N$ be a variety defined through the equations $F_1(X_0, \ldots, X_N) = \ldots = F_m(X_0, \ldots, X_N) = 0$ where the F_i's are homogeneous polynomials. Let S be any k-algebra and define the set $\mathcal{X}(S) := \{(s_0, \ldots, s_N) \in \mathbb{P}^N(S) \mid F_1(s_0, \ldots, s_N) = \ldots = F_m(s_0, \ldots, s_N) = 0\}$. We call $\mathcal{X}(S)$ the set of S-valued points of X.*

Lemma 18. *Let S and S' be k-algebras where k is a field and suppose that there exists a homomorphism of k-algebras $f : S \to S'$. If $(s_0, \ldots, s_N) \in \mathbb{P}^N(S)$ is a S-valued point of a variety $X \subset \mathbb{P}^N$, then $(f(s_0), \ldots, f(s_N)) \in \mathbb{P}^N(S')$ is an S'-valued point of X.*

Proof. Since $(s_0, \ldots, s_N) \in \mathcal{X}(S)$ it follows that $F_i(s_0, \ldots, s_N) = 0$ where the F_i's are the generating homogeneous polynomials of X. Given that f is a homomorphism of k-algebras implies $f(F_i(s_0, \ldots, s_N)) = (fF_i)(s_0, \ldots, s_N) = (Ff)(s_0, \ldots, s_N)) = F(f(s_0, \ldots, s_N)) = f(0) = 0$. Hence, $(f(s_0). \ldots, f(s_N)) \in \mathbb{P}^N(S')$ is an S'-valued point of X. □

As a consequence of Lemma 18 we have the following corollary.

Corollary 11. *Let $f : S \to S'$ be a homomorphism of k-algebras and let $X \subset \mathbb{P}^N$ be a variety. Then f induces a map $\mathcal{X}(f)$ from $\mathcal{X}(S)$ onto $\mathcal{X}(S')$.*

Proposition 15. *Let S, S' and S'' be k-algebras and $f : S \to S'$ and $g : S' \to S''$ be homomorphisms of k-algebras. For a variety $X \subset \mathbb{P}^N$ we have induced maps $\mathcal{X}(f) : \mathcal{X}(S) \to \mathcal{X}(S')$ and $\mathcal{X}(g) : \mathcal{X}(S') \to \mathcal{X}(S'')$ such that $\mathcal{X}(g \circ f) = \mathcal{X}(g) \circ \mathcal{X}(f)$. Here $g \circ f$ is the usual composition of homomorphism of k-algebras.*

Proof. It follows from Corollary 11 that the maps $\mathcal{X}(f)$ and $\mathcal{X}(g)$ exists. Since $g \circ f : S \to S''$ is a homomorphism of k-algebras it also follows that the map $\mathcal{X}(g \circ f)$ exists. Results to prove that $\mathcal{X}(g \circ f) = \mathcal{X}(g) \circ \mathcal{X}(f)$. As this needs only be checked for a set of generators, it suffices to verify this for an arbitrary S-valued point. Therefore, let $(s_0, \ldots, s_N)$ be a point of $\mathcal{X}(S)$. We compute $(\mathcal{X}(g) \circ \mathcal{X}(f))(s_0, \ldots, s_N) = \mathcal{X}(g)(\mathcal{X}(f)(s_0, \ldots, s_N)) = \mathcal{X}(g)(f(s_0), \ldots, f(s_N)) = (g(f(s_0)), \ldots, g(f(s_N))) = ((g \circ f)(s_0), \ldots, (g \circ f)(s_N)) = \mathcal{X}(g \circ f)(s_0, \ldots, s_N)$. Hence, $\mathcal{X}(g \circ f) = \mathcal{X}(g) \circ \mathcal{X}(f)$. □

Lemma 19. *Let $\mathcal{A}_k$ be the category of k-algebras, let $(\mathcal{S}ets)$ be the category of sets and let $X \subset \mathbb{P}^N$ be a variety. Then $\mathcal{X} : \mathcal{A}_k \to (\mathcal{S}ets)$ given by $S \mapsto \mathcal{X}(S)$ is a covariant functor.*

Proof. The collection $\mathcal{A}_k$ in which the objects are k-algebras and the morphisms between objects are k-algebra homomorphisms is in an obvious way a category. The collection $(\mathcal{S}\text{ets})$ in which the objects are sets and the morphisms between the objects are set theoretical maps is in an obvious way a category. Let S, S' be objects in $\mathcal{A}_k$ and let $f, g \in \mathrm{Hom}_{\mathcal{A}_k}(S, S')$. It follows from the construction that $\mathcal{X}$ assigns to S and S' the S-valued points $\mathcal{X}(S)$ and the S'-valued points $\mathcal{X}(S')$, respectively. Obviously, $\mathcal{X}(S)$ and $\mathcal{X}(S')$ are objects in $(\mathcal{S}\text{ets})$. It follows from Proposition 18 that $\mathcal{X}$ assigns to any $f \in \mathrm{Hom}_{\mathcal{A}_k}(S, S')$ a set theoretical map $\mathrm{Hom}_{(\mathcal{S}\text{ets})}(\mathcal{X}(S), \mathcal{X}(S'))$. Clearly, $\mathcal{X}(\mathrm{Id}) = \mathrm{Id}$ and it follows from Proposition 15 that $\mathcal{X}(g \circ f) = \mathcal{X}(g) \circ \mathcal{X}(f)$. Hence, $\mathcal{X} : \mathcal{A}_k \to (\mathcal{S}\text{ets})$ is a covariant functor. □

So far in this section, we obtained results summarized as follows.

Theorem 7 (Functorial character of a projective variety)**.** *We defined for every pair (X, S) where $X \subset \mathbb{P}^N$ is a variety and S an k-algebra a set $\mathcal{X}(S)$, the S-valued points of X and for every k-algebra homomorphism $f : S \to S'$ a set theoretical map $\mathcal{X}(f) : \mathcal{X}(S) \to \mathcal{X}(S')$ with the property that if $f : S \to S'$ and $g : S' \to S''$ are k-algebra homomorphisms, then $\mathcal{X}(f \circ g) = \mathcal{X}(f) \circ \mathcal{X}(g)$. Moreover, the identity of S induces the identity map on $\mathcal{X}(S)$.*

Notice, the covariant functor $\mathcal{X}$ from Theorem 7 is, by construction, induced by a variety $X \subset \mathbb{P}^N$. In other words, we can consider X as a covariant functor which assigns to a k-algebra a set instead of the common zero set of some homogeneous polynomials.

Theorem 8. *Let X be a projective variety. Then contravariant functor $\mathcal{X} : \mathcal{A}_k \to (\mathcal{S}ets)$ is independent of the choice of coordinates on X.*

Proof. Let $(x_1, \ldots, x_n)$ be coordinates of the variety $X \subset \mathbb{P}^N$ which is the set of points $(p_1 : \ldots : p_N) \in \mathbb{P}^N$ satisfying $F_1(p_1, \ldots, p_N) = \ldots = F_m(p_1, \ldots, p_N) = 0$ where F_i is a homogeneous polynomial for $i = 1, \ldots, m$ and let $\wp \subset k[x_1, \ldots, x_N]$ the prime ideal generated by $F_1, \ldots, F_m$. Define A to be the coordinate ring $k[x_1, \ldots, x_N]/\wp$ and consider $\mathrm{Spec}(A)$. The map $\varphi : k[x_1, \ldots, x_N] \to S$ given by $\varphi(f) = f(s_1, \ldots, s_N)$ where $(s_0, \ldots, s_N) \in \mathcal{X}(S)$ is clearly a homomorphism since the map is defined point wise. Moreover, $f(s_0, \ldots, s_N) = 0$ iff $(s_0, \ldots, s_N) \in \wp$. Equivalently, from the definition of S-valued points it follows that $f(s_0, \ldots, s_N) = 0$ iff $(s_0, \ldots, s_N) \in \mathcal{X}(S)$. Hence, $\ker(\varphi) = \mathcal{X}(S)$. By one of the *Homomorphism Theorems of Algebra* we obtain a homomorphism $A \to S$. That is, we can identify $\mathcal{X}(S)$ with $\mathrm{Hom}_k(A, S)$. Hence, $\mathcal{X}(S) = \mathrm{Mor}_{\mathbb{C}}(\mathrm{Spec}\, S, X)$. Since any homomorphism $f : S \to S'$ of k-algebras induces a morphism $f_* : \mathrm{Spec}\, S' \to \mathrm{Spec}\, S$ and since the map, on the level of sets, $\mathcal{X}(f) : \mathrm{Mor}_k(\mathrm{Spec}\, S, X) \to \mathrm{Mor}_k(\mathrm{Spec}\, S', X)$ is given by $g \mapsto g \circ f_*$ it follows that we can extend the covariant functor $\mathcal{X}$ from affine varieties to algebraic varieties. Moreover, the (projective) variety X is obtained by 'glueing' affine varieties U_i where $i \in I$ an index set and we obtain a functor $\mathcal{X}$ from the functors $\mathcal{U}_i$. It directly follows from the definition of a variety that the functor obtained in this manner is independent of the choice of the collection $\{U_i\}_{i \in I}$ which is an open cover of X. Since this choice defines coordinates on X it follows that the contravariant functor $\mathcal{X}$ is independent of the choice of coordinates on X. □

In our proof of Theorem 8 we obtained a functor $\mathrm{Hom}(-, X)$ from varieties to sets defined by $S \mapsto \mathrm{Hom}(S, X)$. This functor is called the *functor of points*. This terminology is suggested by the following observation. If S is a point then $\mathrm{Hom}(S, X) = X$ as set. However, $\mathrm{Hom}(S, X)$ in case S is a point is more than just a set, e.g., it reveals which points of X are related to each other through a(n) (iso) morphisms. That is, if we know the functor $\mathrm{Hom}(-, X)$ we can reconstruct X from it. In other words, doing geometry on the variety X is equivalent to doing geometry on the functor $\mathrm{Hom}(-, X)$. In essence, this is what a formal moduli space is. One substitute so-called moduli functors for moduli spaces. Moduli problems consist of two things. The first thing is that we have to specify a class of objects - in our case (hyperelliptic) curves of genus g - together with a notion of what it means to have a family of these objects over a variety X. The second thing we have to choose is an equivalence relation on the set of all families of objects under consideration over each variety X. In other words, our moduli problem amounts a parameterization of algebraic objects (hyperelliptic curves of genus g) through an affine variety $\mathrm{Spec}\, S$ where $S \in \mathcal{A}_k$. We say that the moduli problem of algebraic objects (hyperelliptic curves of genus g) has a solution if there exists a variety X with the property that any family of algebraic objects (hyperelliptic curves of genus g) over $\mathrm{Spec}\, S$ is uniquely induced through the pullback of a morphism $\mathrm{Spec}\, S \to X$. Such a solution will be called the variety of moduli of algebraic objects (hyperelliptic curves of genus g). Formally, the moduli problem of algebraic objects is a functor

$$\mathcal{F} : \mathcal{A}_k \to (\mathcal{S}\mathrm{ets}),\ S \mapsto \text{ set of families of algebraic objects over } S \text{ up to isomorphism.}$$

In the particular case of hyperelliptic curves of genus g the moduli problem is a functor

$$\mathcal{F}_{\mathcal{M}_g} : \mathcal{A}_k \to (\mathcal{S}\mathrm{ets}),\ S \mapsto \text{ set of families of curves of genus } g \text{ over } S \text{ up to isomorphism.}$$

Our main interest is in $\mathcal{F}_{\mathcal{M}_g}$ for the specific case $g = 2$. The general case $\mathcal{F} : \mathcal{A}_k \to (\mathcal{S}\mathrm{ets})$ is extraordinary and very abstract. In the Appendix (Chapter 8) we defined isomorphic functors using natural transformations. We define formal moduli spaces using the notion of isomorphic functors. To deduce uniqueness properties of these formal moduli spaces we need two more results concerning natural transformations.

Lemma 20. *Let $X, Y \subset \mathbb{P}^N$ where $N \in \mathbb{N}$ be varieties. Then any $f \in \mathrm{Mor}(X, Y)$ induces a natural transformation of functors $\mathcal{X} \to \mathcal{Y}$. Conversely, if $\theta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation of functors, then θ is induced by an element $f \in \mathrm{Mor}(X, Y)$. Here $\mathcal{X}, \mathcal{Y} : \mathcal{A}_k \to (\mathcal{S}ets)$ are as before.*

Proof. For a proof we refer to Mac Lane [15]. □

Now we can define the solution to our moduli problem which is called the moduli space. In our case the space structure is that of a variety. Formally, a moduli space is an isomorphism of the functor $\mathcal{F}$ - and in particular $\mathcal{F}_{\mathcal{M}_g}$ - with the functor $\mathcal{X}(S)$ given by

$$\mathcal{X} : \mathcal{A}_k \to (\mathcal{S}\mathrm{ets}),\ S \mapsto \mathrm{Mor}(\mathrm{Specm}\, S, X).$$

This suggests the following definition.

Definition 14 (Fine moduli space)**.** *We call X a fine variety of moduli for the moduli functor $\mathcal{F}$ if $\mathcal{F}$ is isomorphic to the functor of S-valued points of X, i.e., $\mathcal{F} \cong \mathrm{Hom}(S, X)$.*

Theorem 10 suggests why we define fine moduli spaces in the general setting instead of restricting to the particular case of curves of given genus. Moreover, in standard *Category Theory* a functor $\mathcal{F} : \mathcal{A}_k \to (\mathcal{S}ets)$ is called *representable by a variety* X if it is isomorphic to the functor $\mathcal{X}$ induced through the variety X. For more details we refer to J. Harrison and I. Morrison [10] or to S. Mac Lane [15] for general *Category Theory.*

Theorem 9. *A fine variety of moduli X for the functor $\mathcal{F}$ is unique.*

Proof. Let X and Y be fine varieties of moduli for the functor $\mathcal{F}$. Then $\mathcal{F} \cong \mathrm{Hom}(-, X)$ and $\mathcal{F} \cong \mathrm{Hom}(-, Y)$. Combining these two relations yields $\mathrm{Hom}(-, X) \cong \mathrm{Hom}(-, Y)$. It follows from Lemma 20 that the natural transformation $\mathrm{Hom}(-, X) \xrightarrow{\cong} \mathrm{Hom}(-, Y)$ is induced by a morphism $X \to Y$. Since the natural transformation $\mathrm{Hom}(-, X) \xrightarrow{\cong} \mathrm{Hom}(-, Y)$ is an isomorphism it must follows that the morphism $X \to Y$ is an isomorphism. This implies $X \cong Y$. Hence, the fine variety of moduli X for the functor $\mathcal{F}$ is unique up to isomorphisms. □

In standard literature one often reads the quote *"As some curves posses non-trivial automorphisms it follows that the variety of moduli of curves cannot be a fine variety of moduli"*. If we take this for granted then we know from section 2.2 that the fine moduli space of hyperelliptic curves of given genus does not exist since any hyperelliptic curve of given genus has a non-trivial automorphism. Moreover, it follows from Theorem 3 that the situation in case of genus two even becomes more horrible. For all cases of hyperelliptic curves of genus $g > 2$ we do not obtain that many of automorphisms as we did for the case $g = 2$. However, this advantage is negligible compared with the existence of hyperelliptic and non-hyperelliptic curves of genus g when $g > 2$. According to D. Mumford, J. Fogarty and F. Kirwan [16] we define a smooth curve of genus g over a variety as follows.

Definition 15 (Smooth curve over a variety)**.** *Let X be a variety. We define a smooth curve of genus g over X as a proper, smooth morphism $\pi : C \to X$ whose geometric fibers are smooth, connected 1-dimensional varieties (curves) of genus g.*

Notice, we will speak of a proper, smooth morphism $\pi : C \to X$ whose fibers are smooth, connected curves of genus g and of a smooth algebraic family of curves of genus g interchangeable. The latter will frequently be abbreviated by family of curves of genus g or family of curves depending on whether or not the genus becomes clear from the context.

Theorem 10. *The fine variety of moduli of curves of genus g does not exist for all $g \in \mathbb{Z}$.*

Proof. Suppose $\pi : C \to X''$ is a smooth algebraic family of curves of genus g. Consider a curve X which has a non-trivial group G of automorphisms. Let X' be a variety on which G acts freely, let $X'' = X'/G$ the quotient and let $C' = X' \times X$. Obviously, the group G acts on C' similarly as on X' on the first component and G acts on C' by automorphism on the second component. Hence, the quotient C'/G is a family of curves over X'' and each fiber is isomorphic to X. That is, we constructed a non-trivial morphism $\pi : C \to X''$ which is isomorphic to X and in general not isomorphic to $X \times X''$. Moreover, saying that a family $\tilde{\pi} : Y \to Z$ is trivial means that $Y = Z \times \tilde{C}$. Hence, we obtained a non-trivial morphism $C \to X''$ such that each fiber has the same isomorphism class. Furthermore, the image of X'' onto the moduli space is a point. Therefore, if the moduli space represents the functor $\mathcal{F}_{\mathcal{M}_g}$ it follows that $C \to X''$ is isomorphic to the trivial product family. Hence, $\mathcal{F}_{\mathcal{M}_g}$ cannot be a fine moduli space. □

In the following example we will give an explicit construction of a family of curves used in Theorem 10 to prove that fine moduli spaces of curves of given genus do not exist.

Example Let C be a curve in $\mathbb{C}$ and suppose that there exists a non-trivial $\sigma \in \mathrm{Aut}(C)$. Notice, in case of hyperelliptic curves of given genus this is always possible since the hyperelliptic involution is a non-trivial automorphism of any hyperelliptic curve of given genus. Let $\mathbb{Z}$ act on $\mathbb{C}$ by

$$kz \mapsto z + 2\pi i k,\ k \in \mathbb{Z} \text{ and } z \in \mathbb{C}.$$

Let $\mathbb{Z}$ also act on the product $\mathbb{C} \times C$ by

$$k(z, p) \mapsto (z + 2\pi i k, \sigma^k(p)),\ k \in \mathbb{Z},\ z \in \mathbb{C} \text{ and } p \in C.$$

Clearly, the action of $\mathbb{Z}$ on $\mathbb{C}$ and on $\mathbb{C} \times C$, as defined as above, commute with the projection $\mathrm{pr} : \mathbb{C} \times C \twoheadrightarrow \mathbb{C}$ and the exponential map, $\exp : \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{C}^*$, is an isomorphism. Moreover, the quotient $(\mathbb{C} \times C)/\mathbb{Z}$ induces a family of curves over $\mathbb{C}^*$, i.e., $\tilde{\pi} : (\mathbb{C} \times C)/\mathbb{Z} \to \mathbb{C}^*$. Notice, $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C}/\mathbb{Z}$

and $\mathbb{C}/2\pi i\mathbb{Z} \cong \mathbb{C}^*$ via the exponential map. Therefore, $\mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$. Since the action of $\mathbb{Z}$ on $\mathbb{C}$ and on $\mathbb{C} \times C$, as defined as above, commutes with the projection map pr, it follows that the fiber of $\tilde{\pi}$ of a point of $\mathbb{C}^*$ is isomorphic to C. Since the action of $\mathbb{Z}$ on C is non-trivial it follows that the family over $\mathbb{C}^*$ induced through $(\mathbb{C} \times C)/\mathbb{Z}$ is not isomorphic to $\mathbb{C}^* \times C$. That is, we constructed, explicitly, a family of curves used in Theorem 10 to prove that fine moduli spaces of curves of given genus do not exist.

Corollary 12. *The set $\mathcal{M}_2$ does not admit the structure of a fine variety of moduli.*

Theorem 10 suggests an obstruction. We can proceed, at least, in two different manners. A first natural approach is to weaken the definition of a fine moduli space. This leads to the notion of coarse moduli spaces. As it will turn out, we again obtain an obstruction. Namely, coarse moduli spaces do not parameterize all families of curves of given positive genus. A second approach, to avoid any obstruction, is to stackify. Actually, this is a bit too enthusiastically argued. First of all, stacks are very technical objects. Indeed, stacks generalize the notion of schemes. That is, one extends the category of schemes in such a manner that if a moduli problem is not representable by a variety nor a scheme, then it is - hopefully - representable by some other algebraic object. These algebraic objects are (algebraic) stacks. Moreover, stackification allows to consider the problem of moduli in a different space in which the obstructions (the existence of additional automorphisms on the objects of study) do not appear. However, we start with defining coarse moduli spaces. The definition of a coarse moduli space is best given by D. Mumford, J. Fogarty and F. Kirwan [16] or J. Harrison and I. Morrison [10]. To mimic this definition, we first have to define a particular kind of projective varieties. A ring R without zero divisors is integrally closed if every element of its field of fractions, $\mathrm{Frac}(R)$, which is integral over R, lives in R.

Definition 16 (Normal variety)**.** *An irreducible, affine variety X is normal if $k[X]$ is integrally closed. Here k is any field and $k[X]$ the polynomial ring of X over k. An irreducible, quasi-projective variety X is normal if every point has a normal affine neighborhood.*

We define the coarse moduli space only in case of curves of given genus. This suggests that we already know that such moduli space exists. That the coarse moduli space of curves of given genus exists can be found in D. Mumford, J. Fogerty and F. Kirwan [16] and in J. Harrison and I. Morrison [10]. We will frequently make use of this fact.

Definition 17 (Coarse moduli space)**.** *A normal, quasi-projective variety M_g is the coarse moduli space of $\mathcal{M}_g$ if there exists a natural transformation of functors $\theta_{M_g} : \mathcal{F}_{\mathcal{M}_g} \to Hom(\cdot, M_g)$ such that*

1. *the map $\theta_{M_g}(\mathrm{Specm}\,\mathbb{C}) : \mathcal{F}_{\mathcal{M}_g}(\mathrm{Specm}(\mathbb{C})) \to \mathrm{Hom}(\mathrm{Specm}(\mathbb{C}), M_g)$ is a bijection and*
2. *given another normal, quasi-projective variety M_g' and a natural transformation $\theta_{M_g'} : \mathcal{F}_{\mathcal{M}_g} \to \mathrm{Hom}(-, M_g')$ there exists a unique morphism of varieties $f : M_g \to M_g'$ which induces a natural transformation $\mathrm{Hom}(f) : \mathrm{Hom}(-, M_g) \to \mathrm{Hom}(-, M_g')$ such that $\theta_{M_g'} = \mathrm{Hom}(f) \circ \theta_{M_g}$.*

Notice, Definition 17 can easily be generalized to the coarse moduli space of general algebraic objects by replacing the M_g and M_g' by arbitrary normal, quasi-projective varieties X and Y, respectively and by replacing the functor $\mathcal{F}_{\mathcal{M}_g}$ by the moduli functor $\mathcal{F}$. We now prove some general facts about fine moduli spaces, coarse moduli spaces and about their relations with respect to each other.

Theorem 11. *Any fine variety of moduli is a coarse variety of moduli.*

Proof. First observe that the definition of coarse varieties of moduli of curves of given genus can be extended in a similar manner to coarse varieties of moduli of algebraic objects. This means that this theorem makes sense. Let X be a fine variety of moduli of $\mathcal{F}$. By definition of a fine variety of moduli there exists an isomorphic natural transformation $\theta_X : \mathcal{F} :\to \mathrm{Hom}(-, X)$. Hence, the set theoretical map $\theta_X(\mathrm{Specm}\,\mathbb{C}) : \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \to \mathrm{Hom}(\mathrm{Specm}\,\mathbb{C}, X)$ is a bijection. This implies condition

1 of Definition 17. Let $f : X \to Y$ be a morphism of varieties. It follows from Proposition 20 that there exists a natural transformation $\mathrm{Hom}(f) : \mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)$. It then follows, by construction, that $\theta_Y := \mathrm{Hom}(f) \circ \theta_X$ is a natural transformation $\mathcal{F} \to \mathrm{Hom}(-, Y)$. The commutativity criterion of condition 2 of definition 17 is automatically fulfilled. Hence, any fine variety of moduli is a coarse variety of moduli. □

Certainly the converse of Theorem 11, a coarse variety of moduli is a fine variety of moduli, does not hold true. Explicit counter examples are immediately obtained by combining Theorem 10 with Theorem 13, 14 and 15. These latter three theorems can be found in Section 4.2.

Theorem 12. *A coarse variety of moduli of curves of genus g is unique up to isomorphisms of varieties.*

Proof. Let M_g and M_g' be coarse varieties of moduli of curves of genus g. By definition of coarse moduli spaces it then follows that there exists a natural transformation $\theta_{M_g} : \mathcal{F}_{\mathcal{M}_g} \to \mathrm{Hom}(-, M_g)$ and $\theta_{M_g'} : \mathcal{F}_{\mathcal{M}_g} \to \mathrm{Hom}(-, M_g)$ which induces unique morphisms of varieties $f : M_g \to M_g'$ and $f' : M_g' \to M_g$ such that $\theta_{M_g'} = \mathrm{Hom}(f) \circ \theta_X$ and $\theta_X = \mathrm{Hom}(f') \circ \theta_y$. Combining these two relations gives $\theta_{M_g'} = \mathrm{Hom}(f) \circ \mathrm{Hom}(f') \circ \theta_{M_g'}$. Hence, $\mathrm{Hom}(f)$ and $\mathrm{Hom}(f')$ are each others inverse. Hence, $M_g \cong M_g'$. □

Notice, the result of Lemma 12 is that we can speak of the coarse variety of moduli of curves of genus g. By definition, fine moduli spaces are unique up to isomorphisms. Having a formal definition of a coarse moduli space of curves of given genus and some of its properties we prove in the following section $\mathcal{M}_2$ induces the coarse moduli space of hyperelliptic curves of genus two.

4.2 The variety of moduli of curves

In section 3 we obtained the sets $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$. In this section we will give $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ the structure of a coarse variety of moduli.

4.2.1 The variety of moduli of curves of genus zero

In subsection 3.1 we proved that $\mathcal{M}_0$ is a singleton subset of the complex numbers. We claim that $M_0 := \mathrm{Spec}\,\mathbb{C}$ is the coarse variety of moduli of curves of genus zero.

Lemma 21. *The variety M_0 is a normal, quasi-projective variety.*

Proof. It is well-known that $\mathrm{Spec}\,\mathbb{C}$ is a non-singular, open subset of some projective space. Hence, $\mathrm{Spec}\,\mathbb{C}$ is a normal, quasi-projective variety. □

Theorem 13. *The variety M_0 is the coarse variety of moduli for the functor $\mathcal{F}_{\mathcal{M}_0}$.*

Proof. From Lemma 21 it follows that M_0 is a normal, quasi-projective variety. Let $S := \mathrm{Spec}\,\mathbb{C}$. Then there exists an unique map $S \to \mathbb{P}^1$ which induces a bijection $\mathcal{F}_{\mathcal{M}_0}(S) \to \mathrm{Hom}(S, M_0)$. Define $M_0' := R \subset \mathbb{C} - \mathrm{Spec}\,\mathbb{C}$ a singleton subspace. Then there exists an unique morphism $\varphi : R \to P$ where P is some complex (sub) space. Obviously, the morphism φ induces an unique map $\mathrm{Hom}(\varphi) : \mathrm{Hom}(-, M_0) \to \mathrm{Hom}(-, M_0')$. From the uniqueness it follows that $\theta_{M_0'} = \mathrm{Hom}(\varphi) \circ \theta_{M_0}$ where $\theta_{M_0} : \mathcal{F}_{\mathcal{M}_0} \to \mathrm{Hom}(-, M_0)$ and $\theta_{M_0'} : \mathcal{F}_{\mathcal{M}_0} \to \mathrm{Hom}(-, M_0')$ are natural transformations. Hence, M_0 is the coarse variety of moduli for the functor $\mathcal{F}_{\mathcal{M}_0}$. □

In the following subsection we construct the coarse variety of moduli of elliptic curves out of $\mathcal{M}_1$.

4.2.2 The variety of moduli of elliptic curves

In this section we give the set of isomorphism classes of elliptic curves $\mathcal{M}_1$ the structure of a coarse moduli space M_1. Given an elliptic curve $E/\operatorname{Spec}\mathbb{C}$ in its Legendre form $y^2 = x^3 + ax + b$ there is a map $j : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}[j]$ which we call the *j-line* of $E/\operatorname{Spec}\mathbb{C}$ (See J. H. Silverman [18] or R. Hartshorne [11]). We claim that $M_1 := \operatorname{Spec}\mathbb{C}[j]$.

Lemma 22. *The variety M_1 is a normal, quasi-projective variety.*

Proof. In each point $p \in \operatorname{Spec}\mathbb{C}[j]$ the regular local ring $\mathcal{O}_p$ is integrally closed. Hence, M_1 is normal. Since $\operatorname{Spec}\mathbb{C}[j]$ is homeomorphic to an open subset of $\mathbb{P}^1(\mathbb{C})$ it follows that M_1 is quasi-projective. □

Theorem 14. *The variety M_1 is the coarse variety moduli for the functor $\mathcal{F}_{\mathcal{M}_1}$.*

Proof. We first prove condition 1 of Definition 17. We are given an elliptic curve E/S where $S := \operatorname{Spec}\mathbb{C}$. Locally on S we can produce a Weierstrass equation for E, i.e., $E : y^2 = x^3 + ax + b$ for some rational functions a and b. According to J. H. Silverman [18] we than have the j-invariant $j = 1728(4a^3 \cdot (4a^3 + 27b^2)^{-1})$. Since we work over $\mathbb{C}$ it follows that $4a^3 + 27b^2$ is invertible on our base. Therefore, the elliptic curve E is non-degenerated. Hence, j defines a morphism $S \to$ j-line which induces a bijection $\mathcal{F}_{\mathcal{M}_1}(S) \to \operatorname{Hom}(S, M_1)$.
Remains to prove condition 2 of Definition 17. Given a normal, quasi-projective variety M_1'. Let $S = \operatorname{Spec}(\mathbb{C}[\lambda, (\lambda(\lambda-1)^{-1})])$ and consider the Legendre family $y^2 = x(x-1)(x-\lambda)$ over S. We than have two morphisms $j : S \to M_1$ and $\varphi : S \to M_1'$. The degree of the map j equals six (See J. H. Silverman). Hence, the map j is finite. This implies that the map $(j, f) : S \to M_1 \times M_1'$ is also finite. Hence, the image of (j, f), say Σ, in $M_1 \times M_1'$ is closed. Consider the projection $\operatorname{pr}_{M_1} : \Sigma \twoheadrightarrow M_1$. From Subsection 3.2 it follows that for two elliptic curves E and E' we have that $j(E) = j(E')$ implies that $E \cong E'$. This implies that $f(E) = f(E')$. Hence, the projection $\operatorname{pr}_{M_1} : \Sigma \twoheadrightarrow M_1$ is a bijection. Otherwise stated, the projection fixes the choice of f which implies uniqueness. Furthermore, $\operatorname{pr}_{M_1} : \Sigma \twoheadrightarrow M_1$ is separable. But than the projection $\operatorname{pr}_{M_1}$is an isomorphism. Hence, if we let $\operatorname{pr}_{M_1'} : \Sigma \twoheadrightarrow M_1'$ be a similar defined projection it follows that $\operatorname{pr}_{M_1'} \circ \operatorname{pr}_{M_1}^{-1} : M_1 \twoheadrightarrow M_1'$ is the unique morphism which induces a natural transformation $\operatorname{Hom}(S, M_1) \to \operatorname{Hom}(-, M_1)$ such that $\theta_{M_1'} = \operatorname{Hom}(\operatorname{pr}_{M_1'} \circ \operatorname{pr}_{M_1}^{-1}) \circ \theta_{M_1}$ where $\theta_{M_1} : \mathcal{F}_{\mathcal{M}_1} \to \operatorname{Hom}(S, M_1)$ and $\theta_{M_1'} : \mathcal{F}_{\mathcal{M}_1} \to \operatorname{Hom}(S, M_1')$ are the given natural transformations. From Lemma 22 we know that M_1 is a normal, quasi-projective variety. Hence, $M_1 = \operatorname{Spec}\mathbb{C}[j]$ is a coarse moduli space for the functor $\mathcal{F}_{\mathcal{M}_1}$. □

Even as in Subsection 3.2, we used much of the results of J. H. Silverman [18] while all our results are not stated in this work. Finally, we spend some words towards the problem that the definition of a coarse moduli space of elliptic curves is not sufficient. Given a family of elliptic curves $\pi : E \to X$ and the map $\psi : X \to \mathbb{C}$ given by $x \mapsto j(\pi^{-1}(x))$ from the proof of Theorem 14. Since $\operatorname{Spec}\mathbb{C}[j] \leftrightarrow \mathbb{A}^1_j(\mathbb{C})$ is the coarse moduli space of elliptic curves it follows that $\psi(x) = \psi(y)$ where $x, y \in X$ iff $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are isomorphic elliptic curves. Usually, one wants to require that a moduli space has a so-called *universal family* over it. That is, if $\operatorname{Spec}\mathbb{C}[j]$ is the moduli space of elliptic curves, then there exists a family of elliptic curves $\overline{E} \to \operatorname{Spec}\mathbb{C}[j]$ such that every family of elliptic curves can be obtained by pulling back this universal family via the map ψ. From the Weierstrass equation of an elliptic curve it is easily seen that every elliptic curve contains an automorphism $(x, y) \mapsto (x, -y)$. This implies that there exists non-trivial families of elliptic curves $\pi : E \to X$ such that $\pi^{-1}(x) \cong E'$ where E' is an elliptic curve for all $x \in X$. Then the map ψ is constant which contradicts the existence of an *universal family*, i.e., $\psi : X \to h\operatorname{Spec}\mathbb{C}[j]$ corresponding to the family $E' \times X \to X$ is constant. To overcome this problem one wishes to *stackify* the moduli space of elliptic curves (See Appendix 8). The idea of *stackification* works for M_g and all $g \in \mathbb{Z}$ where M_g is the coarse moduli space of curves of genus g.
Now that we have the variety of moduli of elliptic curves we state in the following subsection the variety of moduli of hyperelliptic curves of genus two.

4.2.3 The variety of moduli of hyperelliptic curves of genus two

We claim that the coarse variety of moduli of hyperelliptic curves of genus two, M_2, equals $\operatorname{Spec}\mathbb{C}[A^5 D^{-1}, B^5D^{-2}, C^5D^{-3}]$. Here A, B, C and D are the classical invariants.

Lemma 23. *The variety M_2 is a normal variety.*

Proof. We have to prove that M_2 is integrally closed in its field of fractions. The invariants A, B, C and D can be considered as independent variables over $\mathbb{C}$. From Corollary 9 it follows that B is a root of $-X^3 + 9C^2 + A^2X^2 - 6AXC = 0$. Therefore, $\mathbb{C}[X] = \mathbb{C}[A, B, C, D]$. Since A, B, C and D are independent variables over $\mathbb{C}$ it follows that $\mathbb{C}[X]$ is normal. From Jun-Ichi Igusa [13] Lemma 2 it follows that M_2 is isomorphic, over $\mathbb{C}$, to the subring of $\{A^aB^bC^cQ^qD^{-d} \mid a, b, c, d \in \mathbb{Z}_{>0}$ and $a+2b+3c+4Q = 5d\} \subset \mathbb{C}(X)$. Here Q is some additional 'coordinate'. Hence, if $m \in \operatorname{Frac}(M_2)$ is integral over M_2 then there exists an element $n \in \mathbb{Z}_{>0}$ such that $D^n \cdot m$ is integral over $\mathbb{C}[X]$. Since $\mathbb{C}[X]$ is normal it follows that $D^n \cdot m \in \mathbb{C}[x]$. Therefore, $m \in M_2$. Hence, M_2 is normal. □

Lemma 24. *The variety M_2 is a quasi-projective variety.*

Proof. By construction we have that $M_2 = \operatorname{Spec}\mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}] = D_+(D)$. Since $D_+(D)$ is the complement of the closed Zariski subset $Z(D)$ in projective space it follows that M_2 is an open subset of projective space. Hence, M_2 is a quasi-projective variety. □

Theorem 15. *The variety M_2 is the coarse variety of moduli for the functor $\mathcal{F}_{\mathcal{M}_2}$.*

Proof. We first prove condition 1 of Defintion 17. Let $S := \operatorname{Spec}\mathbb{C}$. As before, a hyperelliptic curve $\tilde{C}$ of genus two over S, $\tilde{C}/S$, gives locally $\tilde{C} : y^2 = f(x)$ where $f \in \mathcal{B}_6$. Furthermore, we have three invariants A^5D^{-1}, B^5D^{-2} and C^5D^{-3} which we will call the J-values of hyperelliptic curves of genus two or simply the J-values. Since we work over the complex number field it follows that the J-values are invertible. That is, $\tilde{C}$ is non-degenerated. Hence, the J-values define a map $S \to \operatorname{Spec}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$. That is, the J-values parameterize the isomorphism classes of hyperelliptic curves of genus two over the complex number field. Obviously, the map $S \to \operatorname{Spec}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$ induces a bijection $\mathcal{F}_{\mathcal{M}_2} \to \operatorname{Hom}(S, \mathcal{M}_2)$. Remains to prove the condition 2 of Definition 17. From the very beginning we have for hyperelliptic curves of genus two a family $y^2 = x \cdot (x-1) \cdot (x-\lambda_1)(x-\lambda_2) \cdot (x-\lambda_3)$ over S. This family lies in a finite manner over the J-values. Analogous to the case of elliptic curves it than follows that the family is universal which proves condition two of Definition 17. Hence, M_2 is a coarse moduli space. □

Theorem 15 completes so far our construction of the moduli space of hyperelliptic curves of genus two. In the following chapter we study the singularities of the coarse variety of moduli of hyperelliptic curves of genus two.

Chapter 5

The problem of singularities of the variety of moduli of hyperelliptic curves of genus two

In this chapter we study the singularities of the variety of moduli of hyperelliptic curves of genus two. In the first section we will follow the lines of Jun-Ichi Igusa [13]. As a consequence, we obtain the dimension of M_g for $g \in \mathbb{Z}_{\geq 2}$. Here M_g is the coarse variety of moduli of curves of genus g. In the second section we adopt our own conventions and study the singularities of the variety of moduli of hyperelliptic curves of genus two in more detail.

5.1 The problem of singularities according to Jun-Ichi Igusa

In this section we sketch the main result of Jun-Ichi Igusa [13] about singularities of the variety of moduli of hyperelliptic curves of genus two. Jun-Ichi Igusa [13] uses the following formal definition of a singularity of a variety.

Definition 18 (Singularities)**.** *Let X be a variety, let p be a point of X and let T_pX be the* Zariski-tangent space *to X at p. Then $\dim T_pX \geq \dim X$. We call p a smooth point of X if equality holds. If there is inequality we call p a singularity or singular point of X.*

As a consequence of Igusa's method we obtain the complex dimension of M_g for $g \in \mathbb{Z}_{\geq 2}$. Consider the variety M_g for certain $g \in \mathbb{Z}_{\geq 2}$. Using *Deformation Theory* (See R. Hartshorne [12]) one obtains

$$T_m M_g^\vee \cong H^0(C, \omega_C^{\otimes 2})$$

where $m \in M_g$, ω_C is a divisor of C, $\omega_C^{\otimes 2}$ the, two times, tensored divisor ω_C on C and $T_m M_g^\vee$ is the dual tangent space to M_g at m. Notice, the point $m =$ [curve of genus g]. Therefore, we have to consider $H^0(C, \omega_C^{\otimes 2})$. Furthermore, we assume that the curve C is sufficiently general. As we know from the very beginning, $H^0(M_g, \mathcal{L}(2K_{M_g}))$ is the set of holomorphic 1-forms on M_g. Using the 'standard' definition of the *Zariski-tangent space* it follows that $T_m M_g = (T_m M_g^\vee)^\vee$. Moreover, the isomorphism $T_m M_g^\vee \cong H^0(C, \omega_C^{\otimes 2})$ is not as natural as we stated. The 'problem' is that M_g is a coarse moduli space. Anyway, using the *Theorem of Riemann-Roch* it will follows that

$$\begin{aligned}
\dim_{\mathbb{C}} T_m M_g &= \dim_{\mathbb{C}} H^0(C, \omega_C^{\otimes 2})) \\
&= 1 - g + \deg(2 \cdot \omega_C) + \dim_{\mathbb{C}} H^1(C, \omega_C^{\otimes 2})) \\
&= 1 - g + 2(2g-2) + 0 \\
&= 3g - 3
\end{aligned}$$

iff $g \in \mathbb{Z}_{\geq 2}$. This proves the following well-known theorem.

Theorem 16. *Let $g \in \mathbb{Z}_{\geq 2}$. Then the complex dimension of M_g equals $3g-3$.*

Corollary 13. *The complex dimension of M_2 equals three.*

Proof. This is a direct consequence of Lemma 16 in case of $g=2$. □

Notice, the result of Corollary 13 is intuitively clear from the very beginning. We proved that a hyperelliptic curve $\tilde{C}$ of genus two can be given as $\tilde{C} : y^2 = x \cdot (x-1) \cdot (x-\mu_1) \cdot (x-\mu_2) \cdot (x-\mu_3)$ where $\mu_1, \mu_2\mu_3 \in \mathbb{C} - \{0,1\}$ such that $\mu_i \neq \mu_j$ for all $i \neq j$. Therefore, there lies a three dimensional family of hyperelliptic curves of genus two over the space classifying all hyperelliptic curves of genus two. Hence, the variety of moduli of hyperelliptic curves of genus two is of complex dimension three.
Jun-Ichi Igusa [13] proves that there exists an unique point $[\tilde{C}] \in M_2$ such that $\dim_{\mathbb{C}} T_{[\tilde{C}]} M_2 > 3$, i.e., that M_2 contains an unique singularity. In terms of Jun-Ichi Igusa [13] this unique singular point corresponds to the requirement $J_2 = J_6 = J_8 = 0$. Here J_2, J_4 and J_8 are invariants consistent with the method adopted by Jun-Ichi Igusa to obtain M_2. The other two invariants are denoted by Jun-Ichi Igusa [13] as J_4 and J_{10}. A long and tedious computation yields the Igusa's invariants J_2, J_4, J_6, J_8 and J_{10} in terms of the classical invariants A, B, C and D. Explicitly,

$$\begin{aligned} J_2 &= \frac{A}{8}, \\ J_4 &= \frac{A^2 - 64B}{16384}, \\ J_6 &= \frac{59A^3 - 320AB - 4096C}{2359296}, \\ J_8 &= \frac{7543A^4 + 42112A^2B - 36864B^2 - 524288AC}{9663976416} \text{ and} \\ J_{10} &= \frac{D}{4096}. \end{aligned}$$

Remark 4. *We studied hyperelliptic curves and their moduli for the particular case where the hyperelliptic curves of genus g are taken over the complex number field. As one can prove, the (classical) invariants A, B, C, D and E are not well-defined in the case of fields of characteristic two. In contrast, the invariants J_2, J_4, J_6, J_8 and J_{10} are well-defined for curves taken over any field; even for fields of characteristic two.*

It is now easily seen that $J_2 = 0$ iff $A = 0$. Substitute $A = 0$ into J_6. Then $J_6 = 0$ iff $C = 0$. Finally, substitute $A = C = 0$ into J_8. Then $J_8 = 0$ iff $B = 0$. Since any hyperelliptic curve C of genus two is of the form $C : y^2 = (x-\mu_1) \cdot \ldots \cdot (x-\mu_6)$ where $\mu_i \neq \mu_j$ for all $i \neq j$ it follows from the definition of the discriminant that $D \neq 0$. Hence, $J_2 = J_6 = J_8$ implies that $(A, B, C, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$. So the result about singularities of the variety of moduli of hyperelliptic curves according to Jun-Ichi Igusa, but stated in our conventions, yields the following theorem.

Theorem 17. *The variety of moduli of hyperelliptic curves of genus two contains an unique singularity. Moreover, the unique singularity corresponds to $A = B = C = 0$ and $D \in \mathbb{C} - \{0\}$. Here A, B, C and D are the (classical) invariants.*

In the next section we will prove Theorem 17 in our own conventions. Furthermore, we will state the relation between the singularity of M_2, the corresponding hyperelliptic curve of genus two, the invariants corresponding to this hyperelliptic curve and its group of automorphisms.

5.2 Our approach to the problem of singularities

In Section 5.2.1 we state two quick observations from which it follows that M_2 contains an unique singularity. In Section 5.2.2 we use the failure of the *Implicit Function Theorem* to obtain and study the unique singualrity of M_2.

5.2.1 Singularities from the point of view of the projective and affine structure of M_2

First we use the structure of weighted projective spaces. It follows from I. Dolgachev [6] that $(0,1,0,0),(0,0,1,0)$ and $(0,0,0,1)$ are the singular points of $\mathbb{P}(1,2,3,5)$ which at his turn equals $\operatorname{Proj}\mathbb{C}[A,B,C,D]$. An explicit computation using a computer software package yields that (i) the point $(0,1,0,0)$ corresponds to the binary sextic $y^2 = x^6 - x^2$, (ii)the point $(0,0,1,0)$ corresponds to the binary sextic $y^2 = x^6 - x^3$ and (iii) the point $(0,0,0,1)$ corresponds to the binary sextic $y^2 = x^6 - x$. Using Theorem 3 these curves have a reduced automorphism group of order $4, 3$ and 5, respectively. Using the requirements of Theorem 17 of the previous section, it will follows that $[y^2 = x^6 - x] \in M_2$ is the only singularity. This is, one calculates the zero's of $x^6 - x^2, x^6 - x^3$ and $x^6 - x^2$ and substitute these values into the classical invariants A, B, C and D and checks wether or not these classical invariants vanish. Moreover, we obtained that $\operatorname{Aff}(M_2)$ yields $\mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ where $\mathbb{Z}/5\mathbb{Z}$ acts on $\mathbb{A}^3(\mathbb{C})$ by $(x,y,z) \mapsto (\zeta x, \zeta^2 y, \zeta^3 z)$ where ζ is the fifth root of unity. The affine 3-space $\mathbb{A}^3(\mathbb{C})$ is smooth but obviously $\mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ is smooth accept at the point $(0,0,0)$. Hence, M_2 contains an unique singularity.

5.2.2 Singularities from the point of view of the Implicit Function Theorem

In Section 3.3 we obtained that $\mathcal{M}_2$ is parameterized through a subspace of $\operatorname{Proj}\mathbb{C}[A,B,C,D,E]/(E^2 - F(A,B,C,D))$ where $F(A,B,C,D)$ is an irreducible homogeneous polynomial of degree 30 which we gave explicitly in Lemma 11. The *Implicit Function Theorem* states that points for which $\frac{\partial F}{\partial A} = \frac{\partial F}{\partial B} = \frac{\partial F}{\partial C} = \frac{\partial F}{\partial D} = \frac{\partial F}{\partial E} = 0$ is a singular point of $\mathcal{M}_2$. As $\mathcal{M}_2 = \operatorname{Set}(M_2)$ it follows that these are the singularities of the variety of moduli of hyperelliptic curves of genus two. Let us make this idea mathematical rigorous.

Definition 19 (Singularity)**.** *Let X be a variety defined through an equation $F(X_1, \ldots, X_n)$. A point p of X is called a singularity or singular point of X if all partial derivatives of F vanish at p. If not all partial derivatives of F vanish at p we call p a non-singular point of X.*

In Section 3.3 we obtained that $\mathcal{B}_6 = \operatorname{Proj}\mathbb{C}[A,\ldots,E]/(E^2 - F(A,B,C,D))$ and $\operatorname{Set}(M_2) = \mathcal{M}_2$ parameterized through $Z(D)^C \subset \mathcal{B}_6$. Here $F(A,B,C,D)$ is a homogeneous polynomial of degree 30 explicitly known to us (See Lemma 11). This observation together with Definition 19 suggests the following lemma.

Lemma 25. *A singularity of M_2 corresponds to $(A,B,C,D) = (0,0,0,\mathbb{P}^1 - \{0\})$ where A, B, C and D denote the classical invariants.*

Proof. Suppose $[\tilde{C}]$ is a singular point of M_2. Since $\tilde{C}$ is a hyperelliptic curve of genus two it follows that $D \neq 0$. Since we work over the complex number field this implies that $D \in \mathbb{P}^1 - \{0\}$. We may assume $D = 1$. Then solving the system

$$\begin{aligned}
\frac{\partial F}{\partial A}(A,B,C,D) &= 0 \\
\frac{\partial F}{\partial B}(A,B,C,D) &= 0 \\
\frac{\partial F}{\partial C}(A,B,C,D) &= 0 \\
\frac{\partial F}{\partial D}(A,B,C,D) &= 0
\end{aligned}$$

using a computer software package yields $A = B = C = 0$ and $D \neq 0$. Hence, if $[\tilde{C}] \in M_2$ is a singularity then $(A,B,C,D) = (0,0,0,\mathbb{P}^1 - \{0\})$. □

Lemma 26. *There exists an unique point $[\tilde{C}] \in M_2$ corresponding to $(A,B,C,D) = (0,0,0,\mathbb{P}^1 - \{0\})$.*

Proof. The hyperelliptic curve $\tilde{C}$ of genus 2 is of the form $\tilde{C} : y^2 = (x - \mu_1) \cdot \ldots \cdot (x - \mu_6)$ where $\mu_1, \ldots, \mu_6 \in \mathbb{P}^1$ such that $\mu_i \neq \mu_j$ for all $i \neq j$. Since there exists an automorphism of $\mathbb{P}^1$ such that, without loss of generality, $(\mu_1, \ldots, \mu_6) \mapsto (0, 1, \infty, \mu_4, \mu_5, \mu_6)$ it follow that $\tilde{C} : y^2 = x(x-1)(x-\mu_4)(x-\mu_5)(x-\mu_6)$. Expanding this last expression yields $\tilde{C} : y^2 = x^5 + (1 - \mu_4 - \mu_5 - \mu_6)x^4 + (\mu_4 + \mu_5 + \mu_6 + \mu_4\mu_5 + \mu_4\mu_6 + \mu_5\mu_6)x^3 - (\mu_4\mu_5 + \mu_4\mu_6 + \mu_5\mu_6 + \mu_4\mu_5\mu_6)x^2 + \mu_4\mu_5\mu_6 x$. Substitute the coefficients of this binary sextic into the invariants i_2, i_4 and i_6 as given as on *http://www.uni-due.de/ hx0013/ generators/6I2.php*. Then the scheme S over the affine three space in the variables μ_4, μ_5 and μ_6 with respect to the ideal $< i_2, i_4, i_6 >$ is the locus corresponding to $(i_2, i_4, i_6, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$. Equivalently, S is the locus corresponding to $(A, B, C, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$. Computing S with help of *Magma* it follows that S consists of 146 points. Since we consider hyperelliptic curves it must follows that $\mu_4, \mu_5, \mu_6 \in \mathbb{P}^1 - \{0, 1, \infty\}$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Substituting this additional condition into the obtained points of S gives that there exists an unique point P of S satisfying these conditions. Hence, there is an unique point $[\tilde{C}] \in M_2$ such that $(A, B, C, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$. □

We are now arrived at the same point as Jun-Ichi Igusa [13], i.e., M_2 contains an unique singularity. But from the data of the proof of Lemma 26 we can compute the form of the hyperelliptic curve corresponding to the singularity of M_2.

Corollary 14. *The hyperelliptic curve $\tilde{C}$ of genus two which corresponds to the condition $(A, B, C, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$ is given by*

$$\tilde{C} : y^2 = x^5 - 1.$$

Proof. Rewriting the point P of S from our proof of Lemma 26 gives $P = (0, -1, -(-1)^{\frac{1}{4}}, (-1)^{\frac{2}{5}}, -(-1)^{\frac{3}{5}}, (-1)^{\frac{4}{5}})$. Hence,

$$\tilde{C} : y^2 = x \cdot (x + 1) \cdot (x + (-1)^{\frac{1}{5}}) \cdot (x - (-1)^{\frac{2}{5}}) \cdot (x + (-1)^{\frac{3}{5}}) \cdot (x - (-1)^{\frac{4}{5}}).$$

Factorizing this expression using a computer software package yields

$$\tilde{C} : y^2 = x^6 - x.$$

The curve $\tilde{C}$ defined through the binary sextic $x^6 - x$ is the 'same' as the curve, by abuse of notation, $\tilde{C}$ defined through the binary sextic $x^5 - 1$ since $x^6 - x = x(x^5 - 1)$. □

Combining the results of this section so far and using Theorem 3 proves the following theorem.

Theorem 18. *The variety M_2 contains exactly one singular point, say $[\tilde{C}]$. Then $\tilde{C} : y^2 = x^5 - x$. Moreover, $\mathrm{Aut}(\tilde{C})/ < \iota > \cong \mathbb{Z}/5\mathbb{Z}$ and $(A, B, C, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$. Here A, B, C and D are the (classical) invariants of $\tilde{C}$.*

Remark 5. *One can prove that the variety of moduli contains a unique singularity by using covering spaces. We give a very rough sketch of this idea. Fix a genus g. Our idea is to construct a fine moduli space, say F_g, over the coarse moduli space M_g. This is a 'mapping' $F_g \to M_g$ which forget about the level structure of F_g. The difficult part is that we do not know when a point of F_g induces a singularity of M_g. What will help is* Deformation Theory*. More precise, if a point q on M_g do not allow no non-trivial deformations then q is a singularity of M_g. Making this idea mathematically rigorous gives an intrinsic description of the singularities of M_g.*

In the following chapter we will give suggestions for further research and we will give a summary of our work written down in this thesis.

Chapter 6

Conclusion

6.1 Suggestions for further research

We give nine suggestions for further research. This can be done in the form of a Master's thesis or in the form of an article.

1. In Chapter 2 we studied properties of (non)-hyperelliptic curves of given genus. It is possible to significantly extend this list of properties.

2. In Chapter 3 we proved that the number of hyperelliptic curves of genus $g \in \mathbb{Z}_{\geq 3}$ is negligible compared with the number of non-hyperelliptic curves of genus. It is therefore desirable to have an explicit description of the moduli space of non-hyperelliptic curves of genus $g \in \mathbb{Z}_{\geq 3}$.

3. Given an explicit description of the moduli space of hyperelliptic and non-hyperelliptic curves of genus $g \in \mathbb{Z}_{\geq 3}$. How can we 'glue' these spaces to obtain, in an explicit and elementary manner, the moduli space of curves of genus g.

4. In Chapter 3 we gave an explicit description of $\mathcal{M}_2$ using *Invariant Theory*. Give an explicit description of $\mathcal{M}_2$ using *Theta-functions*. The *Teichmüller* approach, the *Hodge Theory* approach and the construction of $\mathcal{M}_2$ which uses the *Torelli-map* can be found in almost all 'standard' books on Moduli Spaces.

5. Give an explicit stackification of M_2 from the point of view of our thesis. Moreover, put a level three structure (a hyperelliptic curve of genus two admits three degrees of freedom, i.e., is parameterized by three points of $\mathbb{C} - \{0, 1\}$) on M_2 to obtain the fine moduli space M_2' of hyperelliptic curves of genus two. Let G be a finite group. Then M_2'/G can be considered as a stack.

6. In Chapter 3 we obtained that $\mathcal{B}_6 = \operatorname{Proj} \mathbb{C}[A, \ldots, E]/(E^2 - F(A, B, C, D)) = \operatorname{Proj} \mathbb{C}[A, B, C, D] = \mathbb{P}(1, 2, 3, 5)$ and that $\mathcal{M}_2$ is a parameterized, in terms of (classical) invariants, through a closed subset of $\mathcal{B}_6$. That is, the (classical) invariant E seems to be irrelevant in constructing the moduli space of hyperelliptic curves of genus two. We 'conjecture' that the geometry of the stack structure on M_2 depends on E.

7. Use coverings to prove that M_2 contains an unique singular point and, hence, obtain an intrinsic description of the singularities of M_2.

8. We obtained that $\operatorname{Aff}(M_2) = \mathbb{A}^3(\mathbb{C})/(\mathbb{Z}/5\mathbb{Z})$ and an unique singularity $[C] \in M_2$ such that $\operatorname{Aut}(C)/ < \iota > \cong \mathbb{Z}/5\mathbb{Z}$. Can these results be related to each other. Moreover, explain the appearance of the action of $\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{A}^3(\mathbb{C})$, i.e., we obtained it simply from some formulas.

9. Most of our suggestions for further research concerns M_2. Of course it is interesting to study how these techniques generalizes in the case M_g where $g \in \mathbb{Z}$.

6.2 Summary

Let $C/\mathbb{C}$ a curve of genus $g \in \mathbb{Z}_{\geq 2}$ and let $\omega_1, \ldots, \omega_g$ be a basis for $H^0(C, \Omega_C)$. We defined the *canonical map* $\varphi_K : C \to \mathbb{P}^{g-1}(\mathbb{C})$ given by $p \mapsto [\omega_1(p) : \ldots : \omega_g(p)]$. We defined C to be a *hyperelliptic curve* of genus g iff φ_K is injective. If φ_K is not injective we called C a non-hyperelliptic curve of genus g. We proved that C is a hyperelliptic curve of genus g iff there exits a holomorphic double covering map $C \to \mathbb{P}^1$ iff $\varphi_K : C \to \mathbb{P}^{g-1}(\mathbb{C})$ is not an isomorphism onto its image. The main ingredients of these proves where (i) the *Theorem of Riemann-Roch*, (ii) the *Riemann-Hurwitz Formula* and (iii) *Serre Duality*. In case $g = 2$ we proved that all curves are hyperelliptic curves and if $g \geq 3$ that there exists hyperelliptic curves and non-hyperelliptic curves. Moreover, we showed that if $g \geq 3$, then the number of hyperelliptic curves of genus g is negligible compared with the number of non-hyperelliptic curves of genus g. For what will follows, let $C/\mathbb{C}$ be a hyperelliptic curve of genus g. Then there exists a holomorphic, double covering map $x : C \to \mathbb{P}^1$ such that $x(\infty) = p + q =: D \in \operatorname{Div} C$ where $p \neq q$. Applying the *Theorem of Riemann-Roch* to $\mathcal{L}(D), \mathcal{L}((g+1)D)$ and to $\mathcal{L}((2g+2)D)$ gives

$$C : y^2 = \prod_{i=1}^{2g+2} (x - \mu_i)$$

where $\mu_i \in \mathbb{P}^1$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Moreover, if C is of the above form, we proved that C is a hyperelliptic curve of genus g. It follows naturally that C can be associated to an unordered $(2g+2)$-tuple, write $C = C_\mu$ where $\mu := (\mu_1, \ldots, \mu_{2g+2})$. Conversely, every unordered $(2g+2)$-tuple in $\mathbb{P}^{2g+2}(\mathbb{C})$ can be associated to a hyperelliptic curve of genus g. We obtained

$$C \leftrightsquigarrow (\mu_1, \ldots, \mu_{2g+2}) \in \mathbb{P}^{2g+2}(\mathbb{C}).$$

If $M \in \operatorname{Aut}(\mathbb{P}^1)$ then $M\mu \sim \mu'$. Hence, we obtained a natural isomorphism

$$C = C_\mu \cong C_{\mu'} = C'.$$

This gives reasons to study hyperelliptic curves of genus g all at once instead of studying them separately. We defined $\mathcal{M}_g := \{[C] \mid C \text{ a (hyperelliptic) curve of genus } g\}$. Then we obtained $\mathcal{M}_0 = P \subset \mathbb{C}$ a singleton subset and $\mathcal{M}_1 = \mathbb{A}^1(\mathbb{C})$. From this point we restricted ourselves to the case $g = 2$. Then $C : y^2 = f(x, 1)$ where $f \in \mathcal{B}_6 := \{f \in \mathbb{C}[x, y] \mid ; f(x, y) = \lambda_0 x^6 + \lambda_1 x^5 y + \lambda_2 x^4 y^2 + \lambda_3 x^3 y^3 + \lambda_4 x^2 y^4 + \lambda_5 x y^5 + \lambda_6 y^6\}$. We defined a linear transformation Gl_2 action on $\mathcal{B}_6$ and defined 'invariants' of this action on $\mathcal{B}_6$. The set of invariants of $\mathcal{B}_6$ with respect to the action of Sl_2 is written like $\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2}$ which we called the classical ring of invariants. We proved that the classical ring of invariants is a graded ring and that

$$\mathbb{C}[\lambda_0, \ldots, \lambda_6]^{\mathrm{Sl}_2} = \mathbb{C}[A, \ldots, E]/(E^2 - F(A, B, C, D)).$$

Here $A, \ldots, E$ are classical invariants and we obtained an explicit description of $A, \ldots E$ and $F(A, B, C, D)$. Introducing the *Projective Scheme*, Proj, we proved

$$\operatorname{Proj} \mathbb{C}[A, \ldots, E]/(E^2 - F(A, B, C, D)) = \operatorname{Proj} \mathbb{C}[A, B, C, D] =: \mathbb{P}(1, 2, 3, 5).$$

Then we obtained $\mathcal{M}_2$ as parameterization in terms of classical invariants of $\mathcal{B}_6$ through an open subset of $\mathcal{B}_6$, i.e., the complement to the set where the discriminant, D, vanishes. We proved that $Z(D) := \{D = 0\} = \mathbb{P}(1, 2, 3)$ for which we gave an affine description. Computing $Z(D)^C$ gives

$$\mathcal{M}_2 \leftrightarrow \operatorname{Spec} \mathbb{C}[A^{-5}D, B^{-5}D^2, C^{-5}D^3].$$

Moreover, we obtained an affine description of $\mathcal{M}_2$

$$\mathcal{M}_2 \leftrightarrow \mathbb{A}^3(\mathbb{C})/\left(\mathbb{Z}/5\mathbb{Z}\right).$$

Furthermore, we proved that $\mathcal{M}_2$ can be embedded into $\mathcal{A}^8(\mathbb{C})$. This was easily proved by observing how $\mathbb{Z}/5\mathbb{Z}$ acts on the coordinates of $\mathbb{A}^3(\mathbb{C})$. Having obtained $\mathcal{M}_2$ as set we put structure on it sch that we can do geometry on it. Otherwise stated, we defined formally the problem of moduli. As we are only interested whether or not a moduli space admits the structure of a variety we describe a general variety X as (contravariant) functor and proved that this functor is independent of the choice of coordinates on X. That is, the (contravariant) functor gives rise to the functor of points. Notice, in essence we used the two degrees of freedom which a variety contains to describe it as a (contravariant) functor. This is a way of describing the moduli problem as functor

$$\mathcal{F} : \mathcal{A}_k \to (\mathcal{S}\text{ets})$$

which assigns to a k-algebra, say S, the set of families of algebraic objects over S. That is, we obtained a 'space' which parameterizes all algebraic objects under study. At his turn, this reveals the definition of a *fine moduli space* (the functor $\mathcal{F}$ is representable) and a *coarse moduli space* (the functor $\mathcal{F}$ is a best approximation with an additional constraint concerning universality). In case the algebraic objects are curves of genus g we write $\mathcal{F}_{\mathcal{M}_g}$ instead of $\mathcal{F}$. We proved uniqueness (up to isomorphism) of $\mathcal{F}$ and proved that $\mathcal{F}_{\mathcal{M}_g}$ is a coarse moduli space in case of $g = 0, 1, 2$, i.e., $M_0 = \operatorname{Spec}\mathbb{C}, M_1 = \operatorname{Spec}\mathbb{C}[j]$ and M_2 is isomorphic to $\operatorname{Spec}\mathbb{C}[A^5D^{-1}, B^5D^{-2}, C^5D^{-3}]$. Since we obtained $\operatorname{Set}(M_2) = \mathcal{M}_2$ as parameterization of an open subset of $\mathcal{M}_2 \subset \operatorname{Proj}\mathbb{C}[A, \ldots, E]/(E^2 - F(A, B, C, D))$ and an explicit description of $F(A, B, C, D)$ we used the failure of the *Implicit Function Theorem* to obtain the singularities of M_2. We proved that $[\tilde{C}] \in M_2$ is a singularity iff $A(\tilde{C}) = B(\tilde{C}) = C(\tilde{C}) = 0$ and $D(\tilde{C}) \neq 0$. Furthermore, we proved that the condition $(A, B, C, D) = (0, 0, 0, \mathbb{P}^1 - \{0\})$ implies that $[\tilde{C}] \in M_2$ is the unique singularity of M_2, that $\tilde{C} : y^2 = x^5 - 1$ and that $\operatorname{Aut}(\tilde{C})/ < \iota > \cong \mathbb{Z}/5\mathbb{Z}$. Moreover, we discussed the work of Jun-Ichi Igusa [13] concerning singularities of the variety of moduli of hyperelliptic curves of genus two. As a consequence, we obtained the so-called *Igusa invariants* in terms of our invariants A, B, C, D and E. Meaning, instead of considering curves of genus two over the complex number field it is possible to consider curves of genus two over an arbitrary field, even of characteristic two, in our conventions. Additionally, we obtained the dimension of the variety of moduli of hyperelliptic curves of genus two by using *Deformation Theory.* We also gave a gentle, but effective, introduction to the theory of Algebraic Geometry concerning *Categories*, *Functors*, *Sheaves*, *Ringed Spaces*, *Schemes* and *Stacks.*

Chapter 7

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Martijn van der Valk, summer 2011

Chapter 8

Appendix

8.1 Algebraic Geometry

In this section we introduce some important notions from Algebraic Geometry. For an extensive discussion of these topics we refer to D. Eisenbud and J. Harris [7], O. Forster [9], R. Hartshorne [11] and S. Mac Lane [15].

8.1.1 Categories

In this section we summarize basic definitions from *Category Theory* which are necessary but sufficient to define formally the problem of moduli.

Definition 20 (Categories)**.** *A category $\mathcal{C}$ consist of three things:*

1. *a class of objects,*
2. *for each pair X, Y of objects a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose elements are called morphisms* and
3. *for each triple X, Y, Z of objects a set $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ denoted by $(f, g) \mapsto g \circ f$.*

The morphisms are required to be (i) associative: $(f \circ g) \circ h = f \circ (g \circ h)$ and (ii) there exist identities: for any object X in $\mathcal{C}$ there exists an identity morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ satisfying $Id_Y \circ f = f = f \circ \mathrm{Id}_X$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Furthermore, a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called an isomorphism in $\mathcal{C}$ if there exists a morphism $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $f \circ f^{-1} = \mathrm{Id}_Y$ and $f^{-1} \circ f = Id_X$.

We define the category $\mathcal{H}$ for which the objects are hyperelliptic curves C of genus 2 over a scheme S, $\pi : C \to S$, such that the fibers are curves. Here the hyperelliptic curves are irreducible, non-singular and are taken over the field of complex numbers. For the morphisms in $\mathcal{H}$ we take *Cartesian diagrams.*

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

Equivalently, we require $C \to C' \times_{S'} S$ to be an isomorphism in the natural way. An important notion in *Category Theory* is the notion of a functor. In essence, a functor is a rule which describes how we get from a given category into another category.

Definition 21 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a rule $\mathcal{F}$ that assigns to each object X in $\mathcal{C}$ an object $\mathcal{F}(X)$ in $\mathcal{D}$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y))$ such that (i) $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$ and (ii) $\mathcal{F}(g \circ h) = \mathcal{F}(g) \circ \mathcal{F}(h)$. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a rule $\mathcal{F}$ that assigns to each object X in $\mathcal{C}$ an object $\mathcal{F}(X)$ in $\mathcal{D}$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y),\mathcal{F}(X))$ such that (i) $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$ and (ii) $\mathcal{F}(g \circ h) = \mathcal{F}(h) \circ \mathcal{F}(g)$.*

Definition 22 (Natural transformation)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be functors from $\mathcal{C}$ to $\mathcal{D}$. We say that θ is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$ if it assigns to each object X in $\mathcal{C}$ a morphism θ_X from $\mathcal{F}(X)$ to $\mathcal{G}(X)$ in $\mathcal{D}$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ yields that $\mathcal{F}(f) \circ \theta_X = \theta_Y \circ \mathcal{F}(f)$. Equivalently, the diagram*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ {\scriptstyle \theta_X}\big\downarrow & & {\scriptstyle \theta_Y}\big\downarrow \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

commutes. Furthermore, the natural transformation is a natural isomorphism if each θ_X is an isomorphism.

In the next section we state the notion of a scheme. We introduce this notion since everywhere we write variety in our work we can replace it by scheme in an appropriate manner.

8.1.2 Schemes

As our work in this thesis is over the field of complex numbers, we will use this to define schemes. Schemes were introduced to broaden the notion of a variety. More precise, there exists a bijective correspondence between (affine) varieties and reduced finitely generated $\mathbb{C}$-algebras. In scheme theory one obtains a bijective correspondence between (affine) schemes and arbitrary rings. That is, we loose the structure of a module. Indeed, schemes admit doing geometry over any ring.

Definition 23 (Presheaf)**.** *Let X be a topological space. Let $\mathrm{Open}\,X$ be the category consisting of objects which are given by open subsets of X and unique morphisms defined by inclusion of subsets $V \subset U$ of X. A presheaf on X with values in a category $\mathcal{C}$ is a functor $\mathcal{F} : \mathrm{Open}\,X \to \mathcal{C}$.*

As a guiding example we consider an algebraic variety X over $\mathbb{C}$. As usual, we put the Zariski topology on X. Consider the set of morphism, $\mathcal{O}_X$, of X onto $\mathbb{C}$, i.e., $\mathcal{O}_X(U) := \{f : U \to \mathbb{C} \mid f \text{ a morphism}\}$. Obviously, $\mathcal{O}_X$ is a ring by point wise addition and multiplication. If we take the morphisms in our category to be the obvious restriction maps induced by the obvious inclusions $V \subset U$ it follows that $\mathcal{O}_X$ is a presheaf of rings. Similarly, the ring of meromorphic functions $\mathcal{M}(U) := \{f : U \to \mathbb{C} \mid f \text{ meromorphic}\}$ together with the natural restrictions of meromorphic functions induced by the obvious inclusions $V \subset U$ implies that $\mathcal{M}_X$ is a presheaf of rings. Even so, the ring of holomorphic differentials Ω_X on X together with the natural restriction induced by the obvious inclusions $V \subset U$ is a presheaf of rings. The latter two are used frequently in Chapter 2.

Definition 24 (Sheaf)**.** *Let $\mathcal{F}$ be a presheaf on a topological space X. We define $\mathcal{F}$ to be a sheaf if for any open subset $U \subset X$ and any open covering $\{U_i\}_{i \in I}$ of U where I an index set the following properties hold:*

1. *if for all $f_1, f_2 \in \mathcal{F}(\mathcal{U})$ and for all $i \in I$, $\rho_U^{U_i}(f_1) = \rho_U^{U_i}(f_2)$, then $f_1 = f_2$*
2. *for a collection of elements $f_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i}^{U_i \cap U_j}(f_j) = \rho_{U_j}^{U_i \cap U_j}(f_j)$ for all $i \neq j$ there exists an element $f \in \mathcal{F}(\mathcal{U})$ such that $\rho_{\mathcal{U}}^{U_i}(f) = x_i$ for all $i \in I$.*

Here ρ_X^Y is the restriction map $\mathcal{F}(Y) \to \mathcal{F}(X)$ induced by $Y \subset X$.

Notice, $\mathcal{O}(U)$ and $\mathcal{M}(U)$ can be considered as functions on a subset U of a topological space X. From that point of views, the first condition of Definition 24 states that a function on U is uniquely determined by its restriction to open subsets from the open covering $\{U_i\}_{i\in I}$ of U. That is, we obtain from global data the local data. The second condition of Definition 24 states that given a bunch of functions on each open subset from the open covering $\{U_i\}_{i\in I}$ of U such that these functions coincide on all possible intersections of these subsets, then the locally defined functions can be glued to functions defined on U. That is, we obtain from local data global data.

Proposition 16. *The presheaves $\mathcal{O}_X, \mathcal{M}_X$ and Ω_X are sheafs.*

Proof. First consider $\mathcal{O}_X$. Let $\mathcal{U} := \{U_i\}_{i\in I}$ be an open cover of $U \subset X$ where I an index set and let $\varphi_i \in \mathcal{O}(U)$ for $i \in I$ such that $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$. Define $\varphi : \mathcal{U} \to \mathbb{C}$ by $\varphi_i(u) = \varphi(u)$ for all $i \in I$ and for all $u \in U_i$. Obviously, this morphism is well-defined and $\varphi|_{U_i} = \varphi_i$ for all $i \in I$. Hence, condition 2 of Definition 24 is satisfied. Let $\varphi, \psi \in \mathcal{O}(\mathcal{U})$ such that the restriction $U_i \cap \mathcal{U}$ on φ equals the restriction of $U_i \cap \mathcal{U}$ on ψ for all $i \in I$. Then $\varphi|_{U_i} = \psi|_{U_i}$ for all $i \in I$ since the restrictions are the natural ones; so $\varphi = \psi$. Hence, condition 1 of Definition 24 is satisfied. This proves that $\mathcal{O}_X$ is a sheaf of rings with respect to the Zariski topology. Similarly, one proves that $\mathcal{M}_X$ and Ω_X are sheafs with respect to their topologies. □

Definition 25 (Morphisms of sheaves)**.** *A morphism of sheaves $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a collection of morphisms $f(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ such that for every inclusion $V \subset U$ the diagram*

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{f(U)} & \mathcal{F}_2(U) \\ {\scriptstyle \rho^V_{1,U}}\downarrow & & {\scriptstyle \rho^V_{2,U}}\downarrow \\ \mathcal{F}_1(V) & \xrightarrow{f(V)} & \mathcal{F}_2(V) \end{array}$$

commutes. Here $\rho^V_{1,U}$ and $\rho^V_{2,U}$ are the restriction maps induced by the inclusion $V \subset U$.

A scheme can be defined in terms of so-called (locally) ringed spaces. One encounters (locally) ringed spaces in many fields within mathematics without paying attention to it For example, smooth, differentiable and complex manifolds are all examples of locally ringed spaces. In Algebraic Geometry we find out that there exists a bijective correspondence between points of subset U of an affine variety X and the maximal ideals of $\mathcal{F}(U)$. However, if we consider a projective variety X, this bijective correspondence does not hold true any longer. Otherwise stated, this distinguish Algebraic Geometry from Commutative Ring Theory as it is in essence if we only consider affine varieties. However, the intuitive idea of ringed spaces is that if we take an arbitrary small open subset U of a variety X, we always obtain a bijective correspondence between points of U and maximal ideals of $\mathcal{F}(U)$.

Definition 26 (Stalk)**.** *Let X be a topological space, let $\mathcal{F}$ be a presheaf and let $x \in X$ a point. The stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is the set*

$$\{(U, f) \mid U \text{ open neighborhood of } x \text{ and } f \in \mathcal{F}(U)\}/\sim$$

where the equivalence relation $\sim$ is given by $(U, f) \sim (U, g)$ for $f, g \in \mathcal{F}(U)$ iff $f(U \cap V) = g(U \cap V)$.

It is easily verified that the equivalence relation in definition 26 is (i) reflexive, (ii) symmetric and (iii) transitive. Equivalently, the stalk of a sheaf defined as in definition 26 is well-defined. Intuitively, the stalk at a point x of a sheaf is what remains if we consider arbitrary small neighborhoods of some point x.

Definition 27 (Ringed space)**.** *A ringed space is a pair $(X, \mathcal{F})$ consisting of a topological space X and a sheaf $\mathcal{F}$. The pair $(X, \mathcal{F})$ is a locally ringed space if for any point $x \in X$ the stalk $\mathcal{F}_x$ consists of a unique maximal ideal.*

Formally, the requirement $\mathcal{F}_x$ consist of a unique maximal ideal in Definition 27 means that $\mathcal{F}_x$ is a local ring. Obviously, from Definition 27 it follows that if we take an arbitrary small open neighborhood of some point x of a topological space X, then the stalk $\mathcal{F}_x$ consist of the unique maximal ideal which corresponds to the point x itself. In the case of $\mathcal{O}_X$ the stalk $\mathcal{O}_x$ at a point $x \in X$ is the ideal of morphisms which vanishes at x. Similarly, $\mathcal{M}_x$ is the stalk consisting of meromorphic functions which vanish at x. Even so, Ω_x is the stalk of holomorphic differential 1-forms which vanish at x. This proves the following proposition.

Proposition 17. *The pairs $(X, \mathcal{O}_X), (X, \mathcal{M}_X)$ and (X, Ω_X) are locally ringed spaces.*

Definition 28 (Morphism of locally ringed spaces)**.** *A morphism of ringed spaces is a pair $(f, f') : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is a continuous map between topological spaces and $f' : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves. The pair (f, f') is a morphism of locally ringed spaces if for all $x \in X$ the map f' induces a local ring homomorphism $f'_x : \mathcal{O}_{f(x)} \to \mathcal{O}_x$, i.e., f' maps the unique maximal ideal of $\mathcal{O}_{f(x)}$ onto the unique maximal ideal of $\mathcal{O}_x$.*

Proposition 18. *A morphism $\varphi : X \to Y$ of varieties induces a morphism of locally ringed spaces $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.*

Proof. The morphism $\varphi : X \to Y$ is by definition continuous with respect to the Zariski topology. Obviously, we have a canonical morphism of sheaves $\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ given by $f \mapsto f \circ \varphi$. The maximal ideal of a point $x \in X$ in the stalk $(\mathcal{O}_X)_x$ consist of all morphisms which vanish at x. If a morphism $f : Y \to \mathbb{C}$ vanishes at a points $\varphi(x)$ it follows that $f \circ \varphi$ vanishes at x. Therefore, the map $\mathcal{O}_Y(-) \to \mathcal{O}_X(\varphi^{-1}(-))$ induces a morphism of local rings $(\mathcal{O}_Y)_{\varphi(x)} \to (\mathcal{O}_X)_x$. Hence, the pair (φ, φ^*) is a morphism of locally ringed spaces. □

Definition 29 (Prime spectrum)**.** *Let R be a commutative ring. The spectrum of R, which we denote by* $\operatorname{Spec} R$*, is a pair consisting of a topological space denoted, by abuse of notation, by* $\operatorname{Spec} R$ *and a sheaf $\mathcal{O}_{\operatorname{Spec} R}$ on* $\operatorname{Spec} R$*. Furthermore, the set* $\operatorname{Spec} R$ *is the set of prime ideals of R on which we put the Zariski topology defined through closed sets $Z(I) := \{\wp \in \operatorname{Spec} R \mid \wp \supset I\}$ over ideals $I \subset R$.*

To be rigorous, one must prove that the defined closed sets of Definition 29 induces a topology on $\operatorname{Spec} R$. For details we refer to R. Hartshorne [11]. Furthermore, one has to define the so-called structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$. Naive spoken, $\mathcal{O}_{\operatorname{Spec} R}$ is the sheaf of morphisms $\mathcal{O}_X$ in which $X = \operatorname{Spec} R$. For each prime ideal $\wp \subset R$, let $R_\wp$ be the localization at $\wp$. For an open set $U \subset \operatorname{Spec} R$ we define $\mathcal{O}(U)$ to be the set of functions $f : U \to \bigsqcup_{\wp \in U} R_p$ such that for each $\wp \in U$ there exists a neighborhood $V \subset U$ of $\wp$ and elements $r, s \in R$ such that for each ideal $v \in V$ it holds that $s \notin v$ and $f(q) = \frac{r}{s}$ in R_v. One proves that sums and products of these functions f are again functions of the same type and that the element 1 which gives 1 in each localization $R_\wp$, is the identity. Hence, $\mathcal{O}(U)$ is a commutative ring. The inclusions $W \subset U$ induces naturally a restriction map $\mathcal{O}(U) \to \mathcal{W}$. Results to prove that $\mathcal{O}$ is a sheaf (See D. Eisenbud [7]). This proves the following proposition.

Proposition 19. *Let R be a commutative ring. The pair* $\operatorname{Spec} R := (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ *is a locally ringed space.*

Another obstruction in Definition 29 might be that we define a Zariski topology on $\operatorname{Spec} R$ while we already have a Zariski topology defined on varieties.

Lemma 27. *The Zariski topology on an affine algebraic variety X over $\mathbb{C}$ coincide with the Zariski topology on* $\operatorname{Spec} X$*.*

Proof. Points of X corresponds with maximal ideals of the coordinate ring $\mathbb{C}[X]$. Any maximal ideal is a prime ideal. Hence, we obtain an injective map $X \hookrightarrow \operatorname{Spec} \mathbb{C}[X]$. Closed sets G of X are zero sets of an ideal I generated by polynomials. Furthermore, the points of G are the maximal ideals containing I. This implies that the Zariski topology on X is the induced subspace topology from the Zariski topology on $\operatorname{Spec} X$ under the map $X \hookrightarrow \operatorname{Spec} X$. □

Any maximal ideal is prime. In general, not every prime ideal is a maximal ideal. This observation suggest that we have to distinguish between points of $\operatorname{Spec} R$ which corresponds to maximal ideals and points of $\operatorname{Spec} R$ which corresponds to prime ideals which are not maximal ideals.

Definition 30 (Closed points)**.** *Let R be a commutative ring. Points of $\operatorname{Spec} R$ which corresponds to maximal ideals are called closed points of $\operatorname{Spec} R$. Points of $\operatorname{Spec} R$ which corresponds to prime ideals which are not maximal ideals are called non-closed points of $\operatorname{Spec} R$.*

Proposition 20. *Let $f : R \to S$ be a ring homomorphism of commutative rings. Then f induces a continuous map $\operatorname{Spec}(f) : \operatorname{Spec} S \to \operatorname{Spec} R$. Furthermore, the map $\operatorname{Spec}(f)$ is functorial.*

Proof. This follows from the fact that the inverse image of a prime ideal $\wp \subset S$ is a prime ideal in R. Moreover, if T is a commutative ring and $g : S \to T$ a ringhomomorphism it is easily seen that $\operatorname{Spec}(g \circ f) = \operatorname{Spec}(g) \circ \operatorname{Spec}(f)$ and $\operatorname{Spec}(\mathrm{Id}) = \mathrm{Id}$. □

Proposition 21. *The prime ideal spectrum of $\mathbb{C}$, $\operatorname{Spec} \mathbb{C}$, is a single closed point.*

Proof. Since $\mathbb{C}$ is a field we have a single maximal ideal $(0) \subset \mathbb{C}$. Hence, $\operatorname{Spec} \mathbb{C}$ is a single closed point. □

Definition 31 (Scheme)**.** *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets of X and I an index set such that $(U, \mathcal{O}_{X|U})$ is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.*

Intuitively, a scheme is obtained by glueing pairs $(\mathrm{R}, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R along open subsets in the Zariski topology.

Definition 32 (Morphism of schemes)**.** *A morphism of schemes is a morphism of locally ringed spaces which are schemes.*

Definition 33 (Scheme over a scheme)**.** *Let X be a scheme. A scheme X over a scheme S is a pair (X, π) where $\pi : X \to S$ a morphism of schemes.*

Instead of saying that X is a scheme over a scheme S we will say that X is a family over S. Both terminology will be used interchangeably.

Definition 34 (Morphisms of schemes over a scheme)**.** *Let X and Y be schemes over a scheme S. A morphism $X \to Y$ is a morphism of schemes X, Y over a scheme S if the composite $X \to Y \to S$ coincide with the morphism $X \to S$.*

In the next subsection we will sketch the concept of a stack.

8.1.3 Stacks

Intuitively, an algebraic stack is a category which is also assumed to be a stack. That is, there exists an op cover $\mathcal{U}$ of schemes such that on each scheme S of $\mathcal{U}$ there exists a family F_S with the following property. If T of $\mathcal{U}$ is an other scheme and F_T a family on T such that S and T overlap, then $F_S = F_T$ on the overlap of S and T. Hence, glueing all schemes of $\mathcal{U}$ gives a family on the whole scheme. However, the failure of being a scheme is measured geometrically. Therefore, algebraic stacks are geometric objects and they turn out to be the best spaces for classifying (algebraic) objects. We will introduce the notion of a stack, weak and vulgar, in the specific case of elliptic curves. This can be considered as complementary to our work in Section 4.2.2.

Definition-proposition 1 (Stack of elliptic curves)**.** *Let $\mathcal{E}$ be the category with objects families of elliptic curves $\pi : E \to X$ and morphisms by pairs (φ, ψ) where $\varphi : E \to E'$ and $\psi : X' \to X$ such that (i) the diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\varphi} & E \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ X' & \xrightarrow{\psi} & X \end{array}$$

commutes and (ii) the elliptic curve E' is isomorphic to the elliptic curve E via ψ. The category $\mathcal{E}$ is called the stack of elliptic curves.

To prove Definition-proposition 1 one have to prove that the category $\mathcal{E}$ is well-defined. The claim is now that there exists an universal family of elliptic curves over $\mathcal{E}$. Notice, M_1 lacks this condition. A variety V can be identified with a category $\mathcal{V}$ in the following way. Take for the objects morphisms of varieties $m : W \to V$ and let the morphisms of $\mathcal{V}$ be a maps $f : W' \to W$ such that $m' = m \circ f$. Then a family of elliptic curves $\pi : E \to V$ is the same thing as a functor $\mathcal{V} \to \mathcal{E}$. Let $\mathcal{E}'$ be the category with objects families of elliptic curves with a section and let the morphisms be given as for $\mathcal{E}$. That is, $\mathcal{E}'$ is obtained as the stack $\mathcal{E}$ with some additional structure on the objects. Forgetting about this structure induces a functor $\mathcal{E}' \to \mathcal{E}$. Now for any family of elliptic curves $\pi : E \to V$ the pullback via $\mathcal{E}$ corresponding to the functor $\mathcal{V} \to \mathcal{E}$ is the variety E, i.e., the variety E corresponds to the categorization of E as described as before. Hence, $\mathcal{E}$ is the moduli space of elliptic curves and $\mathcal{E}' \to \mathcal{E}$ is an universal family over it. This construction can be generalized to curves of arbitrary genus. Hence, stacks solve the problem of moduli space of curves of given genus which do not have a universal family lying over it. In the next section we describe explicit the (classical) invariants.

8.2 Invariants: an explicit description

We defined $\mathcal{B}_6 := \{f \in \mathbb{C}[x,y] \mid f(x,y) = \lambda_0 x^6 + \lambda_1 x^5 y + \lambda_2 x^4 y^2 + \lambda_3 x^3 y^3 + \lambda_4 x^2 y^4 + \lambda_5 x y^5 + \lambda_6 y^6\}$ and called an element of $\mathcal{B}_6$ a binary sextic. Let $f \in \mathcal{B}_6$ and let, by the *Fundamental Theorem of Algebra*, $\mu_1, \ldots, \mu_6$ be its zeros. As is suggested by Jun-Ichi Igusa [13], we abbreviate $(\mu_i - \mu_j)$ by (ij) for $1 \leq i, j \leq 6$. We consider the following three arrangements.

1. Arranging $\mu_1, \ldots, \mu_6$ in three groups each consisting of two of them gives: $(12)\{(34)(56), (35)(46), (36)(45)\}, (13)\{(24)(56), (25)(46), (26)(45)\}, (14)\{(23)(56), (25)(36), (26)(35)\}, (15)\{(23)(46), (24)(36), (26)(34)\}$ and $(16)\{(23)(45), (24)(35), (25)(34)\}$; so there exist 15 possibilities.

2. Arranging $\mu_1, \ldots, \mu_6$ in two groups each consisting of three of them and then arrange each of these groups in groups consisting of two of them gives: $(12)\{(23)(31)(45)(56)(64), (24)(41)(35)(56)(63), (25)(51)(34)(46)(63), (26)(61)(34)(45)(53)\}, (13)\{(34)(41)(25)(56)(62), (35)(51)(24)(46)(62), (36)(61)(24)(45)(52)\}, (14)\{(45)(51)(23)(36)(62), (46)(61)(23)(34)(42)\}$ and $(15)(56)(61)(23)(34)(42)$; so there exist 10 possibilities.

3. In ad. 2 there exists six 'pairings' between any arrangement. These pairings are: $[(14)\{(25)(36), (26)(35)\}, (15)\{(24)(26), (26)(34)\}, (16)\ \{(24)(35), (25)(34)\}], [(13)\{(25)(46), (26)(45)\}, (15)\{(23)(46), (26)(43)\}, (16)\{(25)(46), (26)(45)\}], [(13)\{(24)(56), (26)(54)\}, (14)\{(23)(56), (26)(53)\}, (16)\{(23)(54), (24)(53)\}], [(13)\{(24)(65), (25)(64)\}, (14)\{(23)(65), (25)(63)\}, (15)\{(23)(64), (24)(63)\}], [(12)\{(35)(46), (36)(45)\}, (15)\{(32)(46), (36)(42)\}, (16)\{(32)(45), (35)(42)\}], [(12)\{(34)(56), (36)(54)\}, (14)\{(32)(56), (36)(52)\}, (16)\{(32)(54), (34)(52)\}], [(12)\{(34)(56), (35)(64)\}, (14)\{(32)(65), (35)(62)\}, (15)\{(32)(64), (34)(61)\}], [(12)\{(43)(56), (46)(53)\}, (13)\{(42)(56), (46)(52)\}, (16)\{(42)(53), (43)(52)\}], [(12)\{(43)(56), (45)(63)\}, (13)\{(24)(56), (45)(26)\}, (15)\{(42)(63), (34)(26)\}]$ and $[(12)\{(53)(64), (54)(63)\}, (13)\{(52)(64), (54)(62)\}, (14)\{(52)(63), (53)(61)\}]$; so there exist 60 possibilities.

Consider $\mathcal{B}_6 \ni f = (x-a)(x-b)(x-c)(x-d)(x-f)(x-g)$. Using Definition 8, Lemma 9 and Lemma 10 it is now easy to give an explicit description of the invariants A, B, C, D and E of f. Since A yields the smallest expression, except D, we will only give A. That is, $A = 6a^2b^2c^2 - 4a^2b^2cd - 4a^2bc^2d - 4ab^2c^2d + 6a^2b^2d^2 - 4a^2bcd^2 - 4ab^2cd^2 + 6a^2c^2d^2 - 4abc^2d^2 + 6b^2c^2d^2 - 4a^2b^2cf - 4a^2bc^2f - 4ab^2c^2f -$

$4a^2b^2df+12a^2bcdf+12ab^2cdf-4a^2c^2df+12abc^2df-4b^2c^2df-4a^2bd^2f-4ab^2d^2f-4a^2cd^2f+12abcd^2f-$
$4b^2cd^2f-4ac^2d^2f-4bc^2d^2f+6a^2b^2f^2-4a^2bcf^2-4ab^2cf^2+6a^2c^2f^2-4abc^2f^2+6b^2c^2f^2-4a^2bdf^2-$
$4ab^2df^2-4a^2cdf^2+12abcdf^2-4b^2cdf^2-4ac^2df^2-4bc^2df^2+6a^2d^2f^2-4abd^2f^2+6b^2d^2f^2-4acd^2f^2-$
$4bcd^2f^2+6c^2d^2f^2-4a^2b^2cg-4a^2bc^2g-4ab^2c^2g-4a^2b^2dg+12a^2bcdg+12ab^2cdg-4a^2c^2dg+12abc^2dg-$
$4b^2c^2dg-4a^2bd^2g-4ab^2d^2g-4a^2cd^2g+12abcd^2g-4b^2cd^2g-4ac^2d^2g-4bc^2d^2g-4a^2b^2fg+12a^2bcfg+$
$12ab^2cfg-4a^2c^2fg+12abc^2fg-4b^2c^2fg+12a^2bdfg+12ab^2dfg+12a^2cdfg-120abcdfg+12b^2cdfg+$
$12ac^2dfg+12bc^2dfg-4a^2d^2fg+12abd^2fg-4b^2d^2fg+12acd^2fg+12bcd^2fg-4c^2d^2fg-4a^2bf^2g-$
$4ab^2f^2g-4a^2cf^2g+12abcf^2g-4b^2cf^2g-4ac^2f^2g-4bc^2f^2g-4a^2df^2g+12abdf^2g-4b^2df^2g+12acdf^2g+$
$12bcdf^2g-4c^2df^2g-4ad^2f^2g-4bd^2f^2g-4cd^2f^2g+6a^2b^2g^2-4a^2bcg^2-4ab^2cg^2+6a^2c^2g^2-4abc^2g^2+$
$6b^2c^2g^2-4a^2bdg^2-4ab^2dg^2-4a^2cdg^2+12abcdg^2-4b^2cdg^2-4ac^2dg^2-4bc^2dg^2+6a^2d^2g^2-4abd^2g^2+$
$6b^2d^2g^2-4acd^2g^2-4bcd^2g^2+6c^2d^2g^2-4a^2bfg^2-4ab^2fg^2-4a^2cfg^2+12abcfg^2-4b^2cfg^2-4ac^2fg^2-$
$4bc^2fg^2-4a^2dfg^2+12abdfg^2-4b^2dfg^2+12acdfg^2+12bcdfg^2-4c^2dfg^2-4ad^2fg^2-4bd^2fg^2-4cd^2fg^2+$
$6a^2f^2g^2-4abf^2g^2+6b^2f^2g^2-4acf^2g^2-4bcf^2g^2+6c^2f^2g^2-4adf^2g^2-4bdf^2g^2-4cdf^2g^2+6d^2f^2g^2$.
The invariants A, B, C, D and E of a binary sextic f can also be given not in terms of the zeros of f. If $\mathcal{B}_6 \ni f(x,y) = \lambda_0 x^6 + \lambda_1 x^5 y + \lambda_2 x^4 y^2 + \lambda_3 x^3 y^3 + \lambda_4 x^2 y^4 + \lambda_5 x y^5 + \lambda_6 y^6$ then one verifies that

$$A' = -240\lambda_0\lambda_6 + 40\lambda_1\lambda_5 - 16\lambda_2\lambda_4 + 6\lambda_3^2$$

is an invariant of degree two of f. One can find a Maple program on *http://www.uni-due.de/ hx0013/ generators/6I2.php* which computes the invariants B', C', D' and E' of f in terms of the λ's. Obviously, there exists an unique relation between (A,B,C,D,E) and (A',B',C',D',E') up to scalar multiplication. Moreover, let $C : y^2 = x^5 - 1 =: f(x)$. Then an direct computation using *Magma* yields $A(f) = B(f) = C(f) = A'(f) = B'(f) = C'(f) = 0$ and $D(f) = D'(f) = 3125$. Naturally, we ask ourselves whether or not $A(g) = A'(g), B(g) = B'(g), C(g) = C'(g)$ and $D(g) = D'(g)$ for all $g \in \mathcal{B}_6$. To answer that question, consider $g = (x-a)\cdot(x-b)\cdot(x-c)\cdot(x-d)\cdot(x-f)\cdot(x-g)$. Then, using a computer software package, it follows that $g = abcdfg-(abcdf+abcdg+abcfg+abdfg+acdfg+bcdfg)x+(abcd+abcf+abdf+acdf+bcdf+abcg+abdg+acdg+bcdg+abfg+acfg+bcfg+adfg+bdfg+cdfg)x^2-(abc+abd+acd+bcd+abf+acf+bcf+adf+bdf+cdf+abg+acg+bcg+adg+bdg+cdg+afg+bfg+cfg+dfg)x^3+(ab+ac+bc+ad+bd+cd+af+bf+cf+df+ag+bg+cg+dg+fg)x^4-(a+b+c+d+f+g)x^5+x^6$. Substituting $\lambda_0 = 1, \lambda_1 = a+b+c+d+f+g, \ldots, \lambda_5 = abcdf+abcdg+abcfg+abdfg+acdfg+bcdfg$ and $\lambda_6 = abcdfg$ into $f(x,1)$ and computing $A'(f)$, then $A'(f) = 6a^2b^2c^2-4a^2b^2cd-4a^2bc^2d-4ab^2c^2d+$
$6a^2b^2d^2-4a^2bcd^2-4ab^2cd^2+6a^2c^2d^2-4abc^2d^2+6b^2c^2d^2-4a^2b^2cf-4a^2bc^2f-4ab^2c^2f-4a^2b^2df+$
$12a^2bcdf+12ab^2cdf-4a^2c^2df+12abc^2df-4b^2c^2df-4a^2bd^2f-4ab^2d^2f-4a^2cd^2f+12abcd^2f-4b^2cd^2f-$
$4ac^2d^2f-4bc^2d^2f+6a^2b^2f^2-4a^2bcf^2-4ab^2cf^2+6a^2c^2f^2-4abc^2f^2+6b^2c^2f^2-4a^2bdf^2-4ab^2df^2-$
$4a^2cdf^2+12abcdf^2-4b^2cdf^2-4ac^2df^2-4bc^2df^2+6a^2d^2f^2-4abd^2f^2+6b^2d^2f^2-4acd^2f^2-4bcd^2f^2+$
$6c^2d^2f^2-4a^2b^2cg-4a^2bc^2g-4ab^2c^2g-4a^2b^2dg+12a^2bcdg+12ab^2cdg-4a^2c^2dg+12abc^2dg-4b^2c^2dg-$
$4a^2bd^2g-4ab^2d^2g-4a^2cd^2g+12abcd^2g-4b^2cd^2g-4ac^2d^2g-4bc^2d^2g-4a^2b^2fg+12a^2bcfg+12ab^2cfg-$
$4a^2c^2fg+12abc^2fg-4b^2c^2fg+12a^2bdfg+12ab^2dfg+12a^2cdfg-120abcdfg+12b^2cdfg+12ac^2dfg+$
$12bc^2dfg-4a^2d^2fg+12abd^2fg-4b^2d^2fg+12acd^2fg+12bcd^2fg-4c^2d^2fg-4a^2bf^2g-4ab^2f^2g-$
$4a^2cf^2g+12abcf^2g-4b^2cf^2g-4ac^2f^2g-4bc^2f^2g-4a^2df^2g+12abdf^2g-4b^2df^2g+12acdf^2g+12bcdf^2g-$
$4c^2df^2g-4ad^2f^2g-4bd^2f^2g-4cd^2f^2g+6a^2b^2g^2-4a^2bcg^2-4ab^2cg^2+6a^2c^2g^2-4abc^2g^2+6b^2c^2g^2-$
$4a^2bdg^2-4ab^2dg^2-4a^2cdg^2+12abcdg^2-4b^2cdg^2-4ac^2dg^2-4bc^2dg^2+6a^2d^2g^2-4abd^2g^2+6b^2d^2g^2-$
$4acd^2g^2-4bcd^2g^2+6c^2d^2g^2-4a^2bfg^2-4ab^2fg^2-4a^2cfg^2+12abcfg^2-4b^2cfg^2-4ac^2fg^2-4bc^2fg^2-$
$4a^2dfg^2+12abdfg^2-4b^2dfg^2+12acdfg^2+12bcdfg^2-4c^2dfg^2-4ad^2fg^2-4bd^2fg^2-4cd^2fg^2+6a^2f^2g^2-$
$4abf^2g^2+6b^2f^2g^2-4acf^2g^2-4bcf^2g^2+6c^2f^2g^2-4adf^2g^2-4bdf^2g^2-4cdf^2g^2+6d^2f^2g^2$. Without loss of generality, we can assume that $a = 0$, i.e., there exists an automorphism of the Riemann sphere which maps an unordered 6-tuple $(\mu_1, \ldots, \mu_2)$ consisting of mutually distinct points of the Riemann sphere onto $(0, 1, \infty, \mu_1', \mu_2', \mu_3')$. Hence, $A'(f) - A(f) = 0$ which implies that $A = A'$ since $f \in \mathcal{B}_6$ is arbitrary chosen. However, putting $b = 1$ and computing $B'(f) - B(f)$ yields $-48c^4dfg+36c^3d^2fg+$
$36c^4d^2fg+36c^2d^3fg-72c^3d^3fg+36c^4d^3fg-48cd^4fg+36c^2d^4fg+36c^3d^4fg-48c^4d^4fg+36c^3df^2g+$
$36c^4df^2g-48c^2d^2f^2g-24c^3d^2f^2g-48c^4d^2f^2g+36cd^3f^2g-24c^2d^3f^2g-24c^3d^3f^2g+36c^4d^3f^2g+$
$36cd^4f^2g-48c^2d^4f^2g+36c^3d^4f^2g+36c^2df^3g-72c^3df^3g+36c^4df^3g+36cd^2f^3g-24c^2d^2f^3g-24c^3d^2f^3g+$
$36c^4d^2f^3g-72cd^3f^3g-24c^2d^3f^3g-72c^3d^3f^3g+36cd^4f^3g+36c^2d^4f^3g-48cdf^4g+36c^2df^4g+36c^3df^4g-$

$48c^4df^4g+36cd^2f^4g-48c^2d^2f^4g+36c^3d^2f^4g+36cd^3f^4g+36c^2d^3f^4g-48cd^4f^4g+36c^3dfg^2+36c^4dfg^2-48c^2d^2fg^2-24c^3d^2fg^2-48c^4d^2fg^2+36cd^3fg^2-24c^2d^3fg^2-24c^3d^3fg^2+36c^4d^3fg^2+36cd^4fg^2-48c^2d^4fg^2+36c^3d^4fg^2-48c^2df^2g^2-24c^3df^2g^2-48c^4df^2g^2-48cd^2f^2g^2+144c^2d^2f^2g^2+144c^3d^2f^2g^2-48c^4d^2f^2g^2-24cd^3f^2g^2+144c^2d^3f^2g^2-24c^3d^3f^2g^2-48cd^4f^2g^2-48c^2d^4f^2g^2+36cdf^3g^2-24c^2df^3g^2-24c^3df^3g^2+36c^4df^3g^2-24cd^2f^3g^2+144c^2d^2f^3g^2-24c^3d^2f^3g^2-24cd^3f^3g^2-24c^2d^3f^3g^2+36cd^4f^3g^2+36cdf^4g^2-48c^2df^4g^2+36c^3df^4g^2-48cd^2f^4g^2-48c^2d^2f^4g^2+36cd^3f^4g^2+36c^2dfg^3-72c^3dfg^3+36c^4dfg^3+36cd^2fg^3-24c^2d^2fg^3-24c^3d^2fg^3+36c^4d^2fg^3-72cd^3fg^3-24c^2d^3fg^3-72c^3d^3fg^3+36cd^4fg^3+36c^2d^4fg^3+36cdf^2g^3-24c^2df^2g^3-24c^3df^2g^3+36c^4df^2g^3-24cd^2f^2g^3+144c^2d^2f^2g^3-24c^3d^2f^2g^3-24cd^3f^2g^3-24c^2d^3f^2g^3+36cd^4f^2g^3-72cdf^3g^3-24c^2df^3g^3-72c^3df^3g^3-24cd^2f^3g^3-24c^2d^2f^3g^3-72cd^3f^3g^3+36cdf^4g^3+36c^2df^4g^3+36cd^2f^4g^3-48cdfg^4+36c^2dfg^4+36c^3dfg^4-48c^4dfg^4+36cd^2fg^4-48c^2d^2fg^4+36c^3d^2fg^4+36cd^3fg^4+36c^2d^3fg^4-48cd^4fg^4+36cdf^2g^4-48c^2df^2g^4+36c^3df^2g^4-48cd^2f^2g^4-48c^2d^2f^2g^4+36cd^3f^2g^4+36cdf^3g^4+36c^2df^3g^4+36cd^2f^3g^4-48cdf^4g^4+1620c^22d^22f^22g^22$ which, in general, is non-zero. Hence, in general $B'(f) \neq B(f)$. Considering binary sextics associated to hyperelliptic curves, i.e., binary sextics which have six different zeros, implies that $B'(f) \neq B(f)$ for all $f \in \{g \in \mathcal{B}_6 \mid g \text{ corresponds to a hyperelliptic curve of genus two and } g \neq x^6 - x\} := \mathcal{B}_6^{\text{hyp}}$. Similarly, one computes that $C'(f) \neq C(f)$ for all $f \in \mathcal{B}_6^{\text{hyp}}$. Notice that D' and D are the discriminant.

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