A_4 family symmetry

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Introduction

This thesis explores the role family symmetries might play in the interactions between elementary particles. In the last decade it has become clear that neutrinos, which come in three varieties called flavor, are able to change from one flavor to another, it is said that they oscillate. This indicates that neutrinos have a (very small) mass. From these oscillations the interactions between neutrinos with a definite mass and the weak force (responsible for radioactive decay) can be deduced. The story for the quarks is similar, although the interactions with the weak force are quite different.

These interactions, especially those for the neutrinos, seem to follow a very specific pattern, called the mixing pattern. This pattern cannot be derived from the Standard Model (SM). In the SM it just represents parameters which should be measured. Without an explanation from the SM, one could either say this pattern is due to chance or there might be a mechanism (beyond the SM) behind it. Family symmetries provide such a mechanism. A model with a family symmetry demands that all physics should be the same if the flavors (of the neutrinos or quarks) are interchanged in a certain way (dictated by the symmetry). It is called a family symmetry because each flavor relates to a different family of particles. A model with a family symmetry constrains the interactions that the model allows for. The goal is to implement a symmetry in such a way that the allowed interactions follow the pattern seen in nature.

This thesis focusses on models which use the group A_4 to explain the mixing patterns. A_4 is the symmetry group of a regular tetrahedron and is one of the simplest groups which can be used to try to reproduce the mixing patterns. The second Chapter discusses this group and why it seems promising.

Before discussing models using this symmetry, first the interactions forming the patterns we are interested in are studied in Chapter 3. We study how these interactions come about and what their relation is to the mass terms of the fermions. Special attention is given to the role of the neutrinos, as their interactions follow the most striking pattern and are different from the other fermions (they are electrically neutral). This Chapter will give us the tools we need to be able to discuss models with a family symmetry.

Many models using A_4 have been put forward in the literature. Some obtain the correct mixing pattern, but often a large number of additional fields (particles) is required to do so. In this thesis some of the simpler cases, with less additional fields, are discussed first. The simple models that will be discussed all have the same Higgs sector. This Higgs sector will be studied at length in Chapter 4. This is a necessity as the Higgs fields influence the mixing patterns.

Chapter 5 looks at the models that can be built using this Higgs sector. It is concluded from this discussion that these models are unsatisfactory; additional ingredients are required to get the right mixing patterns. After this, some examples from the literature are studied. From these examples we then try to see how these simple models might be improved.

In Chapter 6 a new possibility, not yet present in the literature, is considered. In this case A_4 is combined with a so called left-right model.

In the SM the weak force only couples to left-handed particles. The SM offers no explanation as to why the weak force would make a distinction between left- and right-handed particles. In left-right models this question is resolved by restoring the symmetry between left and right at high energies. These models are used to provide additional ingredients so that the correct mixing patterns can be produced in combination with A_4. One of the simplest possibilities of the resulting model is studied and turns out to be unsatisfactory, as the right masses are not reproduced. However, in this simple case the right mixing patterns can be reproduced. This is an encouragement to further explore this type of model.

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Family symmetries

It has been known for some time that the fermions of different the families mix among each other. This is due to the fact that there is a mismatch between the weak and mass eigenstates of the fermions.

There are two often used bases in which we can look at the relevant parts of the Lagrangian (in this case the mass terms and the weak interactions), these are called the mass basis and the weak basis. In the mass basis the Lagrangian is written in terms of the mass eigenstates of the fermions, in other words, all the fields have a definite mass. In this basis the weak charged interaction is not diagonal, which is to say, fermions of different families may interact with each other through this interaction.

In the weak basis the charged weak interaction is diagonal, meaning that the fermions only interact with their family members through this interaction. However, in this basis the mass matrices are no longer diagonal; the fields no longer have a definite mass. This is the case in both the lepton and quark sector.

The two bases are related by the mixing matrices

$$\nu^w = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{PMNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{PMNS}\nu^m, \quad d^w = V_{CKM} d^m, \tag{2.1}$$

where $d = (d, s, b)^T$ and the superscripts m and w stand for the mass and weak basis respectively. The 3 by 3 mixing matrices, U_{PMNS} and V_{CKM} are called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) and the Cabibbo-Kobayashi-Maskawa (CKM) matrix and describe the mixing in the lepton and quark sector respectively. The two mixing matrices are unitary as they describe a transformation between two bases[1].

In the lepton sector the two bases are defined in such a way that the mass matrix of the charged leptons is diagonal in both; only the neutrino fields are transformed when we move from one basis to the other, (2.1). In the quark sector the two bases are defined in such a way that the mass matrix of the up-type quarks is diagonal in both; only the down-type quark fields are transformed when we move from one basis to the other, (2.1).

The SM does not allow one to calculate the elements of V_{CKM} and U_{PMNS} from first principles, they are to be measured experimentally. This is not completely satisfactory,

[1]In models in which the neutrino mass terms are produced through a seesaw mechanism the PMNS matrix is not exactly unitary, although the non-unitary contributions are suppressed (see section 3.3.2).

most of all since these matrices exhibit certain patterns. The moduli of their elements are approximately given by

$$|V_{CKM}| \simeq \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad |U_{PMNS}| \simeq \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}, \tag{2.2}$$

where $\lambda \simeq 0.22$. The quark mixing matrix implies there is small mixing in the quark sector, the off-diagonal elements are close to zero. The neutrino mixing matrix on the other hand exhibits large mixing. It also seems to have a specific pattern called Tri-Bimaximal (TB) mixing. It is named so because the second mass eigenstate is a maximal mix of three of the weak eigenstates and the third is a maximal mix of two of the weak eigenstates

$$\begin{aligned} \nu_2 &= \sqrt{1/3}\nu_e + \sqrt{1/3}\nu_\mu + \sqrt{1/3}\nu_\tau, \\ \nu_3 &= \sqrt{1/2}\nu_\mu - \sqrt{1/2}\nu_\tau. \end{aligned} \tag{2.3}$$

It should be noted that the neutrino mixing pattern is not yet completely clear. Recently, there were indications that the (13) element of the PMNS matrix might be non-zero, [1].

These patterns could either be a coincidence or there might be some mechanism behind them. Using a discrete non abelian group as a family symmetry provides such a mechanism. A model with a family symmetry demands that all physics should be the same if the families (of the neutrinos or quarks) are interchanged in certain ways, dictated by the symmetry. This symmetry constrains the interactions that the model allows for (the way in which this is done will become clear later, Chapter 5). The goal is to implement a symmetry in such a way that the allowed interactions follow the same patterns as seen in nature, (2.2).

When constructing a model with a family symmetry one first has to decide which group to use. A large amount of models have already been put forward, using a variety of groups (usually to try to understand only the neutrino mixing matrix). The correct symmetry is not immediately apparent from the mixing matrices, because the family symmetry will be broken by electroweak symmetry breaking (EWSB). Some of the groups used in the literature are shown in (Table 2.1).

Group	Order	Irreps	Ref.'s
$D_3 \cong S_3$	6	$\mathbf{1}, \mathbf{1'}, \mathbf{2}$	[2]
A_4	12	$\mathbf{1}, \mathbf{1'}, \mathbf{1''}, \mathbf{3}$	[3], [4], [5], [6]
T'	24	$\mathbf{1}, \mathbf{1'}, \mathbf{1''}, \mathbf{2}, \mathbf{2'}, \mathbf{2''}, \mathbf{3}$	[7]
S_4	24	$\mathbf{1}, \mathbf{1'}, \mathbf{2}, \mathbf{3}$	[3], [8]
$\Delta(27) \cong (Z_3 \times Z_3) \ltimes Z_3$	27	$\mathbf{1}_1, ..., \mathbf{1}_9, \mathbf{3}, \overline{\mathbf{3}}$	[9]

Table 2.1: Some of the groups used as family symmetries, for more groups and references see, [3].

The goal in each case is to be able to describe the mixing matrices, (2.2). However, choosing a different group will in principle lead to a different model. A model with A_4 or S_4 for example, can lead to TB mixing to leading order. However, it is also possible for a model with S_4 to naturally lead to (to leading order terms in the Lagrangian) what is called Bi-maximal mixing in the neutrino mixing matrix,

$$U_{BM} = \begin{pmatrix} \sqrt{1/2} & \sqrt{1/2} & 0 \\ 1/2 & -1/2 & \sqrt{1/2} \\ -1/2 & 1/2 & \sqrt{1/2} \end{pmatrix}. \tag{2.4}$$

This pattern is called Bi-maximal because the electron neutrino and the third mass eigenstate are maximally mixed combination of mass and weak eigenstates, respectively.

$$\nu_e = \sqrt{1/2}\nu_1 + \sqrt{1/2}\nu_2, \quad \nu_3 = \sqrt{1/2}\nu_\mu + \sqrt{1/2}\nu_\tau. \tag{2.5}$$

When this is the case next to leading order contributions can be taken into account to obtain a mixing pattern closer to TB mixing, see [3].

When studying family symmetries we should start with the simplest possibility. A_4 seems to fit this role as it is the smallest group with a three dimensional irreducible representation (irrep), which is convenient because it can be used to transform the three families among each other. Furthermore, it looks promising because it has been shown to be able to reproduce TB mixing successfully in various models. Therefore we will consider A_4 in this thesis.

In order to study models using A_4, we will first study the group itself in the next section.

2.1 A_4 symmetry

A_4 is the symmetry group of the tetrahedron and the group of even permutations of four objects. It therefore has $4!/2 = 12$ elements. It can be seen that all twelve elements can be obtained by repeatedly multiplying the two generators, $S = (14)(23)$ and $T = (123)$. These satisfy the relations

$$S^2 = (ST)^3 = T^3 = 1. \tag{2.6}$$

The combination of the generators and their relations is a so-called presentation of A_4. From this the equivalence classes and the number of elements they contain can be derived

$$\begin{aligned} &C_1 : I, \\ &C_2 : T, ST, TS, STS, \\ &C_3 : T^2, ST^2, T^2S, TST, \\ &C_4 : S, TST^2, T^2ST. \end{aligned}$$

A_4 thus has four conjugacy classes, this means that there are four irreps and that their dimensions should satisfy $d_1^2 + d_2^2 + d_3^2 + d_4^2 = 12$. The only integer solution to this is $d_1 = d_2 = d_3 = 1$, $d_4 = 3$, this gives us the first column of the character table. It can be seen that the elements in C_2 and C_3 are of order 3 while the elements in C_4 are of order 2. This means that the characters of the one dimensional irreps will be cubic (square) roots of unity. This gives us the characters of the one dimensional irreps. Using the orthogonality relations, the rest of the characters can be found as well, (see Table 2.2).

Class	C_1	C_2	C_3	C_4
χ^1	1	1	1	1
$\chi^{1'}$	1	ω	ω^2	1
$\chi^{1''}$	1	ω^2	ω	1
χ^3	3	0	0	-1

Table 2.2: The character table of A_4.

Here $\omega = e^{2\pi i/3}$ and $\omega + \omega^2 = -1$. In order to build an A_4 invariant Lagrangian we need to know how to construct singlets from the different products of irreps of A_4. For this we first need the Clebsch-Gordan decomposition of direct products into irreps, which can be obtained from the character table

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'', \tag{2.7}$$

$$\mathbf{1}^{(\prime)(\prime\prime)} \otimes \mathbf{3} = \mathbf{3}, \tag{2.8}$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}. \tag{2.9}$$

In order to calculate the product of two triplets we will need the explicit representations. We will first look at this in the so called Ma-Rajasekaran (MR) basis, [10].

2.1.1 The Ma-Rajasekaran basis

The one-dimensional representations can be read off from the character table

$$\mathbf{1} : S = 1, \quad T = 1, \tag{2.10}$$

$$\mathbf{1}' : S = 1, \quad T = \omega, \tag{2.11}$$

$$\mathbf{1}'' : S = 1, \quad T = \omega^2. \tag{2.12}$$

For the three-dimensional representation we take a basis in which S is diagonal, such that the generators are

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

Following (2.9) we now want to find out how to make singlets from the product of two triplets, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Note that a^* transforms in exactly the same way as a, since the generators are real in this basis. We then have the conditions from the transformation rules under S

$$(ab)'_{\mathbf{1}} = a' M_{\mathbf{1}} (b')^T = a S^T M_{\mathbf{1}} S b^T = (ab)_{\mathbf{1}}, \tag{2.14}$$

$$(ab)'_{\mathbf{1}'} = a' M_{\mathbf{1}'} (b')^T = a S^T M_{\mathbf{1}'} S b^T = (ab)_{\mathbf{1}'}, \tag{2.15}$$

$$(ab)'_{\mathbf{1}''} = a' M_{\mathbf{1}''} (b')^T = a S^T M_{\mathbf{1}''} S b^T = (ab)_{\mathbf{1}''}, \tag{2.16}$$

here $M_{\mathbf{1},\mathbf{1}',\mathbf{1}''}$ stand for 3 by 3 matrices containing the Clebsch-Gordan coefficients. And for T we have

$$(ab)'_{\mathbf{1}} = a' M_{\mathbf{1}} (b')^T = a T^T M_{\mathbf{1}} T b^T = (ab)_{\mathbf{1}}, \tag{2.17}$$

$$(ab)'_{\mathbf{1}'} = a' M_{\mathbf{1}'} (b')^T = a T^T M_{\mathbf{1}'} T b^T = \omega (ab)_{\mathbf{1}'}, \tag{2.18}$$

$$(ab)'_{\mathbf{1}''} = a' M_{\mathbf{1}''} (b')^T = a T^T M_{\mathbf{1}''} T b^T = \omega^2 (ab)_{\mathbf{1}''}. \tag{2.19}$$

The conditions from the generator S lead to

$$M_{\mathbf{1},\mathbf{1}',\mathbf{1}''} = \begin{pmatrix} m^{11}_{\mathbf{1},\mathbf{1}',\mathbf{1}''} & 0 & 0 \\ 0 & m^{22}_{\mathbf{1},\mathbf{1}',\mathbf{1}''} & m^{23}_{\mathbf{1},\mathbf{1}',\mathbf{1}''} \\ 0 & m^{32}_{\mathbf{1},\mathbf{1}',\mathbf{1}''} & m^{33}_{\mathbf{1},\mathbf{1}',\mathbf{1}''} \end{pmatrix}. \tag{2.20}$$

The conditions from the generator T set the off-diagonal elements to zero and in addition

$$m^{11}_{\mathbf{1}} = m^{22}_{\mathbf{1}} = m^{33}_{\mathbf{1}}, \tag{2.21}$$

$$m^{11}_{\mathbf{1}'} = \omega m^{22}_{\mathbf{1}'} = \omega^2 m^{33}_{\mathbf{1}'}, \tag{2.22}$$

$$m^{11}_{\mathbf{1}''} = \omega^2 m^{22}_{\mathbf{1}''} = \omega m^{33}_{\mathbf{1}''}. \tag{2.23}$$

Taking the (11) elements to be one, we have for the singlets

$$(ab)_{\mathbf{1}} = a_1b_1 + a_2b_2 + a_3b_3, \tag{2.24}$$

$$(ab)_{\mathbf{1}'} = a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3, \tag{2.25}$$

$$(ab)_{\mathbf{1}''} = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3. \tag{2.26}$$

According to (2.9), apart from the singlets also two triplets can be constructed from the product of two triplets. We will now try to construct such a triplet, $c = (c_1, c_2, c_3)$, out of a product of two other triplets, a and b

$$(ab)^{\mathbf{3}} = c. \tag{2.27}$$

We first note that c_1 is invariant under the generator S, $Sc = (c_1, -c_2, -c_3)$. Since a and b transform in the same way under S, this implies that c_1 is made up out of (some of) the following terms

$$a_1b_1, a_2b_2, a_3b_3, a_2b_3, a_3b_2. \tag{2.28}$$

Since c_1 transforms to c_2 (c_3) under T (T^2) we know what c_2 and c_3 should be for each of these terms. If c_1 contains one of the first three terms, then c_2 and c_3 contain terms of the form a_ib_i, where $i = 1, 2, 3$. However, under S we have $a_ib_i \to a_ib_i$ whereas $c_2, c_3 \to -c_2, -c_3$. So the first three combinations are excluded and c_1 is made up out of a_3b_2 and a_2b_3. These are exactly the two equivalent three-dimensional representations in the multiplication rule, (2.9). After working out c_2 and c_3 this gives us

$$(ab)_{\mathbf{3}_1} = (a_2b_3, a_3b_1, a_1b_2), \quad (ab)_{\mathbf{3}_2} = (a_3b_2, a_1b_3, a_2b_1). \tag{2.29}$$

2.1.2 The Altarelli-Feruglio basis

All of what was previously discussed was in what is referred to as the MR basis. This basis is used frequently in the literature and will be used most of the time throughout this thesis. However, it is sometimes more convenient to use another basis, the Altarelli-Feruglio (AF) basis. This basis is used in one of the most well-known models attempting to use A_4 to explain neutrino mixing, see for instance [3] [11]. In what follows we will see how singlets and triplets can be constructed from two triplets in this basis.

In the AF basis the generator T' is diagonal instead of S'. The generators in the MR (S and T) and the AF (S' and T') basis are connected by a unitary basis transformation, namely

$$T' = V^\dagger T V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{2.30}$$

$$S' = V^\dagger S V = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \tag{2.31}$$

where

$$V = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \tag{2.32}$$

This means that for a triplet a_{MR} in the MR basis we can construct a triplet in the AF basis, $a_{AF}^T = V^\dagger a_{MR}^T$ such that it transforms properly

$$(a_{AF}^T)' = V^\dagger (a_{MR}^T)' = V^\dagger G a_{MR}^T = V^\dagger G V a_{AF}^T = G' a_{AF}^T, \tag{2.33}$$

where G is some combination of S and T, while G' is some combination of S' and T'. Also, note that a_{AF}^* does not transform in the same way as a_{AF}, this can be seen by noting that the generator T' is not real in this basis. In this section only the Clebsch-Gordan coefficients for the product of two triplets will be discussed, the product of a triplet and the complex conjugate of a triplet will not be used in this thesis. In the previous section we had expressions for the singlets in the MR basis (2.24-2.26), using these and then switching basis we obtain

$$(a_{MR} b_{MR})_{\mathbf{1},\mathbf{1}',\mathbf{1}''} = a_{MR} M_{\mathbf{1},\mathbf{1}',\mathbf{1}''} b_{MR}^T = a_{AF} V^T M^{\mathbf{1},\mathbf{1}',\mathbf{1}''} V b_{AF}^T, \tag{2.34}$$

which implies that $(M_{AF})_{\mathbf{1},\mathbf{1}',\mathbf{1}''} = V^T (M_{MR})_{\mathbf{1},\mathbf{1}',\mathbf{1}''} V$. And thus

$$\mathbf{1}_{AF}: \quad (ab)_{\mathbf{1}} = a_1 b_1 + a_2 b_3 + a_3 b_2, \tag{2.35}$$

$$\mathbf{1}'_{AF}: \quad (ab)_{\mathbf{1}'} = a_1 b_2 + a_2 b_1 + a_3 b_3, \tag{2.36}$$

$$\mathbf{1}''_{AF}: \quad (ab)_{\mathbf{1}''} = a_1 b_3 + a_3 b_1 + a_2 b_2. \tag{2.37}$$

The case for the triplets is somewhat more difficult. In order to put the basis transformation to use, we will write the expressions in (2.29) as follows

$$(a_{MR} b_{MR})^i_{\mathbf{3}_1} = a_{MR} M_i b_{MR}^T = c_{MRi}, \tag{2.38}$$

$$(a_{MR} b_{MR})^i_{\mathbf{3}_2} = a_{MR} M_i^T b_{MR}^T = d_{MRi}, \tag{2.39}$$

where a_{MRi}, b_{MRi}, c_{MRi} and d_{MRi} are triplets in the MR basis and

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.40}$$

Now changing our basis to the AF basis, we obtain

$$V_{ij}c_{AFj} = a_{AF}V^TM_iVb_{AF}^T, \quad (2.41)$$
$$V_{ij}d_{AFj} = a_{AF}V^TM_i^TVb_{AF}^T. \quad (2.42)$$

This gives us the following

$$c_{AF1} = \frac{1}{\sqrt{3}}a_{AF}V^T(M_1 + M_2 + M_3)Vb_{AF}^T, \quad (2.43)$$
$$c_{AF2} = \frac{1}{\sqrt{3}}a_{AF}V^T(M_1 + \omega M_2 + \omega^2 M_3)Vb_{AF}^T, \quad (2.44)$$
$$c_{AF3} = \frac{1}{\sqrt{3}}a_{AF}V^T(M_1 + \omega^2 M_2 + \omega M_3)Vb_{AF}^T, \quad (2.45)$$

To obtain the expressions for d_{AFi}, use the above expressions for c_{AFi} and replace M_j with M_j^T. Written out this gives us,

$$c = \frac{1}{\sqrt{3}}(a_1b_1 + \omega a_2b_3 + \omega^2 a_3b_2, a_3b_3 + \omega a_1b_2 + \omega^2 a_2b_1a_2b_2 + \omega a_3b_1 + \omega^2 a_1b_3), \quad (2.46)$$
$$d = \frac{1}{\sqrt{3}}(a_1b_1 + \omega a_3b_2 + \omega^2 a_2b_3, a_3b_3 + \omega a_2b_1 + \omega^2 a_1b_2a_2b_2 + \omega a_1b_3 + \omega^2 a_3b_1), \quad (2.47)$$

here the subscript AF has been dropped. These triplets are more often used in a symmetric and anti-symmetric combination

$$\begin{aligned}\mathbf{3}^S_{AF} &\sim c + d \sim (2a_1b_1 - a_2b_3 - a_3b_2, 2a_3b_3 - a_1b_2 - a_2b_1, 2a_2b_2 - a_3b_1 - a_1b_3),\\ \mathbf{3}^A_{AF} &\sim c - d \sim (a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_3b_1 - a_1b_3).\end{aligned} \quad (2.48)$$

We now know all we need about A_4. However, before putting this knowledge to use by constructing actual models we will first discuss the possibility of deducing a minimal family symmetry from the mass matrices.

2.2 Constraining the family symmetry by the mass matrices

In [12] an interesting idea has been put forward. Namely, that by studying the symmetries of the mass matrices a minimal family symmetry can be deduced. In the case of exact TB mixing S_4 would then be the minimal family symmetry.

The argument uses the traces of the family symmetry that are still apparent in the mass matrices. Firstly, operations which leave the mass matrices invariant (the residual symmetry operators) are identified. Since the mass matrices came to be after EWSB they will not be invariant under the complete family symmetry only under what is left of the family symmetry (the residual operators). It is then argued that these operators all originated from the family symmetry, meaning that the group they generate should be included in the original family symmetry. This would give us a minimal group which could be used as a family symmetry; any family symmetry should have this group as a subgroup.

More concretely, let us look at the lepton mass matrices in the weak basis. We have for the lepton mass terms

$$-\mathcal{L}^l_m = \overline{l_L} m_l l_R + \frac{1}{2}\overline{(\nu^w_L)^c} M_\nu \nu^w_L + h.c. \ , \tag{2.49}$$

here $l = (e, \mu, \tau)^T$, these terms will be discussed in more detail in Chapter 3. We will look at this in the basis where the charged lepton mass matrix, m_l, is diagonal and the neutrino mass matrix is given by $M_\nu = U^*_{PMNS} m_\nu U^\dagger_{PMNS}$, where m_ν is diagonal. Since the charged lepton mass matrix is diagonal it will be invariant under $m_l = F^\dagger m_l F$, where F is a unitary and diagonal matrix of phases with det $F = 1$. If we want F to be a part of a finite group then we should have $F^n = \mathbb{1}$. If we take F to have three different values on the diagonal then $n \geq 3$. Similarly, for the neutrino mass matrix, $M_\nu = G^T M_\nu G$. There are three possibilities for G,

$$G_1 = u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger, \ \ G_2 = -u_1 u_1^\dagger + u_2 u_2^\dagger - u_3 u_3^\dagger, \ \ G_3 = -u_1 u_1^\dagger - u_2 u_2^\dagger + u_3 u_3^\dagger, \tag{2.50}$$

here u_i is the ith column of U_{PMNS}. These matrices satisfy $M_\nu = G_i^T M_\nu G_i$, since $M_\nu = U^*_{PMNS} m_\nu U^\dagger_{PMNS}$ and $U^\dagger_{PMNS} G_i = D_i U^\dagger_{PMNS}$ where D_i are diagonal matrices of ± 1. Now taking TB mixing to be exact, the residual operators G_i are

$$G_1 = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \ G_2 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \ G_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.51}$$

Note that multiplying two of these matrices gives the third. Taking the simplest case for F, (with three different entries on the diagonal) we have $F = \text{diag}(1, w^2, \omega)$. Then it can be seen that the matrices (G_1, G_2, G_3, F) generate S_4 [12].

According to the argument laid out previously, this should lead us to conclude that S_4 is the minimal family symmetry group (in the case of exact TB mixing). However, this argument depends on the assumption that each of these residual operators (G_1, G_2, G_3, F) originates from the family symmetry. This need not be the case.

As an example, let us look at part of an A_4 model describing TB mixing which we will study in more detail later (section 5.3, [3]). We will focus on the neutrino mass terms. In the model we assign the neutrinos to the A_4 triplet, $\nu_L \sim \mathbf{3}$, and we introduce the scalar fields, $\varphi = (\varphi_1, \varphi_2, \varphi_3) \sim \mathbf{3}$ and $\xi \sim \mathbf{1}$, which we assign to an A_4 triplet and singlet respectively. In this model the terms contributing to neutrino masses are of the form

$$\begin{aligned}\mathcal{L}_m =& - a\frac{1}{2}\overline{\nu^c_L}\varphi\nu_L - b\frac{1}{2}\overline{\nu^c_L}\xi\nu_L + h.c. \\ =& - a\frac{1}{2}[\varphi_1(\overline{\nu^c_e}\nu_e - \overline{\nu^c_\mu}\nu_\tau) + \varphi_2(\overline{\nu^c_\mu}\nu_\mu - \overline{\nu^c_e}\nu_\tau) + \varphi_3(\overline{\nu^c_\tau}\nu_\tau - \overline{\nu^c_\mu}\nu_e)] \\ &-b\frac{1}{2}\xi(\overline{\nu^c_e}\nu_e + 2\overline{\nu^c_\mu}\nu_\tau) + h.c. \ ,\end{aligned} \tag{2.52}$$

where the upper index c stands for charge conjugation and the AF basis was used to write the terms out explicitly in components of the fields. By construction, this is invariant under the operations F and G_2 since these are exactly the generators of A_4 in the AF basis. In order for $\mathcal{L}_m$ to be invariant under S_4 it should be invariant under G_3 as well (we do not need to consider G_1 since $G_2G_3 = G_1$). Applying G_3 is equivalent to performing the transformations $(\nu_e, \nu_\mu, \nu_\tau) \to -(\nu_e, \nu_\tau, \nu_\mu)$ and $(\varphi_1, \varphi_2, \varphi_3) \to -(\varphi_1, \varphi_3, \varphi_2)$. Note that $\mathcal{L}_m$ is not invariant under this, effectively it transforms a to $-a$. Thus $\mathcal{L}_m$ is symmetrical under A_4, not S_4. If we now let the scalar fields acquire vacuum expectation values (VEVs), $\langle\varphi\rangle = (v, v, v)$ and $\langle\xi\rangle = u$, we have

$$\begin{aligned}\mathcal{L}_m = &-\frac{va}{2}[\overline{\nu_e^c}\nu_e - \overline{\nu_\mu^c}\nu_\tau + \overline{\nu_\mu^c}\nu_\mu - \overline{\nu_e^c}\nu_\tau \\ &+\overline{\nu_\tau^c}\nu_\tau - \overline{\nu_\mu^c}\nu_e] - \frac{bu}{2}(\overline{\nu_e^c}\nu_e + 2\overline{\nu_\mu^c}\nu_\tau) + h.c. \ ,\end{aligned} \tag{2.53}$$

This expression is still invariant under G_2 but no longer under F. Additionally, it has become invariant under G_3, this is an accidental symmetry which did not come from A_4. As we will see in section 5.3, the neutrino mass matrix is now of the form,

$$M_\nu = \begin{pmatrix} 2A+B & -A & -A \\ -A & 2A & B-A \\ -A & B-A & 2A \end{pmatrix}, \tag{2.54}$$

which leads to exact TB mixing in this model. This means that we cannot assume that each of the residual operators originated from the family symmetry. It is possible, as in this case, that one of these is accidental and not related to the family symmetry at all. In this case we should not include G_3 as a generator of the family symmetry. Taking just F and G_2 as generators gives us A_4, which would be the correct answer in this case. All of this means that this line of argument does not give us a minimal group as a candidate for a family symmetry.

In what follows we will consider family symmetry models based on A_4 only. The knowledge of the multiplication of irreps of A_4 should allow us to build A_4 invariant terms which we can use to construct models. Given the irrep assignment of the fields involved the most general A_4 invariant Lagrangian can be written down. From this we will proceed to construct the mass matrices and infer the mixing matrices, which hopefully agree with experiment. To be able to do this we shall first discuss the different ways (not A_4 specific) in which we can build mass terms and see how these lead to mixing matrices. We will do this in the next Chapter.

3

Mass terms

In this Chapter we will look at the possible mass terms for the fermions and how these are connected to the mixing matrices. We will pay special attention to the neutrinos, as they can have mass terms which are different from those for the other fermions. In doing so, we will study the different types of seesaw mechanism. The knowledge gained here will be put to use when we start building models.

3.1 Dirac and Majorana terms

In general, there are two types of mass terms, which can be used to generate masses for fermions. Firstly, we have the Dirac mass term, which is used to generate all the fermion masses in the SM, it has the following form

$$\mathcal{L}_m^D = -\overline{\psi_{L\alpha}} m_{\alpha\beta} \psi_{R\beta} + h.c. \ , \tag{3.1}$$

here ψ is some fermion field and α and β are flavor indices. This type of mass term always couples the left-handed part to the right-handed part of the field. In general the matrix $m_{\alpha\beta}$ is not diagonal and so members of different generations can mix among each other. The Dirac mass term is invariant under a global transformation, $\psi_\alpha \to e^{i\phi}\psi_\alpha$. The whole kinetic part of the Lagrangian for ψ is then invariant under this transformation. Through Noether's theorem this leads to a conserved current, which is $j^\mu = \sum_\alpha \overline{\psi_\alpha}\gamma^\mu\psi_\alpha$. In the quark (lepton) sector such a transformation leads to baryon (lepton) number conservation. Note that the different flavor numbers are conserved when summed, but not separately.

The second type of mass term is the Majorana mass term and is of the form

$$\mathcal{L}_{m,L(R)}^M = -\frac{1}{2}\overline{\psi_{L(R)\alpha}^c} m_{L(R)\alpha\beta}\psi_{L(R)\beta} - \frac{1}{2}\overline{\psi_{L(R)\alpha}} m_{L(R)\alpha\beta}^\dagger \psi_{L(R)\beta}^c \ , \tag{3.2}$$

where the superscript c again stands for charge conjugation. This type of mass terms couples left- (right-) handed fields to the charge conjugates of the same left- (right-) handed fields. Again, the mass matrix, $m_{L(R)\alpha\beta}$ does not have to be diagonal, so that members of different generations can mix among each other. However, the Majorana mass matrix has to be symmetric, this can be seen by taking the transpose of the mass term

$$\overline{\psi_\alpha^c} m_{\alpha\beta}\psi_\beta = \psi_\beta^T m_{\alpha\beta}(\psi_\alpha^T C^\dagger)^T = -\psi_\beta^T m_{\alpha\beta} C^\dagger \psi_\alpha = \overline{\psi_\alpha^c} m_{\beta\alpha}\psi_\beta, \tag{3.3}$$

here we used some of the properties of the charge conjugation matrix, C, and the fact that fermions anti-commute. Majorana fields have this kind of mass term, these are fields which satisfy the relation $\psi = \psi^c$. The kinetic term for a Majorana field will be $\mathcal{L}_{kin} = \frac{1}{2}(\overline{\psi^c} + \overline{\psi})\,[i\gamma^\mu\partial_\mu - m]\,(\psi^c + \psi)$. It is clear that if ψ is an electrically charged field the mass term will not conserve electric charge. This implies that the fermion fields which have such mass terms should be neutral, this leaves the neutrinos as the only candidates. It can be seen that the Majorana mass term, (3.2), is not invariant under the transformation, $\psi_{L(R)\alpha} \to e^{i\phi}\psi_{L(R)\alpha}$. Thus, a Majorana mass term for the neutrinos would not conserve lepton number.

As said before, in the SM the fermion masses are generated by Dirac mass terms. All fermions apart from the neutrinos, which are massless in the SM, obtain a mass this way. This means that the SM alone cannot explain the neutrino masses now that they are experimentally shown to be non-zero , see [13] and references therein. We will therefore try to understand how neutrino masses might be constructed in extensions of the SM using these two types of mass terms. To this end we will first study the masses and the mixing matrices they lead to in the SM after which we will discuss how it might be extended to incorporate neutrino masses.

3.2 Quark sector

In the SM the fermions get their masses through a Dirac mass term. This mass term is generated by the interaction of the fermions with the Higgs field. After spontaneous symmetry breaking the Higgs doublet acquires a VEV which, together with Yukawa coupling constants, act as Dirac mass terms for the fermions.

We will first look at the mass terms and their consequences in the quark sector. In the SM the most general mass term for the quarks is given by

$$\mathcal{L}_m^{quark} = -\overline{Q}_L\phi\Lambda_d d_R - \overline{Q}_L\tilde{\phi}\Lambda_u u_R + h.c. \ , \tag{3.4}$$

where $\overline{Q}_L$ are the three left-handed quark $SU(2)$ doublets, ϕ is the Higgs doublet, while $\tilde{\phi} = i\sigma_2\phi^*$ where σ_2 is the second Pauli matrix and $u_R = (u_R, c_R, t_R)^T$, $d_R = (d_R, s_R, b_R)^T$. After spontaneous symmetry breaking the mass matrices become $M_i = \frac{v}{\sqrt{2}}\Lambda_i$, with $\frac{v}{\sqrt{2}}$ the VEV acquired by the Higgs particle and $i = (u, d)$. These matrices are in general complex and are not diagonal in the basis where the weak interactions are diagonal. In order to find the particles with a definite mass, we will need to find a basis in which the mass matrices are diagonal.

By the Polar Decomposition theorem we can write the matrix M_i as $M_i = H_iS_i$, where H is a hermitian matrix and S is unitary. It can then be seen that the mass matrix is diagonalized as follows

$$U_i^\dagger M_i V_i = m_i, \tag{3.5}$$

where m_i is a real and positive diagonal matrix, $V_i = S_i^\dagger U_i$ and U_i is the matrix diagonalising H_i, so that $U_i^\dagger H_i U_i = m_i$. Since V is the product of two unitary matrices it is unitary itself. So we can now diagonalize the mass matrices by two unitary matrices, this means that diagonal mass matrices can be achieved by a unitary basis transformation of the fields.

This new basis is usually called the mass basis. These two bases are related by $u_R = V_u u_R^m$ and $d_R = V_d d_R^m$, for the right-handed fields and for the left-handed fields $d_l = U_d d_l^m$ and $u_l = U_u u_l^m$. In the mass basis the mass matrices are indeed diagonal and given by m_i, however other parts of the Lagrangian also change because of this basis transformation. In fact if we write the weak charged current in this new basis we obtain the following

$$\mathcal{L}_{CC} \sim W_\mu^+ \overline{u}_L \gamma^\mu d_L = W_\mu^+ \overline{u}_L^m \gamma^\mu U_u^\dagger U_d d_L^m = W_\mu^+ \overline{u}_L^m \gamma^\mu d_L^w, \tag{3.6}$$

where we defined $d_L^w = V_{CKM} d_L^m$ and $U_u^\dagger U_d = V_{CKM}$ is the well known CKM (Cabibbo-Kobayashi-Maskawa) matrix. d_L^w are the eigenstates in which the weak charged current is diagonal and the down-type quark mass matrix is not. This tells us that the eigenstates of down-type quarks which take part in the weak charged current are linear combinations of the down-type quark mass eigenstates. In other words, the weak eigenstates are a mixture of the mass eigenstates. The CKM matrix contains all the information about this mixing, it is therefore interesting to take a closer look at it.

In the general case of N generations the CKM matrix is a $N \times N$ matrix, made up out of two unitary matrices and so is unitary itself. A $N \times N$ unitary matrix will have N^2 real independent parameters, these can be divided into $\frac{N(N-1)}{2}$ real parameters leading to real quantities (angles) and $\frac{N(N+1)}{2}$ real parameters leading to complex quantities (phases). Not all of these phases are physical. This is because the CKM matrix determines the coupling between $\overline{u}_l^m$ and d_l^m, see (3.6). We can redefine these $2N$ fields so that, $u_i^m \to e^{i\phi_i} u_i^m$ and $d_i^m \to e^{i\varphi_i} d_i^m$, everything in the Lagrangian apart from the charged current is invariant under this transformation. We now have $2N$ independent phases which we could use to absorb $2N$ of the phases in the CKM matrix. However, redefining all the quark fields with the same phase does leave the charged current invariant, meaning that we can only absorb $2N-1$ of the CKM phases. This leaves us with

$$\frac{N(N-1)}{2} \quad \text{angles,} \tag{3.7}$$

$$\frac{N(N+1)}{2} - (2N-1) = \frac{(N-1)(N-2)}{2} \quad \text{phases.} \tag{3.8}$$

For three generations this gives three angles and one phase. The standard parametrization for this matrix is

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{3.9}$$

where $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$. The CKM matrix turns out to be nearly diagonal, in fact $\theta_{12} \simeq 0.224$, $\theta_{23} \simeq 0.042$, $\theta_{13} \simeq 0.0035$ and $\delta \simeq 0.021$, [14].

In the next section we will venture into the lepton sector.

3.3 Lepton sector

In the lepton sector the mass terms of the charged leptons are analogous to the mass terms of the quarks. This is different for the neutrinos. For a long time they were assumed to be massless in the SM, simply because there was no experimental evidence for neutrino masses. But also because it is more difficult to construct a mass term for the neutrinos than for the other fermions. This is in part due to the absence of right-handed neutrinos. Right-handed neutrinos are absent in the SM simply because they have never been observed. Since a Dirac mass term consists of the coupling between a left- and right-handed field, it is then impossible to construct a Dirac mass term for the neutrinos without right-handed neutrinos. A Majorana mass term is possible. With only the SM fields at our disposal, a coupling between left-handed neutrino fields and Higgs fields is the simplest Majorana mass term we can write down. As we will see in the next section, such a mass term is not renormalizable, because it has dimension 5. Assuming the neutrinos have no mass then indeed seems to be the simplest option, since it does not require any unobserved fields or non-renormalizable terms.

To see what this assumption would mean for the mixing matrix in the lepton sector we follow the same procedure as in the previous section. First we diagonalize the charged lepton mass matrix and then look at the weak charged current Lagrangian

$$\mathcal{L}_{CC} \sim W_\mu^- \bar{l}_L \gamma^\mu \nu_L = W_\mu^- \bar{l}_L^m \gamma^\mu U^\dagger \nu_L = W_\mu^- \bar{l}_L^m \gamma^\mu \nu_L^w, \tag{3.10}$$

where $l_L = (e_L, \mu_L, \tau_L)^T = U l_L^m$, $\nu_L^w = (\nu_{e,L}, \nu_{\mu,L}, \nu_{\tau,L})^T$ and U is one of the matrices diagonalizing the charged lepton mass matrix. In order to diagonalize the charged lepton mass matrix and the weak charged current we have had to define weak neutrino eigenstates. Since there is no neutrino mass term we can do this without consequence and there is no physical mixing.

As mentioned before, experimentally there is evidence for (very small) neutrino masses and a physical mixing matrix (called the Pontecorvo-Maki-Nakagawa-Sakata matrix) [13]. To accurately describe these features we will need a mass term for the neutrinos. To achieve this we can either add a new type of field which produces a mass term by coupling to the neutrino fields or allow for non-renormalizable terms in the Lagrangian. We will consider the latter first.

3.3.1 Dimension-5 operator

The lowest dimensional operator that can be written down, by using just the SM fields is the following dimension-5 operator

$$\mathcal{L}_5 = -\frac{g_{\alpha\beta}}{2M} \overline{(\tilde{\phi}^\dagger L_{L,\alpha})^c} (\tilde{\phi}^\dagger L_{L,\beta}) + h.c. \ , \tag{3.11}$$

here L_L stands for the left-handed lepton $SU(2)$ doublet and the indices α and β label the generation. $g_{\alpha\beta}$ is a matrix of dimensionless constants and M is a constant with the dimension of mass. Since this term is non-renormalizable it will only give valid results for energies up to the scale of $E \ll M$. The idea is that this is not the final theory, but only the effective theory at low energy. At the energy scale of M this mass term is generated by unknown interactions, hopefully described by a new theory. In the same way as the non-renormalizable Fermi theory was a low energy description of weak interaction, which is now described by the renormalizable mediation of vector bosons.

Taking a closer look at the dimension-5 operator, we see that after the Higgs field acquires its VEV, $\langle\phi\rangle = v$, we obtain the following mass term

$$\mathcal{L}_m = -\frac{v^2}{2M} g_{\alpha\beta} \overline{\nu^c_{L,\alpha}} \nu_{L,\beta} + h.c. \ , \tag{3.12}$$

which is a Majorana mass term. This should not be much of a surprise since without a right-handed neutrino field we cannot build a Dirac mass term. The neutrino masses will be of the order of $m_\nu \sim \frac{v^2}{M}$. If we take the scale of the new theory, M, to be large this could explain the small neutrino masses. We will see an example of an interaction at a high energy scale which can generate this effective mass term at low energy, called the see-saw mechanism, later (section 3.3.2).

In order to find the mixing matrix we need to diagonalize the mass matrix. Because the Majorana mass matrix, $M^\nu_{\alpha\beta} = \frac{v^2}{M} g_{\alpha\beta}$ is symmetric, the diagonalization is slightly different from the case in the quark sector. As we have seen any complex matrix can be diagonalized by two unitary matrices U and V

$$U^\dagger M^\nu V = m, \tag{3.13}$$

where m is real, positive and diagonal. Since M^ν is symmetric we can write

$$M^\nu (M^\nu)^\dagger = U m^2 U^\dagger, \quad (M^\nu)^T ((M^\nu)^T)^\dagger = V^* m^2 V^T, \tag{3.14}$$

$$\rightarrow V^T U m^2 = m^2 V^T U. \tag{3.15}$$

Since $V^T U$ is unitary, the last relation implies that it is a diagonal matrix of phases. This means that there is a diagonal matrix of phases, D, such that $V_\nu = VD, U_\nu = UD \rightarrow V^T_\nu U_\nu = \mathbb{1}, V^T_\nu = U^\dagger_\nu$. From this we can see that the mass matrix can be diagonalized as follows

$$D U^\dagger_\nu M^\nu V_\nu D^* = m \rightarrow V^T_\nu M^\nu V_\nu = m. \tag{3.16}$$

So the Majorana mass matrix can be diagonalized by just one unitary matrix. After diagonalising the charged lepton mass matrix we have for the charged current Lagrangian

$$\mathcal{L}_{CC} \sim \overline{l_L} \gamma^\mu \nu_L + h.c. = \overline{l^m_L} \gamma^\mu V^\dagger_l V_\nu \nu^m_L + h.c. \ , \tag{3.17}$$

here V_l is one of the matrices diagonalizing the charged lepton mass matrix; $V^\dagger_l M_l U_l = \text{diag}(m_e, m_\mu, m_\tau)$. So we now have

$$\nu^w_L = V^\dagger_l V_\nu \nu^m_L = U_{PMNS} \nu^m_L. \tag{3.18}$$

The 3 by 3 unitary matrix U_{PMNS} is in the neutrino sector what V_{CKM} is for the quark sector. This means the parametrization is similar to the quark case, there is only one difference. Since in this case the neutrinos are Majorana particles we can not absorb phases in a redefinition of the neutrino field. This would then also redefine $\overline{\nu^c_L}$ and therefore not leave the mass term invariant. This means we will have $N-1$ extra phases. The correct parametrization then is

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.19}$$

where $c_{ij} = \cos(\theta_{ij})$, $s_{ij} = \sin(\theta_{ij})$ and α_1 and α_2 are the phases due to the Majorana nature of the neutrinos. If neutrinos turn out to be Dirac particles instead, the phases can be absorbed and we effectively have $\alpha = \beta = 0$. As of yet it is not clear whether neutrinos are Dirac or Majorana particles. Oscillation experiments cannot distinguish between the two as they are not sensitive to the Majorana phases [15]. Neutrinoless double beta decay would be evidence of lepton number violation and the Majorana nature of neutrinos [13].

The experimental values of the angles seem to be close to $s_{12} \approx \sqrt{1/3}$, $s_{23} \approx \sqrt{1/2}$ and $s_{13} \approx 0$, see [13] and references therein. Although recently, there were indications that s_{13} is non-zero, [1]. These values (with $s_{13} \approx 0$) seem to point to a PMNS matrix exhibiting TB mixing. The TB mixing matrix has the following form, in a particular phase convention

$$U_{TB} = \begin{pmatrix} -\sqrt{2/3} & \sqrt{1/3} & 0 \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \end{pmatrix}. \tag{3.20}$$

Because this is the pattern we will eventually want to reproduce (assuming $s_{13} \approx 0$) it will be useful to know what the form of the neutrino mass matrix is when the PMNS matrix is close to the TB mixing pattern. To see this will take a slightly more general mixing matrix matrix U and absorb any phases in the masses

$$U = \begin{pmatrix} -c_{12} & s_{12} & 0 \\ \frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \sqrt{1/2} \\ \frac{c_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\sqrt{1/2} \end{pmatrix}, \quad m = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{3.21}$$

where the m_i are now in general complex. The neutrino mass matrix then takes the form

$$M_\nu = U^* m U^\dagger = \begin{pmatrix} x & y & y \\ y & w & v \\ y & v & w \end{pmatrix}, \tag{3.22}$$

here x, y, w and v are functions of θ_{12} and the masses. In the case of exact TB mixing, $s_{12} = \sqrt{1/3}$, we have $x + y = w + v$ and $m_1 = x - y$, $m_2 = x + 2y$ and $m_3 = w - v$.

Oscillation experiments are not only sensitive to the mixing angles but also to the difference between the squares of the neutrino masses [13], [15]. They are given by $\Delta m_{21}^2 = m_2^2 - m_1^2 \simeq 7.7 * 10^{-5} eV^2$ and $|\Delta m_{31}^2| = |m_3^2 - m_1^2| \simeq 2.4 * 10^{-3} eV^2$, [13]. Any model describing the neutrinos should be able to reproduce these differences. These results also imply that at least two of the neutrinos have a non-zero mass. Note that the sign of Δm_{31}^2 is unknown, this gives rise to two possibilities; normal ($\Delta m_{31}^2 > 0$) and inverse ($\Delta m_{31}^2 < 0$) hierarchy.

Instead of allowing dimension-5 operators, as we did here, we can add a number of new fields to be able to construct neutrino mass terms. The mass term are usually obtained through a so called seesaw mechanism. In some of these mechanisms the neutrinos now also mix with the added fields. The mixing matrix is affected as a result of this. The effect on the PMNS matrix is very small so that the parametrization (3.19) remains valid to a good approximation. Also the form of the neutrino mass matrix when the PMNS matrix is near TB mixing is still valid. We will study the different seesaw mechanisms next.

3.3.2 The seesaw mechanism

We will now see how we can achieve a neutrino mass term by introducing additional fields. These fields will lead to neutrino mass terms through their interactions with the neutrino fields. Usually this requires the new fields to have heavy masses in order to produce the light neutrino masses that are observed. This feature of a heavy mass producing a small mass is called the seesaw mechanism. There are three basic types of seesaw mechanisms, each relating to the introduction of a different kind of field. Type I seesaw, [15], introduces a number of fermion $SU(2)$ singlets, usually interpreted as the right-handed neutrinos. Type II seesaw, [4], instead adds a scalar $SU(2)$ triplet, whereas type III seesaw, [16], adds a fermion $SU(2)$ triplet to the SM. These are not the only possible mechanisms, a mix of the three different types is certainly a possibility. For now, we will discuss the three types separately, starting with type I.

Type I seesaw mechanism

In this type of seesaw mechanism N_R right-handed sterile neutrino fields, $\nu_R = (\nu_{R_1}, ..., \nu_{R_{N_R}})^T$, are introduced. These are called sterile because they do not participate in any interaction; they are singlets under the whole gauge group. With these we can write down a number of neutrino mass terms

$$\mathcal{L}_m^\nu = -\overline{\nu}_R M_D \nu_L - \frac{1}{2}\overline{\nu_L^c} M_L \nu_L - \frac{1}{2}\overline{\nu_R} M_R \nu_R^c + h.c. \ , \tag{3.23}$$

the first term here is the usual Dirac mass term, the last two terms are Majorana mass terms. In general M_L is a 3×3 matrix, M_D is a $N_R \times 3$ matrix and M_R is $N_R \times N_R$. The left-handed Majorana term is not invariant under the symmetries of the SM, and as stated before there is no neutrino mass term (with dimension 4) possible with just the SM fields. This is why in most cases of type I seesaw M_L is considered to be zero, we will keep it as it also appears when combining type I and II. The right-handed Majorana mass term is allowed since ν_R is a singlet under all the SM symmetries.

There are some properties of these matrices which will be useful. Firstly, the Majorana mass matrices are symmetric as discussed before. While for the Dirac mass matrix we have

$$\overline{\nu_{R\alpha}} M_{\alpha\beta} \nu_{L\beta} = -\nu_{L\beta}^T M_{\alpha\beta} \overline{\nu_{R\alpha}}^T = -\nu_{L\beta}^T C^\dagger M_{\alpha\beta} C \overline{\nu_{R\alpha}}^T = \overline{\nu_{L\alpha}^c} M_{\beta\alpha} \nu_{R\beta}^c. \tag{3.24}$$

With this in mind we can write the mass terms in a much more compact way

$$\mathcal{L}_m^\nu = -\frac{1}{2}\overline{n_L^c} M^{D+M} n_L + h.c. \ , \tag{3.25}$$

where $n_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}$ and $M^{D+M} = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}$. This mass matrix is a N by N symmetric matrix, where $N = N_R + 3$. To diagonalize this we change our basis in order to get a diagonal mass matrix. The new basis is defined by $n_L = V_\nu N_L$, where $N_L = \begin{pmatrix} \nu_{L,1} \\ \vdots \\ \nu_{L,(N+3)} \end{pmatrix}$ are the states with a definite mass. The Lagrangian now becomes

$$\mathcal{L}_m^\nu = -\frac{1}{2}\sum_i m_i \overline{N_{L,i}^c} N_{L,i} + h.c. \ . \tag{3.26}$$

This can be rewritten by introducing $\nu_i = \nu_{L,i} + \nu_{L,i}^c$, which are Majorana fields; $\nu = \nu^c$. It is then clear that the mass term is a Majorana mass

$$\mathcal{L}_m^\nu = -\frac{1}{2}\sum_i m_i \overline{\nu_i^c}\nu_i. \tag{3.27}$$

In general the combination of Dirac masses and Majorana masses will lead to Majorana-neutrinos. Only when $M_L = M_R = 0$ they will be Dirac-neutrinos.

To be able to say something about the mass scale of the neutrinos, we will need the mass scale of these different mass terms. Since the Dirac mass terms are thought to be generated through the VEV of the Higgs, the natural size of these terms is of the order of the mass of the other SM particles, i.e. the electroweak scale. We imagine the masses of the right-handed neutrino fields to be generated by physics beyond the SM at a large energy scale, so we expect them to have a large mass. To sum up, we have $\mathcal{O}(M_L) \ll \mathcal{O}(M_D) \ll \mathcal{O}(M_R)$ and the neutrino mass matrix is

$$M^{D+M} = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{3.28}$$

This is a N by N matrix as before. To get an idea of the mass scales this results in, we will block diagonalize the mass matrix (up to corrections of order $M_D M_R^{-1}$) with an $N \times N$ unitary matrix W

$$W^T M^{D+M} W \simeq \begin{pmatrix} M_L - M_D^T M_R^{-1} M_D & 0 \\ 0 & M_R \end{pmatrix}, \tag{3.29}$$

where

$$W = \begin{pmatrix} 1 - \frac{1}{2} M_D^\dagger (M_R^\dagger)^{-1} M_R^{-1} M_D & M_D^\dagger (M_R^\dagger)^{-1} \\ -M_R^{-1} M_D & 1 - \frac{1}{2} M_R^{-1} M_D M_D^\dagger (M_R^\dagger)^{-1} \end{pmatrix}. \tag{3.30}$$

We now have the 3×3 mass matrix for the light neutrinos

$$M_\nu \simeq M_L - M_D^T M_R^{-1} M_D \tag{3.31}$$

And the $N_R \times N_R$ mass matrix, M_R for the heavy neutrinos. So we expect to have N_R neutrinos with masses of the order of the energy scale of the new physics. And we have three neutrinos with much lighter masses, lower than the SM masses, since we expect the elements of $M_R^{-1} M_D$ to be small. We would expect $\mathcal{O}(M_D^T M_R^{-1} M_D) \sim v^2/M$, where v is the VEV of the Higgs doublet and M is the energy scale of the new physics. As we will see in the case of type II seesaw, the scale of M_L will be similar to v^2/M, although M may have a different origin in this case.

When we put $M_L = 0$ we need at least two right-handed neutrinos to give masses to all the left-handed neutrinos. In the case of just one right-handed neutrino, the Dirac mass matrix is just a row vector, $M_D = (m_{D1}, m_{D2}, m_{D3})$ and the right-handed Majorana matrix becomes a scalar. The mass matrix for the left-handed neutrinos is then given by

$$M_\nu \simeq -M_D^T M_D \frac{1}{M_R}. \tag{3.32}$$

The eigenvalues of $M_\nu M_\nu^\dagger$ should give the squared masses of the light neutrinos,

$$M_\nu M_\nu^\dagger = \frac{1}{M_R^2} M_D^T M_D M_D^\dagger M_D^* = \frac{|m_{D1}|^2 + |m_{D2}|^2 + |m_{D3}|^2}{M_R^2} M_D^T M_D^*. \tag{3.33}$$

Since the columns of this matrix are clearly not linearly independent it will have zero eigenvalues, indeed just one mass is non-zero: $m_1 = \frac{(|m_{D1}|^2+|m_{D2}|^2+|m_{D3}|^2)^2}{M_R^2}$, $m_{2,3} = 0$.

Type II seesaw

In this kind of model, instead of introducing a right-handed neutrino field, a scalar $SU(2)$ triplet is introduced. This scalar triplet is usually written as $\Delta = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix}$, so that it transforms under $SU(2)$ as $\Delta \to U\Delta U^\dagger$. We can then write down a coupling between the scalar triplet and the left-handed lepton doublets

$$\mathcal{L}_m^\nu = \frac{1}{2} g_{\alpha\beta} \overline{L_{L\alpha}^c} i\sigma_2 \Delta L_{L\beta} + h.c. \ . \tag{3.34}$$

After the Δ^0 field acquires a VEV, this would result in a Majorana mass term for the neutrinos. Note that in order to conserve lepton number the Δ fields should have lepton number -2. Since we added a scalar field, we now have an enlarged Higgs potential. The Higgs potential that is usually taken [4] is generally of the form

$$\begin{aligned} V(\phi, \Delta) =& \mu\phi^\dagger\phi + \frac{1}{2}\lambda_\phi(\phi^\dagger\phi)^2 + M^2\text{Tr}(\Delta^\dagger\Delta) + \frac{1}{2}\lambda_\Delta \text{Tr}(\Delta^\dagger\Delta)^2 \\ &+ \lambda_{\phi\Delta}\phi^\dagger\phi\text{Tr}(\Delta^\dagger\Delta) + M_{\phi\Delta}\phi^T i\sigma_2\Delta^\dagger\phi + h.c. \ . \end{aligned} \tag{3.35}$$

If we take the VEVs, $\langle\Delta^0\rangle = u$ and $\langle\phi^0\rangle = v$, and look at the minimum conditions we obtain,

$$0 = \frac{\partial V_0}{\partial v} = 2v(\mu + \lambda_\phi v^2 + \lambda_{\phi\Delta}u^2 + M_{\phi\Delta}u), \tag{3.36}$$

$$0 = \frac{\partial V_0}{\partial u} = 2u(M^2 + \lambda_\Delta u^2 + \lambda_{\phi\Delta}v^2) + M_{\phi\Delta}v^2. \tag{3.37}$$

Since we expect the mass of the triplet to originate from physics at a high energy scale, we will take $M \sim M_{\phi\Delta} \gg \sqrt{\mu}$. We can then obtain the VEVs, $v^2 \sim \mu$ and $u^2 \sim \frac{\mu^2}{M^2} \ll \mu$. As was the case in type I seesaw, taking one mass to be large means having small neutrino masses of the order of $m_\nu \sim \frac{v^2}{M}$. In this case a large scalar triplet mass forces its VEV to be small and thus generates a small neutrino mass. If we were to combine type I and II, we could assume that the additional fields, Δ and ν_R, have masses of the same high energy scale, M. In that case both contributions to the neutrino masses are of the same order $\mathcal{O}(M_L) \sim \mathcal{O}(M_D^T M_R^{-1} M_D \sim v^2/M$. Note that the $M_{\phi\Delta}$ term in the potential is important in order to get a small u, and does not conserve lepton number.

Type III seesaw

In this last type of seesaw, first proposed in [16], we will discuss, N_Σ lepton triplets are introduced. The scheme is then very similar to type I seesaw as we will see. We will write the lepton triplets as follows, $\Sigma_\alpha = \begin{pmatrix} \Sigma^0_\alpha/\sqrt{2} & \Sigma^+_\alpha \\ \Sigma^-_\alpha & -\Sigma^0_\alpha/\sqrt{2} \end{pmatrix}$. The terms generating the masses are

$$-\mathcal{L}_m = y_{\alpha\beta}\tilde{\phi}^\dagger \overline{\Sigma_\alpha} L_{L\beta} + \frac{1}{2}\mathrm{Tr}(M_{\alpha\beta}\overline{\Sigma_\alpha}\Sigma^c_\beta) + h.c. \ , \tag{3.38}$$

where the $M_{\alpha\beta}$ term is a Majorana mass term for the Σ_α fields and is generated by physics at some higher energy scale. After spontaneous symmetry breaking a Dirac mass term is generated between the neutrinos and the neutral component of the triplet, Σ^0_α. Indeed, we get a $3 \times N_\Sigma$ Dirac mass matrix $M_D = \frac{v}{\sqrt{2}} y_{\alpha\beta}$ and a $N_\Sigma \times N_\Sigma$ heavy Majorana mass matrix $(M_{\Sigma^0})_{\alpha\beta} = M_{\alpha\beta}$, so that we can write

$$\mathcal{L}^\nu_m = -\frac{1}{2}\overline{n^c} M^{D+M} n + h.c. \ , \tag{3.39}$$

here $\mathcal{L}^\nu_m$ contains all the terms from $\mathcal{L}_m$ which involve Σ^0. We have for the mass matrix $M^{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_{\Sigma^0} \end{pmatrix}$ and $n = \begin{pmatrix} \nu_L \\ (\Sigma^0)^c \end{pmatrix}$, with $\Sigma^0 = (\Sigma^0_1, ..., \Sigma^0_{N_\Sigma})^T$. We can now see that this is completely analogous to type I seesaw and that the Σ^0_α fields play the role of the right-handed neutrino fields. Following the line of reasoning from the type I case, we obtain for the neutrino masses

$$M_\nu \simeq -M_D^T M_\Sigma^{-1} M_D. \tag{3.40}$$

As before, taking a large mass for the introduced fields results in a small neutrino mass.

It is interesting to see that in each of these seesaw scenarios the effective operator at low energy is similar to the dimension 5 operator (3.11) discussed earlier. In the case of type II seesaw this is rather clear. After the neutral component of the scalar triplet fields has acquired its VEV, $u \sim \frac{v^2}{M}$, the mass term (3.34) has exactly the same form as the dimension 5 operator after spontaneous symmetry breaking, (3.12). In the case of type I and III seesaw we can see this by integrating out the introduced fermion fields. The relevant piece of the Lagrangian, before spontaneous symmetry breaking, for both cases, is

$$-\mathcal{L} = \overline{\chi} m_D \tilde{\phi}^\dagger L_L + \frac{1}{2}\overline{\chi} M_R \chi^c + h.c. \ , \tag{3.41}$$

where χ represents either the right-handed neutrinos or the Σ^0 fields from the triplet fermions and m_D is a matrix of dimensionless coupling constants. Assuming that at low energy we can neglect the kinetic term for χ we obtain for the equation of motion

$$0 \simeq -\frac{\partial \mathcal{L}}{\partial \overline{\chi}} = m_D \tilde{\phi}^\dagger L_L + M_R \chi^c. \tag{3.42}$$

We now have for the introduced fermion fields

$$\chi^c \simeq -M_R^{-1} m_D \tilde{\phi}^\dagger L_L, \quad \overline{\chi} \simeq -\overline{(\tilde{\phi}^\dagger L_L)^c} m_D^T M_R^{-1}. \tag{3.43}$$

Inserting this into the Lagrangian, we get

$$\mathcal{L} \simeq \frac{1}{2}\overline{(\tilde{\phi}^\dagger L_L)^c} m_D^T M_R^{-1} m_D (\tilde{\phi}^\dagger L_L) + h.c. \ . \tag{3.44}$$

This is equal to (3.11) when $\frac{g_{\alpha\beta}}{M} = -(m_D^T M_R^{-1} m_D)_{\alpha\beta}$. The dimension 5 operator thus seems to describe each of these scenarios effectively at low energy.

We now move on to discuss the neutrino mixing matrix for these different scenarios.

Mixing matrix

We will discuss the diagonalization of the most general (seesaw) neutrino mass matrix. We will take the neutrino mass matrix to be the most general mass matrix that can be produced by seesaw type I, II or III or a combination of these. Such a matrix can still be written as $M^{D+M} = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}$, where we imagine the Dirac mass matrix to result from the coupling of neutrinos to singlet or triplet fermion fields or both. M_R then results from the coupling of these fermion fields among themselves. Note that for type II seesaw we have $M_D = M_R = 0$. In this case the neutrino mixing matrix comes about in exactly the way as in the case of the dimension 5 operator.

In what follows $n_L = \begin{pmatrix} \nu_L \\ \chi^c \end{pmatrix}$, where χ again stands for the newly introduced fermion fields, N_R is the number of fields that are introduced and $N = N_R + 3$.

To see what happens to the mixing matrix we diagonalize the neutrino mass matrix by changing our basis as before, so that

$$\mathcal{L}_m^\nu = -\frac{1}{2}\overline{n_L^c} M^{D+M} n_L + h.c. = -\frac{1}{2}\sum_i \overline{N_{L,i}^c} m_i N_{L,i} + h.c. \ , \tag{3.45}$$

here $n_L = V_\nu N_L$ in which V_ν diagonalizes M^{D+M}, n_L contains the left-handed weak eigenstates and N_L the mass eigenstates. We can write the left-handed neutrinos as linear combinations of the mass eigenstates as follows

$$\nu_{L,\alpha} = (V_\nu)_{\alpha k} N_{L,k}, \tag{3.46}$$

where $\alpha = e, \mu, \tau$ and k ranges from 1 to N. Now diagonalizing the charged lepton mass matrix as we did before, we have the following weak charged current

$$\mathcal{L}_{CC} \sim W_\mu^+ \overline{l_L} \gamma^\mu \nu_L + h.c. = W_\mu^+ \overline{l_{L,\alpha}^m} \gamma^\mu (V_l^\dagger)_{\alpha\beta} (V_\nu)_{\beta k} N_{L,k} + h.c. \ , \tag{3.47}$$

where V_l is one of the 3×3 matrices diagonalizing the charged lepton matrix as before and α and β range over the flavors while k ranges from 1 to N. We can conclude that the weak basis and the mass basis are related as follows

$$\nu_{L,\alpha}^w = (V_l^\dagger)_{\alpha\beta} (V_\nu)_{\beta k} \nu_{L,k} = U_{\alpha k} \nu_{L,k}, \quad \alpha = e, \mu, \tau, \ \ \beta = 1, 2, 3, \ \ k = 1, ..., N, \tag{3.48}$$

$$\nu_{R,\alpha}^{c,w} = (V_\nu)_{\alpha k} \nu_{L,k}, \qquad \alpha = 1, ..., N_R, \ \ k = 1, ..., N. \tag{3.49}$$

It can be seen that $U_{\alpha k} = (V_l^\dagger)_{\alpha\beta}(V_\nu)_{\beta k}$ is in general not a unitary matrix. Although V_l and V_ν are unitary matrices, the rectangular matrix $(V_\nu)_{\alpha k},\ \ \alpha = 1, 2, 3,\ \ k = 1, ..., N$ in general is not. This means that even though $UU^\dagger = I$, generally $U^\dagger U \neq I$.

In the cases of pure type I or III seesaw it can be seen that the diagonalizing matrix V_ν will be of the form

$$V_\nu \simeq W \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{3.50}$$

where A a 3×3 matrix and B a $N_R \times N_R$ matrix diagonalize $-M_D^T M_R^{-1} M_D$ and M_R, respectively. Looking at W we see that the upper left block is of order one while the upper right block is of order $M_D M_R^{-1}$. This tells us that up to order $M_D M_R^{-1}$ the matrix U is block diagonal. So, up to order $M_D M_R^{-1}$ we have

$$\nu_{L,\alpha}^w = U_{\alpha k} \nu_{L,k}, \quad i = e, \mu, \tau,\ \ k = 1, 2, 3. \tag{3.51}$$

Which means that the light and heavy states mostly mix among themselves and that mixing between light and heavy states is suppressed by a factor $M_D M_R^{-1}$. In most models the mixing between the heavy and light states is neglected and the PMNS matrix is taken to be (nearly unitary) $U_{PMNS} = V_l^\dagger V_\nu'$ where V_ν' is the matrix diagonalizing the upper-left block of $W^T M^{D+M} W$, which is $-M_D^T M_R^{-1} M_D$. Under these approximations the parametrization of U_{PMNS} is the same as in equation (3.19).

The non-unitarity of U has an effect on the neutral part of the weak interaction. If we rewrite the neutrino neutral current in mass eigenstates we obtain

$$\overline{\nu_L} \gamma^\mu \nu_L = \overline{N_L} \gamma^\mu V_\nu^\dagger V_\nu N_L = \overline{N_L} \gamma^\mu U^\dagger U N_L. \tag{3.52}$$

The neutrino neutral current is no longer diagonal in the mass basis, though the off-diagonal terms are of the order $M_D M_R^{-1}$.

Now that we have studied the mass terms and resulting mixing matrices there is one more part we will need in order to construct an A_4 model. This part is the Higgs sector. When the Higgs fields acquire vacuum expectation values (VEVs) they provide mass terms for the fermions. A different VEV will in general provide different mass terms. Since it is these mass terms we are particularly interested we will have to study the possible VEVs of the Higgs fields. This is exactly the goal of the next Chapter.

4

Higgs sector

In this thesis the goal is to reproduce the mixing matrices, in order to do so we will need to be able to construct the mass matrices of the fermions. For this we not only need to know about the symmetry group and the possible mass terms, but also about the VEVs of the Higgs doublets. These help determine the mass matrices and so it is important to know which VEVs are possible. The Higgs masses are also of interest since these should obey experimental constraints. In order to find the possible vacua and the masses of the Higgs particles, we need the scalar potential. In search of the minima and the Higgs masses we will follow the reasoning of [5] and [17]. In this Chapter we will try to find the possible VEVs and see which of them are acceptable and which are not.

4.1 Multiple Higgs doublets

Throughout most of this thesis we will assume two additional Higgs $SU(2)$ doublets, so that we have three Higgs doublets acting as a triplet under A_4. The reason for this is that in order to get viable mass matrices for the quarks and leptons, the Higgs doublet needs to be something other than a singlet, see [18]. In A_4 this leaves only a triplet assignment for the Higgs field. This means that we will need two additional Higgs doublets.

Before we start discussing the Higgs potential we will look at what changes in the SM when we introduce two additional doublets. The Higgs mechanism does not only gives rise to the fermion masses but also provides the $W^{\pm}$ and Z bosons with masses. To see how this goes when we have three doublets instead of one, we will take a look at the relevant part in the Lagrangian. This is the term involving the covariant derivative, in the case of three doublets, $\phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix}$, $(i = 1, 2, 3)$, and A_4 invariance it is given by

$$\mathcal{L} = \sum_i (D_\mu \phi)_i^\dagger (D^\mu \phi)_i, \tag{4.1}$$

here $(D_\mu \phi)_i = (\partial_\mu - i\frac{g}{2}\vec{\tau}.\vec{A_\mu} - i\frac{g'}{2}B_\mu)\phi_i$. Where g (g') and A_μ^a (B_μ) are the coupling constant and gauge field of $SU(2)$ $(U(1))$ respectively and τ^a are the Pauli matrices. We assume the

Higgs fields acquire the following neutral VEVs

$$\langle\phi_i\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}v_i \end{pmatrix}. \tag{4.2}$$

The v_i are in general complex, the covariant derivative in the vacuum is then as follows

$$(D_\mu\langle\phi\rangle)_i = \frac{v_i}{\sqrt{2}} \begin{pmatrix} -i\frac{g}{2}(A^1_\mu - iA^2_\mu) \\ \frac{i}{2}(gA^3_\mu - g'B_\mu) \end{pmatrix}. \tag{4.3}$$

The minimum of the Lagrangian of (4.1) then becomes

$$\mathcal{L}_0 = \frac{gv^2}{8}\left[(A^1_\mu)^2 + (A^2_\mu)^2\right] + \frac{v^2}{8}\left[gA^3_\mu - g'B_\mu\right]^2, \tag{4.4}$$

here $v^2 = \sum_i |v_i|^2$. This means that we have the usual fields with slightly modified (compared to the SM) expressions for the masses

$$\begin{aligned} W^\pm =& \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu), \;\; M^2_W = \frac{g^2v^2}{4}, \\ Z =& \frac{gA^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}}, \;\; M^2_Z = v^2\frac{g^2+g'^2}{4}, \\ A_\mu =& \frac{gA^3_\mu + g'B_\mu}{\sqrt{g^2+g'^2}}, \;\; M_A = 0. \end{aligned} \tag{4.5}$$

We see that the main difference with the one doublet case of the SM is that the squared VEV of the old doublet has been replaced by the sum of the squares of the new ones, $v^2_{SM} \to |v_1|^2 + |v_2|^2 + |v_3|^2$. Fortunately, there is still a massless combination of the gauge fields so that the photon field remains massless. Note however, that the vacuum alignment we chose is an assumption since the charged components, ϕ^+_i, may obtain a VEV as well. If one of these charged components would obtain a VEV there is no longer a massless combination of the gauge fields available. This, together with the fact that this would produce an electrically charged vacuum (when it is not compensated by the VEV of another, oppositely charged, field), makes these VEVs undesirable. This is the reason that in what follows we will assume that the Higgs doublets acquire the following VEVs

$$\phi_1 = \begin{pmatrix} 0 \\ v_1e^{-i\alpha/2} \end{pmatrix}, \;\; \phi_2 = \begin{pmatrix} 0 \\ v_2e^{i\beta/2} \end{pmatrix}, \;\; \phi_3 = \begin{pmatrix} 0 \\ v_3e^{i\gamma/2} \end{pmatrix}. \tag{4.6}$$

We will denote this as $\boldsymbol{(v_1e^{-i\alpha/2}, v_2e^{i\beta/2}, v_3e^{i\gamma/2})}$. The v_i are now real, note that we can set one of these phases to zero by a global rotation.

Our goal is to see what kind of vacua are possible and to calculate the Higgs masses in these cases. From this we can then conclude which of the vacua are suitable to use in a model and which we can discard. Towards this end, in the next section we will construct to most general A_4 invariant potential.

4.2 The Higgs potential

If we have three Higgs doublets the most general potential, invariant under the SM symmetries and A_4, is

$$\begin{aligned}V =&\mu\phi^\dagger\phi + \lambda_1(\phi^\dagger\phi)^{\mathbf{1}}(\phi^\dagger\phi)^{\mathbf{1}} + \lambda_2(\phi^\dagger\phi)^{\mathbf{1}'}(\phi^\dagger\phi)^{\mathbf{1}''} + \lambda_3(\phi^\dagger\phi)^{\mathbf{3}_1}(\phi^\dagger\phi)^{\mathbf{3}_2}\\ &+\{\lambda_4(\phi^\dagger\phi)^{\mathbf{3}_1}(\phi^\dagger\phi)^{\mathbf{3}_1} + \lambda_5(\phi^\dagger\phi)^{\mathbf{3}_2}(\phi^\dagger\phi)^{\mathbf{3}_2} + h.c.\}\\ =&\mu(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3) + \lambda_1(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3)^2\\ &+\lambda_2(\phi_1^\dagger\phi_1 + \omega^2\phi_2^\dagger\phi_2 + \omega\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1 + \omega\phi_2^\dagger\phi_2 + \omega^2\phi_3^\dagger\phi_3)\\ &+\lambda_3\left[(\phi_3^\dagger\phi_2)(\phi_2^\dagger\phi_3) + (\phi_1^\dagger\phi_3)(\phi_3^\dagger\phi_1) + (\phi_2^\dagger\phi_1)(\phi_1^\dagger\phi_2)\right]\\ &+\left[\lambda_4[(\phi_2^\dagger\phi_3)^2 + (\phi_3^\dagger\phi_1)^2 + (\phi_1^\dagger\phi_2)^2] + \lambda_5[(\phi_3^\dagger\phi_2)^2 + (\phi_1^\dagger\phi_3)^2 + (\phi_2^\dagger\phi_1)^2] + h.c.\right],\end{aligned} \tag{4.7}$$

here we used the MR basis and all the parameters are real, apart from λ_4 and λ_5 which are in general complex. Note that although we wrote down all the possible A_4 contractions, we only used one $SU(2)$ contraction, $\phi^\dagger\phi$. This is because all the other possible contractions can be written in terms of this one. To see this we will look into these other contractions. The multiplication rules of $SU(2)$ are

$$\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{3}, \quad \mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}. \tag{4.8}$$

This means we can build singlets from the doublets, $a^{(\prime)} = (a_1^{(\prime)}, a_2^{(\prime)})$ and $b^{(\prime)} = (b_1^{(\prime)}, b_2^{(\prime)})$, as follows $\mathbf{1} \sim (ab)^{\mathbf{1}}, ((ab)^{\mathbf{3}}(a'b')^{\mathbf{3}})^{\mathbf{1}}$. The superscript denotes the irrep the combination inside the brackets is in. We have two types of doublets in our potential, ϕ_i and $\tilde{\phi}_i$. The first term then allows us to construct the singlets $(\phi_i\phi_j)^{\mathbf{1}} = \phi_i^T\tilde{\phi}_j^*$, $(\tilde{\phi}_i\tilde{\phi}_j)^{\mathbf{1}} = \phi_i^\dagger\tilde{\phi}_j$ and $(\tilde{\phi}_i\phi_j)^{\mathbf{1}} = \phi_i^\dagger\phi_j$, where the Clebsch-Gordan coefficients were used to write them out explicitly. The only non-zero coupling that can be constructed from this which we have not yet used is

$$(\phi_i^\dagger\tilde{\phi}_j)(\phi_k^T\tilde{\phi}_l^{\,*}). \tag{4.9}$$

A_4 restricts this term some more, among other things it demands that each index should appear an even number of times. To see this, note that the three elements of the class C_4 (section 2.1) imply that the potential should be invariant under

$$\phi_a \to -\phi_a,\ \ \phi_b \to -\phi_b,\ \ \phi_c \to \phi_c, \tag{4.10}$$

with a, b, c any permutation of $1, 2, 3$. Since we only have three Higgs doublets, it is clear that two of the indices of the quartic term (4.9) should be equal, say $i = j$ for clarity. If we now apply the transformation (4.10) with $a = i$ we see that the term goes to minus itself unless $k = l$, in which case it is invariant. So indeed, every index should appear an even number of times, this holds for every term in the Higgs potential. This means we are left with just one term which is of the form

$$\begin{aligned}(\phi_i^\dagger\tilde{\phi}_j)(\phi_i^T\tilde{\phi}_j^{\,*}) =&(\phi_i^\dagger\tilde{\phi}_j)(\phi_i^T\tilde{\phi}_j^{\,*})^T = -(\phi_i^\dagger\tilde{\phi}_j)(\phi_j^T\tilde{\phi}_i^{\,*})\\ =&-\phi_i^-(\phi_j^0)^*\phi_j^+\phi_i^0 + \phi_i^-(\phi_j^0)^*\phi_j^0\phi_i^+ + (\phi_i^0)^*\phi_j^-\phi_j^+\phi_i^0 - (\phi_i^0)^*\phi_j^-\phi_j^0\phi_i^+\\ =&(\phi_i^\dagger\phi_i)(\phi_j^\dagger\phi_j) - (\phi_i^\dagger\phi_j)(\phi_j^\dagger\phi_i),\end{aligned} \tag{4.11}$$

where we used the fact that the second Pauli matrix is anti-symmetric in the first line.

Now, for terms of the form $((ab)^{\mathbf{3}}(a'b')^{\mathbf{3}})^{\mathbf{1}}$. The Clebsch-Gordan coefficients for two $SU(2)$ doublets, a and b, and two triplets, $c = (c_1, c_2, c_3)$ and $d = (d_1, d_2, d_3)$ are

$$(ab)^{\mathbf{3}} \sim (a_1 b_1, \sqrt{1/2}(a_1 b_2 + a_2 b_1), a_2 b_2), \quad (cd)^{\mathbf{1}} \sim c_1 d_3 + c_3 d_1 - c_2 d_2. \tag{4.12}$$

By using the two doublets ϕ_i and $\tilde{\phi}_i$ we can construct three kinds of triplets $(\phi_i \phi_j)^{\mathbf{3}}$, $(\tilde{\phi}_i \tilde{\phi}_j)^{\mathbf{3}}$ and $(\phi_i \tilde{\phi}_j)^{\mathbf{3}}$. From this it can be seen that from all the possible terms only the following contractions are allowed

$$(\phi_i \tilde{\phi}_j)^{\mathbf{3}} (\phi_k \tilde{\phi}_l)^{\mathbf{3}}, \quad (\tilde{\phi}_i \tilde{\phi}_j)^{\mathbf{3}} (\phi_k \phi_l)^{\mathbf{3}}. \tag{4.13}$$

Again, A_4 requires that every index appears an even number of times. This restricts the number of terms somewhat, by writing these out we find that they can be written as

$$(\tilde{\phi}_i \tilde{\phi}_i)^{\mathbf{3}} (\phi_j \phi_j)^{\mathbf{3}} = (\phi_i^\dagger \phi_j)^2, \ (\tilde{\phi}_i \tilde{\phi}_j)^{\mathbf{3}} (\phi_i \phi_j)^{\mathbf{3}} = \tfrac{1}{2}\left[(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + (\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i)\right], \tag{4.14}$$

$$(\phi_i \tilde{\phi}_j)^{\mathbf{3}} (\phi_i \tilde{\phi}_j)^{\mathbf{3}} = -\tfrac{1}{2}(\phi_j^\dagger \phi_i)^2, \ (\phi_i \tilde{\phi}_j)^{\mathbf{3}} (\phi_j \tilde{\phi}_i)^{\mathbf{3}} = \tfrac{1}{2}(\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i) - (\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j), \tag{4.15}$$

$$(\phi_i \tilde{\phi}_i)^{\mathbf{3}} (\phi_j \tilde{\phi}_j)^{\mathbf{3}} = \tfrac{1}{2}(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) - (\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i). \tag{4.16}$$

So it can now be concluded that any $SU(2)$ contraction invariant under A_4 can indeed be written in terms of $\phi_i^\dagger \phi_j$. This implies that the potential discussed above is indeed the most general one, after renaming the parameters it can be written as

$$\begin{aligned} V =& \mu(\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3) + \lambda_1 (\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3)^2 \\ &+ \lambda_2 \left[(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + (\phi_1^\dagger \phi_1)(\phi_3^\dagger \phi_3) + (\phi_3^\dagger \phi_3)(\phi_2^\dagger \phi_2)\right] \\ &+ (\lambda_3 - \lambda_2)\left[(\phi_3^\dagger \phi_2)(\phi_2^\dagger \phi_3) + (\phi_1^\dagger \phi_3)(\phi_3^\dagger \phi_1) + (\phi_2^\dagger \phi_1)(\phi_1^\dagger \phi_2)\right] \\ &+ \frac{1}{2}\lambda_4 \left[e^{i\epsilon}\left[(\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2 + (\phi_1^\dagger \phi_2)^2\right] + h.c.\right], \end{aligned} \tag{4.17}$$

here the parameters μ, λ_i and ϵ are all real. We renamed the parameters as follows

$$\begin{aligned} \lambda_1 &= \lambda_1' + \lambda_2', \quad \lambda_2 = -3\lambda_2', \\ \lambda_3 &= \lambda_3' - 3\lambda_2', \quad \frac{1}{2}\lambda_4 e^{i\epsilon} = \lambda_4' + (\lambda_5')^*, \end{aligned} \tag{4.18}$$

where the primes denote the old parameters.

The parameters are constrained by the demand that the potential should be bounded from below. This should hold in the limit that we let any combination of the fields go to infinity as well. Since μ will be negative in order to facilitate a minimum away from the origin, the contribution of the quartic terms should be positive when we take such a limit. We will write the fields as

$$\phi_i = \begin{pmatrix} |\phi_i^+| e^{i\beta_i} \\ |\phi_i^0| e^{i\alpha} \end{pmatrix}. \tag{4.19}$$

The constraints we find this way are given in Table (4.1). Adding the $(\pi/2)$ in the last row corresponds to the minus sign and without it the condition with the plus sign is obtained. Whether this is the complete set of constraints is uncertain. In any case, each of these constraints we found should be satisfied by the potential.

Constraint	$(\lvert\phi_1^+\rvert, \lvert\phi_2^+\rvert, \lvert\phi_3^+\rvert)$	$(\lvert\phi_1^0\rvert, \lvert\phi_2^0\rvert, \lvert\phi_3^0\rvert)$	Phases
$\lambda_1 > 0$	$(\infty, 0, 0)$	$(\infty, 0, 0)$	-
$3\lambda_1 + \lambda_3 + \lambda_4 \cos\epsilon > 0$	(∞, ∞, ∞)	(∞, ∞, ∞)	$\alpha_i = \beta_i = 0$
$3\lambda_1 + \lambda_3 > 0$	(∞, ∞, ∞)	(∞, ∞, ∞)	$\alpha_i = \beta_i$, $\beta_1 = \beta_3 + \epsilon/4 + 3\pi/8$, $\beta_2 = \beta_3 - \epsilon/4 + \pi/8$
$4\lambda_1 + \lambda_2 > 0$	$(\infty, 0, \infty)$	$(\infty, 0, \infty)$	$\alpha_1 = \beta_1 - \beta_3 + \alpha_3 + \pi$
$4\lambda_1 + \lambda_3 \pm \lambda_4 > 0$	$(\infty, 0, \infty)$	$(\infty, 0, \infty)$	$\alpha_i = \beta_i$, $\beta_1 = \beta_3 - \epsilon/2 + (\pi/2)$

Table 4.1: The constraints on the parameters of the potential obtained by demanding that the potential is bounded from below.

4.3 CP violation

Now that we know the potential, we can try to see whether or not there is CP violation in the Higgs sector. Under the simplest CP transformation the Higgs doublets transform as $\phi_i \to \phi_i^*$, where $\phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix}$. The Higgs potential (4.17) is clearly not invariant under this transformation, the transformation effectively changes ϵ to $-\epsilon$. However, one can also perform a basis transformation under which the physical content of the Lagrangian does not change. Such a transformation can be written as

$$\phi_i \xrightarrow{basis} \phi_i' = V_{ij}\phi_j, \tag{4.20}$$

where V_{ij} is unitary. We can then define a more general CP transformation as follows, [19],

$$\phi_i \xrightarrow{CP} U_{ij}\phi_j^*, \tag{4.21}$$

where U_{ij} is unitary. The Higgs potential conserves CP if there exists a matrix U such that the CP transformation (4.21) leaves the Higgs potential invariant, [19]. We can now see that the Higgs potential (4.17) is invariant when we combine the simplest CP transformation with the exchange of two Higgs doublets. For instance, if we exchange the first and second Higgs doublets the matrix U becomes, $U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the doublets transform as follows

$$\phi_1 \xrightarrow{CP} \phi_2^*, \;\; \phi_2 \xrightarrow{CP} \phi_1^*, \;\; \phi_3 \xrightarrow{CP} \phi_3^*. \tag{4.22}$$

The Higgs potential is indeed invariant under this CP transformation. In this way we can conclude that there is no explicit CP violation in the Higgs potential.

However, there is still the possibility of spontaneous CP violation. Near the minimum the VEVs are acquired and the Higgs doublets can be written as $\phi_{0i} = \begin{pmatrix} \phi_i^+ \\ v_i + \phi_i^0 \end{pmatrix}$, where v_i are in general complex. For the potential to be CP invariant in this case it can be shown that the CP transformation should satisfy [20]

$$\langle \phi_i \rangle = \sum_j U_{ij} \langle \phi_j \rangle^*. \tag{4.23}$$

So we get $(v_1, v_2, v_3)^T = U(v_1^*, v_2^*, v_3^*)^T$. Using the same CP transformation on the Higgs fields as before, (4.22), we see that (4.23) is satisfied when[1] $v_3 \in \mathrm{Re}$ and $v_1 = v_2^*$. When these constraints are satisfied, the A_4 Higgs potential will be invariant under the CP transformation (4.22) even after the VEVs are acquired. This is the case for all minima looked at in [17], these will be discussed in section 4.4.

4.3.1 Invariants

Instead of searching for a transformation that leaves the potential invariant, it is often simpler to calculate so called invariants. These are quantities which depend on the parameters of the Higgs potential and are invariant under a basis transformation. They are zero when CP is conserved and when any of them is non-zero CP is violated. If the A_4 Higgs potential is indeed CP conserving these invariants should all be zero.

Explicit CP violation

There are two kinds of invariants; those that deal with the potential before the VEVs are acquired and those that deal with the potential afterwards. We will first look at the case before the VEVs are acquired, i.e. we will see if there is explicit CP violation. In doing so, we will follow [19].

To see how such invariants can be constructed, consider a general multi-Higgs doublet potential

$$V = \phi_i^\dagger Y_{ij} \phi_j + \phi_i^\dagger \phi_j Z_{ijkl} \phi_k^\dagger \phi_l, \tag{4.24}$$

where $Y_{ij} = Y_{ji}^*$, $Z_{ijkl} = Z_{jilk}^*$ and $Z_{ijkl} = Z_{klij}$ by hermiticity. Under a basis transformation, (4.20), we have

$$Y'_{ij} = V_{ia} Y_{ab} V_{bj}^\dagger, \quad Z'_{abcd} = V_{ai} V_{ck} Z_{ijkl} V_{jb}^\dagger V_{ld}^\dagger. \tag{4.25}$$

If the potential is invariant under a CP transformation, (4.21), then there should be a matrix, U, such that

$$Y_{ab}^* = U_{ai}^\dagger Y_{ij} U_{jb}, \quad Z_{abcd}^* = U_{ai}^\dagger U_{ck}^\dagger Z_{ijkl} U_{jb} U_{ld}. \tag{4.26}$$

We will write the matrices as $Z_{ac}^{bd} = Z_{abcd}$ and $Y_a^b = Y_{ab}$.

These relations give us the means to construct invariants. By combining a number of these matrices and contracting each upper index with a lower index, we obtain scalars, I_i, which are invariant under a basis transformation. This can be seen by applying a basis transformation, (4.25), to such a scalar. Since every upper index is contracted with a lower index, every matrix V (which appears through the basis transformation) is contracted with its hermitian conjugate. Thus, a basis transformation will leave the quantities I_i invariant.

[1] If we had chosen to exchange another pair of Higgs doublets, say, ϕ_i and ϕ_j $(i \neq j)$ in the CP transformation (4.22), these constraints would change to $v_i = v_j^*$ and $v_k \in \mathrm{Re}$, where $i \neq k \neq j$.

If the potential is invariant under a CP transformation, (4.21), the relations (4.26) should hold. Because in the invariants, I_i, every upper index is contracted with a lower index, each matrix U_{ai} we obtain through use of (4.26) will be contracted with another, $U^\dagger_{ib}$. Thus (4.26) implies that such a scalar will be equal to its conjugate.

So, we now have quantities, $\mathrm{Im}\, I_i$, which are invariant under a basis transformation and zero when CP is conserved (i.e. when (4.26) holds). The simplest invariants we could construct in this way are

$$\mathrm{Im}\, I_1 = \mathrm{Im}\, Y_i^i, \quad \mathrm{Im}\, I_2 = \mathrm{Im}\, Z_{kl}^{kl}, \quad \mathrm{Im}\, I_3 = \mathrm{Im}\, Z_{kl}^{lk},$$

Unfortunately, these simple examples are identically zero since Y and Z are both hermitian. However, it is possible to construct useful invariants in this way.

For three Higgs doublets one of the simplest invariant which is relevant is given by, [19],

$$\mathrm{Im}\, I_4 = \mathrm{Im}\left[Z_{ak}^{nq} Y_n^m Z_{mq}^{rs} Z_{rx}^{tx} Z_{ts}^{ka}\right] \sim (Y_{33}-Y_{22}) f(Z) + (Y_{33}-Y_{11}) g(Z) + (Y_{22}-Y_{11}) h(Z), \tag{4.27}$$

where $f(Z)$, $g(Z)$ and $h(Z)$ are functions of the elements of Z. From the explicit form of this invariant it can be seen that it will be zero in the A_4 case. For the A_4 Higgs potential, $Y = \mu \mathbb{1}$ in any basis, which means $\mathrm{Im}\, I_4$ vanishes. This was to be expected since in the A_4 Higgs potential we have already identified a CP transformation (4.21) which leaves the potential invariant.

Spontaneous CP violation

In the case of spontaneous CP violation we will follow [21]. We now have two conditions for CP invariance in the vacuum: 1) before the VEVs were acquired there should exist a CP transformation, (4.21), which leaves the potential invariant. 2) the CP transformation matrix, U, should satisfy (4.23). In this case invariants will be developed from a specific basis. Namely, the basis in which just one of the doublets acquires a VEV.

We will start in a general basis with the general VEVs $\langle\phi\rangle = (v_1, v_2, v_3)$. Assuming there is a CP transformation, (4.21), which leaves the potential invariant and (4.23) is satisfied, we have

$$U \begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{4.28}$$

From here we move to the basis, ϕ', in which just one doublet acquires a (real) VEV, we have

$$\phi' = V\phi, \tag{4.29}$$

such that $\langle\phi'\rangle = (v, 0, 0)$, $(v \in \mathrm{Re})$ and thus $V \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$. In this basis the CP transformation is

$$\phi' \xrightarrow{CP} U' \phi'^*, \tag{4.30}$$

where $U' = VUV^T$. It can be seen that (4.23) is satisfied in this basis as well

$$U' \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} = VU \begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \end{pmatrix} = V \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}. \tag{4.31}$$

For this to hold U' should have the form $U' = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \multirow{2}{*}{$U_{2\times 2}$} \\ 0 & \end{pmatrix}$, where $U_{2\times 2}$ is a unitary 2 by 2 matrix.

In the ϕ' basis the Higgs fields $\operatorname{Im}\phi_1^{0\prime}$ and $\phi_1^{+\prime}$ are the Goldstone bosons, which are eaten by the W and Z bosons. Thus the CP transformation in this basis

$$\phi' \xrightarrow{CP} U'\phi'^*, \tag{4.32}$$

reduces to

$$\begin{pmatrix} \phi_2' \\ \phi_3' \end{pmatrix} \xrightarrow{CP} U_{2\times 2} \begin{pmatrix} \phi_2'^* \\ \phi_3'^* \end{pmatrix}, \tag{4.33}$$

while leaving $\operatorname{Re}\phi_1^{0\prime}$ invariant. Thus, if the potential is invariant under a general CP transformation, (4.21), and (4.23) holds then in the ϕ' basis the potential is invariant under a CP transformation of the form of (4.33).

When we write the neutral Higgs fields and $U_{2\times 2}$ in their real and imaginary components, $\Phi^0 = (\operatorname{Re}\phi_2^{0\prime}, \operatorname{Re}\phi_3^{0\prime}, \operatorname{Im}\phi_2^{0\prime}, \operatorname{Im}\phi_3^{0\prime})^T$ and $U_{2\times 2} = \operatorname{Re}U_{2\times 2} + i\operatorname{Im}U_{2\times 2}$, the transformation of (4.33) for the neutral fields can be written as

$$\Phi^0 \xrightarrow{CP} \begin{pmatrix} \operatorname{Re}U_{2\times 2} & \operatorname{Im}U_{2\times 2} \\ \operatorname{Im}U_{2\times 2} & -\operatorname{Re}U_{2\times 2} \end{pmatrix} \Phi^0 = U_{4\times 4}\Phi^0, \tag{4.34}$$

where $U_{4\times 4}$ is a orthogonal matrix. A similar transformation holds for the charged fields, which can be obtained by letting $\Phi^0 \to \Phi^+ = (\operatorname{Re}\phi_2^{+\prime}, \operatorname{Re}\phi_3^{+\prime}, \operatorname{Im}\phi_2^{+\prime}, \operatorname{Im}\phi_3^{+\prime})^T$. Similarly, a basis transformation, $\begin{pmatrix} \phi_2' \\ \phi_3' \end{pmatrix} \xrightarrow{basis} K \begin{pmatrix} \phi_2' \\ \phi_3' \end{pmatrix}$, where K is a unitary 2 by 2 matrix, can be written as

$$\Phi^0 \xrightarrow{basis} \begin{pmatrix} \operatorname{Re}K & -\operatorname{Im}K \\ \operatorname{Im}K & \operatorname{Re}K \end{pmatrix} \Phi^0 = K_{4\times 4}\Phi^0, \tag{4.35}$$

where $K_{4\times 4}$ is orthogonal. Again analogously for Φ^+. The matrices $U_{4\times 4}$ and $K_{4\times 4}$ have the properties

$$\begin{aligned} K_{4\times 4}\varepsilon(K_{4\times 4})^T &= \varepsilon, \\ U_{4\times 4}\varepsilon(U_{4\times 4})^T &= -\varepsilon, \end{aligned} \tag{4.36}$$

where $\varepsilon = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}$. With these properties we can construct quantities which are invariant under a basis transformation but uneven under a CP transformation, as was done in [21].

This can be achieved by going back to the potential which can now be expressed in terms of Φ^0, Φ^+ and $\operatorname{Re}\phi_1^{0\prime}$. We can imagine the parameters of the Higgs potential to be transforming under a CP transformation. Tensors of parameters connecting a vector, Φ^0 or Φ^+, with invariants will transform as a vector, tensors connecting two vectors (some combination of Φ^0 and Φ^+) will transform as a second rank tensor, etc. For example, the terms making up the neutral and charged mass matrices are

$$\begin{aligned} V_m =& \frac{1}{2}M_{11}^2(\operatorname{Re}\phi_1^{0\prime})^2 + \operatorname{Re}\phi_1^{0\prime}b^T\Phi^0 + \frac{1}{2}(\Phi^0)^T c\Phi^0 + \frac{1}{2}(\Phi^+)^T h\Phi^+ \\ \xrightarrow{CP}& \frac{1}{2}M_{11}^2(\operatorname{Re}\phi_1^{0\prime})^2 + \operatorname{Re}\phi_1^{0\prime}b^T U_{4\times 4}\Phi^0 + \frac{1}{2}(\Phi^0)^T U_{4\times 4}^T c U_{4\times 4}\Phi^0 + \frac{1}{2}(\Phi^+)^T U_{4\times 4}^T h U_{4\times 4}\Phi^+ \end{aligned} \tag{4.37}$$

where the neutral mass matrix is given by $M^2 = \begin{pmatrix} M_{11}^2 & b^T \\ b & c \end{pmatrix}$, c is a 4 by 4 matrix and b a vector of length 4. The 4 by 4 charged mass matrix is $M_C^2 = h$. Thus, under a CP transformation $b \xrightarrow{CP} U_{4\times4}^T b$, $c \xrightarrow{CP} U_{4\times4}^T c U_{4\times4}$ and $h \xrightarrow{CP} U_{4\times4}^T h U_{4\times4}$. While under a basis transformation we have $b \xrightarrow{basis} K_{4\times4}^T b$, $c \xrightarrow{basis} K_{4\times4}^T c K_{4\times4}$ and $h \xrightarrow{basis} K_{4\times4}^T h K_{4\times4}$. With this we can construct invariants, the simplest ones would be

$$b^T \varepsilon b, \quad \mathrm{Tr}[c\varepsilon], \quad \mathrm{Tr}[h\varepsilon],$$

all of these quantities are invariant under a basis transformation but uneven under a CP transformation, however, they are also identically zero. This can be seen by recalling that c and h are symmetric while ε is antisymmetric. Thus, the simplest useful invariants are

$$J_1 = b^T c\epsilon b, \quad J_2 = b^T cc\epsilon b, \tag{4.38}$$

these are both invariant under a basis transformation but odd under a CP transformation (by equations (4.36)). Of course, more invariants can be constructed by replacing c by h in J_1 and J_2. Furthermore, additional invariants can be obtained by sandwiching ε between more and more elements of M.

If there exists a matrix $U_{4\times4}$ such that the potential is invariant under a CP transformation, (4.34), then from (4.37) we must have $b = U_{4\times4}^T b$, $c = U_{4\times4}^T c U_{4\times4}$ and $h = U_{4\times4}^T h U_{4\times4}$. Since the invariants are uneven under a CP transformation these equalities imply $J_i = -J_i$. For example, if there exists a matrix $U_{4\times4}$ such that the potential is invariant under (4.34) then we can write

$$J_1 = b^T c\varepsilon b = b^T U_{4\times4} U_{4\times4}^T c U_{4\times4} \varepsilon U_{4\times4}^T b = -J_1, \tag{4.39}$$

so that $J_1 = 0$.

In conclusion, if there exists a CP transformation (4.21), in any basis, under which the potential is invariant before the VEVs are acquired and the transformation matrix satisfies (4.23), this will lead to a CP transformation of the form (4.34) which leaves the potential invariant in the ϕ' basis. The existence of such a CP transformation which leaves the potential invariant in this basis then implies that the kind of invariants we have just constructed and are mentioned in [19] are zero.

These conditions are met in the A_4 Higgs sector. Namely, there is a CP transformation, (4.22) which leaves the potential invariant. In this case the CP transformation matrix satisfies (4.23) for all the vacua as we will see in the next section. This implies that there is no CP violation in the Higgs sector, explicit nor spontaneous. If we check this explicitly for the A_4 Higgs potential, then both J_1 and J_2 are indeed zero. In principle however, the VEVs of the Higgs doublets could still contribute to CP violation in the CKM and PMNS matrices. If or when this happens remains to be seen.

We will now return to the Higgs potential and derive the minimum conditions to see what the possible and interesting vacua are.

4.4 The vacuum solutions

Before we look at the possible vacua, it should be mentioned that there are some configurations of the parameters we will not be interested in. These are the configurations for which the potential has additional accidental symmetries. These are unwanted because if we are studying an A_4 invariant theory we want the Higgs potential to be invariant under A_4 and not invariant under some larger symmetry group, $G > A_4$. Also, when a continuous symmetry is present in the potential and is spontaneously broken when the VEVs are acquired we will obtain additional Goldstone bosons. Since the potential is invariant under $SU(2)$ which gets spontaneously broken we already have the usual three Goldstone bosons. This is exactly the amount that can be eaten by the gauge bosons so that they obtain a mass, as they did in (4.5). This means that we will not be able to get rid of the massless particles generated by the spontaneous breaking of the larger symmetry. For these reasons we will not use any vacuum which constrains the parameters in such a way that the potential has additional symmetries.

There are several accidental symmetries possible in this potential. For example, setting $\lambda_4 = 0$ leaves the potential invariant under two $U(1)$ symmetries, $\phi_{1,2} \to e^{i\alpha_{1,2}}\phi_{1,2}$. Note that there is another transformation which leaves the potential invariant, $\phi_i \to e^{i\alpha}\phi_i$, but it is already present in the most general potential and is part of the $SU(2) \times U(1)$ symmetry.

Another continuous symmetry that could be possible for some value of the parameters is an $SO(3)$ symmetry. In other words the potential might become invariant under $\vec{\phi} \to O\vec{\phi}$ where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)^T$ and O is a 3 by 3 orthogonal matrix with determinant 1. This can be checked by writing down the most general $SO(3)$ symmetric potential. Since $SO(3)$ contains the transformations $\phi_i \to -\phi_i$, we know that every index, i, should appear an even number of times in every $SO(3)$ invariant term. The argument from the A_4 case then applies here as well and we again conclude that we only need to consider the $SU(2)$ contraction $\phi_i^\dagger\phi_j$. If $\vec{\phi}$ is a triplet of $SO(3)$ then so is $\vec{\phi}^\dagger$ and the most general $SO(3)$ invariant potential is

$$V_{SO(3)} = g_1(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{1}} + g_2(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{1}}(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{1}} + g_3(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{3}}(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{3}} + g_4(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{5}}(\vec{\phi}^\dagger\vec{\phi})^{\mathbf{5}} \tag{4.40}$$

This can be worked out with the help of the Clebsch-Gordan coefficients, after some rewriting and renaming of the parameters this can be written as

$$\begin{aligned} V =& g_1(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3) + g_2(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3)^2 \\ &+ g_3'[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + (\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + (\phi_3^\dagger\phi_3)(\phi_2^\dagger\phi_2) \\ &- [(\phi_3^\dagger\phi_2)(\phi_2^\dagger\phi_3) + (\phi_1^\dagger\phi_3)(\phi_3^\dagger\phi_1) + (\phi_2^\dagger\phi_1)(\phi_1^\dagger\phi_2)]] \\ &+ g_4'[(\phi_2^\dagger\phi_3)^2 + (\phi_3^\dagger\phi_1)^2 + (\phi_1^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_2)^2 + (\phi_1^\dagger\phi_3)^2 + (\phi_2^\dagger\phi_1)^2 \\ &- 2[(\phi_3^\dagger\phi_2)(\phi_2^\dagger\phi_3) + (\phi_1^\dagger\phi_3)(\phi_3^\dagger\phi_1) + (\phi_2^\dagger\phi_1)(\phi_1^\dagger\phi_2)]] \end{aligned} \tag{4.41}$$

We see that the most general A_4 potential (4.17) takes this form when $\lambda_3 + \lambda_4 = 0$ and $\varepsilon = 0$. Thus, we now have identified two cases we will not consider. This is not to say that all the other vacua will be without extra symmetries and Goldstone bosons, but at least we have ruled out two uninteresting cases which we can now avoid altogether. Some vacua with these extra symmetries were looked at in [17] in which case extra massless particles were indeed found.

Now that we know two cases which we should avoid, we will try to find the minima which are interesting to us. To find the minima we derive the vacuum expectation value of the scalar potential with respect to the VEVs of the Higgs doublets, this leads to

$$\begin{aligned}
0 &= \frac{\partial V_0}{\partial v_1} = v_1\left(\mu + 2\lambda_1 v^2 + \lambda_3(v_2^2+v_3^2) + \lambda_4(v_2^2\cos\theta_3 + v_3^2\cos\theta_2\right), \\
0 &= \frac{\partial V_0}{\partial v_2} = v_2\left(\mu + 2\lambda_1 v^2 + \lambda_3(v_1^2+v_3^2) + \lambda_4(v_1^2\cos\theta_3 + v_3^2\cos\theta_1\right), \\
0 &= \frac{\partial V_0}{\partial v_3} = v_3\left(\mu + 2\lambda_1 v^2 + \lambda_3(v_2^2+v_1^2) + \lambda_4(v_2^2\cos\theta_1 + v_1^2\cos\theta_2\right), \\
0 &= \frac{\partial V_0}{\partial \alpha} = \lambda_4 v_1^2(v_3^2\sin\theta_2 - v_2^2\sin\theta_3), \\
0 &= \frac{\partial V_0}{\partial \beta} = \lambda_4 v_2^2(v_3^2\sin\theta_1 - v_1^2\sin\theta_3), \\
0 &= \frac{\partial V_0}{\partial \gamma} = \lambda_4 v_3^2(v_1^2\sin\theta_2 - v_2^2\sin\theta_1),
\end{aligned} \tag{4.42}$$

where

$$\begin{aligned}
\theta_1 =& \varepsilon - \beta + \gamma, \quad \theta_2 = \varepsilon - \alpha - \gamma, \\
\theta_3 =& \varepsilon + \alpha + \beta, \quad v^2 = v_1^2 + v_2^2 + v_3^2.
\end{aligned} \tag{4.43}$$

We will now see what the solutions to these equations are, we start by looking at the case where the vacuum alignment is in general complex.

A: complex vacua

In this case we have two possible vacua, $(\boldsymbol{v_1 e^{-i\alpha/2}, v_2, 0})$ and $(\boldsymbol{v_1 e^{-i\alpha/2}, v_2 e^{i\beta/2}, v_3})$ (or permutations thereof). This is because we need at least two VEVs in order to have a phase which cannot be absorbed by a global rotation. Working out the minimum conditions for the first option we obtain:

1A : $(\boldsymbol{\frac{1}{\sqrt{2}} v e^{-i\alpha/2}, \frac{1}{\sqrt{2}} v, 0})$
Plugging the first vacuum alignment into the minimum conditions we obtain $\varepsilon = -\alpha + N\pi$ and $(\lambda_3+\lambda_4)(v_1^2 - v_2^2) = 0$. After picking $v_1 = v_2$ there are two constraints

$$\mu = -\frac{v^2}{2}(4\lambda_1 + \lambda_3 + \lambda_4), \quad \varepsilon = -\alpha + N\pi, \tag{4.44}$$

here we have absorbed the possible sign of $\cos\theta_3 = \pm 1$ in a redefinition of λ_4.

As was shown in [5] for the second option we get the vacuum:

2A : $(\boldsymbol{w e^{-i\alpha/2}, w e^{i\alpha/2}, r w})$
Here $w = \frac{v}{\sqrt{2+r^2}}$, r is a dimensionless constant and we have absorbed the phase of the third

VEV by a global rotation. This vacuum alignment then has the following constraints

$$0 = \mu + v^2(2\lambda_1 + \frac{1}{2+r^2}(\lambda_3(1+r^2) + \lambda_4(\cos\theta_3 + r^2\cos\theta_1), \tag{4.45}$$

$$0 = \mu + 2v^2(\lambda_1 + \frac{1}{2+r^2}(\lambda_3 + \lambda_4\cos\theta_1), \tag{4.46}$$

$$0 = \frac{v^4}{(2+r^2)^2}(r^2\sin\theta_1 - \sin\theta_3). \tag{4.47}$$

B: real vacua

For real vacua we of course have the special cases of the complex vacua; $\alpha = \beta = \gamma = 0$. These give two possibilities:

1B : $\boldsymbol{(\frac{1}{\sqrt{2}}v, \frac{1}{\sqrt{2}}v, 0)}$
This vacuum is obtained by taking $\alpha = 0$ in the first complex vacuum. This leads to the following conditions on the parameters

$$\mu = -v^2(4\lambda_1 + \lambda_3 + \lambda_4)/2, \quad \varepsilon = 0. \tag{4.48}$$

2B : $\boldsymbol{(\frac{1}{\sqrt{3}}v, \frac{1}{\sqrt{3}}v, \frac{1}{\sqrt{3}}v)}$
This vacuum is obtained by taking $\alpha = 0$ which forces $r = 1$ in the second complex vacuum. The conditions on the parameters of the potential then reduce to

$$\mu = -\frac{v^2}{3}(6\lambda_1 + 2\lambda_3 + 2\lambda_4\cos\varepsilon). \tag{4.49}$$

The fact that this vacuum alignment places less conditions on the parameters has a large effect on the masses of the Higgs particles, as we will see later.

Since we are not interested in vacua with $\lambda_4 = 0$ or $\lambda_4 + \lambda_3 = 0$, there is just one other possible vacuum:

3 : $\boldsymbol{(v, 0, 0)}$
or permutations thereof. We did not see a complex version of this vacuum, because a possible phase can be absorbed by a global rotation. Again there is just one condition on the parameters of the potential

$$\mu = -2\lambda_1 v^2. \tag{4.50}$$

Note that all the discussed vacua indeed satisfy the relation (4.23) for CP invariance after the VEVs are acquired. At first sight vacuum 2A may seem to violate this condition, however this vacuum can be written as $\boldsymbol{(\frac{1}{\sqrt{2}}ve^{-i\alpha/4}, \frac{1}{\sqrt{2}}ve^{i\alpha/4}, 0)}$ by a global rotation.

In the upcoming section we will discuss the masses of the Higgs particles for each of these vacua. From this we hope to be able to say something about whether or not a particular vacuum is suitable for modelling.

4.5 The Higgs masses

By inspecting the potential after the VEVs are acquired, $\phi_i = \begin{pmatrix} \frac{1}{\sqrt{2}}(\text{Re }\phi_i^+ + i\text{Im }\phi_i^+) \\ v_i + \frac{1}{\sqrt{2}}(\text{Re }\phi_i^0 + i\text{Im }\phi_i^0) \end{pmatrix}$, we can write the mass terms as follows

$$\mathcal{L}_m = \frac{1}{2}\Phi^\dagger M^2 \Phi + \frac{1}{2}(\Phi^+)^\dagger M_C^2 \Phi^+, \tag{4.51}$$

here $\Phi^T = (\text{Re }\phi_1^0, ..., \text{Re }\phi_3^0, \text{Im }\phi_1^0, ..., \text{Im }\phi_3^0)$ and $(\Phi^+)^T = (\phi_1^+, \phi_2^+, \phi_3^+)$. Furthermore, M^2 and M_C^2 are a 6 by 6 and a 3 by 3 matrix respectively. By hermiticity of the Lagrangian, M^2 is hermitian. It is also easy to see that it is symmetric by taking the transpose of the mass term, $\Phi^T M^2 \Phi = \Phi^T (M^2)^T \Phi$. Thus, M^2 is a real symmetric matrix and can be diagonalized by an orthogonal matrix.

Similarly, by hermiticity we can see that M_C is hermitian. It is diagonalized by a unitary matrix. The elements of this matrix are real when the VEVs are real and $\varepsilon = 0$, this is because the elements of M_C^2 are combinations of the parameters of the potential and the VEVs. The matrix is then real and symmetric and diagonalized by an orthogonal matrix.

Another thing happens when the VEVs are real and $\varepsilon = 0$; the imaginary parts and the real parts of the neutral fields no longer mix among each other in the neutral mass matrix. In other words it becomes a block diagonal matrix consisting of two 3 by 3 matrices. Such a matrix is of course diagonalized by a block diagonal matrix. This means that there are three mass eigenstates which are linear combinations of Re ϕ_i^0. And there are three mass eigenstates which are linear combinations of Im ϕ_i^0 of which one is the Goldstone boson which will be eaten by the Z boson.

Now that we are somewhat familiar with the mass matrices, we will discuss the masses. Each of the vacuum alignments and their conditions discussed above leads to different mass matrices. Since we have avoided vacua with additional symmetries we expect the right number of Goldstone bosons to come out, which means we expect one charged and one neutral mass state to be zero. After calculating the eigenvalues for each of these vacua, as was done in [17], one obtains:

$1A:\ (\frac{1}{\sqrt{2}}ve^{-i\alpha/2}, \frac{1}{\sqrt{2}}v, 0)$

This vacuum has the right amount of massless states. However, when $\alpha = 0$ we again obtain the completely real case, 1A : $(v, v, 0)$, in which case tachyonic states ensue. This can be seen by noting that m_1^2 and m_3^2 cannot both be positive when $\alpha = 0$. In general however, we can have all the m_i^2 positive.

$$m_1^2 = -v^2(\lambda_3 + \lambda_4), \qquad m_2^2 = v^2(4\lambda_1 + \lambda_3 + \lambda_4), \tag{4.52}$$

$$m_3^2 = \frac{v^2}{2}(\lambda_3 - \lambda_4 + 2\lambda_4\cos(3\alpha/2)), \qquad m_4^2 = -2\lambda_4 v^2, \tag{4.53}$$

$$m_5^2 = \frac{v^2}{2}(\lambda_3 - \lambda_4 - 2\lambda_4\cos(3\alpha/2)), \qquad m_6^2 = 0, \tag{4.54}$$

$$m_{C_1}^2 = \frac{v^2}{2}(2\lambda_2 - \lambda_3 - \lambda_4), \qquad m_{C_2}^2 = -v^2(\lambda_3 - \lambda_2 + \lambda_4), \tag{4.55}$$

$$m_{C_3}^2 = 0. \tag{4.56}$$

$2A:\ (we^{-i\alpha/2}, we^{i\alpha/2}, rw)$
In the general complex case, 2A, the mass matrices become rather complicated, which is why only the following cases of r will be considered:
$r \sim 0$
The minimum conditions now become

$$\varepsilon \approx -2\alpha + N\pi, \quad \lambda_4 \approx \tfrac{\lambda_3}{1-2\cos 3\alpha}, \tag{4.57}$$
$$\mu \approx -v^2(2\lambda_1 + \lambda_3 \tfrac{1-\cos 3\alpha}{1-2\cos 3\alpha}). \tag{4.58}$$

This leads to the following masses

$$\begin{aligned}
m_1^2 \sim&\ v^2\mathcal{O}(r^2), \ \ m_2^2 \sim -2v^2\frac{\lambda_3}{1-2\cos 3\alpha},\\
m_3^2 \sim& 2v^2(-2\lambda_1 + (4\lambda_1+\lambda_3)(1-\cos 3\alpha)/(1-2\cos 3\alpha)),\\
m_4^2 \sim& -2v^2\lambda_3\cos(3\alpha)/(1-2\cos 3\alpha),\\
m_5^2 \sim& -4v^2\lambda_3\sin^2(3\alpha/2)/(1-2\cos 3\alpha),\\
m_{C_1}^2 \sim& -v^2(\lambda_3-\lambda_2+(\lambda_2+(\lambda_3-\lambda_2)\cos 3\alpha)/(1-2\cos 3\alpha),\\
m_{C_2}^2 \sim& -v^2(2\lambda_3-2\lambda_2+(\lambda_2+2(\lambda_3-\lambda_2)\cos 3\alpha)/(1-2\cos 3\alpha),\\
m_6^2 =& 0, \quad m_{C_3}^2 = 0.
\end{aligned} \tag{4.59}$$

Apart from the usual Goldstone bosons we now have another mass, m_1^2, which is nearly massless. Therefore this option is discarded.

$r \gg 1$
In this case the conditions for the minimum become

$$\varepsilon \approx \alpha + N\pi, \quad \lambda_3 \approx -\lambda_4, \tag{4.60}$$
$$\mu \approx -2\lambda_1 v^2. \tag{4.61}$$

Since $\lambda_3 + \lambda_4 \approx 0$ we now have an approximate $O(3)$ symmetry and so one might expect extra (nearly) massless states. Indeed, one finds $m_{1,2}^2$ to be very small

$$m_{1,2}^2 \sim v^2\mathcal{O}(1/r^2), \qquad m_3^2 \sim 4\lambda_1 v^2, \tag{4.62}$$
$$m_{4,5}^2 \sim 2\lambda_3 v^2, \qquad m_6^2 = 0, \tag{4.63}$$
$$m_{C_1,C_2}^2 \sim \lambda_2 v^2, \qquad m_{C_3}^2 = 0. \tag{4.64}$$

Since $m_{1,2}^2$ are nearly massless this case is discarded.
$2B:\ (\sqrt{\tfrac{1}{3}}v, \sqrt{\tfrac{1}{3}}v, \sqrt{\tfrac{1}{3}}v)$
The real case, 2B, is quite different. The mass matrices are simpler again and the masses

can be calculated analytically. In this case we get the right number of massless states

$$
\begin{aligned}
m_1^2 = \frac{4v^2}{3}(3\lambda_1 + \lambda_3 + \lambda_4 \cos\varepsilon),\ \ m_6^2 = 0,\\
m_{2,3}^2 = \frac{v^2}{3}(-\lambda_3 - 4\lambda_4\cos\varepsilon + \sqrt{\lambda_3^2 + 4\lambda_4^2(2-\cos 2\varepsilon) + 4\lambda_3\lambda_4\cos\varepsilon}),\\
m_{4,5}^2 = \frac{v^2}{3}(-\lambda_3 - 4\lambda_4\cos\varepsilon - \sqrt{\lambda_3^2 + 4\lambda_4^2(2-\cos 2\varepsilon) + 4\lambda_3\lambda_4\cos\varepsilon}),\\
m_{C_1,C_2}^2 = -\frac{v^2}{3}(3(\lambda_3-\lambda_2) + 3\lambda_4\cos\varepsilon \pm \sqrt{3}\lambda_4\sin\varepsilon),\\
m_{C_3}^2 = 0.
\end{aligned}
\tag{4.65}
$$

3 : $(v,0,0)$
The last vacuum again gives us the right amount of massless states.

$$m_1^2 = 4v^2\lambda_1, \qquad m_{2,3}^2 = v^2(\lambda_3 - \lambda_4), \tag{4.66}$$

$$m_{4,5}^2 = v^2(\lambda_3 + \lambda_4), \qquad m_6^2 = 0, \tag{4.67}$$

$$m_{C_1,C_2}^2 = \lambda_2 v^2, \qquad m_{C_3}^2 = 0. \tag{4.68}$$

4.6 Final remarks

With the masses calculated we can already say a lot about the different vacua. If we are going to build a model with one of these vacuum alignments we cannot allow for tachyonic states, simply because they are not observed. So, we can discard the vacuum with tachyonic states, $(v, v, 0)$. Looking at the vacuum alignment $(ve^{-i\alpha/2}, ve^{i\alpha/2}, rv)$ we see that it produces masses of the order of $m^2 \sim r^2v^2$ ($m^2 \sim \frac{v^2}{r^2}$) where $r \sim 0$ ($r \gg 1$). Unlike most other vacuum alignments which result in masses of the order of $m_i^2 \sim v^2$. This means that instead of a lightest Higgs mass of the order of the VEV (as it should be), the lightest Higgs mass will be some orders of magnitude smaller. The numerical analysis performed in [17] confirms this. Not only in the limiting cases, but in general, the lightest Higgs mass seems to be very small. For this reason we should discard this solution as well, since nearly massless Higgs particles are not observed. However, by adding a small A_4 breaking term in the Higgs potential this problem can be overcome, as was shown in [22]. Nonetheless, if we assume that the A_4 symmetry is not explicitly broken we can conclude that just a few phenomenologically acceptable solutions remain. Discarding the solutions with tachyonic states and too low Higgs masses we are left with

$$\mathcal{S}_1 : \quad (\tfrac{1}{\sqrt{3}}v, \tfrac{1}{\sqrt{3}}v, \tfrac{1}{\sqrt{3}}v), \tag{4.69}$$

$$\mathcal{S}_2 : \quad (v, 0, 0), \tag{4.70}$$

$$\mathcal{S}_3 : \quad (\tfrac{1}{\sqrt{2}}ve^{-i\alpha/2}, \tfrac{1}{\sqrt{2}}v, 0). \tag{4.71}$$

In the next Chapter we will first try to see whether we can build a viable model with this Higgs sector and using these three acceptable VEVs. It appears the options are limited. This is due to the fact that in two of these VEVs, $\mathcal{S}_2$ and $\mathcal{S}_3$, some of the Higgs doublets do not acquire a VEV. The remaining VEV, $\mathcal{S}_1$, has just one parameter in both cases this does not allow for a lot of freedom when constructing a model.

Therefore, we will also look at a model (by Lavoura and Kühböck (LK), [5]) which uses the VEV, 2A. This case had a very light Higgs mass, but this could remedied by adding a small A_4 breaking term in the potential. This LK model has other problems which were discussed in [22], but the use of the 2A VEV does give more freedom when constructing a model.

We will also discuss another model (by Altarelli and Feruglio (AF), [3]), which uses a different Higgs sector and adds a number of additional scalar fields. From this we can see that these additional fields allow for more freedom while constructing a model. Firstly, however, we will discuss the possible models using the Higgs sector of this Chapter.

5

A_4 models

In this Chapter we will take a look at the fermion mass matrices we can construct in an A_4 symmetric Lagrangian. We will consider models using the previously discussed Higgs sector, (4.17). This basically means that we will study the possible Yukawa couplings, $\mathcal{L}_Y$, in combination with the Higgs sector of (4.17). When discussing the lepton sector we will employ the dimension-5 operator and the type I seesaw mechanism (by adding right-handed neutrinos) to generate light neutrino masses. Type II and III seesaw will not be considered. Type III seesaw leads to the same physics for the neutrinos and type II seesaw would force us to change the Higgs potential. We will see what kinds of models are possible with this minimum of additional Higgs and neutrino fields. We will call these models 'simplest A_4 models' for their small amount of additional fields.

After concluding that these options are not satisfactory for one reason or another, we will discuss two models: the LK, [5], and the AF, [3], [11], model. The LK model uses a VEV, 2A, which requires small A_4 breaking terms in the potential in order to produce acceptable Higgs masses. The AF model uses a different Higgs sector with a number of additional scalar fields. In both cases we will see that this allows for more freedom when constructing a model.

However, we will start by looking at the possible Dirac mass matrices in models using the Higgs sector, (4.17), of the previous Chapter.

5.1 Simplest A_4 models

The Dirac mass matrices do not only depend on the VEVs but also on the A_4 representation assignment of the fermions. We will be looking for viable combinations of assignments and vacua.

There are some restrictions to the choices one can make in the assignment of the fermions. Because the Higgs fields constitute an A_4 triplet, we will need at least one fermion A_4 triplet in order to construct A_4 invariant terms of the form $\overline{\psi_L}\phi\psi_R + h.c.$, where $\psi = (\psi_1, \psi_2, \psi_3)$ is some fermion field of three generations. ψ_L are $SU(2)$ doublets, whereas ψ_R are $SU(2)$ singlets. These Dirac terms are necessary since they produce the mass terms after spontaneous symmetry breaking. The mass terms of the charged leptons and down-type quarks are of this form. The Dirac mass terms of the neutrinos and the up-type quarks have a similar form, with ϕ replaced by $\tilde{\phi}$.

Furthermore, we would like members of the same $SU(2)$ doublet to be in the same A_4 representation and members of an A_4 triplet to be in the same $SU(2)$ representation. This also restrict our field assignment.

5.1.1 Possible Dirac mass matrices

Let us begin by seeing what the consequences are for a Dirac mass matrix when it is generated by the coupling between the Higgs fields, one A_4 fermion triplet and fermion singlets. We will be interested in the diagonalized matrices to see whether or not the correct charged fermion masses can be produced. The masses that should be reproduced have a strong hierarchy, $m_3 \gg m_2 \gg m_1$, and none of the masses should be zero. If either demand is not met by an assignment, the assignment is unacceptable and cannot be used to model the charged fermions.

We start by considering the assignment $\psi_{Li} \sim \mathbf{3}$ and $\psi_{Ri} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$.

$\psi_{Li} \sim \mathbf{3}$, $\psi_{Ri} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$

Field	ψ_L	ψ_R	ϕ
A_4	**3**	**1**, **1′**, **1″**	**3**
$SU(2)$	**2**	**1**	**2**

In this case the most general A_4 invariant Lagrangian is given by

$$-\mathcal{L}_Y = y_1(\overline{\psi_L}\phi)^{\mathbf{1}}\psi_{R1} + y_2(\overline{\psi_L}\phi)^{\mathbf{1}''}\psi_{R2} + y_3(\overline{\psi_L}\phi)^{\mathbf{1}'}\psi_{R3} + h.c. \tag{5.1}$$

Using the relations (2.24)-(2.26) we find for the resulting mass matrix

$$M = \begin{pmatrix} y_1v_1 & y_2v_1 & y_3v_1 \\ y_1v_2 & \omega y_2v_2 & \omega^2 y_3v_2 \\ y_1v_3 & \omega^2 y_2v_3 & \omega y_3v_3 \end{pmatrix}, \tag{5.2}$$

here y_i are Yukawa coupling constants and v_i the VEVs of the Higgs doublets (in the case of neutrino or up-type quark masses they should be replaced by v_i^*).

Note that if we permute the assignment of the singlets, say $\mathbf{1}', \mathbf{1}, \mathbf{1}''$ instead of $\mathbf{1}, \mathbf{1}', \mathbf{1}''$, we obtain a mass matrix which has the form of the previous matrix with its columns permuted similarly as the singlet assignment and redefined Yukawa coupling constants. In this case the new matrix would be given by $M' = \begin{pmatrix} y_1v_1 & y_2v_1 & y_3v_1 \\ \omega y_1v_2 & y_2v_2 & \omega^2 y_3v_2 \\ \omega^2 y_1v_3 & y_2v_3 & \omega y_3v_3 \end{pmatrix} = M(y')E$ where $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y_1' = y_2$, $y_2' = y_1$ and $y_3' = y_3$. This means that the expressions for the masses remain of the same form. Since, generally, if the mass matrix is diagonalized as follows $U(y)M(y)V(y) = \text{diag}(m_1, m_2, m_3)$ then the new matrix is, $M' = M(y')E$ where E is a permutation matrix. This new matrix can be diagonalized by $U(y')M'V' =$

$U(y')M(y')V(y') = \text{diag}(m_1(y'), m_2(y'), m_3(y'))$, where $V' = E^T V(y')$. V' is still unitary because E is a permutation matrix and unitary itself. Thus, the only difference in the masses is that the Yukawa coupling constants have been redefined.

If we were to assign the left-handed fields to singlets and the right-handed fields to a triplet we would obtain the following mass matrix $M' = \begin{pmatrix} y_1v_1 & y_1v_2 & y_1v_3 \\ y_2v_1 & \omega^2 y_2v_2 & \omega y_2v_3 \\ y_3v_1 & \omega y_3v_2 & \omega^2 y_3v_3 \end{pmatrix} = M^\dagger(y', v^*)$, where y' represent redefined coupling constants. So, if the old matrix is diagonalized as follows, $VMU = \text{diag}(m_1, m_2, m_3)$, then the new matrix (which has the form $M' = M^\dagger(y', v^*)$) is diagonalized by $U^\dagger(y', v^*)M'V^\dagger(y', v^*) = \text{diag}(m_1(y', v^*), m_2(y', v^*), m_3(y', v^*))$. Again the form of the masses is not affected.

Note that if we would use the same singlet twice, e.g. $\mathbf{1}, \mathbf{1}', \mathbf{1}'$, we would obtain zero masses. The rows (or columns) of the mass matrix, M, would no longer be linearly independent and the same would be true for $MM^\dagger$ which would then obtain some zero as an eigenvalue. Since the eigenvalues of this matrix correspond to the squares of the masses, one of them will be zero.

So, any Dirac mass matrix produced through a coupling of the form $\overline{\psi_L}\phi\psi_R + h.c.$ (with $\psi_L \sim \mathbf{3}$ and $\psi_R \sim$ **any combination of singlets** or vice versa) will either result in at least one mass being zero, if the same singlet was used more than once, or it will result in the same form of the masses as the (5.2) case. We can now see what the masses are for the VEVs we had concluded were viable in the previous Chapter, (4.69)-(4.71).

$\mathcal{S}_1 : (\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v})$
In this case the masses that follow from the matrix are $m_1^2 = 3y_1^2v^2$, $m_2^2 = 3y_2^2v^2$ and $m_3^2 = 3y_3^2v^2$. This can be seen by noting that in this case the matrix (5.2) is diagonalized by $VM = \text{diag}(3y_1^2v^2, 3y_2^2v^2, 3y_3^2v^2)$, where $V = \sqrt{1/3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$.
This is a good possibility, none of the charged fermion masses are forced to be zero and all masses can be fitted by the three Yukawa constants.

$\mathcal{S}_2 : (\boldsymbol{v}, \mathbf{0}, \mathbf{0})$
In this case there is just one non-zero mass, $m^2 = (|y_1|^2 + |y_2|^2 + |y_3|^2)v^2$, thus it cannot be used to model the charged fermions.

$\mathcal{S}_3 : (\boldsymbol{v}e^{-i\alpha/2}, \boldsymbol{v}, \mathbf{0})$
The masses now become $m_3 = 0$, $m_{1,2}^2 = v^2(a \pm |b|)$, where $a = |y_1|^2 + |y_2|^2 + |y_3|^2$ and $b = |y_1|^2 + \omega|y_2|^2 + \omega^2|y_3|^2$. The zero mass makes this possibility unacceptable as well.

$\psi_L \sim \mathbf{3}$, $\psi_R \sim \mathbf{3}$

Field	ψ_L	ψ_R	ϕ
A_4	**3**	**3**	**3**
$SU(2)$	**2**	**1**	**2**

We will now see whether we can construct a Dirac mass matrix with acceptable masses when we assign $\psi_L \sim \mathbf{3}$, $\psi_R \sim \mathbf{3}$. Using the relations, (2.24) and (2.29) one obtains

$$\begin{aligned}-\mathcal{L}_Y =& y_1(\overline{\psi_{L1}}\phi_2\psi_{R3} + \overline{\psi_{L2}}\phi_3\psi_{R1} + \overline{\psi_{L3}}\phi_1\psi_{R2}) \\ &+ y_2(\overline{\psi_{L3}}\phi_2\psi_{R1} + \overline{\psi_{L1}}\phi_3\psi_{R2} + \overline{\psi_{L2}}\phi_1\psi_{R3}) \quad + h.c.\end{aligned} \tag{5.3}$$

This gives the following mass matrix

$$M = \begin{pmatrix} 0 & y_1v_3 & y_2v_2 \\ y_2v_3 & 0 & y_1v_1 \\ y_1v_2 & y_2v_1 & 0 \end{pmatrix}, \tag{5.4}$$

where y_i are Yukawa coupling constants and v_i again the VEVs of the Higgs fields. Again, we look at the masses for the different VEVs.

$\mathcal{S}_1 : \boldsymbol{(v, v, v)}$

In this case the mass matrix takes the form $M = \begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}$, where $a = y_1 v$ and $b = y_2 v$. The matrix is diagonalized as follows $V^\dagger M V = m_{diag}$, with

$$\begin{aligned} m_1^2 =& |a|^2 + |b|^2 + 2|a||b|\cos\alpha, \;\; m_2^2 = |a|^2 + |b|^2 + 2|a||b|\cos(\alpha + 2\pi/3), \\ m_3^2 =& |a|^2 + |b|^2 + 2|a||b|\cos(\alpha - 2\pi/3), \end{aligned} \tag{5.5}$$

where $\alpha = \mathrm{Arg}(y_1/y_2)$. At first this seems viable, although the angle α may have to be fine tuned to obtain the right masses. However, it can be seen that the hierarchy that is present in the up- and down- type quarks and the charged lepton cannot be reproduced in this case. To see this, we will solve the parameter $|b|$ as a function of the masses. We start with

$$\begin{aligned} |a|^2 + |b|^2 =& \frac{m_1^2 + m_2^2 + m_3^2}{3} \equiv m^2, \;\; |a||b|\cos\alpha = \frac{2m_1^2 - m_2^2 - m_3^2}{6} \equiv M^2, \\ m_3^2 - m_2^2 =& 2\sqrt{3}|a||b|\sin\alpha. \end{aligned} \tag{5.6}$$

From this one can obtain $\cos^2\alpha = \frac{12M^4}{12M^4 + (m_3^2 - m_2^2)^2}$ and for $|b|$,

$$|b|^2 = \frac{1}{2}m^2 \pm \frac{1}{2}\sqrt{m^4 - 4M^4 - (m_3^2 - m_2^2)^2/3}. \tag{5.7}$$

Writing out the square root,

$$\sqrt{\dots} = \left[(-m_1 + m_2 + m_3)(m_1 - m_2 + m_3)(m_1 + m_2 - m_3)(m_1 + m_2 + m_3)\right]^{1/2}, \tag{5.8}$$

we see that $|b|^2$ will be complex for $m_3 > m_2 + m_1$ (or permutations thereof). Thus, only masses which obey $m_3 \leq m_2 + m_1$ (and permutations thereof) can be realized, meaning that the hierarchy $m_1 \ll m_2 \ll m_3$ present in nature cannot be reproduced in this case.

$\mathcal{S}_2 : \boldsymbol{(v, 0, 0)}$

In this case the masses turn out to be, $m_1^2 = 0$, $m_2^2 = |y_1|^2 v^2$, $m_3^2 = |y_2|^2 v^2$. This is not a viable option.

$\mathcal{S}_3 : \boldsymbol{(ve^{-i\alpha/2}, v, 0)}$
In this case the masses are given by, $m_1^2 = 0$, $m_{2,3}^2 = (|y_1|^2 + |y_2|^2)v^2$. This again is not a viable option.

We can conclude that given this 'simplest' Higgs sector (4.17) a mass matrix that is generated by the coupling between the Higgs fields and two A_4 fermion triplets cannot reproduce the mass hierarchy seen in nature. A mass matrix that is generated by the coupling between the Higgs fields, one A_4 fermion triplet and fermion singlets can only reproduce the correct masses when the $\mathcal{S}_1$ vacuum is chosen. This is of course only true for models using this Higgs sector (4.17). These conclusions restrict the assignment of the fermion fields greatly. We will now see what kind of model we can produce with this in mind.

5.1.2 Constructing a model

As mentioned before, we will consider models which assume this 'simplest' Higgs sector. In order to be able to produce neutrino masses we will employ the dimension-5 operator. Additionally, we will we will introduce three right-handed neutrinos, ν_R, which will contribute to the neutrino masses through type I seesaw. The considerations of the previous section require that the Dirac mass matrices for the quarks and charged leptons[1] are produced by the coupling between the Higgs fields, one A_4 fermion triplet and fermion singlets. And the only vacuum that is viable is $\mathcal{S}_1 = \boldsymbol{(v, v, v)}$.

Quark sector

For the quark sector this means we have two options, $Q_{Li} \sim \mathbf{3}$ and $u_{Ri}, d_{Ri} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$ or $Q_{Li} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$ and $u_{Ri}, d_{Ri} \sim \mathbf{3}$. As mentioned before, in both cases and for any permutation of the singlets the masses will be of the same form.

In the case that we assign $Q_{Li} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$ and $u_{Ri}, d_{Ri} \sim \mathbf{3}$ the mass matrices are diagonalized by, $M_d V = \mathrm{diag}(m_d, m_s, m_b)$ and $M_u V = \mathrm{diag}(m_u, m_c, m_t)$. So to diagonalize these matrices we only have to transform the basis of the right-handed fields, which means the weak charged current stays diagonal and thus $V_{CKM} = \mathbb{1}$.

When we have $Q_{Li} \sim \mathbf{3}$ and $u_{Ri}, d_{Ri} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$, the matrices are diagonalized by $VM_d = \mathrm{diag}(m_d, m_s, m_b)$ and $VM_u = \mathrm{diag}(m_u, m_c, m_t)$. The CKM matrix then becomes, $V_{CKM} = VV^\dagger = \mathbb{1}$. Permuting the singlets in this case does not have an affect on the CKM matrix. After permuting the assignment of the singlets, new matrices would become $M_d' = M_d(y')E_d$ and $M_u' = M_u(y')E_u$. The $E_{u,d}$ matrices can be absorbed by a redefinition of the right-handed fields. These new mass matrices are now diagonalized by, $V(y') = V$, since V does not depend on the Yukawa couplings. So that the CKM matrix does not change. This is actually the model discussed in [6], here a more realistic CKM matrix is obtained by allowing for small A_4 breaking terms in the Lagrangian.

[1] This is not necessarily the case for the neutrinos since the neutrino mass matrix also depends on the right-handed Majorana mass terms, through type I seesaw.

To conclude, the best possible outcome for the CKM matrix seems to be the identity matrix, as long as we do not break A_4 explicitly.

Lepton sector

The general Lagrangian from which all the mass terms for the leptons originate will be useful in what follows. It is of the form

$$\begin{aligned} -\mathcal{L}_Y^l =& \overline{L_L}\phi l_R + \overline{\nu_R}\tilde{\phi}L_L \\ &+ \frac{1}{2\Lambda}\overline{(\tilde{\phi}^\dagger L_L)^c}(\tilde{\phi}^\dagger L_L) + \frac{1}{2}\overline{\nu_R}M_R\nu_R^c + h.c. \ , \end{aligned} \tag{5.9}$$

where Λ is some large mass coming from a high energy scale. After the Higgs fields have acquired their VEVs these couplings are given by

$$-\mathcal{L}_Y^l = \overline{l_L}M_l l_R + \overline{\nu_R}M_D\nu_L + \frac{1}{2}\overline{\nu_L^c}M_L\nu_L + \frac{1}{2}\overline{\nu_R}M_R\nu_R^c + h.c. \tag{5.10}$$

These mass matrices of course depend on the A_4 representations of all the fields and the VEVs of the Higgs fields. We will take the $\mathcal{S}_1$ vacuum as it is the only one which could produce acceptable Dirac mass matrices.

For the irrep assignment we have again two choices. We can assign $L_{Li} \sim \mathbf{3}$ or $L_{Li} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$, we will start with the latter.

$L_{Li} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$

Field	L_L	l_R	ν_R
A_4	$\mathbf{1}, \mathbf{1}', \mathbf{1}''$	$\mathbf{3}$	$\mathbf{3}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$

If we pick $L_{Li} \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$, we are then forced to assign $l_{Ri}, \nu_{Ri} \sim \mathbf{3}$ (no A_4 invariant mass terms are possible otherwise). The mass matrices are given by

$$\begin{aligned} M_l =& v\begin{pmatrix} y_1 & y_1 & y_1 \\ y_2 & \omega^2 y_2 & \omega y_2 \\ y_3 & \omega y_3 & \omega^2 y_3 \end{pmatrix} , \ M_D = v\begin{pmatrix} y_4 & y_5 & y_6 \\ y_4 & \omega y_5 & \omega^2 y_6 \\ y_4 & \omega^2 y_5 & \omega y_6 \end{pmatrix} , \\ M_L =& \frac{3v^2}{\Lambda}\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{pmatrix} , \ M_R = M\mathbb{1}, \end{aligned} \tag{5.11}$$

where M is a heavy mass (of the right-handed neutrinos) and y_i, a and b are dimensionless coupling constants. To diagonalize the charged lepton mass matrix we only have to transform the basis of the charged right-handed fields, so that the diagonalization of M_l does not contribute to the mixing matrix. The matrices are diagonalized by $M_l V^\dagger = \sqrt{3}v\text{diag}(y_1, y_2, y_3)$ and $VM_D = \sqrt{3}v\text{diag}(y_4, y_5, y_6)$. In the basis where the charged lepton mass matrix and the charged current are diagonal, we obtain the light neutrino mass matrix, (through the seesaw relation (3.31))

$$M_\nu = M_L - M_D^T M_R^{-1} M_D = 3v^2 \begin{pmatrix} \frac{a}{\Lambda} - \frac{y_4^2}{M} & 0 & 0 \\ 0 & 0 & \frac{b}{\Lambda} - \frac{y_5 y_6}{M} \\ 0 & \frac{b}{\Lambda} - \frac{y_5 y_6}{M} & 0 \end{pmatrix}. \tag{5.12}$$

This is not the pattern that leads to TB mixing additionally, we have two degenerate masses. M_ν is diagonalized by the PMNS matrix which is given by

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\sqrt{1/2} & \sqrt{1/2} \\ 0 & -i\sqrt{1/2} & \sqrt{1/2} \end{pmatrix}. \tag{5.13}$$

Assigning the singlets differently would lead to the mass matrices $M_l' = EM_l(y'_{1,2,3})$, $M_D' = M_D(y'_{4,5,6})E^T$ and $M_L' = EM_L(a', b')E^T$. This gives for the neutrino mass matrix (in the basis where M_l is diagonal) $M_\nu' = M_\nu(a', b', y'_{4,5,6})$, so that the form of the masses nor the mixing pattern changes.

$L_{Li} \sim \mathbf{3}$

Choosing the left-handed fields as a triplet forces us to take $l_R \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$. However, we can still assign ν_R to a triplet or singlets. We first look at the case $\nu_R \sim \mathbf{3}$.

Field	L_L	l_R	ν_R
A_4	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \mathbf{1}''$	$\mathbf{3}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$

In this case the left-handed Majorana mass term is quite large; there are many ways to construct a A_4 singlet from four triplets. This mass term was used in combination with the 2A vacuum in a model trying to explain the neutrino mixing matrix [23]. The resulting mass matrix is simpler for the vacuum we are considering. The mass matrices are given by

$$\begin{aligned} M_l &= v \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1 & \omega y_2 & \omega^2 y_3 \\ y_1 & \omega^2 y_2 & \omega y_3 \end{pmatrix}, \quad M_D = v \begin{pmatrix} 0 & y_4 & y_5 \\ y_5 & 0 & y_4 \\ y_4 & y_5 & 0 \end{pmatrix}, \\ M_L &= \frac{v^2}{\Lambda} \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}, \quad M_R = M\mathbb{1}. \end{aligned} \tag{5.14}$$

The charged lepton mass matrix is diagonalized by $VM_l = \sqrt{3}v\text{diag}(y_1, y_2, y_3)$. In this basis where the charged leptons and the weak charged current are diagonal the left-handed Majorana and Dirac neutrino mass matrices are given by $M_L' = V^* M_L V^\dagger$ and $M_D' = M_D V^\dagger$

respectively. The neutrino mass matrix is then given by

$$\begin{aligned} M_\nu =& V^* M_L V^\dagger - V^* M_D M_R^{-1} M_D V^\dagger \\ =& v^2 \begin{pmatrix} \frac{a+2b}{\Lambda} - \frac{(y_4+y_5)^2}{M} & 0 & 0 \\ 0 & 0 & \frac{a-b}{\Lambda} - \frac{y_4^2+y_5^2-y_4y_5}{M} \\ 0 & \frac{a-b}{\Lambda} - \frac{y_4^2+y_5^2-y_4y_5}{M} & 0 \end{pmatrix}. \end{aligned} \tag{5.15}$$

So there are two degenerate masses and the neutrino mixing matrix is again given by

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\sqrt{1/2} & \sqrt{1/2} \\ 0 & -i\sqrt{1/2} & \sqrt{1/2} \end{pmatrix}. \tag{5.16}$$

Assigning the singlets differently leads to $M_l = M_l(y')E$. Since the diagonalizing matrix, V, does not depend on the Yukawa constants it is not affected by this and E can be absorbed through a redefinition of the l_R fields.

Now the case $\nu_R \sim \mathbf{1}, \mathbf{1}', \mathbf{1}''$.

Field	L_L	l_R	ν_R
A_4	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \mathbf{1}''$	$\mathbf{1}, \mathbf{1}', \mathbf{1}''$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$

Now M_l and M_L have the same form as in the previous case, whereas M_D and M_R are now given by

$$M_D = v \begin{pmatrix} y_4 & y_4 & y_4 \\ y_5 & \omega^2 y_5 & \omega y_5 \\ y_6 & \omega y_6 & \omega^2 y_6 \end{pmatrix}, \quad M_R = M \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & d \\ 0 & d & 0 \end{pmatrix}. \tag{5.17}$$

In the basis where the charged lepton mass matrix and the charged current are diagonal the neutrino mass matrix becomes

$$\begin{aligned} M_\nu =& M_L' - V^* M_D^T M_R^{-1} M_D V^\dagger \\ =& v^2 \begin{pmatrix} \frac{a+2b}{\Lambda} - \frac{3y_4^2}{cM} & 0 & 0 \\ 0 & 0 & \frac{a-b}{\Lambda} - \frac{3y_5y_6}{dM} \\ 0 & \frac{a-b}{\Lambda} - \frac{3y_5y_6}{dM} & 0 \end{pmatrix}. \end{aligned} \tag{5.18}$$

So that again

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\sqrt{1/2} & \sqrt{1/2} \\ 0 & -i\sqrt{1/2} & \sqrt{1/2} \end{pmatrix}. \tag{5.19}$$

In this case changing the singlet assignment can be absorbed by a redefinition of the right-handed fields, ν_R, l_R, and so does not change the form of the masses or the mixing.

Despite introducing additional Higgs fields and three right-handed neutrinos, the models we can build are rather unsatisfying. In the quark sector the best result seems to be six independent quarks masses and a CKM matrix equal to the identity matrix. The latter might not be too bad as a first approximation. In the lepton sector we can obtain independent masses for the charged leptons, but the neutrinos have two degenerate masses in each case, this is excluded by phenomenology [13]. The PMNS matrix in particular seems far from TB mixing in each case.

Of course, we could see whether a viable model can be constructed if we add more or less right-handed neutrino fields. Less ν_R fields is a bad option. In this case we cannot have $\nu_R \sim \mathbf{3}$, thus the right-handed neutrinos must be A_4 singlets. Having just one right-handed neutrino leads to just one light neutrino mass from pure type I seesaw, as discussed before. The resulting neutrino mass matrix can be obtained by setting $y_4 = y_5 = 0$ in (5.18). Two right-handed neutrino fields lead to a neutrino mass matrix which is obtained by setting $y_4 = 0$ in (5.18). Both options do not improve the situation. Adding more than three right-handed neutrinos will not be considered.

In this section we used type I seesaw. If we had used type III, the situation would be similar. As was discussed in section 3.3.2 the neutral component of the fermion $SU(2)$ triplet would play the role ν_R did in this section. Using type II seesaw would give rise to a different situation. The reason for not discussing this option is because introducing a scalar $SU(2)$ triplet would not only allow for Majorna mass terms for the left-handed neutrinos, but also alter our Higgs potential. Models with a different Higgs potential are not in the class of models which we called 'simplest A_4 models'.

In order to construct a viable model we need more ingredients. One way of introducing new ingredients would be to allow for small A_4 breaking terms. As stated in the previous Chapter, such terms would make additional VEVs acceptable (it remedies the small Higgs masses for the $2A$ vacuum [22]). We will examine a model (the LK model) for the quark sector with such a VEV later on. Another way to obtain more ingredients is to introduce new fields. A_4 models which successfully produce TB mixing for the neutrinos usually employ two scalar A_4 triplets, each acquiring different VEVs [3]. Such a model, (the AF model), will also be discussed.

5.2 The Lavoura-Kühböck model

In this model A_4 is used to explain the quark mass and mixing matrices, the lepton sector is not discussed. The most general A_4 invariant Higgs potential is used (4.17). In the model the $2A$ vacuum, $\boldsymbol{(ve^{-i\alpha/2}, ve^{i\alpha/2}, rv)}$, is chosen. As was noted while discussing the Higgs sector, this particular vacuum alignment produces some very light Higgs masses. This can be remedied when A_4 is softly broken [22]. In the paper by Lavoura and Kühböck [5] the masses and mixing angles are fitted to the experimental values. Their best fit shows r to be large, we will assume this from the start in order to be able to obtain the CKM matrix approximately analytically.

The model assigns the left- and right-handed quarks to A_4 triplets and singlets respectively, $Q_L \sim \mathbf{3}$, $u_{r,i}, d_{r,i} \sim \mathbf{1}, \mathbf{1'}, \mathbf{1''}$. In the MR basis we then get the following couplings

$$\mathcal{L}_Y = -\, y_1(\overline{Q_L}\phi)^{\mathbf{1}} d_{R1} - y_2(\overline{Q_L}\phi)^{\mathbf{1}''} d_{R2} - y_3(\overline{Q_L}\phi)^{\mathbf{1}'} d_{R3} \tag{5.20}$$

$$-y_4(\overline{Q_L}\tilde{\phi})^{\mathbf{1}} u_{R1} - y_5(\overline{Q_L}\tilde{\phi})^{\mathbf{1}''} u_{R2} - y_6(\overline{Q_L}\tilde{\phi})^{\mathbf{1}'} u_{R3} \tag{5.21}$$

Plugging in the vacuum alignment gives us the mass matrices,

$$M_d = D \begin{pmatrix} y_1 v & y_2 v & y_3 v \\ y_1 v & \omega y_2 v & \omega^2 y_3 v \\ y_1 r v & \omega^2 y_2 r v & \omega y_3 r v \end{pmatrix}, \tag{5.22}$$

$$M_u = D^* \begin{pmatrix} y_4 v & y_5 v & y_6 v \\ y_4 v & \omega y_5 v & \omega^2 y_6 v \\ y_4 r v & \omega^2 y_5 r v & \omega y_6 r v \end{pmatrix}, \tag{5.23}$$

where $D = \mathrm{diag}(e^{-i\alpha/2}, e^{i\alpha/2}, 1)$ and the Yukawa coupling constants can be chosen real since their phases can be absorbed in a redefinition of the right-handed fields. The mass and mixing matrices depend on the following eight real quantities, $y_1 v, ..., y_6 v, \alpha, r$. Thus, these eight quantities determine ten observables, the six quark masses, the three mixing angles and one phase of the CKM matrix. We can already say something about the CP violating phase in the CKM matrix. The Jarlskog invariant, [24], $J = [M_u M_u^\dagger, M_d M_d^\dagger]$ turns out to be zero, [5], meaning that there is no CP violation in the CKM matrix and so $\delta = 0$ or π. Any CP violation will have to originate from sources other than the CKM matrix.

The square of the masses of the quarks can then be calculated by solving the eigenvalue equations for $M_u M_u^\dagger$ and $M_d M_d^\dagger$. These matrices are very similar, they are of the form

$$M_d M_d^\dagger = D \begin{pmatrix} a_d & b_d & r b_d^* \\ b_d^* & a_d & r b_d \\ r b_d & r b_d^* & r^2 a_d \end{pmatrix} D^*, \quad M_u M_u^\dagger = D^* \begin{pmatrix} a_u & b_u & r b_u^* \\ b_u^* & a_u & r b_u \\ r b_u & r b_u^* & r^2 a_u \end{pmatrix} D \tag{5.24}$$

Here $a_d = v^2(y_1^2 + y_2^2 + y_3^2)$ and $b_d = v^2(y_1^2 + \omega^2 y_2^2 + \omega y_3^2)$, while for the up quarks, $a_u = v^2(y_4^2 + y_5^2 + y_6^2)$ and $b_u = v^2(y_4^2 + \omega^2 y_5^2 + \omega y_6^2)$. $a_{d,u}$ are clearly real while $b_{d,u}$ are not. It is also clear that the diagonal D matrices have no influence on the masses, which means that the eigenvalue equations have the same form for the up- as the down-type quarks

$$\lambda^3 - a(2 + r^2)\lambda^2 + (1 + 2r^2)(a^2 - |b|^2)\lambda - r^2(a^3 + b^3 + (b^*)^3 - 3a|b|^2) = 0, \tag{5.25}$$

where a, b are either a_u, b_u or a_d, b_d, but r is the same for both up- and down-type quarks. For each of these two equations λ will have three solutions, each relating to one of the three quark masses $\lambda_i = m_i^2$. Because the absolute value of each element of a matrix is smaller than the sum of the eigenvalues, we can see from (5.24) that $ar^2 < \lambda_1 + \lambda_2 + \lambda_3 \simeq \lambda_3$ (since $m_3^2 \gg m_2^2 \gg m_1^2$ for both quark types). Using this and $a \geq |b|$ in (5.25) and neglecting terms which are a factor r^2 smaller, we get for the third eigenvalue

$$\lambda_3 \simeq ar^2. \tag{5.26}$$

For the smaller two eigenvalues we now have $\lambda_{1,2} \ll ar^2$. Using this in the eigenvalue equation we get two solutions. In the case that we have $\lambda_1 \ll \lambda_2$ and thus a hierarchical ordering of the masses, these solutions can be approximated by

$$\lambda_1 \simeq r^2 \frac{a^3 + b^3 + (b^*)^3 - 3a|b|^2}{2(2 + r^2)(a^2 - |b|^2)}, \quad \lambda_2 \simeq \frac{(1 + 2r^2)(a^2 - |b|^2)}{2a(2 + r^2)}. \tag{5.27}$$

We can now also express the parameters a and b in terms of the masses, which will be useful later on

$$\begin{aligned} a =& \frac{(2+r^2)(m_3^2+m_1^2) - \sqrt{4m_1^2m_3^2 - 4(m_3^4+m_1^4)r^2 + (m_3^2+m_1^2)^2r^r + 4(1+2r^2)^2|b|^2}}{2+4r^2} \simeq m_3^2/r^2, \\ |b|^2 =& \frac{(m_1^2+m_2^2+m_3^2)^2}{(2+r^2)^2} - \frac{m_1^2m_2^2+m_3^2m_2^2+m_1^2m_3^2}{1+2r^2}, \qquad \cos 3\phi \simeq 3\frac{a}{2|b|} - \frac{a^3}{2|b|^3}, \end{aligned} \tag{5.28}$$

here ϕ stands for $\phi_{u,d} = \mathrm{Arg}(b_{u,d})$.

The CKM matrix can be constructed by the matrices diagonalizing the mass matrices. As seen before, if we have unitary matrix U_γ, ($\gamma = u, d$), such that $U_\gamma^\dagger M_\gamma M_\gamma^\dagger U_\gamma = m_\gamma^2$ then there is another unitary matrix V_γ so that $U_\gamma^\dagger M_\gamma V_\gamma = m_\gamma$. For the CKM matrix we then get, $V_{CKM} = U_u^\dagger U_d$. We use the first of these equations to find the U_γ matrices. The columns of the diagonalizing matrices are given by,

$$W_\gamma^i = \begin{pmatrix} \frac{r[b^2+(m_i^2-a)b^*]}{\left[(1+2r^2)|b|^4+(a-m_i^2)((a-m_i^2)^3-2r^2(b^3+(b^*)^3)+2|b|^2(a-m_i^2)(r^2-1))\right]^{1/2}} \\ \frac{r[(b^*)^2+(m_i^2-a)b]}{\left[(1+2r^2)|b|^4+(a-m_i^2)((a-m_i^2)^3-2r^2(b^3+(b^*)^3)+2|b|^2(a-m_i^2)(r^2-1))\right]^{1/2}} \\ \frac{(m_i^2-a)^2-|b|^2}{\left[(1+2r^2)|b|^4+(a-m_i^2)((a-m_i^2)^3-2r^2(b^3+(b^*)^3)+2|b|^2(a-m_i^2)(r^2-1))\right]^{1/2}} \end{pmatrix}_\gamma, \tag{5.29}$$

where $U_d = DW_d$ and $U_u = D^*W_u$. As we have seen, $m_3^2 \simeq ar^2$, using this we can approximate the third column by the following

$$W_\gamma^3 \simeq \begin{pmatrix} \frac{rb^*}{m_3^2} \\ \frac{rb}{m_3^2} \\ 1 \end{pmatrix}_\gamma, \tag{5.30}$$

here we used $\frac{rb^*}{m_3^2} \leq \mathcal{O}(1/r)$. This and the unitarity conditions then lead to

$$\begin{aligned} W_\gamma =& \begin{pmatrix} \frac{e^{i\varphi}}{\sqrt{2}} & \frac{ie^{i\varphi}}{\sqrt{2}} & e^{-i\phi}|\frac{rb}{m_3^2}| \\ \frac{e^{i-\varphi}}{\sqrt{2}} & \frac{-ie^{-i\varphi}}{\sqrt{2}} & e^{i\phi}|\frac{rb}{m_3^2}| \\ \frac{\sqrt{2}rb}{m_3^2}\cos(\phi+\varphi) & \frac{\sqrt{2}rb}{m_3^2}\sin(\phi+\varphi) & 1 \end{pmatrix}_\gamma \\ &+ \begin{pmatrix} \mathcal{O}(1/r^2) & \mathcal{O}(1/r^2) & \mathcal{O}(1/r^3) \\ \mathcal{O}(1/r^2) & \mathcal{O}(1/r^2) & \mathcal{O}(1/r^3) \\ \mathcal{O}(1/r^3) & \mathcal{O}(1/r^3) & \mathcal{O}(1/r^2) \end{pmatrix}, \end{aligned} \tag{5.31}$$

where φ stands for $\varphi_\gamma + \pi/2 = \mathrm{Arg}(b_\gamma^2 + (m_{2,\gamma}^2 - a_\gamma)b_\gamma^*)$. From this we can obtain the CKM matrix

$$
\begin{aligned}
&V_{CKM} = U_u^\dagger U_d \\
&= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & \frac{\sqrt{2}}{r}[\frac{|b_d|}{a_d}\cos(\theta_1+\theta_2) - \frac{|b_u|}{a_u}\cos(\theta_1+\theta_3)] \\ -\sin\theta_1 & \cos\theta_1 & \frac{\sqrt{2}}{r}[\frac{|b_u|}{a_u}\sin(\theta_1+\theta_3) - \frac{|b_d|}{a_d}\sin(\theta_1+\theta_2)] \\ \frac{\sqrt{2}}{r}[\frac{|b_u|}{a_u}\cos\theta_3 - \frac{|b_d|}{a_d}\cos\theta_2] & \frac{\sqrt{2}}{r}[\frac{|b_d|}{a_d}\sin\theta_2 - \frac{|b_u|}{a_u}\sin\theta_3] & 1 \end{pmatrix} \\
&+ \begin{pmatrix} \mathcal{O}(1/r^2) & \mathcal{O}(1/r^2) & \mathcal{O}(1/r^3) \\ \mathcal{O}(1/r^2) & \mathcal{O}(1/r^2) & \mathcal{O}(1/r^3) \\ \mathcal{O}(1/r^3) & \mathcal{O}(1/r^3) & \mathcal{O}(1/r^2) \end{pmatrix}, \qquad (5.32)
\end{aligned}
$$

where $\theta_1 = \alpha + \varphi_u - \varphi_d$, $\theta_2 = \varphi_d + \phi_d$ and $\theta_3 = \varphi_d + \phi_u - \alpha$. Now using (5.28), the CKM matrix can be written in terms of the quark masses, r, φ_γ and α. Of course φ_γ are not independent parameters, it turns out that φ_γ are constants. From the expressions for φ_γ one can obtain, $\tan(\varphi_\gamma + \pi/2) = \frac{x_\gamma \sin 2\phi_\gamma - (x_\gamma^2 - 1)|b_\gamma| \sin\phi_\gamma}{x_\gamma \cos 2\phi_\gamma - (x_\gamma^2 - 1)|b_\gamma| \cos\phi_\gamma}$ using the expressions (5.28) this can be written in terms the ratio $x_\gamma = \frac{a_\gamma}{|b_\gamma|}$ only. Deriving this expression with respect to x_γ gives $\frac{\partial\varphi_\gamma}{\partial x_\gamma} \simeq 0$. Thus φ_γ is approximately a constant.

Using the expression for $\tan(\varphi_\gamma + \pi/2)$ at $a_\gamma = 2|b_\gamma|$, one obtains $\tan(\varphi_\gamma + \pi/2) \simeq \tan\phi_\gamma/2$. When $x_\gamma = 2$ we have $\phi_\gamma \simeq (1 + 2n_\gamma)\pi/3$, with $n_\gamma \in \mathbb{N}$. There is an ambiguity in the angles ϕ_γ, so will be an ambiguity in φ_γ as well; $\varphi_\gamma = \frac{\pi}{3}(k_\gamma - 1)$ with $k_\gamma \in \mathbb{N}$.

This leaves just α and r to parametrize the CKM matrix, whereas in general it is parametrized by three angles. We can see that θ_1 should be equal to the Cabibbo angle $\theta_1 \simeq \theta_C$. Taking s_{13} and s_{23} to be small, we have $s_{13} \simeq \frac{\sqrt{2}}{r}[\frac{|b_d|}{a_d}\cos(\theta_1+\theta_2) - \frac{|b_u|}{a_u}\cos(\theta_1+\theta_3)]$ and $s_{23} \simeq \frac{\sqrt{2}}{r}[\frac{|b_u|}{a_u}\sin(\theta_1+\theta_3) - \frac{|b_d|}{a_d}\sin(\theta_1+\theta_2)]$. In this case s_{13} and s_{23} are not independent, they are both functions of r and the quark masses. These equations can not be solved algebraically, instead we will fit for r, If we try to fit for r, using the experimental values for the mixing angles we see that $\chi^2 = \chi_{13}^2 + \chi_{23}^2$ has a minimum at $r = 42.49$, ($\chi^2 = 0.102$) and

$$
\begin{aligned}
&\phi_u \simeq -1/3 \arccos\frac{3x_u - x_u^3}{2}, \;\; \phi_d \simeq 1/3 \arccos\frac{3x_d - x_d^3}{2} - 2\pi/3, \\
&\varphi_u = -2\pi/3, \;\; \varphi_d = -2\pi/3 \to \alpha = \theta_C = 0.225166,
\end{aligned} \qquad (5.33)
$$

where $\chi_{13} = (s_{13} - \frac{\sqrt{2}}{r}[\frac{|b_d|}{a_d}\cos(\theta_1+\theta_2) - \frac{|b_u|}{a_u}\cos(\theta_1+\theta_3)])/\sigma_{s_{13}}$, $\chi_{23} = s_{23} - \frac{\sqrt{2}}{r}[\frac{|b_u|}{a_u}\sin(\theta_1+\theta_3) - \frac{|b_d|}{a_d}\sin(\theta_1+\theta_2)]/\sigma_{s_{23}}$. This fit is not as good as the fit performed in [5] (since we solved all the parameters apart from r and did not fit them), however, the results are quite close. Although these values fit the mixing angles well, not all parameters of the CKM matrix are reproduced in this way since this model predicts $\delta = 0$ which is not the case. The CKM matrix for this fit is given by

$$
V_{CKM} \simeq \begin{pmatrix} 0.975 & 0.223 & -0.0036 \\ -0.223 & 0.975 & -0.041 \\ 0.0058 & 0.041 & 1 \end{pmatrix}. \qquad (5.34)
$$

This agrees quite well with the matrix we get when inserting the experimental values for the angles and phase, [14], into the parametrisation (3.9) even though the phase, $\delta \neq 0$, cannot be reproduced. This is due to the fact that δ is small.

Of course, other choices for the ambiguities in the phases, $n_{u,d}$, can be made, these option do not seem to lead to a better fit of r.

Summarizing, in this model 8 parameters were used; 6 Yukawa constants, α and r, to fit 9 observables, the 6 masses and three mixing angles. The vacuum, $2A$, provided more freedom, in the form of the parameters r and α. With it the quark mixing pattern can qualitatively be understood. It is clear from (5.32) that the mixing among the first two generations will be largest. However, not all the features of the CKM matrix can be reproduced in this model. The mixing angles can be fitted, but the CP violating phase, δ, is predicted to be zero. Also, it was pointed out in [22] that this model predicts flavor changing neutral currents which appear to be far above the experimental bounds.

Instead of using more complicated VEVs, more freedom might be achieved by introducing more scalar fields. In the next section we will look at a model which takes this approach in trying to explain the mixing in the lepton sector.

5.3 Tri-bimaximal mixing in the Altarelli-Feruglio model

This model, [3] [11], introduces additional scalar fields (flavons) in order to explain neutrino mixing. The model is supersymmetric and allows for dimension- 5 and higher (non-renormalizable) terms. This opens up the possibility for a number of extra terms with respect to the case in which only dimension-4 terms are allowed. Among these terms are not only those which will be useful while constructing this model, but also those which could spoil the model if we would allow for them. We will be forced to find some mechanism which sets these to zero, in this case an additional Z_3 symmetry will be used.

Apart from the symmetries, A_4 and Z_3, an $U(1)_R$ symmetry is introduced. This contains the R-parity, present in supersymmetric models, as a subgroup. R-parity ensures that supersymmetric particles only appear in pairs. The field assignment is given in Table 5.1. In this section the neutrino masses are obtained through a dimension-6 operator, analogous to the dim-5 operator discussed previously. Later on it will be shown that the model can be modified so that the masses arise through the seesaw mechanism.

Field	L_L	l_R	$h_{u,d}$	φ_S	φ_T	ξ	$\tilde{\xi}$	φ_0^S	φ_0^T	ξ_0
A_4	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \mathbf{1}''$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$
Z_3	ω	ω	1	ω	1	ω	ω	1	ω	ω
$U(1)_R$	1	1	0	0	0	0	0	2	2	2

Table 5.1: The irrep assignment of the AF model is shown.

In Table 5.1 φ_S, φ_T, ξ, $\tilde{\xi}$ are the flavons and φ_0^S, φ_0^T, ξ_0 are the driving fields. The driving fields are so named because they 'drive' the VEVs of the flavons through their interactions with them. All the newly introduced fields are singlets under $SU(2)$. Following the symmetry assignment in Table 5.1 we can write down the allowed Yukawa terms

$$\begin{aligned} -\mathcal{L}_Y =& \frac{y_e}{\Lambda}\overline{e_R}h_d(L_L\varphi_T)^{\mathbf{1}} + \frac{y_\mu}{\Lambda}\overline{\mu_R}h_d(L_L\varphi_T)^{\mathbf{1}'} + \frac{y_\tau}{\Lambda}\overline{\tau_R}h_d(L_L\varphi_T)^{\mathbf{1}''} \\ &+ \frac{x_u}{\Lambda^2}\xi(\overline{(L_Lh_u)^c}(h_uL_L))^{\mathbf{1}} + \frac{x_S}{\Lambda^2}\varphi_S\overline{((L_Lh_u)^c}(h_uL_L))^{\mathbf{3}} + h.c. + ... \end{aligned} \tag{5.35}$$

where Λ is the cut-off of the theory (it is assumed that $v_{u,d} \ll \Lambda$) and the dots stand for terms at higher order in $\frac{1}{\Lambda}$. Throughout the discussion of this model we will use the AF basis. As we will see later, the flavon field $\tilde{\xi}$ does not acquire a VEV, which is why it was neglected in the Yukawa terms, the reason for this field will become apparent later on. It can be seen that by introducing the additional Z_3 symmetry, 5 couplings are effectively put to zero. These couplings are those which are acquired by letting $\varphi_T \Leftrightarrow \varphi_S$ and the previously discussed dimension-5 operator.

The allowed vacua can be deduced from the driving term (terms involving the interaction of the driving fields with the flavons)

$$\begin{aligned} w_d =& M(\varphi_0^T\varphi_T)^{\mathbf{1}} + g\varphi_0^T(\varphi_T\varphi_T)^{\mathbf{3}} + g_1\varphi_0^S(\varphi_S\varphi_S)^{\mathbf{3}} \\ &+ g_2\tilde{\xi}(\varphi_0^S\varphi_S)^{\mathbf{1}} + g_3\xi_0(\varphi_S\varphi_S)\tilde{\xi} + g_4\xi_0\xi^2 + g_5\xi_0\xi\tilde{\xi} + g_6\xi_0\tilde{\xi}^2 . \end{aligned} \tag{5.36}$$

Note that here too, the flavon fields φ_S and φ_T do not mix. There is in principle no difference between the ξ and $\tilde{\xi}$ flavons (they are members of the same representations), so $\tilde{\xi}$ can be defined as the combination coupling to the φ_S field. From the driving term the scalar potential can be obtained, $V = \sum_i |\frac{\partial w_d}{\partial \phi_i}| + m_i^2|\phi_i|^2$. Here ϕ_i stand for the scalar fields and m_i are soft masses, which are expected to be small with regard to the scales in w_d. The minima can then be calculated as follows

$$\begin{aligned} 0 =& \frac{\partial w_d}{\partial \varphi_{01}^T} = M\varphi_{T1} + \frac{2g}{3}(\varphi_{T1}^2 - \varphi_{T2}\varphi_{T3}), \\ 0 =& \frac{\partial w_d}{\partial \varphi_{02}^T} = M\varphi_{T3} + \frac{2g}{3}(\varphi_{T2}^2 - \varphi_{T1}\varphi_{T3}), \\ 0 =& \frac{\partial w_d}{\partial \varphi_{03}^T} = M\varphi_{T2} + \frac{2g}{3}(\varphi_{T3}^2 - \varphi_{T1}\varphi_{T2}), \\ 0 =& \frac{\partial w_d}{\partial \varphi_{01}^S} = g_2\tilde{\xi}\varphi_{S1} + \frac{2g_1}{3}(\varphi_{S1}^2 - \varphi_{S2}\varphi_{S3}), \\ 0 =& \frac{\partial w_d}{\partial \varphi_{02}^S} = g_2\tilde{\xi}\varphi_{S3} + \frac{2g_1}{3}(\varphi_{S2}^2 - \varphi_{S1}\varphi_{S3}), \\ 0 =& \frac{\partial w_d}{\partial \varphi_{03}^S} = g_2\tilde{\xi}\varphi_{S2} + \frac{2g_1}{3}(\varphi_{S3}^2 - \varphi_{S2}\varphi_{S1}), \\ 0 =& \frac{\partial w_d}{\partial \xi_0} = g_4\xi^2 + g_5\xi\tilde{\xi} + g_6\tilde{\xi}^2 + g_3(\varphi_{S1}^2 + 2\varphi_{S2}\varphi_{S3}). \end{aligned} \tag{5.37}$$

The first three equations can be solved by setting $\langle\varphi_T\rangle = (v_T, 0, 0)$ where $v_T = -\frac{3M}{2g}$. The next three are solved for $\langle\varphi_S\rangle = (v_S, v_S, v_S)$ and $\langle\tilde{\xi}\rangle = 0$. The last equation then gives $\langle\xi\rangle = u$, where $u^2 = -\frac{3g_3v_S^2}{g_6}$. The reason for the flavon field, $\tilde{\xi}$ now becomes apparent, with just one singlet scalar (reproduced by setting $\xi = 0$), the acquired VEV, $\langle\varphi_S\rangle$, would not be possible.

With these VEVs we can obtain the mass matrices by using the multiplication rules (in the Altarelli basis (section 2.1.2))

$$M_l = v_d \frac{v_T}{\Lambda}\begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad M_\nu = \frac{2v_u^2}{3\Lambda^2}\begin{pmatrix} 3x_u u + 2x_S v_S & -x_S v_S & -x_S v_S \\ -x_S v_S & 2x_S v_S & 3x_u u - x_S v_S \\ -x_S v_S & 3x_u u - x_S v_S & 2x_S v_S \end{pmatrix}. \tag{5.38}$$

The charged lepton mass matrix is diagonal and the three independent Yukawa couplings ensure that any mass spectrum can be explained this way. Any phases of y_e, y_μ, y_τ can be absorbed by a redefinition of the l_R fields. The neutrino mass matrix is of the TB mixing form discussed earlier (section 3.3), thus the mixing matrix will be tri-bimaximal. The neutrino masses are

$$m_1 = a + b, \;\; m_2 = a, \;\; m_3 = b - a, \tag{5.39}$$

where the Majorana phases have been absorbed in these masses and $a = 2x_u u\frac{v_u^2}{\Lambda^2}$ and $b = 2x_S v_S\frac{v_u^2}{\Lambda^2}$. If we take experimental data into account, only the normal hierarchy is possible in this model [25]. Demanding that $\Delta m_{21}^2 > 0$ implies that $-|b|^2 > ab^* + a^*b$ and thus $ab^* + a^*b < 0$. Using this we see that $\Delta m_{31}^2 = -2(ab^* + a^*b) > 0$ and thus we have a normal hierarchy.

It can now also be seen that there is a good reason for enforcing the Z_3 symmetry. Consider for instance the following Z_3 violating terms

$$(\overline{L_L}\varphi_S)^{\mathbf{1}}e_R + (\overline{L_L}\varphi_S)^{\mathbf{1}'}\mu_R + (\overline{L_L}\varphi_S)^{\mathbf{1}''}\tau_R, \tag{5.40}$$

due to the different direction of the VEV of φ_S these terms would guarantee that the charged lepton matrix is not diagonal, which would make things more complicated. Furthermore, if the $\varphi_T(\overline{(L_L h_u)^c}(h_u L_L))^{\mathbf{3}}$ term were allowed the resulting M_ν matrix would break the TB mixing requirement $(M_\nu(1,1) + M_\nu(1,2) = M_\nu(2,2) + M_\nu(2,3))$. Finally, the contributions from the dimension-5 operator, $\frac{x}{\Lambda}((Lh_u)(h_u L))^{\mathbf{1}}$, will be of the same form as the x_u term, a possible problem is that they will be of the order $\mathcal{O}(\frac{v_u^2}{\Lambda})$. The other neutrino mass terms are a factor $\frac{u}{\Lambda}$ or $\frac{v_S}{\Lambda}$ smaller than this, so the dimension-5 operator might be the dominant contribution. Also the neutrino masses are now of the order $\mathcal{O}(\frac{v_u^2}{\Lambda})$, and the charged lepton masses are of the order $\mathcal{O}(\frac{v_d v_T}{\Lambda})$. This means that the smallness of the neutrino masses compared to the charged lepton masses no longer arises naturally from the large cut-off, but must now come from the ratios of the VEVs. The smallness of the ratio $\frac{v_u}{v_T}$ may be realized by considering the origins of the different VEVs. The VEVs of the Higgs doublets, $v_{u,d}$, should be of the electroweak scale, whereas the VEVs of the flavons, v_S, v_T and u are expected to be coming from a much higher energy scale, still below Λ [11]. Thus, $\frac{v_u^2}{v_d v_T}$ is expected to be small.

We will now see that a similar model can be achieved through the seesaw mechanism (type I).

5.3.1 Seesaw

The seesaw mechanism can be incorporated in the same model with a few minor changes. Obviously a right handed neutrino field should be introduced and apart from that the Z_3

Field	ν_R	φ_S	ξ	$\tilde{\xi}$	φ_0^T	ξ_0
A_4	**3**	**3**	**1**	**1**	**3**	**1**
Z_3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2

Table 5.2: The changes in assignment of the model are shown.

assignment changes somewhat. These changes are summarized in Table 5.2
The new Lagrangian is given by

$$\begin{aligned}-\mathcal{L}_Y =&\frac{y_e}{\Lambda}\overline{e_R}h_d(L_L\varphi_T)^{\mathbf{1}} + \frac{y_\mu}{\Lambda}\overline{\mu_R}h_d(L_L\varphi_T)^{\mathbf{1}'} + \frac{y_\tau}{\Lambda}\overline{\tau_R}h_d(L_L\varphi_T)^{\mathbf{1}''} \\ &+yh_u(\overline{L_L}\nu_R)^{\mathbf{1}} + x_u\xi(\overline{\nu_R^c}\nu_R)^{\mathbf{1}} + x_S\varphi_S(\overline{\nu_R^c}\nu_R)^{\mathbf{3}} + h.c. + ...\end{aligned} \tag{5.41}$$

The charged lepton mass matrix is left unchanged, the neutrino Dirac and right-handed mass matrices are now given by

$$M_D = yv_u\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad M_R = \frac{2}{3}\begin{pmatrix}3x_u^*u^*+2x_S^*v_S^* & -x_S^*v_S^* & -x_S^*v_S^* \\ -x_S^*v_S^* & 2x_S^*v_S^* & 3x_u^*u^*-x_S^*v_S^* \\ -x_S^*v_S^* & 3x_u^*u^*-x_S^*v_S^* & 2x_S^*v_S^*\end{pmatrix}. \tag{5.42}$$

The right handed mass matrix has exactly the same form as the neutrino mass matrix did previously. It is then not too hard to see that this should also work. Consider the neutrino mass matrix in the see-saw mechanism, with no left-handed Majorana term, $M_L = 0$, it is given by $M_\nu = -M_D^T M_R^{-1} M_D$. M_R satisfies the TB condition, $m_{diag} = U_{TB}M_R U_{TB}^T$, this implies that its inverse will be diagonalized as follows $m_{diag}^{-1} = U_{TB}^* M_R^{-1} U_{TB}^\dagger$. Since M_D is proportional to the identity matrix, we have

$$M_D M_R^{-1} M_D^T = y^2 v_u^2 M_R^{-1}. \tag{5.43}$$

Thus, M_ν is of the TB mixing form. The masses will be proportional to the inverse of the eigenvalues of M_R, which implies that we can immediately write down the neutrino masses

$$m_1 = \frac{y^2v_u^2}{A+B}, \;\; m_2 = \frac{y^2v_u^2}{A}, \;\; m_3 = \frac{y^2v_u^2}{B-A}, \tag{5.44}$$

where $A = 2x_u^*u^*$ and $B = 2x_S^*v_S^*$. The neutrinos now have masses of the order $\mathcal{O}(\frac{v_u^2}{u})$, which means that the smallness of the neutrino masses no longer comes from the large cut-off of the theory but is now achieved through a ratio of the VEVs. This ratio, $\frac{v_u^2}{v_d u}$, is expected to be small as u should originate from a energy scale far above that of $v_{u,d}$.

Note that even if the Yukawa couplings x_u, x_S are taken to be real, the inverted hierarchy can be achieved this time. $\Delta m_{21}^2 > 0$ now implies that $(A+B)^2 > A^2$ which allows B to have the same or opposite sign of A. For the other squared mass difference we have $\Delta m_{31}^2 = \frac{4AB}{(B^2-A^2)^2}$, since AB can be either positive or negative, both normal and inverted hierarchy are possible.

In this model the TB mixing pattern can be explained. Interestingly, the charged lepton mass matrix can be parametrized by three and the neutrino mass matrix by four (real) parameters. So, in this model, knowledge of seven parameters gives the values of eleven observables; six masses, three mixing angles and two Majorana phases. This seems like a better result than previous model, [5], in which eight parameters were needed for nine observables. However, this model was more modest in its use of additional fields and symmetries.

5.4 Final remarks

From our discussion of the 'simplest' A_4 models we concluded that in order to explain neutrino mixing the 'simplest' Higgs potential, (4.17), is not enough. To see how this problem is solved in the literature the LK and AF models were discussed. In the first more freedom was achieved by using a VEV which requires small A_4 breaking terms in the Higgs potential in order to produce acceptable Higgs masses. The second used a different Higgs sector and different scalar fields altogether. So there are three ways to proceed; we can allow for (small) A_4 breaking terms, introduce more (other) fields or use a different symmetry group.

We will not pursue the last option. It could be argued that the reason the 'simplest' A_4 models are not able to explain the mixing matrices is because they do not allow enough freedom in constructing the neutrino mass matrix. This mass matrix should be quite different from the mass matrix of the charged fermions in order to explain the large mixing in the PMNS matrix. Since the mass terms of all the fermions arise through an interaction with the same Higgs fields, ϕ, we are rather restricted in constructing a neutrino mass matrix which is sufficiently different from the mass matrix of the charged leptons. If we were to consider 'simple' models which use other groups but a similar Higgs sector we would expect to encounter the same problem.

In the next Chapter, we will look into a model which follows the second route; we will introduce more (scalar) fields. The models we will look at are models with a left-right symmetry. These have an enlarged Higgs sector which allows for Type I and II seesaw. So we would expect to gain more freedom in constructing the neutrino mass matrix.

6

Left-Right models

In the SM the weak interaction is asymmetrical between left- and right-handed fields, only the left-handed fields participate. The reason for this is unknown. In left-right models the answer is that there is a symmetry between left and right at high energies which is spontaneously broken. In order to enable such a left-right symmetry the gauge group of the SM is enlarged and the Higgs sector is extended.

Left-right models are of interest to us for several reasons. They have an enlarged Higgs sector; this means additional fields which may help in reproducing the correct mixing patterns. Also, left-right models allow for type I and type II seesaw, giving more freedom in constructing neutrino mass matrices. Lastly, left-right models allow for family dependent couplings. It seems interesting to combine this family dependence with a family symmetry, so that these couplings may be determined by the family symmetry. For these reasons we will explore the possibility of combining left-right models with an A_4 family symmetry.

6.1 Left-Right symmetry

In the SM the $W^{\pm}$ and Z gauge bosons couple to left-handed fields only. In left-right symmetric models there are similar gauge bosons, $W_L^{\pm}$ and Z_L, which couple to left-handed fields only. In addition, $W_R^{\pm}$ and Z_R gauge bosons are added which only couple to right-handed fields, so that a symmetry between left- and right-handed fields can be achieved. In the limit that the energy at which this left-right symmetry is present goes to infinity ($M_{W_R} \to \infty$) the $W_L^{\pm}$ and Z_L bosons are equivalent to the SM fields, $W^{\pm}$ and Z. However, generally the bosons of the different $SU(2)$ gauge groups mix among each other. They mix into the SM bosons, $W^{\pm}$ and Z, and their heavier partners, $W^{\pm\prime}$ and Z'. The lower bound on the masses of the additional gauge bosons, $W_R^{\pm}$ and Z_R, is in the TeV range, $M_{W_R,Z_R} \gtrsim 1$ TeV [26].

These gauge bosons come from an additional gauge group; the electroweak gauge group of the SM is extended to $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. As said previously, the Higgs sector is also extended. The Higgs doublet of the standard model is replaced by a bi-doublet, which has $(2, 2, 0)$ as $(SU(2)_L, SU(2)_R, U(1)_{B-L})$ quantum numbers, and a left- and a right-handed

Higgs triplet, $(3,1,2)$ and $(1,3,2)$ respectively, are introduced,

$$\phi = \begin{pmatrix} \phi_A^0 & \phi_A^+ \\ \phi_B^- & \phi_B^0 \end{pmatrix}, \quad \Delta_{L,R} = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix}. \tag{6.1}$$

The bi-doublet transforms under $SU(2)_L$ and $SU(2)_R$ as $\phi \to U_L \phi U_R^\dagger$, whereas the triplets transform as $\Delta_{L,R} \to U_{L,R} \Delta_{L,R} U_{L,R}^\dagger$.

There are two possibilities for the left-right symmetry transformation [26],

$$\mathcal{P} : \begin{cases} \psi_L \leftrightarrow \psi_R \\ \phi \leftrightarrow \phi^\dagger \end{cases}, \quad \mathcal{C} : \begin{cases} \psi_L \leftrightarrow (\psi_R)^c \\ \phi \leftrightarrow \phi^T \end{cases}, \tag{6.2}$$

here ψ stands for the fermion fields, but the scalar $SU(2)$ triplets transform in the same way. The transformations are named $\mathcal{P}$ and $\mathcal{C}$ because they are related to parity and charge conjugation supplemented by the exchange of the left and right $SU(2)$ groups. We will be focussing on the $\mathcal{P}$ case.

Of course, a left-right model should be similar to the SM at low energies. This is achieved by the VEV of the Higgs triplets [27]

$$\langle \Delta_R \rangle = \begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix}, \quad \langle \Delta_L \rangle = 0. \tag{6.3}$$

These VEVs break the left-right symmetry and $SU(2)_R$, which means we are back to the gauge group of the SM. Spontaneous electroweak symmetry breaking occurs when the bi-doublet acquires a VEV,

$$\langle \phi \rangle = \begin{pmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{pmatrix}. \tag{6.4}$$

Subsequently, the left-handed Higgs triplet acquires a small VEV, $\langle \Delta_L \rangle = v_L$, where $v_L \sim \frac{\kappa_A^2 + \kappa_B^2}{v_R}$.

To see which of these VEVs can be complex, we need to look at the amount of phases we can get rid of, [27]. One might be inclined to use a global rotation at this point. However, in principle the $\mathcal{P}$ transformation (6.2) may have phases associated with it

$$\psi_L \leftrightarrow e^{i\theta_\psi} \psi_R, \quad \phi \leftrightarrow e^{i\theta_\phi} \phi^\dagger. \tag{6.5}$$

Such a transformation would require these phases to appear in the Lagrangian, luckily they can be absorbed by an appropriate global rotation. This just leaves us with the freedom of $SU(2)_{L,R}$ transformations to rotate the phases of the Higgs VEVs away. The only useful transformations to do this are of the form

$$U_{L,R} = \begin{pmatrix} e^{i\theta_{L,R}} & 0 \\ 0 & e^{-i\theta_{L,R}} \end{pmatrix}. \tag{6.6}$$

There are two parameters in these matrices, so we should be able to make two VEVs completely real. Applying these transformations we see that

$$\begin{aligned} &\kappa_A \to \kappa_A e^{i(\theta_L - \theta_R)}, \;\; \kappa_B \to \kappa_B e^{-i(\theta_L - \theta_R)}, \\ &v_L \to v_L e^{-2i\theta_L}, \;\; v_R \to v_R e^{-2i\theta_R}. \end{aligned} \tag{6.7}$$

It is clear that we can absorb two phases, but not both of the phases in $\langle\phi\rangle$. A common choice is to take κ_A and v_R to be real and allow κ_B and v_L to be complex. In this choice the phase of κ_B is not completely free, it can be constrained by the heaviest quark masses and the ratio κ_B/κ_A, [26].

Now that we know that in the low energy limit the SM can be obtained, let us look at the mass terms in such a model

$$\begin{aligned} -\mathcal{L}_Y =& \overline{Q_L}(Y_q\phi + \tilde{Y}_q\tilde{\phi})Q_R + \overline{L_L}(Y_l\phi + \tilde{Y}_l\tilde{\phi})L_R \\ &+ i\frac{1}{2}\left[\overline{L_L^c}F\sigma_2\Delta_L L_L + \overline{L_R^c}F\sigma_2\Delta_R L_R\right] + h.c. \ , \end{aligned} \tag{6.8}$$

here $L_{L,R} = \begin{pmatrix}\nu \\ e\end{pmatrix}_{L,R}$, $Q_{L,R} = \begin{pmatrix}u \\ d\end{pmatrix}_{L,R}$, $\tilde{\phi} = \sigma_2\phi^*\sigma_2$ and Y and F are 3 by 3 matrices. Demanding that this is invariant under $\mathcal{P}$ of (6.2) implies that $Y = Y^\dagger$ and $\tilde{Y} = \tilde{Y}^\dagger$. Although the Yukawa matrices are hermitian the Dirac mass matrices are in general not. Consider for instance the up- and down-type quark mass matrices

$$M_u = \kappa_A Y_u + \kappa_B^* \tilde{Y}_u, \quad M_d = \kappa_B Y_d + \kappa_A^* \tilde{Y}_d, \tag{6.9}$$

a phase in the VEVs of ϕ could spoil the hermiticity of these mass matrices. However, the constraints on the phase of κ_B imply that the mass matrices are always approximately hermitian [26]. Since we have Higgs triplets and right-handed neutrinos (which are needed since we also have left-handed neutrinos) the neutrino masses are generated through a combination of type I and type II seesaw mechanism (section 3.3.2). In this case the light neutrino mass matrix will be given by

$$M_\nu = M_L - M_D^T M_R^{-1} M_D = v_L F - (\kappa_A^* Y + \kappa_B \tilde{Y})^T (v_R F^\dagger)^{-1} (\kappa_A^* Y + \kappa_B \tilde{Y}). \tag{6.10}$$

Since $v_L \sim \frac{\kappa_A^2+\kappa_B^2}{v_R}$, both contributing terms, and thus the order of the neutrino masses, will be of the order of $m_\nu \sim \frac{\kappa_A^2+\kappa_B^2}{v_R}$.

We see that the neutrino mass terms are produced through interaction with three types of scalar fields, ϕ and $\Delta_{L,R}$, whereas the other fermion mass terms are produced solely through the interaction with the ϕ fields. This seems promising with respect to the freedom we will have in constructing the neutrino mass matrix.

Now that we have seen the general build-up of a left-right symmetric model, we will try to see what happens when it is combined with A_4.

6.2 Combining Left-Right and A_4

When building a model, as before, first the field assignments should be decided upon. In a model combining left-right symmetry with A_4 there are fewer possible field assignments than one might at first think. As before we cannot have the Higgs field, the bi-doublet in this case, as an A_4 singlet as it would produce degenerate masses, [18]. In order to be able to construct singlets for terms of the form of (6.8), we need either Q_L or Q_R to be a triplet under A_4. The left-right symmetry then forces both to be a triplet, similarly for L_L and

L_R. This just leaves the assignment of the scalar $SU(2)$ triplets, for these holds the same; picking the assignment of one of them determines the assignment of the other as well. We will take $\Delta_{L,R} \sim \mathbf{3}$ for the upcoming model. Summarizing we have, in addition to the usual left-right model,

$$\phi \to (\phi_1, \phi_2, \phi_3), \qquad \Delta \to (\Delta_1, \Delta_2, \Delta_3). \tag{6.11}$$

The field assignment is summarized in the following table.

Field	$Q_{L,R}$	$L_{L,R}$	ϕ	$\Delta_{L,R}$
A_4	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$

Table 6.1: The assignment of the left-right model is shown.

The usual procedure from here on would be to study the Higgs sector to see which VEVs are acceptable and then study the possible mass and mixing matrices. However, the Higgs sector is very large in this case, making it difficult to see which VEVs are acceptable and which are not. To see whether it is possible to obtain the right mixing matrices we will choose VEVs which seem natural,

$$\langle\phi_i\rangle = \begin{pmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{pmatrix}, \quad \langle\Delta_{Ri}\rangle = \begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix},$$

$$\langle\Delta_{L1}\rangle, \langle\Delta_{L2}\rangle, \langle\Delta_{L3}\rangle = \begin{pmatrix} 0 & 0 \\ v_L & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

these VEVs are the simplest ones which were found to be acceptable in the case of the simpler Higss sector (Chapter 4). These VEVs allow for the same considerations as before, meaning that κ_A and v_R can be picked real and κ_B and v_L are in general complex. Using the Yukawa interactions (6.8), the multiplication rules of A_4 in the MR basis, (2.29), and following the definitions for the mass matrices, (3.23), we have

$$\begin{aligned}
M_u &= \begin{pmatrix} 0 & y_q\kappa_A + \tilde{y}_q\kappa_B^* & y_q^*\kappa_A + \tilde{y}_q{}^*\kappa_B^* \\ y_q^*\kappa_A + \tilde{y}_q{}^*\kappa_B^* & 0 & y_q\kappa_A + \tilde{y}_q\kappa_B^* \\ y_q\kappa_A + \tilde{y}_q\kappa_B^* & y_q^*\kappa_A + \tilde{y}_q{}^*\kappa_B^* & 0 \end{pmatrix}, \\
M_d &= \begin{pmatrix} 0 & y_q\kappa_B + \tilde{y}_q\kappa_A^* & y_q^*\kappa_B + \tilde{y}_q{}^*\kappa_A^* \\ y_q^*\kappa_B + \tilde{y}_q{}^*\kappa_A^* & 0 & y_q\kappa_B + \tilde{y}_q\kappa_A^* \\ y_q\kappa_B + \tilde{y}_q\kappa_A^* & y_q^*\kappa_B + \tilde{y}_q{}^*\kappa_A^* & 0 \end{pmatrix}, \\
M_l &= \begin{pmatrix} 0 & y_l\kappa_B + \tilde{y}_l\kappa_A^* & y_l^*\kappa_B + \tilde{y}_l{}^*\kappa_A^* \\ y_l^*\kappa_B + \tilde{y}_l{}^*\kappa_A^* & 0 & y_l\kappa_B + \tilde{y}_l\kappa_A^* \\ y_l\kappa_B + \tilde{y}_l\kappa_A^* & y_l^*\kappa_B + \tilde{y}_l{}^*\kappa_A^* & 0 \end{pmatrix}, \\
M_D &= \begin{pmatrix} 0 & y_l\kappa_A^* + \tilde{y}_l\kappa_B & y_l^*\kappa_A^* + \tilde{y}_l{}^*\kappa_B \\ y_l^*\kappa_A^* + \tilde{y}_l{}^*\kappa_B & 0 & y_l\kappa_A^* + \tilde{y}_l\kappa_B \\ y_l\kappa_A^* + \tilde{y}_l\kappa_B & y_l^*\kappa_A^* + \tilde{y}_l{}^*\kappa_B & 0 \end{pmatrix},
\end{aligned} \tag{6.12}$$

$$M_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & fv_L \\ 0 & fv_L & 0 \end{pmatrix}, \quad M_R = \begin{pmatrix} 0 & f^*v_R & f^*v_R \\ f^*v_L & 0 & f^*v_R \\ f^*v_R & f^*v_R & 0 \end{pmatrix}. \tag{6.13}$$

The mass matrices are diagonalized by the familiar matrix $V = \sqrt{1/3}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2\end{pmatrix}$ as follows

$$\begin{aligned} m_u =& VM_uV^\dagger, \quad m_d = VM_dV^\dagger,\\ m_l =& VM_lV^\dagger, \quad m_D = VM_DV^\dagger, \end{aligned} \tag{6.14}$$

where m_α stands for a diagonal matrix with in general complex values. The phases of the diagonal elements can be absorbed by a redefinition of the right-handed fields, $|m_{u,d,l}|K_{u,d,l} = m_{u,d,l}$, where $K_{u,d,l}$ are diagonal matrices of phases. In this basis, the quark and charged lepton mass matrices now being diagonal, real and positive, we have $u_{Lm} = Vu_L$, $d_{Lm} = Vd_L$, $u_{Rm} = K_uVu_R$ and $d_{Rm} = K_dVd_L$. This implies for the charged currents of the quarks

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}\left[W_L^+\overline{u_{Lm}}V\gamma^\mu V^\dagger d_{Lm} + W_R^+\overline{u_{Rm}}K_uV\gamma^\mu V^\dagger K_d^* d_{Rm}\right] + h.c.\ . \tag{6.15}$$

Thus, $V_L^{CKM} = \mathbb{1}$ and $V_R^{CKM} = K_uK_d^*$.

If we now demand that the leptonic charged currents are diagonal (and real), we obtain the weak basis of the neutrino fields; $\nu_{Lw} = V\nu_L$ and $\nu_{Rw} = K_lV\nu_R$. In this basis the diagonalizing matrices will be the mixing matrices. Writing out the neutrino mass matrices in this basis

$$M_L' = V^*M_LV^\dagger,\ \ M_R' = K_lVM_RV^TK_l,\ \ M_D' = K_lm_D, \tag{6.16}$$

here M_D' is clearly diagonal. The resulting mass matrix for the light neutrinos is $M_\nu = M_L' - M_D'^T(M_R')^{-1}M_D' = M_L' - m_D^TV^*M_R^{-1}V^\dagger m_D$ and for the heavy right-handed neutrinos, M_R'. Writing these out explicitly, with $m_D = \text{diag}(m_{D1}, m_{D2}, m_{D3})$, we have

$$M_\nu = \begin{pmatrix} \frac{2fv_L}{3} - \frac{m_{D1}^2}{2f^*v_r} & -\frac{fv_l}{3} & -\frac{fv_l}{3}\\ -\frac{fv_l}{3} & \frac{2fv_l}{3} & -\frac{fv_l}{3} + \frac{m_{D2}m_{D3}}{f^*v_R}\\ -\frac{fv_l}{3} & -\frac{fv_l}{3} + \frac{m_{D2}m_{D3}}{f^*v_R} & \frac{2fv_l}{3} \end{pmatrix},\ \ M_R' = f^*v_RK_l\begin{pmatrix}-2 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}K_l. \tag{6.17}$$

M_ν is of the TB mixing form, (3.22), with $w = -2y$, thus it is diagonalized by

$$V_L = \begin{pmatrix} -c_{12} & s_{12} & 0\\ \frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \sqrt{1/2}\\ \frac{c_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\sqrt{1/2} \end{pmatrix}, \tag{6.18}$$

then $U_L^{PMNS} = V_L\text{diag}(e^{i\alpha_{L1}}, e^{i\alpha_{L2}}, 1)$, where α_{1L}, α_{2L} are the Majorana phases. For exact TB mixing we need $s_{12}^2 = 1/3$, then $x + y = w + v$ holds in (3.22). In our case this means we have exact TB mixing when $m_{D1}^2 = -2m_{D2}m_{D3}$, we then have the neutrino masses

$$m_{\nu 1} = fv_L - \frac{m_{D1}^2}{2f^*v_R},\ \ m_{\nu 2} = -\frac{m_{D1}^2}{2f^*v_R},\ \ m_{\nu 3} = fv_L + \frac{m_{D1}^2}{2f^*v_R}, \tag{6.19}$$

these are in general complex masses as the Majorana phases are absorbed in them. They follow the relation $m_{\nu 1} = 2m_{\nu 2} + m_{\nu 3}$, this allows us to write the masses in terms of the squared mass differences. The definitions $\Delta m_{31}^2 = |m_3|^2 - |m_1|^2$ and $\Delta m_{21}^2 = |m_2|^2 - |m_1|^2$ together with the previous relation give us

$$\cos\alpha_{12} = \frac{\Delta m_{21}^2 + |m_1|^2 - \Delta m_{31}^2/4}{|m_1|\sqrt{\Delta m_{21}^2 + |m_1|^2}}, \tag{6.20}$$

where α_{12} is the angle between m_1 and m_2, $\alpha_{12} = 2(\alpha_{L1} - \alpha_{L2})$. It can be seen that the inverse hierarchy is not possible in this case since $\frac{\Delta m_{21}^2 + |m_1|^2}{|m_1|\sqrt{\Delta m_{21}^2 + |m_1|^2}} \geq 1$, thus whenever $\Delta m_{31}^2 \leq 0$, as is the case for the inverse hierarchy, we have $\cos\alpha_{12} \geq 1$. The normal hierarchy is conceivable however. The fact that $-1 \leq \cos\alpha_{12}$ gives us a lower bound on $|m_1|$

$$|m_1| \geq \frac{\Delta m_{31}^2 - 4\Delta m_{21}^2}{\sqrt{8\Delta m_{31}^2 - 16\Delta m_{21}^2}} \simeq 0.015 \text{ eV}. \tag{6.21}$$

As $|m_1|$ increases goes to infinity, $\cos\alpha_{12}$ tends to one.

It can be seen that two of the heavy neutrino masses are degenerate, we have $M_{\nu 1} = 2|f|v_R$ and $M_{\nu 2} = M_{\nu 3} = |f|v_R$. M_R' is diagonalized by

$$U_R = K_l^* \begin{pmatrix} i & 0 & 0 \\ 0 & \sqrt{1/2} & -i\sqrt{1/2} \\ 0 & \sqrt{1/2} & i\sqrt{1/2} \end{pmatrix}, \tag{6.22}$$

so that $U_R^{PMNS} = U_R \text{diag}(e^{i\alpha_{R1}}, e^{i\alpha_{R2}}, 1)$ where α_{1R}, α_{2R} are the Majorana phases for the right-handed neutrinos.

We now return to the Dirac mass matrices of (6.12), they have similar forms and are diagonalized by $VMV^\dagger = m$. The matrix m is given by $m = \text{diag}(m_1, m_2, m_3)$, with

$$\begin{aligned} m_1 =& (y + y^*)\kappa_A + (\tilde{y} + \tilde{y}^*)\kappa_B^*, \;\; m_2 = (\omega y + \omega^2 y^*)\kappa_A + (\omega\tilde{y} + \omega^2\tilde{y}^*)\kappa_B^*, \\ m_3 =& (\omega^2 y + \omega y^*)\kappa_A + (\omega^2\tilde{y} + \omega\tilde{y}^*)\kappa_B^*), \end{aligned} \tag{6.23}$$

where κ_A and κ_B should be exchanged for the down-type Dirac mass matrix. Note that this is a special case of (5.5), with $b = a^*$ whenever κ_B is real. This would constrain the masses too severely, as we saw in (5.5). However, even when κ_B is complex we have $m_3 + m_2 + m_1 = 0$. Meaning that $|m_3| = |m_2 + m_1|$ and the maximum for $|m_3|$ is $|m_3| = |m_2| + |m_1|$, it is clear that this does not allow for the hierarchy seen in nature.

Note that m_{D1}, m_{D2} and m_{D3} follow the same relation. This does not change the discussion for the neutrino masses however, since the combinations m_{D1}^2 and $m_{D2}m_{D3}$ appearing in the neutrino mass matrix can still be seen as independent parameters.

6.2.1 Hermiticity

The relation between the masses of the charged fermions that resulted from the previous model, is shared by a wider range of models. Each model with the assignment of Table 6.1 and hermitian mass matrices for the charged fermions will lead to the same relation; $m_3 + m_2 + m_1 = 0$.

This can be seen by noting that every hermitian matrix can be diagonalized by a single unitary matrix,

$$U^\dagger M U = m = \mathrm{diag}(m_1, m_2, m_3), \tag{6.24}$$

where M is a hermitian mass matrix and U a unitary matrix. Since we know the A_4 assignment we can write down M for a general VEV, it is given by

$$M = \begin{pmatrix} 0 & y\kappa_{A3} + \tilde{y}\kappa_{B3}^* & y^*\kappa_{A2} + \tilde{y}^*\kappa_{B2}^* \\ y^*\kappa_{A3} + \tilde{y}^*\kappa_{B3}^* & 0 & y\kappa_{A1} + \tilde{y}\kappa_{B1}^* \\ y\kappa_{A2} + \tilde{y}\kappa_{B2}^* & y^*\kappa_{A1} + \tilde{y}^*\kappa_{B1}^* & 0 \end{pmatrix}. \tag{6.25}$$

This is the form of the mass matrices of the up-type quarks and the neutrinos, for the down-type quarks and the charged leptons κ_A should be interchanged with κ_B. The important thing to notice is that there are zeros on the diagonal. We can use this as a constraint on the parameters of the diagonalizing matrix and the masses. We have $M = UmU^\dagger$, the expressions for the elements on the diagonal are

$$\begin{aligned} 0 =& |u_{11}|^2 m_1 + |u_{12}|^2 m_2 + |u_{13}|^2 m_3, \\ 0 =& |u_{21}|^2 m_1 + |u_{22}|^2 m_2 + |u_{23}|^2 m_3, \\ 0 =& |u_{31}|^2 m_1 + |u_{32}|^2 m_2 + |u_{33}|^2 m_3. \end{aligned} \tag{6.26}$$

Adding these three equations and remembering that U is unitary gives exactly $m_3 + m_2 + m_1 = 0$. This means that models with hermitian mass matrices lead to wrong predictions for the charged fermion masses and are not acceptable.

The left-right symmetry constrains the Yukawa matrices to be hermitian as mentioned before. The only way to construct mass matrices for the charged fermions which are not hermitian is through the use of complex VEVs. It can be seen by looking at the general mass matrix, (6.25), that complex VEVs indeed break the hermiticity.

In the previous case however, choosing κ_A and κ_B complex did not have the desired effect. With these complex VEVs we obtained $m_3 + m_2 + m_1 = 0$, where now the masses could be complex. Thus, the constraint on the masses was weakened somewhat but not nearly enough. To see that the right mass hierarchy can indeed be achieved, we will look at an example.

We will choose the following VEVs,

$$\langle\phi\rangle = \begin{pmatrix} r\kappa_A & 0 \\ 0 & r\kappa_B \end{pmatrix}, \begin{pmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{pmatrix}, \begin{pmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{pmatrix}, \tag{6.27}$$

with κ_B complex. The mass matrix of (6.25) then reduces to

$$M = \begin{pmatrix} 0 & a & b \\ b & 0 & ra \\ a & rb & 0 \end{pmatrix}, \tag{6.28}$$

with $a = y\kappa_A + \tilde{y}\kappa_B^*$ and $b = y^*\kappa_A + \tilde{y}^*\kappa_B^*$. Note that $|a|$ and $|b|$ are in principle independent parameters, they are equal when $\mathrm{Re}\, y\tilde{y}^*\kappa_A\kappa_B = \mathrm{Re}\, y^*\tilde{y}\kappa_A\kappa_B$. This is the case when both VEVs are real. Taking a and b to be independent we have the mass matrix which was used in [23] as the charged lepton mass matrix. The three parameters a, b and r were solved in terms of the three masses.

Thus the right masses can be produced in these types of models, however it is necessary to break the hermiticity of the mass matrices.

6.3 Final remarks

In this Chapter we looked at combining A_4 with a left-right symmetric model. The left-right model indeed allows for more freedom. The downside is that the Higgs sector becomes rather large. Even so, using a simple VEV alignment the correct mixing patterns can be reproduced. The model then predicts the normal hierarchy for the neutrino masses. The charged fermion masses are non-degenerate, but not sufficiently so to describe the mass hierarchy seen in nature. A different (complex) VEV alignment can help to overcome this problem, the challenge will be to find an alignment which gives the correct mass hierarchy and satisfactory mixing matrices. In conclusion, the results of this model are encouraging to further study models combining A_4 and a left-right symmetry.

7

Concluding remarks

In this thesis models with a family symmetry were discussed, in particular those using the group A_4. Family symmetries might help us in understanding the flavor mixing patterns. These are apparent in the interactions of the fermions with definite masses with the charged weak interaction. As the SM offers no explanation for these patterns such models often employ new physics. Thus, finding the right model might not only tell us how these patterns come about but might also reveal what lies beyond the SM.

In order to be able to do this, the appropriate tools were developed in the first Chapters. The group of our choice, A_4, the weak interactions of the fermions, their mass terms and their workings were discussed in detail. We also looked into the argument for finding the minimal family symmetry group, [12], which in the end did not appear to be compelling.

In the later Chapters 'simple' A_4 models with a specific Higgs sector were studied. An interesting feature of this Higgs sector is that it does not allow for any CP violation whatsoever, as was shown in Chapter 4. The study of the Higgs sector is necessary to be able to evaluate the possible models properly. Most importantly, studying the Higgs sector allowed us calculate the possible VEVs which, in part, determine the mass matrices. By looking at the Higgs masses in each of these vacua we could then decide which of these are acceptable.

After this the 'simple' A_4 models possible with this Higgs sector and up to three right-handed neutrinos were investigated. It was concluded that none of these were satisfactory for various reasons. Either the masses came out wrong or the correct mixing patterns could not be reproduced. In order to see how this might be remedied, two examples from the literature were discussed. Firstly, the LK model, [5], for which an approximately analytical expression for the CKM matrix was derived, using an assumption (namely, $r \gg 1$). Secondly, the AF model, [3], was discussed. From these models it was concluded that either explicit A_4 breaking terms or additional fields might remedy the problems of the 'simple' models. The possibility of choosing a different family symmetry was not pursued.

In the end A_4 was combined with a left-right symmetry in the hope to be able to describe the mixing patterns. The left-right symmetry uses additional Higgs fields which may help with regard to the problems of the 'simple' models. In this case the Higgs sector was not studied as it is rather large, instead VEVs which seem natural were chosen to base the model on. These simple VEVs gave rise to charged fermion masses which cannot reproduce the hierarchy seen in nature. Nonetheless, the correct mixing patterns were reproduced. This is encouraging to further study this type of model, with a different (complex) VEV alignment.

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Bibliography

[1] K. Abe *et al.* [T2K Collaboration], Phys. Rev. Lett. **107**, 041801 (2011) [arXiv:1106.2822 [hep-ex]].

[2] L. Lavoura and E. Ma, Mod. Phys. Lett. A **20**, 1217 (2005) [arXiv:hep-ph/0502181].

[3] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010) [arXiv:1002.0211 [hep-ph]].

[4] E. Ma and U. Sarkar, Phys. Rev. Lett. **80**, 5716 (1998) [arXiv:hep-ph/9802445].

[5] L. Lavoura and H. Kühböck, Eur. Phys. J. C **55**, 303 (2008) [arXiv:0711.0670 [hep-ph]].

[6] E. Ma, Mod. Phys. Lett. A **17**, 627 (2002) [arXiv:hep-ph/0203238].

[7] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B **775**, 120 (2007) [Erratum-ibid. **836**, 127 (2010)] [arXiv:hep-ph/0702194].

[8] C. Hagedorn, M. Lindner and R. N. Mohapatra, JHEP **0606**, 042 (2006) [arXiv:hep-ph/0602244].

[9] I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B **648**, 201 (2007) [arXiv:hep-ph/0607045].

[10] J. Barry and W. Rodejohann, Phys. Rev. D **81**, 093002 (2010) [Erratum-ibid. D **81**, 119901 (2010)] [arXiv:1003.2385 [hep-ph]].

[11] G. Altarelli and F. Feruglio, Nucl. Phys. B **741**, 215 (2006) [arXiv:hep-ph/0512103].

[12] C. S. Lam, Phys. Rev. D **83**, 113002 (2011) [arXiv:1104.0055 [hep-ph]].

[13] G. Altarelli, Nuovo Cim. C **32N5-6**, 91 (2009) [arXiv:0905.3265 [hep-ph]].

[14] S. Eidelman *et al.* [Particle Data Group], Phys. Lett. B **592**, 1 (2004).

[15] Giunti, Carlo, and Chung W. Kim. ”Fundamentals of Neutrino Physics and Astrophysics,” Oxford: Oxford University Press, 2007. Oxford Scholarship Online. Oxford University Press. 23 March 2011 http://dx.doi.org/10.1093/acprof:oso/9780198508717.001.0001

[16] R. Foot, H. Lew and G. C. Joshi, Phys. Rev. D **39**, 3402 (1989).

[17] R. de Adelhart Toorop, F. Bazzocchi, L. Merlo and A. Paris, JHEP **1103**, 035 (2011) [arXiv:1012.1791 [hep-ph]].

[18] C. I. Low and R. R. Volkas, Phys. Rev. D **68**, 033007 (2003) [arXiv:hep-ph/0305243].

[19] G. C. Branco, M. N. Rebelo and J. I. Silva-Marcos, Phys. Lett. B **614**, 187 (2005) [arXiv:hep-ph/0502118].

[20] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, M. Sher and J. P. Silva, arXiv:1106.0034 [hep-ph].

[21] L. Lavoura and J. P. Silva, Phys. Rev. D **50**, 4619 (1994) [arXiv:hep-ph/9404276];

[22] R. de Adelhart Toorop, F. Bazzocchi, L. Merlo and A. Paris, JHEP **1103**, 040 (2011) [arXiv:1012.2091 [hep-ph]].

[23] S. Morisi and E. Peinado, Phys. Rev. D **80**, 113011 (2009) [arXiv:0910.4389 [hep-ph]].

[24] J. Bernabeu, G. C. Branco and M. Gronau, Phys. Lett. B **169**, 243 (1986).

[25] B. Brahmachari, S. Choubey and M. Mitra, Phys. Rev. D **77**, 073008 (2008) [Erratum-ibid. D **77**, 119901 (2008)] [arXiv:0801.3554 [hep-ph]].

[26] A. Maiezza, M. Nemevsek, F. Nesti and G. Senjanovic, Phys. Rev. D **82**, 055022 (2010) [arXiv:1005.5160 [hep-ph]].

[27] N. G. Deshpande, J. F. Gunion, B. Kayser and F. I. Olness, Phys. Rev. D **44**, 837 (1991).