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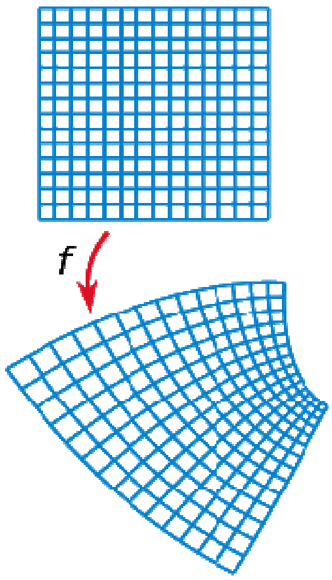
Conformal maps and the theorem of Liouville



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Student: Mirjam Soeten

First supervisor: G. Vegter

Second supervisor: H. de Snoo

Abstract

In this thesis we state and prove the theorem of Liouville. This theorem states that every conformal map in $\mathbb{R}^n$ for $n \geq 3$ is a composition of Möbius transformations.
Before proving this theorem, information is needed about inversion geometry, conformal maps and Möbius transformations. These subjects are discussed in chapters 2 to 4. In the fifth chapter the theorem of Liouville is proven. Two different proves are given. The first just holds in $\mathbb{R}^3$ because triply orthogonal systems are applied. The second is a general proof for $\mathbb{R}^n$.

The picture on the front is taken from [8].

Contents

1 Preface

In 1850 the French mathematician Joseph Liouville discovered and proved a remarkable theorem. In this paper we wish to state his theorem and also prove it. Liouville stated that every conformal map in $\mathbb{R}^n$ for $n \geq 3$ is a composition of Möbius transformations. What makes this theorem remarkable is that it doesn't hold in $\mathbb{R}^2$.
The main goal of this thesis is to explain the theories of inversion geometry, conformal maps and Möbius transformations, eventually leading to the theorem of Liouville. This paper consists of four chapters. A chapter is dedicated to each subject.
Inversion geometry is discussed in the first chapter and describes how to transform lines and circles into lines and circles. So a line can be mapped to a circle and vice versa. In chapter 2 we will discuss the general theory, the construction of inversion points, properties of inversions and at last cross ratios will appear.
After this chapter conformal maps are discussed. First, the general theory of conformal maps will be discussed, to make clear what a conformal map is. Secondly, some examples will be discussed to get familiar with the conformal maps. The examples are given in the form of theorems, and these theorems turn out to be useful in the final chapters.
The third chapter is about Möbius transformations. We will discuss two types of Möbius transformations, the general transformations and the extended transformations, where also ∞ is allowed.
After the first three chapters, we have enough information to prove the theorem of Liouville. Now we will look at two cases. First we will prove the theorem in $\mathbb{R}^3$. But before this can be done, some lemmas have to be discussed. After the proof in $\mathbb{R}^3$ is finished, we will look at the general case $\mathbb{R}^n$ for $n \geq 3$. This proof is very long and technical, and therefore a short summary of the most important steps is given.
In this paper not all the proofs of the lemmas, propositions and theorems are given. The most relevant proofs for the paper are given. When a proof has not been given, there is a reference so the reader will be able to find the proof. We expect the reader to be familiar with the basics of complex function theory and to have some knowledge of analysis. These theories are not explained in this paper.

2 Inversion geometry

In this chapter we will discuss inversion geometry. We will discuss this topic because it is necessary to understand our main theorem, the theorem of Liouville. In section 2.1 the general definitions about inversion geometry will be discussed. After this, the properties of inversions will be discussed in section 2.2. At last, in section 2.3 cross ratios will be discussed.
How the images of points under inversion can be constructed, is shown in appendix A. We will discuss inversion geometry in $\mathbb{R}^n$ because all the theory holds for $\mathbb{R}^n$ for $n \geq 3$.

2.1 Introduction

Inversion geometry is about a point p and its inversion point $I(p)$ with respect to a circle or sphere. The interesting part in the inversion geometry is how points p behave under inversion. Before we are going to talk about the behavior of inversion points, we have to define them. See chapter 5.1 in [2].

Definition 2.1. *The inversion point $I(p)$ of p is defined as the point $I(p)$ on the plane through a and p such that*

$$|p - a|\,|I(p) - a| = r^2 \tag{1}$$

where I is a map $I : \mathbb{R}^n \setminus \{a\} \to \mathbb{R}^n \setminus \{a\}$ (see chapter 4.1 in [3]), and $|p - a|$ denotes the Euclidian distance between p and a. If p lies in the interior of $\mathcal{C}$, then $I(p)$ lies in the exterior of $\mathcal{C}$ and vice versa.

In the rest of this chapter, we will look at the inversion point with respect to the sphere $\mathcal{C}$ with center a and radius r. Here a is a point $(a_1, a_2, ..., a_n) \in \mathbb{R}^n$. Furthermore, p and $I(p)$ also are points in $\mathbb{R}^n$.
With only definition 2.1, we don't know how to find $I(p)$ yet. Therefore, we can find an explicit formula for $I(p)$. To find this formula, we use formula (1).
We know that p and $I(p)$ are on the same line , so $I(p)$ is a multiple of the distance between p and a, so $I(p) = a + \lambda(p - a)$ with λ a constant which we want to know. With this information, a straightforward computation shows that $I(p)$ is given by

$$I(p) = a + \left(\frac{r}{|p - a|}\right)^2 (p - a), \text{ [3], chapter 4.1} \tag{2}$$

Beside this formula there exists an algebraic function that gives $I(p)$ in coordinates in the unit sphere. This function $f : \mathbb{R}^n \setminus \{(0, ..., 0)\} \to \mathbb{R}^n$ is given by

$$\begin{aligned} I(p) &= f(x_1, x_2, ..., x_n) \\ &= \left(\frac{x_1}{x_1^2 + x_2^2 + ... + x_n^2}, \frac{x_2}{x_1^2 + x_2^2 + ... + x_n^2}, ..., \frac{x_n}{x_1^2 + x_2^2 + ... + x_n^2}\right) \end{aligned} \tag{3}$$

The derivation of this formula is again a straightforward computation, where we have used that for the point p with coordinates $(x_1, ..., x_n)$ it must hold that $I(p)$ has coordinates $(kx_1, ..., kx_n)$ with k the unknown constant. See chapter

5.1 in [2].
If $\mathcal{C}$ is not the unit sphere, we can use the same argument to get that

$$\begin{aligned} I(p) &= f(x_1, x_2, ..., x_n) \\ &= \left(\frac{r^2 x_1}{x_1^2 + x_2^2 + ... + x_n^2}, \frac{r^2 x_2}{x_1^2 + x_2^2 + ... + x_n^2}, ..., \frac{r^2 x_n}{x_1^2 + x_2^2 + ... + x_n^2} \right) \end{aligned} \tag{4}$$

2.2 Properties of inversions

In inversion geometry, there are some properties of inversions that we will use in the next chapters. Therefore, they will be given here. The first and second proposition sometimes are referred to as the basic properties of inversions, [1], chapter 2.1. The third proposition is one about the behavior of inversions themselves, [3], chapter 4.1. The first proposition gives us the images under inversions of spheres and planes, so that we know what the inversion of a sphere or a plane looks like.

Proposition 2.1. *For the inversion sphere $\mathcal{C}$ with center* a *we have the following properties about inversions of a sphere or a hyperplane.*

- *The image of a hyperplane through* a *under inversion is the hyperplane itself*
- *The image of a hyperplane not through* a *under inversion is a sphere through* a
- *The image of a sphere through* a *under inversion is a hyperplane not through* a
- *The image of a sphere not through* a *under inversion is a sphere not through* a

The proof of this proposition is not relevant for our main theorem, therefore we refer to [1] for the proof.
The other basic property of inversions is stated in the following proposition.

Proposition 2.2. *Any circle through a pair of inversion points is orthogonal to the circle of inversion, and, conversely, any circle cutting the circle of inversion orthogonally and passing through a point* p *passes through its inversion point* $I(p)$.

For the proof of this proposition we refer again to [1]. The third proposition in this section is about the properties of the inversion itself.

Proposition 2.3. *For a point* p *and its inversion point $I(p)$ with respect to a circle $\mathcal{C}$ with center* a *and radius* r *we have the following properties*

- $I(p) = p$ *iff* $p \in \mathcal{C}(a, r)$
- $I^2(p) = p$ *for all* $x \neq a$
- *For two inversion points $I(p)$ and $I(q)$ we have*

$$|I(p) - I(q)| = r^2 \frac{|p - q|}{|p - a|\,|q - a|}$$

for all $x, y \neq a$

For the proof of this proposition, we refer to [3].
So from these three propositions we know how inversions behave.

2.3 Cross ratios

In this section we will discuss the cross ratio. This cross ratio is relevant for this chapter and the next chapters because cross ratios play a role in inversion geometry and Möbius transformations.
We will define cross ratios in the complex case, since this will return in chapter 4 about Möbius transformations.
First, we give the definition of the cross ratio in the complex case, see [1], chapter 2.4.

Definition 2.2. *The cross ratio of four points z_1, z_2, z_3, $z_4 \in \mathbb{C}$ is given by*

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

An interesting property of the cross ratio in this chapter is the following proposition.

Proposition 2.4. *Let z_i, $i = 1, 2, 3, 4$ be four points in $\mathbb{C}$. Let z_i' be the inversion point of z_i with respect to $\mathcal{C}$. Then*

$$(z_1', z_2', z_3', z_4') = \overline{(z_1, z_2, z_3, z_4)}$$

To see this, it is enough to know that in the complex case, inversion in a point z with respect to a circle $\mathcal{C}$ with center z_0 and radius r is given by $z' = z_0 + \frac{r^2}{\bar{z} - z_0}$. The proposition now follows from direct computations. Another property of the cross ratio is given in the next proposition.

Proposition 2.5. *The cross ratio of four points is real iff the four points are collinear or concyclic*

In this proposition concyclic means that the points lie on the same circle. The proof of this proposition is not relevant here. For the proof we refer to [1]. The cross ratios will appear to be interesting in Möbius transformations, see chapter 4.
For more information about the proofs of proposition 2.4 and lemma 2.5, see [1] chapter 2.4.

3 Conformal maps

In this chapter, we will discuss a special kind of maps, called conformal maps. This type of map is important for our main theorem, Liouvilles theorem.
We will discuss some general theory about conformal maps in section 3.1. The sections 3.2 till 3.7 contain examples of conformal maps. In these examples, some new definitions will be discussed.

3.1 Introduction

Briefly, a conformal map is a map that preserves angles. So if we have two surfaces, S_1 and S_2, take the map $\phi : S_1 \to S_2$. Take two curves $\gamma_1(t)$ and $\gamma_2(t)$ on S_1, where these curves intersect each other with angle θ in point p. Then ϕ is a conformal map if $\phi \circ \gamma_1(t)$ and $\phi \circ \gamma_2(t)$ intersect each other with the same angle θ in the point $\phi(p)$, [8].
When we know this, a question arises. Is there an easier way to see if a map is conformal? To see this, we have the following proposition.

Proposition 3.1. *Let U be an open subset of $\mathbb{R}^n$ with a C^1-function $\phi : U \to \mathbb{R}^n$. Then ϕ is conformal iff there exists a function $\kappa : U \to \mathbb{R}$ such that $\kappa(x)^{-1}\phi'(x)$ is an orthogonal matrix for all x in U, where $\phi'(x)$ is the Jacobian matrix of ϕ in x. We call κ the scale factor of ϕ.*

Before we can use this proposition, we need to know when there exists such a κ. To determine this, we can use lemma 3.1.

Lemma 3.1. *Let A be a real $n \times n$ matrix. Then there exists a positive scalar k such that $k^{-1}A$ is an orthogonal matrix iff the linear map with matrix A preserves angles between nonzero vectors.*

For the proofs of above proposition and lemma we refer to [3], chapter 4.1. In general, this is how can be detected if a map is a conformal map. It is useful to look at some examples, to get familiar with them and with their conformality. The following examples are also useful for our main theorem, Liouvilles theorem.

3.2 Inner products and differential forms

In this short section, we will give a lemma to check if a function f is conformal. We will give this lemma because we will need it in the next section about conformality of inversions.
The lemma in this section is about inner products.

Lemma 3.2. *Let $f : U \to f(U)$ be a one-to-one map where $U \subset \mathbb{R}^n$ is such that df_x is nonsingular for all $x \in U$. Then f is conformal iff for all vectors $v, w \in \mathbb{R}^n$, $< df_x v, df_x w >= e^{2\sigma(x)} < v, w >$ for a real function σ on U.*

The function $e^{2\sigma(x)}$ is called the conformality factor of the function f.
We refer to [1], chapter 3.3, for the proof.

3.3 Inversion in circle and sphere

In chapter 2 we discussed the general theory of inversion in circle and sphere. So in this section, we will only look at the conformality of the inversions.

Theorem 3.1. *Inversion in a circle or a sphere is a conformal map.*

To prove this, we use lemma 3.2 from the previous section. The proof will be given for inversion in a circle and after this, we will explain how it can be extended to higher dimensions.

Proof. Without loss of generality we consider inversion I in the unit circle. We know that in the complex case inversion in the unit circle is given by $I(z) = \frac{1}{\bar{z}} = \frac{z}{|z|^2}$, or in Cartesian coordinates, $I(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$.
The derivative of I at (x,y) has matrix J given by

$$J = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix}$$

Since

$$\begin{aligned} J^TJ &= \frac{1}{(x^2+y^2)^4}\begin{pmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix} \cdot \begin{pmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix} \\ &= \frac{1}{(x^2+y^2)^4}\begin{pmatrix} (x^2+y^2)^2 & 0 \\ 0 & (x^2+y^2)^2 \end{pmatrix} \\ &= \frac{1}{(x^2+y^2)^2} \cdot Id \end{aligned}$$

we see that for two vectors v and w

$$\begin{aligned} \langle Jv, Jw\rangle &= v^TJ^TJw \\ &= v^T \cdot \frac{1}{(x^2+y^2)^2} \cdot Id \cdot w \\ &= \frac{1}{(x^2+y^2)^2}\langle v, w\rangle \end{aligned}$$

And thus we have $\langle Jv, Jw\rangle = \lambda(x,y)\langle v, w\rangle$ with $\lambda(x,y) = \frac{1}{(x^2+y^2)^2}$, so according to lemma 3.2 I is conformal, so inversion in a circle is conformal. □

This proof can be extended to $\mathbb{R}^n$ with $n \geq 3$ by taking

$$\begin{aligned} I(x_1, x_2, ..., x_n) &= \left(\frac{x_1}{x_1^2+x_2^2+...+x_n^2}, \frac{x_2}{x_1^2+x_2^2+...+x_n^2}, ..., \frac{x_n}{x_1^2+x_2^2+...+x_n^2}\right) \\ &= (p_1, p_2, ..., p_n) \end{aligned}$$

and finding a $\lambda = \lambda(x_1, x_2, ..., x_n)$ by computing the matrix of DI again, which is now an $n \times n$ matrix. The rest of the proof remains the same. So indeed inversion in a circle or sphere is a conformal map.

3.4 Stereographic projection

In this section, we will talk about the stereographic projection as a conformal map. Before we can do this we need the definition of a stereographic projection. This requires some knowledge of the extended complex plane and the Riemann sphere, which will be dealt with. At the end of this section we will arrive at

the conformality of the stereographic projection. The theory in this section is mainly coming from [2], chapter 5.2.
Stereographic projection is introduced to give us a way to visualize the point ∞ and to find its image under inversion. Stereographic projection, denoted by the map π, projects the complex plane $\mathbb{C}$ to a so called Riemann sphere. This is a sphere $\mathbb{S}$ with center $(0,0,0)$ and radius 1. To make the projection, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the map $x + iy \mapsto (x, y)$. Now each point p in $\mathbb{C}$ can be related to a point p' on $\mathbb{S}$ by drawing a line from the North Pole $(0,0,1)$ through p. Where this line intersects $\mathbb{S}$, lies p', see figure 1. The only point on the sphere that will never be reached is the North pole itself, this we will relate to ∞ on the complex plane. This means that $\mathbb{C}$ is extended by a point that is related to the North Pole.

Definition 3.1. *The extended complex plane is defined as* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

This process of relating two points can be carried out via the stereographic projection π, which is the map $\pi : \mathbb{S} \to \widehat{\mathbb{C}}$ given by

$$\pi(X, Y, Z) = \frac{X}{1-Z} + i\frac{Y}{1-Z}$$

for a point (X, Y, Z) on $\mathbb{S}$. Conversely, the map $\pi^{-1} : \widehat{\mathbb{C}} \to \mathbb{S}$ is given by

$$\pi^{-1}(x + iy) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)$$

for a point $x + iy$ in $\widehat{\mathbb{C}}$.
These formulas are the algebraic way to say that a line is drawn from the North Pole to p, which was already mentioned above. Furthermore, with these formulas it is easier to see that we can relate the North Pole to ∞. If we take the point $(X, Y, Z) = (0, 0, 1)$, then $\pi(X, Y, Z) = \infty$, so indeed the North Pole is connected to the point ∞.

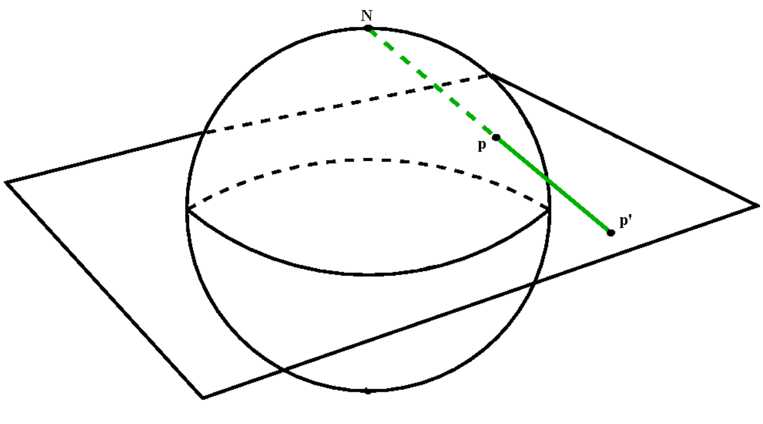

Figure 1: Sterographic projection, [12]

Now we know everything we need about stereographic projection. We are now ready to look at the conformality.

Theorem 3.2. *Stereographic projection is a conformal map.*

In this theorem, we have the term conformal on spheres. Then we mean by conformal that on the sphere we have to look at the tangent lines of two curves in a point, and the images of these tangent lines intersect each other with the same angles as the tangent lines did.

Then we mean by conformal that the tangent lines in a point are angle preserving.
The proof is not relevant for the main theorem. Therefore, the proof will not be given here. For the proof we refer to [1], chapter 2.2.

3.5 Möbius transformation

In this section, we will give a short overview of Möbius transformations and we will see that a Möbius transformation is a conformal map, see [1] chapter 2.3. In chapter 4 we will look at Möbius transformation in more detail.
A Möbius transformation is a transformation of the form

$$M(z) = \frac{az+b}{cz+d}$$

Where a, b, c, $d \in \mathbb{C}$ and with $ad - bc \neq 0$. Furthermore, M is a map $M : \mathbb{C} \to \mathbb{C}$. This map M can be extended to $M : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by defining $M(-\frac{d}{c}) = \infty$ and $M(\infty) = \frac{a}{c}$.
In this section we are only interested in the conformality of Möbius transformations.

Theorem 3.3. *Möbius transformations are conformal maps.*

Before we can prove this theorem, we need the following lemma.

Lemma 3.3. *A Möbius transformation is the composition of a translation, inversion, reflection with rotation, and dilation, [7].*

In this lemma, we see the term dilation. To understand this term, we use the following definition.

Definition 3.2. *A dilation is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ of the form $f(x) = s + \xi(x - s)$ where ξ is a nonzero scalar and s is a fixed point,* [14].

Now we know this, we can proof the lemma.

Proof of lemma 4.2. To see this, take four functions.

$$\begin{aligned}
M_1(z) &= z + \frac{d}{c} \\
M_2(z) &= \frac{1}{z} \\
M_3(z) &= -\frac{(ad - bc)}{c^2} z \\
M_4(z) &= z + \frac{a}{c}
\end{aligned}$$

where

- M_1 is a translation by $\frac{d}{c}$
- M_2 is an inversion and reflection with respect to the real axis
- M_3 is a dilation and rotation
- M_4 is a translation by $\frac{a}{c}$

Now an easy computation shows us that indeed $M_4 \circ M_3 \circ M_2 \circ M_1(z) = \frac{az+b}{cz+d}$. □

Now we can prove that a Möbius transformation is conformal.

Proof of theorem 3.3. To prove the theorem it is enough to show that each M_i, with $i = 1, 2, 3, 4$ is conformal, since the composition of conformal maps is again conformal.
M_1 and M_4 are conformal because a translation is a conformal map, since the angle between two curves doesn't change when these curves are translated.
M_2 is conformal since both inversion and reflection are conformal. That inversion is conformal, is discussed in the previous section. Also reflection is conformal, since it doesn't change the angle between two curves. Finally, M_3 is conformal. This is true because dilation is nothing more than scalar multiplication, and this doesn't change the angle between to curves. Also rotation doesn't change the angle, so also M_3 is conformal. This means that also the composition of the M_i is conformal, and thus $M(z)$ is conformal. □

3.6 Anti-Homographies

In this section, we first will give the definition of an anti-homography. After that, we will look at the conformality of the anti-homographies.
An anti-homography is a transformation that looks like a Möbius transformation, only with $\bar{z}$ instead of z. So an anti-homography $W : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is defined as

$$W(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$$

Since an inversion in the unit sphere in the complex case is given by $w = \frac{1}{\bar{z}}$, inversion is included in the set of anti-homographies. The most important thing we can say about anti-homographies is the next theorem.

Theorem 3.4. *Anti-homographies are conformal maps.*

This result is not very difficult to see, since an anti-homography is a special type of Möbius transformation, and Möbius transformations are conformal as we have shown in section 3.5.
Together with the homographies or Möbius transformations, the anti-homographies form a group, which maps lines and circles to lines and circles, see [1], chapter 2.6. See for more information chapter 4

3.7 (Anti-)Holomorphic functions

In this section, we first give the definitions of a holomorphic function and a anti-holomorphic function. After that, we will look at the conformality of these functions.

A holomorphic function is a complex-valued function that is complex differentiable in every point of $\mathbb{C}$. An anti-holomorphic function z is a function that is differentiable with respect to the complex conjugate $\bar{z}$.
We also can define holomorphic and anti-holomorphic in terms of the Cauchy-Riemann equations. For a function $f = u+iv$, where $u = u(x, y)$ and $v = v(x, y)$ are real valued functions, we can say that

1. f is holomorphic iff f satisfies $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

2. f is anti-holomorphic iff f satisfies $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

This last property can be used to check if a function f is conformal. Therefore, we have the following theorem:

Theorem 3.5. *Take $f : \mathbb{R}^2 \to \mathbb{R}^2$ a function of class C^1 with a nonvanishing Jacobian. Then f as a map is conformal iff f as a function of $z \in \mathbb{C}$ is holomorphic or anti-holomorphic.*

The proof of this theorem uses the Cauchy-Riemann equations. Since the proof of this theorem consists of a lot of computations and isn't relevant for our main theorem, the proof will not be given here. For the proof, see [1] chapter 4.2.

4 Möbius Transformations

In this chapter, we will discuss the Möbius transformations in detail.
In section 4.1, we will discuss the general Möbius transformations. In this section we used [1] chapter 2.3 and 2.4, [2] chapter 5.3 and [3] chapter 4.3. The section is devided in two subsections, the first subsection is about Möbius transformations in $\mathbb{R}^2$, the second subsection is about Möbius transformatons in $\mathbb{R}^n$. In section 4.2, we will see another type of Möbius transformations, the extended Möbius transformations, here we used [1] chapter 2.6.

4.1 General Möbius transformations

4.1.1 Möbius in $\mathbb{R}^2$

We have already defined Möbius transformation in chapter 3.5 and we have seen that this transformations are conformal. In this subsection, we will see the Möbius transformations in more detail.
First, we have the following lemma.

Lemma 4.1. *A Möbius transformation $M(z) = \frac{az+b}{cz+d}$ in $\mathbb{R}^2$ is the composition of inversions in spheres.*

Before we can prove the lemma, we need two new functions, the extended linear function and the extended reciprocal function, [2].

Definition 4.1. *An extended linear function is a function of the form*

$$t(z) = az + b$$

where z, a, $b \in \widehat{\mathbb{C}}$ and $a \neq 0$.
The extended linear function can be decomposed into $t = t_2 \circ t_1$ where

- *$t_1(z) = |a|\, z$ is a scaling*
- *$t_2(z) = \frac{a}{|a|} z + b$ is an isometry*

Definition 4.2. *The extended reciprocal function is a funtion t given by*

$$t(z) = \frac{1}{z}$$

where $z \in \widehat{\mathbb{C}} \setminus \{0\}$.
The extended reciprocal function can be decomposed into $t = t_2 \circ t_1$ where

- *$t_1(z) = \frac{1}{\bar{z}}$ is an inversion*
- *$t_2(z) = \bar{z}$ is a conjugation*

Now we know this, we can prove the lemma, [2].

Proof. We distinguish two cases, the case $c = 0$ and the case $c \neq 0$.
First, if $c = 0$ we can say that M is an extended linear function, and therefore

it is a composition of inversions in spheres.
Now assume $c \neq 0$. Then we can write for $z \in \widehat{\mathbb{C}} \setminus \{-\frac{d}{c}\}$ that

$$\begin{aligned} M(z) &= \frac{a(cz+d) - ad + bc}{c(cz+d)} \\ &= -\left(\frac{ad-bc}{c}\right) \cdot \left(\frac{1}{cz+d}\right) + \frac{a}{c} \end{aligned}$$

So we can write M as the composition $t_3 \circ t_2 \circ t_1$ where t_2 is the extended reciprocal function, and thus a composition of inversions. Furthermore, t_1 and t_3 are the extended linear functions given by

$$t_1(z) = \begin{cases} cz+d & \text{if} \quad z \neq \infty \\ \infty & \text{if} \quad z = \infty \end{cases}$$

and

$$t_2(z) = \begin{cases} -\left(\frac{ad-bc}{c}\right) z + \frac{a}{c} & \text{if} \quad z \neq \infty \\ \infty & \text{if} \quad z = \infty \end{cases}$$

Also the extended linear functions are a composition of inversions, and therefore, since both t_1 and t_2 as well as t_3 are compositions of inversions, it must hold that $M(z)$ is a composition of inversions as well, which we wanted to prove. $\square$

For a Möbius transformation the following lemma holds.

Lemma 4.2. *The composition of two Möbius transformations is again a Möbius transformation, [1].*

To see this, take two Möbius transformations given by

$$M_1(z) = \frac{az+b}{cz+d}$$

and

$$M_2(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

and compute the composition $M_2 \circ M_1$. It is easy to see that this again is a Möbius transformation.
We also can prove this lemma by lemma 4.1. Since M_1 and M_2 are a finite composition of inversions, it must hold that $M = M_2 \circ M_1$ is a finite composition of inversions as well, and therefore a Möbius transformation.
Another way to compute the composition of M_1 and M_2 is to take the associated matrix of the Möbius transformation. This associated matrix is defined as follow.

Definition 4.3. *For a Möbius transformation $M(z) = \frac{az+b}{cz+d}$ the matrix A given by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix associated with $M(z)$, see [2].

To compute the composition of M_1 and M_2 with associated matrices A_1 and A_2 we can just compute the product A_1A_2. This matrix product is now the associated matrix of the composition of M_1 and M_2.
Since this is not completely trivial, we will show that this result is true. Take two Möbius transformations $M_1 = \frac{a_1z+b_1}{c_1z+d_1}$ and $M_2 = \frac{a_2z+b_2}{c_2z+d_2}$ with associated matrices A_1 and A_2 respectively, where A_1 and A_2 are given by

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Then, with an easy computation we can see that the composition of M_1 and M_2 is given by

$$\begin{aligned} M_2 \circ M_1(z) &= M_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right) \\ &= \frac{(a_2a_1+b_2c_1)z+(a_2b_1+b_2d_1)}{(c_2a_1+d_2c_1)z+(c_2b_1+d_2d_1)} \end{aligned} \tag{5}$$

which has associated matrix

$$A = \begin{pmatrix} a_2a_1+b_2c_1 & a_2b_1+b_2d_1 \\ c_2a_1+d_2c_1 & c_2b_1+d_2d_1 \end{pmatrix}$$

With another computation it follows easily that $A_2A_1 = A$, so indeed to compute the compostion of Möbius transformations it is enough to compute the product of the associated matrices.
Now lemma 4.2 immediately yields the following lemma.

Lemma 4.3. *The Möbius transformations form a group*

With this lemma, we can also conclude that the inverse of a Möbius transformation can be computed with help of the associated matrix, and we get

$$M^{-1}(z) = \frac{dz-b}{a-cz}$$

In the theory about Möbius transformations, the cross ratios play a role, because of the following lemma.

Lemma 4.4. *The cross ratio of four points is invariant under a Möbius transformation.*

To prove this lemma, we take four distinct points z_i, $i = 1, 2, 3, 4$ and we take $M(z_i)$ the images of the z_i under a Möbius transformation $M(z)$. The lemma now follows from direct computation of the cross ratio of the $M(z_i)$. For the complete proof, see [1].
So now we have seen the following properties of Möbius transformations:

- Möbius transformations are conformal maps
- The composition of two (or more) Möbius transformations is again a Möbius transformation
- Möbius transformations form a group
- The cross ratio is invariant under a Möbius transformation

4.1.2 Möbius in $\mathbb{R}^n$

In this subsection, we will give the definition of a Möbius transformation in $\mathbb{R}^n$, and we will check that the properties stated in section 4.1.1 also hold for $\mathbb{R}^n$.

Definition 4.4. *A Möbius transformation in $\mathbb{R}^n$ is a finite composition of inversions of $\mathbb{R}^n$ in spheres, [3].*

So with this definition, we have generalized lemma 4.1 to a definition in $\mathbb{R}^n$. Now we want to check if all the properties in section 4.1.1 also hold for this definition of a Möbius transformation.

- Möbius transformations are conformal maps

This property holds for $\mathbb{R}^n$. We know that every Möbius transformation is a composition of inversions in spheres, and every inversion is conformal, so their composition is conformal as well, and thus every Möbius transformation in $\mathbb{R}^n$ is conformal.

- The composition of two (or more) Möbius transformations is again a Möbius transformation

This property is also valid in $\mathbb{R}^n$. Take an arbitrary number of Möbius transformations given by $M_1 = I_{m_1} \circ I_{m_2} \circ ... \circ I_{m_k}$, $M_2 = I_{n_1} \circ I_{n_2} \circ ... \circ I_{n_l}$,..., $M_q = I_{j_1} \circ I_{j_2} \circ ... \circ I_{j_p}$, where all the I_{k_i} are inversions in spheres. Then the composition $M = M_q \circ ... \circ M_1$ is also a composition of inversions, and therefore again a Möbius transformations.

- Möbius transformations form a group under composition

Also this property holds in $\mathbb{R}^n$. That the Möbius transformations form a group yields from the previous property if we can show that a Möbius transformation in $\mathbb{R}^n$ has an inverse. So the only thing we have to do is find the inverse of $M = I_1 \circ I_2 \circ ... \circ I_m$. Then for this Möbius transformation, the inverse is given by $M^{-1} = I_m \circ I_{m-1} \circ ... \circ I_1$, because then it holds that $M^{-1} \circ M = Id$ with Id the identity.

- The cross ratio is invariant under a Möbius transformation

The cross ratio only holds in $\mathbb{R}^2$ or $\widehat{\mathbb{C}}$, and therefore we don't have to check this property in this section.
Therefore, all the necessary properties also hold in $\mathbb{R}^n$.
The associated matrix is very difficult to extend to $\mathbb{R}^n$, and therefore we will not go into this subject here.

4.2 Extended Möbius transformations

In this subsection we will look at a special group containing the Möbius transformations and the anti-homographies. In section 3.6 we have already seen the anti-homographies, but in this section we will look to these maps in more detail, and to the group they form together with the homographies or Möbius transformations.
In the previous section, we have seen the definition of a Möbius transformation, or a homography. The definition of an anti-homography is the following, like we have seen in section 3.6.

Definition 4.5. *A map $W : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by*

$$W(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$$

Where a, b, c, $d \in \mathbb{C}$ and with $ad - bc \neq 0$, is called an anti-homography.

From Möbius transformations we know that they leave the cross ratios invariant. For anti-homographies, this works a little different. If we have an anti-homography $W(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, with (z_1, z_2, z_3, z_4) is the cross ratio of the z_i, and (w_1, w_2, w_3, w_4) is the cross ratio of the image of the z_i under the anti-homography, then

$$(w_1, w_2, w_3, w_4) = \overline{(z_1, z_2, z_3, z_4)}$$

Now we know what anti-homographies are, we can look at the extended Möbius transformations.

Definition 4.6. *The group formed by the set of all homographies and anti-homographies is called the group of extended Möbius transformations.*

So the extended Möbius transformations consists of the homographies and the anti-homographies. Since both Möbius transformations and anti-homographies map lines and circles to lines and circles, we have two results following from the theory of Carathéodory, see [10]. First, every 1-1 circle-preserving map of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ is an extended Möbius transformation. Furthermore, if we have a plane region R and a set R' such that every circle lying in R is a line or circle in R', then every 1-1 map from R to R' is an extended Möbius transformation.

5 Liouville's Theorem

In this section, we will study our main theorem, Liouville's theorem. In section 5.1 we will state the theorem. In section 5.2, the proof of the theorem in $\mathbb{R}^3$ will be given. Before we can prove the theorem, we need some lemmas and definitions. These will also be given in section 5.2. In sections 5.1 and 5.2, the theorem and the proof are in three dimensions. In section 5.3, we will generalize this to n dimensions. In the last section, section 5.4, we will give a counterexample of the theorem of Liouville in $\mathbb{R}^2$.

5.1 Liouville

The theorem of Liouville is stated as follow.

Theorem 5.1 (Liouville's theorem in $\mathbb{R}^3$)**.** *Let $f : U \to f(U)$ be a one-to-one C^3 conformal map, where $U \in \mathbb{R}^3$ is open. Then f is a composition of similarities and inversions.*

In this theorem we see a new term, namely a similarity. So before we proceed to the proof of the theorem, we need to know what a similarity is.

Definition 5.1. *A function f from a metric space to the same metric space is a similarity if*

$$d(f(x), f(y)) = rd(x, y)$$

for a positive scalar r, [13].

So Liouville proved that every conformal map is a composition of Möbius transformations. This is remarkable, since this is not true in two dimensions as we will show in section 5.4.

5.2 Proof

In this section, we will give the proof of Liouville's theorem in $\mathbb{R}^3$. For the proofs of the required lemmas, we used [5] chapter 4 and chapter 2. All lemmas and proofs can be found in here, except the proof of the lemma of Dupin. This can be founded in [1] chapter 6.2.

5.2.1 Lemma of Dupin

In this subsection, we will discuss the first lemma we need, the lemma of Dupin.

Lemma 5.1 (Lemma of Dupin)**.** *The surfaces of a triply orhtogonal system intersect each other in the lines of curvature.*

In this lemma, we see two new terms, namely a triply orthogonal system and lines of curvature.

Definition 5.2. *A triply orthogonal system consists of three families of surfaces in an open set in $\mathbb{R}^3$ with one surface from each family passing through each point and such that the tangent planes at each point are mutually perpendicular.*

So if we look at a point p in a triply orthogonal system with families of surfaces K_1, K_2 and K_3, there is a surface from each of the K_i passing through p. The surface that is coming from the family K_i is called k_i. Furthermore, these surfaces k_i are perpendicular to each other, so k_i is perpendicular to k_j, for $i = 1, 2, 3$, $j = 1, 2, 3$ and $i \neq j$.
To make the concept of a triply orthogonal system more clear, we will give an example. A triply orthogonal system is a system where the first family consists of all planes parallel to the (x, y)-plane, the second family consists of all the circular cylinders having the z-axis as their common axis, and the third family consists of all planes that pass through the z-axis. We then get the following picture for our example.

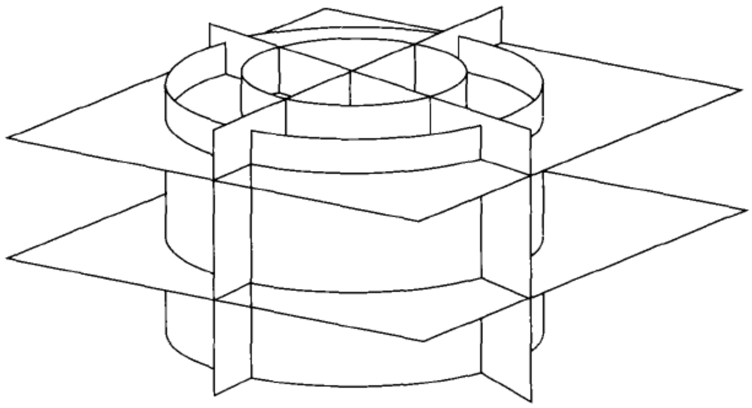

Figure 2: An example of a triply orthogonal system, [5]

The other unknown term is the line of curvature. To define a line of curvature, take a surface $K \subset \mathbb{R}^3$ with curve x on this surface.

Definition 5.3. *A curve x is a line of curvature of a surface K if its derivative always points along a principal direction.*

Furthermore, a curve x is a line of curvature if and only if its geodesic torsion τ_g is zero along the curve, where τ_g is defined as

$$\tau_g = \langle \frac{d\mathbf{n}}{ds}, \mathbf{v} \rangle = -\langle A\mathbf{T}, \mathbf{v} \rangle$$

for $\mathbf{v} = \mathbf{n} \times \mathbf{T}$, $\mathbf{n}$ the surface normal and A the Weingarten map, defined as

$$A\mathbf{v} = -\frac{d\mathbf{n}}{ds}$$

for $A : T_pK \to T_pK$. This map is also called the shape operator, [6] chapter 2 and chapter 5. Now we know what Dupin's lemma says, we can prove this lemma. For the proof we used [1] chapter 6.2.

Proof. First, we take three surfaces K_1, K_2 and K_3, where each surface is coming from a family. Since in a triply orthogonal system each family intersect with the others, we have curves x_i on the intersections of the surfaces, so we have that

- x_1 is the curve parametrized by arc length on $K_2 \cap K_3$
- x_2 is the curve parametrized by arc length on $K_3 \cap K_1$
- x_3 is the curve parametrized by arc length on $K_1 \cap K_2$

To show that the x_i are lines of curvature, we want to show that the geodesic torsion of each x_i is zero. Therefore, define $\mathbf{v}_{ab} = \mathbf{n}_b \times \mathbf{T}_a$ for $a, b = 1, 2, 3$. Here a is the index that refers to the number of the curve, so wich x_i is used, and b is the index that refers to the surface that is used, so b refers to which K_i is used. Since $\mathbf{T}_a$ and $\mathbf{n}_a$ are parallel, we can say that $\mathbf{v}_{ab} = \mathbf{n}_b \times \mathbf{n}_a = \pm \mathbf{n}_c$, where $c \neq a, b$.
Now let's consider $x_1 = K_2 \cap K_3$. First take x_1 as a curve on K_2, then we have

$$\mathbf{v}_{12} = \mathbf{n}_2 \times \mathbf{T}_1 = -\mathbf{n}_3$$

And for x_1 as a curve on K_3 we have

$$\mathbf{v}_{13} = \mathbf{n}_3 \times \mathbf{T}_1 = \mathbf{n}_2$$

Now we can compute the geodesic torsion on K_2.

$$\begin{aligned} \langle \frac{d\mathbf{n}_2}{ds}, \mathbf{v}_{12} \rangle &= \langle \frac{d\mathbf{n}_2}{ds}, -\mathbf{n}_3 \rangle \\ &\overset{(*)}{=} \langle \mathbf{n}_2, \frac{d\mathbf{n}_3}{ds} \rangle \\ &= \langle \mathbf{v}_{13}, \frac{d\mathbf{n}_3}{ds} \rangle \end{aligned} \tag{6}$$

Where in (*) in the second step of (6) we used the fact that $0 = \frac{d}{ds}\langle \mathbf{n}_2, \mathbf{n}_3 \rangle$, the rest follows from an easy computation. So in (6) we can see that the geodesic torsion on K_2 is equal to the geodesic torsion on K_3. We call this torsion τ_g^1. Furthermore, if we use the Weingarten map, we can say that for τ_g^1

$$\tau_g^1 = \langle \mathbf{n}_2, \frac{d\mathbf{n}_3}{ds} \rangle = -\langle A_3 \mathbf{T}_1, \mathbf{n}_2 \rangle$$

Similarly we can say that

$$\begin{aligned} \tau_g^2 &= \langle \frac{d\mathbf{n}_3}{ds}, \mathbf{v}_{23} \rangle \\ &= -\langle \frac{d\mathbf{n}_3}{ds}, \mathbf{n}_1 \rangle \\ &= \langle A_3 \mathbf{T}_2, \mathbf{n}_1 \rangle \end{aligned} \tag{7}$$

So in the point of intersection p we know that

$$\tau_g^1 + \tau_g^2 = -\langle A_3 \mathbf{T}_1, \mathbf{n}_2 \rangle + \langle A_3 \mathbf{T}_2, \mathbf{n}_1 \rangle$$

Furthermore, $\mathbf{T}_a = \mathbf{n}_a$, and with the symmetry of the Weingarten map A_3 we can say that

$$\tau_g^1 + \tau_g^2 = -\langle A_3 \mathbf{n}_1, \mathbf{n}_2 \rangle + \langle A_3 \mathbf{n}_2, \mathbf{n}_1 \rangle = -\langle A_3 \mathbf{n}_1, \mathbf{n}_2 \rangle + \langle A_3 \mathbf{n}_1, \mathbf{n}_2 \rangle = 0$$

In the same way,

$$\tau_g^2 + \tau_g^3 = 0$$
$$\tau_g^3 + \tau_g^1 = 0$$

So $\tau_g^1 = \tau_g^2 = \tau_g^3 = 0$.
So x_1, x_2 and x_3 are lines of curvature, and since the x_i were also lines of intersection, we have now proved that the lines of intersections in a triply orthogonal system are lines of curvature. □

5.2.2 Lemma of Möbius

The second lemma we need to prove Liouville's theorem, is the lemma of Möbius. This lemma will be discussed in this section.

Lemma 5.2 (Lemma of Möbius)**.** *Take U and V open sets with U,$V \subset \mathbb{R}^3$ and U a connected set. If $f : U \to V$ is a map which takes parts of spheres and planes to parts of spheres and planes, then f is a composition of similarities and inversions, in fact at most one of each.*

Before we give the proof of this lemma, we give some general information that we will need in the proof.
Suppose we have a sphere S' with center p and a sphere S with the point $p \in S$, but p not necessarily the center of S. We take I' the inversion in the sphere S'. From the lemma, we now that $I'(S \setminus \{p\})$ is a sphere or a plane. Then we can conclude that $I'(s \setminus \{p\})$ is a plane and not a sphere. To see this, define $I'(S \setminus \{p\}) = H$ and suppose H is a sphere. Then H is compact. If H is compact, then $\infty \in H$. And ∞ is the inversion point of p, so the inversion point of p is in H. But we don't take the inversion of p since p is the center of the circle of inversion. So ∞ can't be in H, so H can't be compact, so H is not a sphere. Therefore, $H = I'(S \setminus \{p\})$ is a plane.
In the same way we can see that for a plane P with a point p such that $p \notin P$ that $I'(p) = S \setminus \{p\}$.
Now we can prove the theorem, where we use the spheres and points above.

Proof. Take p_* a point in U with $p_* \neq p$. Take a sphere Σ_1 around p_* such that every point in the ball B (this is Σ_1 with its interior) is in U, but $p \notin B$. We can do this by taking Σ_1 small enough.
We do the same thing for V, but with $f(p)$, $f(p_*)$, Σ_2 and B'.
Now we take two inversions I_1 and I_2 with

$$I_1 : \mathbb{R}^3 \setminus \{p_*\} \to \mathbb{R}^3 \setminus \{p_*\}$$
$$I_2 : \mathbb{R}^3 \setminus \{f(p_*)\} \to \mathbb{R}^3 \setminus \{f(p_*)\}$$

inversions in Σ_1 and Σ_2 respectively. We can't take the inversions $I_i : \mathbb{R}^3 \to \mathbb{R}^3$ since p_* and $f(p_*)$ are the centers of inversion, so we don't take the inversion points of them, and p_* and $f(p_*)$ are also no inversion points.
Now define a map $F : \mathbb{R}^3 \setminus B \to \mathbb{R}^3$ given by $F = I_2 \circ f \circ I_1$. This map F has three important properties:

1. F is defined everywhere on $\mathbb{R}^3 \setminus B$

2. p is in the domain of F
3. F takes parts of planes and spheres to parts of planes and spheres, as both I_1 and I_2, as well as f, satisfy this property.

Now take a sphere S in Σ_1 met $p_* \in S$. Then $f(S)$ is a sphere in V with $f(p_*) \in f(S)$. Therefore, $I_2(f(S) \setminus \{f(p_*)\})$ is a plane. This is true since we proved this in the first statement above the proof. Since $I_2(f(S) \setminus \{f(p_*)\})$ is a plane, we can say that F brings planes in $\mathbb{R}^3 \setminus B$ to planes in $\mathbb{R}^3$. We can also see this by looking at the maps that form F. Remember that $F = I_2 \circ f \circ I_1$. Then the map I_1 brings a plane in $\mathbb{R}^3 \setminus B$ to a sphere in U. The map f brings this sphere to a sphere through $f(p_*)$. This sphere is inverted to $\mathbb{R}^3$ by the map I_2.
Furthermore, F brings straight lines in $\mathbb{R}^3$ to straight lines in $\mathbb{R}^3$ since these straight lines are the intersection of two planes.
What we also can say about F is that F preserves parallelism of straight lines. To see this, consider two situations, the situation where two lines l_1 and l_2 are in $P \subset \mathbb{R}^3 \setminus B$ and the situation where l_1 and l_2 are at different sides from B. First suppose that l_1 and l_2 are parallel in $P \subset \mathbb{R}^3 \setminus B$. Then $F(l_1)$ and $F(l_2)$ are different straight lines in $F(P)$ with intersection $F(l_1) \cap F(l_2) = \emptyset$, so $F(l_1)$ and $F(l_2)$ are also parallel.
Suppose l_1 and l_2 are parallel lines lying on opposite sides of B. Then choose the line l_3 such that l_1 and l_3 are in the plane $P_1 \subset \mathbb{R}^3 \setminus B$ and l_2 and l_3 are in the plane $P_2 \subset \mathbb{R}^3 \setminus B$. Then $F(l_1)$ is parallel to $F(l_3)$ and $F(l_2)$ is parallel to $F(L_3)$, because of the reason above, so $F(l_1)$ is parallel to $F(l_2)$. So F preserves parallelism of straight lines.
Now we define the translation $T_q : x \mapsto x + q$ and we look at a map G in a convex neighbourhood $\mathcal{U}$ of 0, where $G = T_{-F(p)} \circ F \circ T_p$. We have four properties of G.

1. G maps 0 to 0:

$$\begin{aligned} G(0) &= (T_{-F(p)} \circ F \circ T_p)(0) \\ &= (T_{-F(p)} \circ F)(T_p(0)) \\ &= T_{-F(p)}(F(p)) \\ &= F(p) - F(p) \\ &= 0 \end{aligned}$$

2. G maps straight lines to straigt lines, since both T_q and F have this property
3. G preserves parallelism, since both T_q and F have this property
4. G is a linear map.

To see the fourth point, we have to prove that $G(x + y) = G(x) + G(y)$ and $G(\alpha x) = \alpha G(x)$.
We first prove that $G(x + y) + G(x) + G(y)$. For x, y, $x + y \in \mathcal{U}$ with x and y linearly independent, we know that $G(x + y) = G(x) + G(y)$ because of the parallellogram construction of two vectors. By continuity, the same property holds for x and y linearly dependent. So indeed $G(x + y) = G(x) + G(y)$.

The second thing to prove for linearity of G is that $G(\alpha x) = \alpha G(x)$. To see this, we compute the left side and the right side of the equation and we show that these are the same.

$$\begin{aligned} G(\alpha x) &= (T_{-F(p)} \circ F \circ T_p)(\alpha x) \\ &= (T_{-F(p)} \circ F)(\alpha x + p) \\ &= T_{-F(p)}(F(\alpha x + p)) \\ &= F(\alpha x + p) - F(p) \\ &= F(\alpha x) \\ &= \alpha F(x) \end{aligned}$$

And for the right side, so $\alpha G(x)$ we have

$$\begin{aligned} \alpha G(x) &= \alpha(T_{-F(p)} \circ F \circ T_p)(x) \\ &= \alpha(T_{-F(p)} \circ F)(p + x) \\ &= \alpha(T_{-F(p)})(F(x + p)) \\ &= \alpha(F(x + p) - F(p)) \\ &= \alpha F(x) \end{aligned}$$

And therefore $G(\alpha x) = \alpha G(x)$, so G is linear.
So we know that G is linear, so G is a composition of an orthogonal map and a self-adjoint map. For the proof of this, see Spivak vol.I. But we also know that G takes small spheres around 0 to spheres. Therefore, the self-adjoint factor must be a multiple of the identity. So G is a similarity, with $G = T_{-F(p)} \circ F \circ T_p = T_{-F(p)} \circ I_2 \circ f \circ I_1 \circ T_p$. So f is a composition of inversions and similarities, and that is what we had to prove. Now we only have to prove the uniqueness. To prove this uniqueness, extend f to the so called conformal space, which is $\mathbb{R}^3 \cup \{\infty\}$. Then repeat the proof for $p_* = \infty$. Then the inversion I_1 around p_* is just a similarity and one inversion. Moreover, if $f(\infty) = \infty$, then I_2 is also a similarity, and the composition reduces to just a similarty. With this, the uniqueness is proved, and thus the lemma is proved. □

5.2.3 Umbilic points

The last lemma we need for the proof of Liouville's theorem in $\mathbb{R}^3$ is a lemma about umbilic points. Before we state the lemma, we need the following definition.

Definition 5.4. *An umbilic point is a point where all the directions are principal directions.*

Knowing this, we can state and prove the following lemma.

Lemma 5.3. *If $K \subset \mathbb{R}^3$ is a connected surface such that each point is an umbilic point, then K is part of a plane or a sphere.*

Proof. To make the proof, we will first show that if every point is an umbilic point, then $\kappa = \kappa_1 = \kappa_2 = c$ with c a constant. After this we will look at two situations, $\kappa = 0$ and $\kappa \neq 0$, and we will show that this leads to a part of a plane or part of a sphere.

We know every point of K is an umbilic point, so for the principal curvatures κ_1 and κ_2 it holds that $\kappa = \kappa_1 = \kappa_2$. We also know that $\kappa = \langle \frac{d\mathbf{T}}{ds}, \mathbf{n} \rangle = \langle A\mathbf{w}, \mathbf{w} \rangle$ with A the Weingarten map, $\mathbf{n}$ the surface normal and $\mathbf{w}$ the unit tangent at a point p. If $\mathbf{w} = \frac{x_j}{|x_j|}$, then $A\mathbf{w} = -\frac{1}{|x_j|}\frac{\partial \mathbf{n}}{\partial u_j}$. So

$$\begin{aligned}\kappa &= \langle A\mathbf{w}, \mathbf{w} \rangle \\ &= -\frac{1}{|x_j|}\langle \frac{\partial \mathbf{n}}{\partial u_j}, \frac{x_j}{|x_j|} \rangle \\ &= -\frac{1}{x_j^2}\langle \frac{\partial \mathbf{n}}{\partial u_j}, x_j \rangle\end{aligned}$$

And from this it follows that

$$-\kappa x_j = \frac{\partial \mathbf{n}}{\partial u_j} \tag{8}$$

Differentiating yields

$$\begin{aligned}\frac{\partial^2 \mathbf{n}}{\partial u_j \partial u_i} &= \frac{\partial}{\partial u_i}\left(\frac{\partial \mathbf{n}}{\partial u_j}\right) \\ &= -\frac{\partial \kappa}{\partial u_j} x_i - \frac{\partial^2 x}{\partial u_j \partial u_i}\end{aligned}$$

By interchanging the order of differentiating we get

$$\frac{\partial \kappa}{\partial u_1} x_2 = \frac{\partial \kappa}{\partial u_2} x_1$$

But x_1 and x_2 are linearly independent, so $\frac{\partial \kappa}{\partial u_1} x_2 - \frac{\partial \kappa}{\partial u_2} x_1 = 0$ implies that $\frac{\partial \kappa}{\partial u_i} = 0$, so κ is constant, which is the first part of our proof.
Now first assume that $\kappa = 0$. Then $\frac{\partial \mathbf{n}}{\partial u_i} = -\kappa x_i = 0$, so the field of unit normals is constant on the surface. So the surface is a plane perpendicular to $\mathbf{n}$, so $\kappa = 0$ leads to K is a part of a plane.
Now suppose $\kappa \neq 0$. Then consider $x + \frac{1}{\kappa}\mathbf{n}$. If we differentiate this, we get

$$\frac{\partial}{\partial u_i}(x + \frac{1}{\kappa}\mathbf{n}) = x_i - \frac{1}{\kappa}\kappa x_i = 0$$

Where the second term is true because of (8) and $x_i = \frac{\partial x}{\partial u_i}$. So $x + \frac{1}{\kappa}\mathbf{n}$ is a constant. Call this constant c. Then $\langle x - c, x - c \rangle = \frac{1}{\kappa^2}$, which is the equation of a sphere with center c and radius $\frac{1}{|\kappa|}$, so $\kappa \neq 0$ leads to K is part of a sphere.
So we have proved that K part of a sphere or part of a plane is when all points of K are umbilics. $\square$

5.2.4 Proof of Liouville

Now we have all the lemmas we need to prove Liouville's Theorem. Before we will give the proof, we will repeat the theorem to make the proof more clear.

Theorem 5.2 (Liouville's theorem in $\mathbb{R}^3$)**.** *Let $f : U \to f(U)$ be a one-to-one C^3 conformal map, where $U \in \mathbb{R}^3$ is open. Then f is a composition of similarities and inversions.*

Proof. Take $S \subset U$ a connected surface with S part of a plane or a sphere. We can find a triply orthogonal system with S in one of the families such that the lines of intersection with S are curves with any desired tangent vector in a given point. Since f is conformal, we know that f preserves angles, so the image of the triply orthogonal family under f is again orthogonal. This image forms a new triply orthogonal family, call the families of surfaces M_i.
Now we use the lemma of Dupin. The lines of intersection of this new family M_i with $f(S)$ are lines of curvature on $f(S)$. So we can find lines of curvature that points in every direction in a given point of $f(S)$. So all points of $f(S)$ are umbilics.
Now we know this, we can use lemma 5.3, so we know that $f(S)$ is part of a plane or a sphere.
The last thing we have to do is use the lemma of Möbius. Since $f(S)$ is part of a plane or a sphere, we can conclude that f is a composition of similarities and inversions, and thus is f a composition of Möbius transformations. □

5.3 Liouville in $\mathbb{R}^n$

In section 5.2, we only proved the theorem of Liouville for $\mathbb{R}^3$. In this section, we are going to generalize the theorem and the proof to $\mathbb{R}^n$. We will first give the generalized theorem, after that we will give the proof of the new theorem. The theorem and the proof are coming from [4] chapter 8.5.

Theorem 5.3 (Liouville's theorem in $\mathbb{R}^n$)**.** *Let $f : U \to f(U)$ be a one-to-one C^n conformal map, where $U \in \mathbb{R}^n$ for $n \geq 3$ is open. Then f is a composition of isometries, dilations and inversions.*

This generalized theorem states that every conformal map f in $\mathbb{R}^n$ for $n \geq 3$ is a composition of Möbius transformations. The proof of theorem 5 is different than the proof of the theorem in $\mathbb{R}^3$, because in $\mathbb{R}^n$ we can't make use of a triply orthogonal system, and this system is an essential part of the proof in $\mathbb{R}^3$. So we have to make another proof for theorem 5.3. This proof is very long, so to keep the overview we first will give a pointwise summary which shows the most important steps of the proof, without technical details. After this summary, we will give the proof in detail.

1. Find an expression for the coefficient of conformality λ in terms of the orthonomal frame field $e_1, ..., e_n$
2. After a lot of computations, show that $\frac{\partial^2 \rho}{\partial x_i \partial x_j} = \sigma \delta_{ij}$ for some σ, with $\rho = \frac{1}{\lambda}$ and conclude that σ is constant
3. Distinguish two cases, the case $\sigma \neq 0$ and the case $\sigma = 0$
4. If $\sigma \neq 0$, show that ρ is a kwadratic function where we can write $\rho = a_1 |p - p_0|^2 + k_1$ with k_1 a constant, $a_1 = \frac{\sigma}{2} \neq 0$
5. Assume $k_1 = 0$ and finish the proof by making a map $h = g \circ f^{-1}$ where $g = \frac{p-p_0}{|p-p_0|^2} + p_0$ an inversion
6. Show $k_1 = 0$
7. If $\sigma = 0$, show this implies that λ is constant, and finish the proof.

Now we are ready to prove Liouville's theorem. In the proof we will refer to the steps above, so that it is clear what we are doing and where we want to go.

Proof. We begin the proof by taking the canonical basis for $\mathbb{R}^n$, so $a_1 = (1, 0, 0, ..., 0)$, ..., $a_n = (0, ..., 0, 1)$. Then take $(x_1, ..., x_n)$ the cartesian coordinates of $\mathbb{R}^n$ relative to this basis.
Now let $e_1, ..., e_n$ be parallel differentiable vector fields on U such that $\langle e_i, e_j \rangle = \delta_{ij}$ at each point of U. Now take λ the conformality coefficient of f, i.e the $\lambda : U \to \mathbb{R}$ such that

$$\langle df_p(v_1), df_p(v_2) \rangle = \lambda^2 \langle v_1, v_2 \rangle \tag{9}$$

for all pairs of vectors v_1 and v_2 at a point $p \in U$.
If λ is the conformality factor of f, then we can write

$$\langle df(e_i), df(e_k) \rangle = \lambda^2 \delta_{ik} \tag{10}$$

The next step is to take the second differential $d^2 f$ of f. So $d^2 f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric bilinear map with values in $\mathbb{R}^n$ and $d^2 f(a_i, a_j) = \frac{\partial^2 f}{\partial x_i \partial x_j}$ in the canonical basis. Take the indices i, j and k distinct. If we differentiate equation (10), thus take the d-operator of it, we get the following three equations.

$$\begin{aligned}
\langle d^2 f(e_i, e_j), df(e_k) \rangle + \langle df(e_i), d^2 f(e_k, e_j) \rangle = 0 \\
\langle d^2 f(e_j, e_k), df(e_i) \rangle + \langle df(e_j), d^2 f(e_i, e_k) \rangle = 0 \\
\langle d^2 f(e_k, e_i), df(e_j) \rangle + \langle df(e_k), d^2 f(e_j, e_i) \rangle = 0
\end{aligned}$$

We can see that the equations above are true by applying the d-operator to (10). Then we get

$$d \langle df(e_i), df(e_k) \rangle (e_j) = d(\lambda^2 \delta_{ik})(e_j) = d(\text{scalar}) = 0$$

With the product rule for the left hand side of the equation, we get the three equations we wanted to check. If we now sum the first two equations above and subtract the third, we get

$$\langle d^2 f(e_k, e_j), df(e_i) \rangle = 0$$

if i, j, k are distinct.
Now we fix k and j and let i vary in the $(n-2)$ remaining indices. Then we can conclude that $d^2 f(e_k, e_j)$ belongs to the plane generated by $df(e_j)$ and $df(e_k)$. So we can make an equation of $d^2 f(e_k, e_j)$, given by

$$d^2 f(e_k, e_j) = \mu df(e_k) + \nu df(e_j) \tag{11}$$

Now we want to find μ and ν. From equation (10) we know that $\langle df(e_k), df(e_k) \rangle = \langle df(e_j), df(e_j) \rangle = \lambda^2$. Combining (10) and (11) to compute μ and ν,we get

$$\begin{aligned}
\langle d^2 f(e_k, e_j), df(e_k) \rangle &= \langle \mu df(e_k) + \nu df(e_j), df(e_k) \rangle \\
&= \mu \langle df(e_k), df(e_k) \rangle + \nu \langle df(e_j), df(e_k) \rangle \\
&= \mu \lambda^2
\end{aligned} \tag{12}$$

And thus

$$\begin{aligned}\mu &= \frac{\langle d^2 f(e_k, e_j), df(e_k)\rangle}{\lambda^2} \\ &= \frac{\lambda d\lambda(e_j)}{\lambda^2} \\ &= \frac{d\lambda(e_j)}{\lambda}\end{aligned} \tag{13}$$

The second step here is not obvious, so we will explain this step here.
From (9) we know that

$$\lambda^2 \langle e_j, e_k\rangle = \langle df(e_j), df(e_k)\rangle$$

So if we take the d-operator of both sides we get

$$d(\lambda^2\langle e_j, e_k\rangle)(e_i) = d(\langle df(e_j), df(e_k)\rangle)(e_i)$$

And thus

$$\begin{aligned}2\lambda d\lambda(e_i)\langle e_j, e_k\rangle &= d\langle df(e_j), df(e_k)\rangle(e_i) \\ &= \langle\langle d^2 f(e_j, e_i), df(e_k)\rangle + \langle df(e_j), d^2 f(e_k, e_i)\rangle\end{aligned}$$

Now take $j = k$. Then we get

$$\lambda d\lambda(e_i) = \langle d^2 f(e_k, e_i), df(e_k)\rangle \tag{14}$$

Now we can replace this equation in (13) and then the second step is clear. In the same way we get

$$\nu = \frac{d\lambda(e_k)}{\lambda} \tag{15}$$

If we now fill in (13) and (15) in (11), we get the following equation

$$d^2 f(e_k, e_j) = \frac{1}{\lambda}(df(e_k)d\lambda(e_j) + df(e_j)d\lambda(e_k)) \tag{16}$$

In the rest of the proof, we will take $\rho = \frac{1}{\lambda}$. Now we want to calculate the second differential $d^2(\rho f)$. To do this, we use that $d(\rho f) = d\rho f + \rho df$. So if we replace f by ρf in the left side of equation (16), we get

$$\begin{aligned}d^2(\rho f)(e_k, e_j) &= d\left[d(\rho f)(e_k)\right](e_j) \\ &= d\left[d\rho(e_k)f + \rho df(e_k)\right](e_j) \\ &= d^2\rho(e_k, e_j)f + \rho d^2 f(e_k, e_j) + d\rho(e_k)df(e_j) + d\rho(e_j)df(e_k) \\ &\overset{(*)}{=} d^2\rho(e_k, e_j)f + \frac{1}{\lambda}d^2 f(e_k, e_j) - \frac{1}{\lambda^2}\big(d\lambda(e_k)df(e_j) + d\lambda(e_j)df(e_k)\big) \\ &\overset{(**)}{=} d^2\rho(e_k, e_j)f\end{aligned} \tag{17}$$

where we used the following in (*) and (**)

- In (*) we used that if $\rho = \frac{1}{\lambda}$, then $d\rho = d(\frac{1}{\lambda}) = -\frac{1}{\lambda^2}d\lambda$.

- In (**) we used equation (16)

The proof of the following lemma is step 5 in our list.

Lemma 5.4. *If $k \neq j$, then $d^2\rho(e_k, e_j) = 0$.*

Proof. To prove the lemma, we are going to calculate the third differential, $d^3(\rho f)$, with the third differential a mapping $d^3(\rho f) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that in the canonical basis it holds that $d^3(\rho f)(a_i, a_j, a_k) = \frac{\partial^3(\rho f)}{\partial x_i \partial x_j \partial x_k}$, where the a_i formed the canonical basis. Now we use equation (17) to obtain

$$\begin{aligned} d^3(\rho f)(e_k, e_j, e_i) &= d\left[d^2(\rho f)(e_k, e_j)\right](e_i) \\ &= d\left[d^2\rho(e_k, e_j) f\right](e_i) \\ &= d^3\rho(e_k, e_j, e_i) f + d^2\rho(e_k, e_j) df(e_i) \end{aligned} \tag{18}$$

In equation (18), the left hand side and the first part of the right hand side are symmetric in i, j, k. Therefore, the same thing must happen in the second part of the right side. Therefore we can conclude that

$$d^2\rho(e_k, e_j) df(e_i) = d^2\rho(e_k, e_i) df(e_j) \tag{19}$$

Furthermore, we know that $df(e_i)$ and $df(e_j)$ are linearly independent, and i,j,k are distinct but arbitrary indices. So from (19) we can see that

$$d^2\rho(e_k, e_j) df(e_i) - d^2\rho(e_k, e_i) df(e_j) = 0$$

And since $df(e_i)$ and $df(e_j)$ are linearly independent, it must hold that $d^2\rho(e_k, e_j) = d^2\rho(e_k, e_i) = 0$, so $d^2\rho(e_k, e_j) = 0$ for all $j \neq k$, and that is what was needed to be proven. □

Now we fix a point $p \in U$. Then we can choose the vector fields $e_1, e_2, ..., e_n$ in such a way that they form an orthonormal basis in p. Since these e_i form an orthonormal basis, the claim in lemma 5.4 is valid at p for every orthonormal basis. Because $p \in U$ is arbitrary, the equation $d^2\rho(e_k, e_j) = 0$ is valid at every point of U for every orthonormal basis. Furthermore we know that $d^2\rho$ is a symmetric bilinear form, and we know that

$$0 = d^2\rho\left(\frac{e_j + e_k}{\sqrt{2}}, \frac{e_j - e_k}{\sqrt{2}}\right) = \frac{1}{2}\left[d^2\rho(e_j, e_j) - d^2\rho(e_k, e_k)\right] \tag{20}$$

The first step in this equation is true since we proved in lemma (5.4) that $d^2\rho(e_k, e_j) = 0$ for an orthonormal basis. So in particular this must hold for the orthonormal basis $\{\frac{e_j+e_k}{\sqrt{2}}, \frac{e_j-e_k}{\sqrt{2}}\}$ of $\mathbb{R}^n$. The second step in (20) is true since $d^2\rho$ is a symmetric bilinear form.
And thus from (20) we can conclude that $d^2\rho(e_j, e_j) = d^2\rho(e_k, e_k)$ for all $j \neq k$. This yields that for any orthonomal basis ap p we have that $d^2\rho(e_k, e_j) = \sigma\delta_{jk}$ for some σ. So if we take the canonical basis as orthonormal basis, we have that

$$\frac{\partial^2\rho}{\partial x_i \partial x_j} = \sigma\delta_{ij} \tag{21}$$

Now we want to take the derivative of both sides of (21). Therefore, we first take $i = j$, and we obtain

$$\frac{\partial^2 \rho}{\partial x_j \partial x_j} = \sigma$$

Now we can differentiate with respect to x_i to get for $i \neq j$

$$\begin{aligned}
\frac{\partial \sigma}{\partial x_i} &= \frac{\partial^3 \rho}{\partial x_i \partial x_j \partial x_j} \\
&= \frac{\partial^3 \rho}{\partial x_j \partial x_i \partial x_j} \\
&= \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \rho}{\partial x_i \partial x_j}\right) \\
&= \frac{\partial}{\partial x_j}(0) \\
&= 0
\end{aligned}$$

Since from (21) it follows that $\frac{\partial^2 \rho}{\partial x_i \partial x_j} = 0$ if $i \neq j$ we can conclude that $\frac{\partial \sigma}{\partial x_i} = 0$, so σ is constant.
To prove the theorem, we now consider two cases, $\sigma = 0$ and $\sigma \neq 0$, and we are going to show that f is a composition of isometries, dilatations or inversions.
First consider the case that σ is a nonzero constant. It is easy to see that (21) implies

$$\rho = \frac{\sigma}{2} \sum x_i^2 + \sigma \sum b_i x_i + c_i \tag{22}$$

If we write (22) in another form we get the following formula for ρ.

$$\frac{1}{\lambda} = \rho = a_1 |p - p_0|^2 + k_1 \tag{23}$$

With $a_1 = \frac{\sigma}{2}$, k_1 a constant and $p_0 \in \mathbb{R}^n$. This formula can be seen by completing the squares. Therefore, first rewrite (22) as

$$\rho = \frac{\sigma}{2} \sum (x_i + b_i)^2 + (c - \frac{1}{2}\sigma \sum b_i^2)$$

Now take the point $p \in U$ with coordinates $(x_1, x_2, ..., x_n)$, take $p_0 = (-b_1, ..., -b_n)$ and $k_1 = c - \frac{1}{2}\sigma \sum b_i^2$. Then we have showed that equation (23) is true.
Now the proof is complete for the case $\sigma \neq 0$ if in (23) $k_1 = 0$, because then we can take the map $h = g \circ f^{-1}$ for the inversion in the unit sphere $g(p)$, and we can see that this gives us that f is the composition of an isometry, a dilation and an inversion. We first will give the last step of the proof before we will show that $k_1 = 0$, so we now assume that $k_1 = 0$.
Without loss of generality, take the inversion in the unit sphere centered at p_0, this is the map $g : U \to \mathbb{R}^n$ given by

$$g(p) = \frac{p - p_o}{|p - p_0|^2} + p_0$$

Now we take the composition $h = g \circ f^{-1}$. Then h is a conformal map, because both f^{-1} and g are conformal, since an inversion is a conformal map, and f is conformal, so f^{-1} is conformal. Furhtermore, the composition of two conformal maps is again conformal, and therefore h is conformal. To find the coefficient of conformality of the map h, we use that λ^{-1} is the coefficient of conformality of f^{-1} if λ is the coefficient of conformality of f. So if we take the coefficients of conformality of both g and f^{-1}, we find that the coefficient of conformality of h is given by

$$a_1 |p - p_0|^2 \frac{1}{|p - p_0|^2} = a_1$$

with a_1 defined as in equation (23), so $a_1 = \frac{\sigma}{2}$. Now we can state the following lemma.

Lemma 5.5. *The map h is an isometry followed by a dilation.*

To see this lemma, we use that h is a conformal map, and that every conformal map is an isometry followed by a dilation. This last statement follows from equation (9). So h is an isometry followed by a dilation, and therefore we can say that $f = h^{-1} \circ g$ is an inversion followed by a dilation followed by an isometry, which we wanted to prove.
But in above argument, we assumed $k_1 = 0$. So this remains to prove.
We start with applying equation (23) to f^{-1}. Then we get that

$$\lambda = a_2 |f(p) - q_0|^2 + k_2 \tag{24}$$

With a_2 and k_2 constant.
To see this formula, we use that $f^{-1}(\frac{1}{\lambda}) = \lambda$ and the inverse of (23) is indeed the formula above. So now we get

$$(a_1 |p - p_0|^2 + k_1)(a_2 |f(p) - q_0|^2 + k_2) = 1 \tag{25}$$

Because $\frac{1}{\lambda} \cdot \lambda = 1$. Equation (25) shows us that a sphere with center p_0 is mapped by f into a sphere with center q_0. Furthermore we know that f preserves angles, so the radial segments of the first sphere are mapped into radii of the second sphere. Now take $p(s)$ a radial segment of the first sphere contained in U with $0 \leq s \leq s_0$ and s the arc length. Let $f \circ p(s)$ be the image of $p(s)$. Then the length of the image segment is given by

$$\int_0^{s_0} \left| df\left(\frac{dp}{ds}\right) \right| ds$$

To compute this integral, we use equation (10) and equation (24) and the following equation

$$\begin{aligned}
\left| df\left(\frac{dp}{ds}\right) \right| &= \sqrt{\left\langle df\left(\frac{dp}{ds}\right), df\left(\frac{dp}{ds}\right) \right\rangle} \\
&= \sqrt{\lambda^2} \\
&= |\lambda| \\
&= \left| \frac{1}{\rho} \right|
\end{aligned}$$

If we now use (23), we get that

$$\left| df\left(\frac{dp}{ds}\right) \right| = \frac{1}{a_1 |p(s) - p_0|^2 + k_1} \tag{26}$$

If we fill this in in the integral, we get for the length of the segment

$$\int_0^{s_0} \left| df\left(\frac{dp}{ds}\right) \right| ds = \int_0^{s_0} \frac{ds}{a_1 \left|p(s) - p_0\right|^2 + k_1}$$
$$= \left|f(p(s_0)) - f(p(0))\right| \tag{27}$$

The first part in this integral is the length of the image segment as we already mentioned. The second part is also clear, since we have shown this in equation (26). The last step needs some explanation. The image segment is a straight line, and therefore its length is given by the difference between the end point and the starting point. This difference is given in the right hand side of equation (27).
We will prove that $k_1 = 0$ by contradiction. So suppose $k_1 \neq 0$. Then $|f(p(s_0)) - f(p(0))|$ is a transcedental function. This can be seen by computing

$$\int_0^{s_0} \frac{ds}{a_1 \left|p(s) - p_0\right|^2 + k_1}$$

which will give us a solution with the arccot, so indeed $|f(p(s_0)) - f(p(0))|$ is not an algebraic function of $|p(s_0) - p_0|$. But if we look at equation (25), we can see that this function is indeed an algebraic function of $|p(s_0) - p_0|$ since we can solve (25) for $|p(s_0) - p_0|$ in an algebraic way. Therefore, there is a contradiction in the being algebraic of $|f(p(s_0)) - f(p(0))|$ as a function of $|p(s_0) - p_0|$. So it must hold that $k_1 = 0$, which we wanted to prove, and with this the proof for $\sigma \neq 0$ is finished.
Now the case $\sigma = 0$ is left. In this situation we can rewrite equation (22) to

$$\rho = \frac{1}{\lambda} = \sum a_i x_i + c_1, \tag{28}$$

with c_1 a constant. To make this part of the proof easier, we write $A_1(x) = \sum a_i x_i$ with $x = (x_1, x_2, ..., x_n)$. So then we get

$$\rho = \frac{1}{\lambda} = A_1(x) + c_1 \tag{29}$$

This part of the proof works in the same way as the previous part where $\sigma \neq 0$. So if we take f^{-1} again, and apply this to (29), we get that

$$(A_1(x) + c_1)(A_2(f(x)) + c_2) = 1, \tag{30}$$

with $A_2(f(x)) = \sum a_i f(x_i)$ and c_2 a constant. We can use the same argument as before. In equation (30) we can see that a hyperplane parallel to $A_1 = 0$ is taken by f into a hyperplane parallel to $A_2 = 0$. Because f is conformal, it preserves angles, and thus a line perpendicular to the hyperplane $A_1 = 0$ is taken by f into a line perpendicular to the hyperplane $A_2 = 0$. Now consider a segment $p(s)$ of such a line, with $0 \leq s \leq s_0$ and $p(s)$ parametrized by arc length s. Then we can obtain the following equation, in the same way as the case $\sigma \neq 0$.

$$|f(p(s_0)) - f(p(0))| = \int_0^{s_0} \frac{ds}{A_1(p(s))c_1}$$

Again we use the contradiction. Suppose $A_1(p(s)) \neq 0$. Then equation (30) is an algebraic function, but $|f(p(s_0)) - f(p(0))|$ is not. Therefore we must have

that $A_1(p(s)) = 0$.
If $A_1(p(s)) = 0$, this means that (28) reduces to

$$\rho = \frac{1}{\lambda} = c_1$$

So λ is a constant. This means that the lengths of the tangent vectors are multiplied by a constant λ and thus f is an isometry followed by a dilatation. So the proof is also finished for $\sigma = 0$.
So f is a composition of at most one inversion, dilatation and isometry, and that is what we wanted to prove. $\square$

5.4 Counterexample

In this short section, we will give an example of a conformal map which is not a Möbius transformation, i.e. an example for which Liouville's theorem doesn't hold. We will give this example to show that the criterium of being in $\mathbb{R}^n$ with $n \geq 3$ is necessary in the theorem.
In $\mathbb{R}^2$, look at the group of analytic funtions, also called holomorphic functions, and the anti-holomorphic functions. As we have seen in section 3.7, these functions are conformal. The analytic functions however don't need to be Möbius transformations. Actually, most of the analytic functions are no Möbius transformation. For example, take $f(z) = \sin z$. This function is analytic, because the derivative of $f(z)$ exists everywhere. But $f(z)$ can't be written in the form

$$f(z) = \frac{az + b}{cz + d}$$

with a, b, c, d constants in $\mathbb{C}$. Therefore, in $\mathbb{R}^2$ the analytic functions and anti-holomorphic functions are not all a composition of Möbius transformations, and therefore the theorem of Liouville does not hold in $\mathbb{R}^2$.

6 Discussion

In this thesis we have seen several subjects. We started with the inversion geometry. The inversion geometry we talked about, is only the necessary theory for the theorem of Liouville. There is much more to talk about. In the chapter about inversion geometry we have seen the general theory followed by construction methods, properties and a little part of the cross ratios.
After the inversion geometry we have seen the conformal maps. Also for the conformal maps it holds that we have only discussed a little part of the theory. In the chapter about conformal maps we have seen the general theory followed by six examples which are useful for the rest of the thesis. Some of the theorems in this example are proved, some are not. These proves are left for the reader.
The next small chapter was about Möbius transformations. Also in this chapter we have only discussed the information we needed for the theorem of Liouville. In the chapter about Möbius transformations we first have seen the general Möbius transformations. We have seen how can be detected if a funtion is a Möbius transformation and we have discussed some properties about this type of transformations. After the general Möbius transformations we have seen the extended Möbius transformations.
The most important chapter in this thesis is the chapter about the theorem of Liouville, because this was the goal of the thesis: to prove Liouville's theorem. First we stated the theorem. Then we distinguished two cases. First we looked at the case of the theorem in $\mathbb{R}^3$. Before we could prove this, we had to look at the lemma of Dupin, the lemma of Möbius and a lemma about umbilics. After we proved these lemmas, we were able to prove the theorem of Liouville in $\mathbb{R}^3$. The second case was Liouville's theorem in $\mathbb{R}^n$. We didn't need lemmas to be able to prove the theorem. The proof is very technical tough. But we were able to prove the theorem. We have also given a counterexample of the theorem of Liouville in $\mathbb{R}^2$. In this way, we have proved that the criterium to be in $\mathbb{R}^n$ for $n \geq 3$ is necessary.
We have just given an overview of the necessary theory. Further research can be done in mainly the conformal mapping theory. There is much more literature about this subject, for example applications of the conformal maps.

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References

[1] *David E. Blair*, `Inversion Theory and Conformal Mapping`, AMS, Vol. 9, 2000, chapter 1-6

[2] *David A. Brannan, Matthew F. Esplen, Jeremy J. Gray*, `Geometry`, Cambridge University Press, The Open University, 1999, chapter 5

[3] *John G. Ratcliffe*, `Foundations of Hyperbolic Manifolds`, Springer-Verlag, New York, 1994, chapter 4

[4] *Manfredo P. do Carmo*, `Riemannian Geometry`, Birkhäuser, Boston, 1992, chapter 8

[5] *Michael Spivak*, `A Comprehensive Introduction to Differential Geometry`, Vol. 3, Third edition, Publish or Perish inc, Houston, Texas, 1999, chapter 2 and chapter 4

[6] *Barrett O'Neill*, `Elementary Differential Geometry`, Revised 2nd Edition, Elsevier Academic Press, 2006

[7] *David Gu*, `Möbius Transformation`, Tsinghua University, Mathematics Science Center, June 15, 2010

[8] *David X. Gu, Feng Luo, Shing-Tung Yau*, `Fundamentals of Computational Conformal Geometry`

[9] *Suman Ganguli*, `Conformal Mappings and its Applications`, November 20, 2008

[10] *C. Carathéodory*, `The most general transformations of plane regions which transform circles into circles`, Bull. Amer. Math. Soc. 43 (1937)

[11] http://en.wikipedia.org/wiki/Stereographic_projection

[12] http://www.math.rutgers.edu/ greenfie/mill_courses/math503a/diary2.html

[13] http://en.wikipedia.org/wiki/Similarity_(geometry)

[14] http://en.wikipedia.org/wiki/Homothety

A Construction of images of points under inversion

In this appendix we will give three ways to construct $I(p)$ when p lies inside the circle of inversion $\mathcal{C}$ and two ways to construct $I(p)$ when p is outside $\mathcal{C}$. This will be done in the subsections A.1 and A.2
There are pictures in the sections to visualize the constructions. The pictures are in $\mathbb{R}^2$ to make things more clear. Therefore, the theory in this section will also be in $\mathbb{R}^2$, but things work the same in $\mathbb{R}^n$ with spheres and planes. Furthermore, in this section we will call the inversion point p' in stead of $I(p)$, because p' is an easier notation here.
The theory is coming from Blair chapter 1, [1], Brannan chapter 5.1, [2] Ratcliffe chapter 4.1, [3].

A.1 p inside $\mathcal{C}$

When p lies in the inside $\mathcal{C}$, we have three ways to construct the inversion point. We will discuss each of these ways shortly.

1. We begin the construction of $I(p)$ by drawing the line $\overline{ap}$, since p and $I(p)$ should be on the same line. The next step is to construct a line perpendicular to the line $\overline{ap}$ through p. These lines intersects $\mathcal{C}$ in the points U and V. To find $I(p)$, draw the tangent lines in U and V to $\mathcal{C}$. These tangent lines intersect the line $\overline{ap}$ in the point $I(p)$.

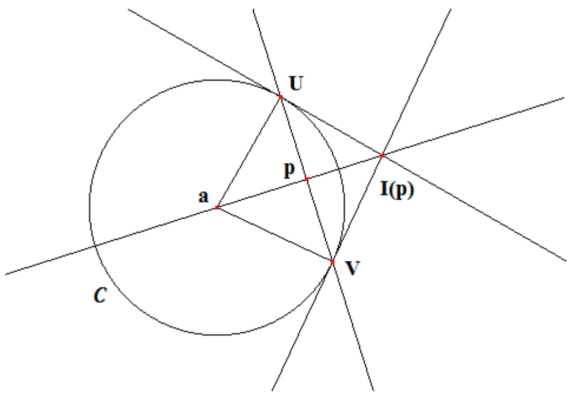

Figure 3: Construction method 1 for p in $\mathcal{C}$

To check if this construction gives us the right point $I(p)$ we have to check that $\overline{ap} \cdot \overline{ap'} = r^2$. In figure 3 we can see that $\triangle aUp \sim \triangle ap'U$, so

$$\frac{\overline{ap}}{\overline{aT}} = \frac{\overline{aU}}{\overline{ap'}}$$

And thus $\overline{ap} \cdot \overline{ap'} = \overline{aU}^2 = r^2$, so this way of construction gives us the correct inversion point p'.

2. For the second way to construct $I(p)$ we begin by drawing the line $\overline{ap}$ again. Then we draw a line perpendicular on $\overline{ap}$ through a This line intersects $\mathcal{C}$ in the points U and V. The next step is to draw the line $\overline{Up}$, which intersects $\mathcal{C}$ in Q. To find $I(p)$, draw the line $\overline{VQ}$, which intersects $\overline{ap}$ in $I(p)$.

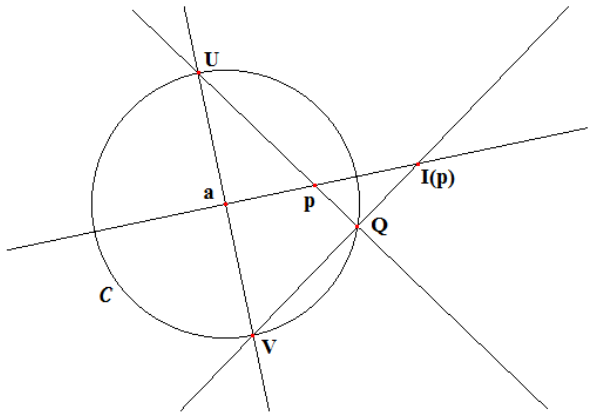

Figure 4: Construction method 2 for p in $\mathcal{C}$

In figure 4 we can see that $\triangle Uap \sim \triangle UQV \sim \triangle p'aV$. We will use the relation $\triangle Uap \sim \triangle p'aV$ such that we can conclude that

$$\frac{\overline{ap}}{\overline{aU}} = \frac{\overline{aV}}{\overline{ap'}}$$

And thus $\overline{ap} \cdot \overline{ap'} = \overline{aU} \cdot \overline{aV} = r^2$, which confirms that $I(p)$ is the correct inversion point.

3. The last way to construct $I(p)$ again begins by drawing the line $\overline{ap}$. Next, draw a line through a perpendicular to $\overline{ap}$ with length the diameter of $\mathcal{C}$. This line intersects $\mathcal{C}$ in U. Now we construct a new circle $\mathcal{D}$ by drawing a circle with diameter $\overline{aU}$ and with the middle of $\overline{aU}$ as center. Next we draw the line $\overline{Up}$, which intersects $\mathcal{D}$ in the point Q. Then construct the parallel of $\overline{aU}$ through Q. This line also intersects $\mathcal{D}$ in the point Q'. The last step is to draw the line $\overline{UQ'}$, which intersects $\overline{ap}$ in the point $I(p)$.

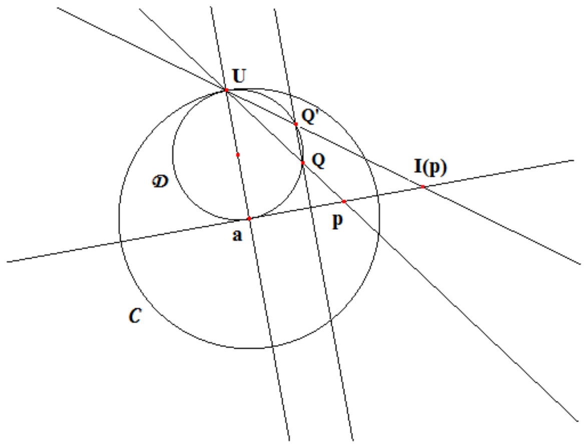

Figure 5: Construction method 3 for p in $\mathcal{C}$

In figure 5 we can see that $\triangle aUp \sim \triangle ap'U$, so

$$\frac{\overline{ap}}{\overline{aU}} = \frac{\overline{aU}}{\overline{ap'}}$$

And thus $\overline{ap} \cdot \overline{ap'} = \overline{aU}^2 = r^2$ so the third way of construction also gives us the correct inversion point p'.

A.2 p outside $\mathcal{C}$

There are two ways to construct the inversion point $I(p)$ when p lies outside the circle $\mathcal{C}$. These ways will also be discussed briefly.

1. To begin the construction of $I(p)$, draw the line $\overline{ap}$. Then construct the tangent line from p to $\mathcal{C}$. This line is tangent to $\mathcal{C}$ in the point U. The last step to find $I(p)$ is to draw the line through U perpendicular to $\overline{ap}$, which intersects $\overline{ap}$ in $I(p)$. This is the opposite way of section A.1 item 1.

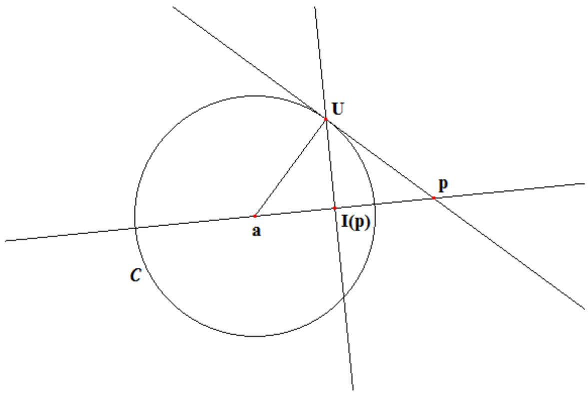

Figure 6: Construction method 1 for p outside $\mathcal{C}$

Again we want to check if the construction gives us the correct point p'. Since this way of construction is the opposite of A.1 item 1, we can use the same argument to show that p' is correct.
We can see in figure 6 that $\triangle ap'U \sim \triangle aUp$ and thus

$$\frac{\overline{ap}}{\overline{aU}} = \frac{\overline{aU}}{\overline{ap'}}$$

And so it follows that $\overline{ap} \cdot \overline{ap'} = \overline{aU}^2 = r^2$, and indeed p' is constructed in the correct way.

2. The second way to construct $I(p)$ starts again with drawing $\overline{ap}$. Take the middle of the line segment between a and p and call this point Q. Now we are going to construct another circle $\mathcal{D}$ with center Q and radius $|p - Q|$. The intersection points of $\mathcal{C}$ and $\mathcal{D}$ are called U and V. The last step to find $I(p)$ is to draw the line $\overline{UV}$, which intersects $\overline{ap}$ in $I(p)$.

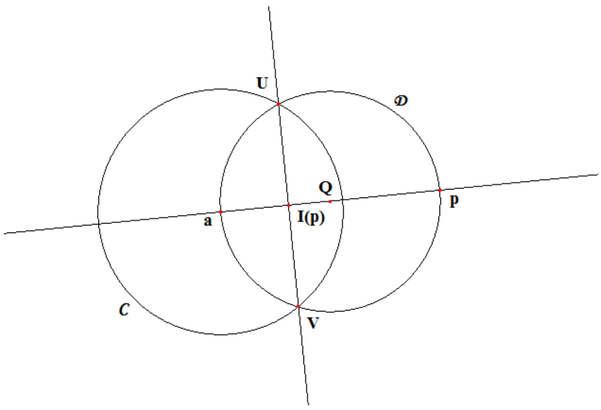

Figure 7: Construction method 2 for p outside $\mathcal{C}$

For this way of construction, we need an extra step to get to the congruent triangles. If we use the theorem of Thales, we can conclude that $\angle aUp = 90°$. With this knowledge we can get to the congruent triangles. In figure 7 we can see that $\triangle ap'U \sim \triangle aUp$, and thus

$$\frac{\overline{ap}}{\overline{aU}} = \frac{\overline{aU}}{\overline{ap'}}$$

And therefore $\overline{ap} \cdot \overline{ap'} = \overline{aU}^2 = r^2$ So indeed this last way of construction gives us the correct point p'.