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Fundamental Polygons for Coverings of the Double-Torus

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Contents

1 Introduction

This text is concerned with the construction of fundamental polygons for coverings of finite multiplicity for the double torus, or the orientable surface of genus 2. We will consider the double torus as a geometric surface of constant curvature. We will see that this means that it is locally isometric to the hyperbolic plane and is therefor called a hyperbolic surface. The surfaces we consider are assumed to be connected, compact and orientable. By the classification of compact surfaces, such a surface is always homeomporphic to a sphere with $n \geq 0$ handles.

A covering of a surface S, is a map from another surface, called the coverings surface, onto S. This map is usually required to be a local homeomorphism such that every point of S is evenly covered. The coverings we are considering are local isometries.

A hyperbolic surface can be given as the quotient of the hyperbolic plane by the action of a discontinuous group of isometries. The points of the surface become the orbit of a point under the group action. A surface can thus be given as a discontinuous group of isometries, and we will see that subgroups of this group correspond to covering surfaces for the original surfaces. Such a covering is automatically a local isometry.

Another way of describing a surface is by identifying the edges of a polygon. This amounts to labeling the edges of the polygon in a particular way, such that edges of the polygon become identified in pairs. The identification of sides automatically identifies the vertices. If this polygon is constructed in the right geometry, the resulting identification space becomes a geometric surface. Constructing a covering for an identification space can be done identifying the edges of seperate polygons.

The methods of describing a surface by a discontinuous group and as an identifiction space come together via the notion of fundamental polygon. For a discontinous group, a fundamental polygon is a polygon containing in it's interior precisely one representative for each orbit. The isometries mapping an edge of the fundamental polygon onto another are called side-pairings and they are seen to generate the group and to yield an edge-labeling for the polygon. On the other hand, if a polygon for the identification space is given in the right geometric setting, the edge-identifications can be realized by isometries of that geometry and the polygon becomes a fundamental domain for the group generated by the side-pairings.

The double torus can be constructed by identifying the edges of a hyperbolic octagon, and can be given as the group generated by the side-pairings of this octagon. First we discuss hyperbolic geometry, then we discuss the construction of surfaces as an identification space of a polygon and finally we discuss the construction via discontinuous groups. The last part considers some constructions for covering spaces of the double torus. The example of the torus will be used as a guide.

2 Hyperbolic Geometry

Hyperbolic geometry is the study of a complete and simply connected space with a metric of constant negative curvature. Originally discovered in the search for a proof for the infamous parallel postulate, when it was realized that it need not hold in order to maintain geometric consitency. Hyperbolic geometry has a strong connection with the theory of Möbius transformations. The group of general Möbius transformations on $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is the group of isometries for hyperbolic $n + 1$ space (the interested reader is referred to[4]). In our discussion we shall omit most proofs, however these (and a more complete overview of hyperbolic geometry) can be found in almost any textbook about hyperbolic geometry and for example in [1, 4].

2.1 The Hyperbolic Plane

The upper-half plane and the open unit disk (which are conformally equivalent by a Möbius transformation) can be used to model the geometry of the hyperbolic plane. By specifying an appropriate metric and assuming a familiarity with a Euclidean description of geometric objects, these sets will enable us to discuss hyperbolic geometry in terms of Euclidean geometry. We will denote the upper-half plane of the complex plane by

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

and the open unit disk by

$$\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}.$$

These are so-called conformal models, meaning that the Euclidean angles in both models equal the hyperbolic angles. The *circle at infinity* is the set of points at infinity: for the $\mathbb{H}^2$ model it is denoted by $\partial\mathbb{H}^2$ and is the set $\{z \in \mathbb{C} \mid \operatorname{Im}(z) = 0\} \cup \{\infty\}$, and for the $\mathbb{D}^2$ model this is denoted by $\partial\mathbb{D}^2$ and is equal to the unit circle in the complex plane. The circle at infinity is *not* a part of the hyperbolic plane, but is a very usefull tool in studying hyperbolic lines and isometries.

The metric for the $\mathbb{H}^2$ model is derived from the differential $ds = \frac{|dz|}{\operatorname{Im} z}$ in the following way. Let $z, w \in \mathbb{H}^2$, and let $\gamma : [a, b] \to \mathbb{H}^2$ be a piecewise continously differentiable curve in $\mathbb{H}^2$ with endpoints z and w. The length $\|\gamma\|$ of γ is defined to be

$$\|\gamma\| = \int_a^b \frac{|\gamma'(t)|}{\operatorname{Im}(\gamma(t))} dt.$$

The distance function ρ for $\mathbb{H}^2$ is then defined to be

$$\rho(z, w) = \inf\{\ \|\gamma\| \mid \gamma \text{ connects } z \text{ and } w\}.$$

It is easily seen that the function ρ is non-negative and symmetric. The triangle inequality is also satisfied, because we take the infimum of the length of curves connecting z to w and a violation of the triangle inequality is an immediate contradiction to the ρ being the infumum. Also it is quite obvious that $\rho(z, z) = 0$. To see that $\rho(z, w) > 0$ if $z \neq w$, choose a neighborhood of z not containing w and note that the integrand in the definition for length of a curve must have a positive lower bound.

The upper-half plane is conformally equivalent to the open unit disk $\mathbb{D}^2$ by the transformation

$$\phi : \mathbb{H}^2 \to \mathbb{D}^2,\ z \mapsto \frac{iz + 1}{z + i},$$

which maps $\partial\mathbb{H}^2$ to $\partial\mathbb{D}^2$ such that the points $0, \infty$ are mapped to the points $-i$ and i, and it maps the semi-circle through $-1, i, 1$ in $\mathbb{H}^2$ to the Euclidean line segment from -1 to 1 in $\mathbb{D}^2$ (see figure 1). Indeed, ϕ becomes an isometry $\mathbb{H}^2 \to \mathbb{D}^2$ if we set ρ' for $\mathbb{D}^2$ to be $\rho(\phi^{-1}(z), \phi^{-1}(w))$ as a metric for $\mathbb{D}^2$. This is equivalent to deriving the metric (as we have done for $\mathbb{H}^2$) from the differential $ds = \frac{2|dz|}{1-|z|^2}$.

Let us explicate: From now on we let denote $\mathbb{H}^2$ and $\mathbb{D}^2$ to be the metric spaces $(\mathbb{H}^2, \rho)$ and $(\mathbb{D}^2, \rho')$. Since we have ϕ as an isometry for them we simply write ρ for both metrics, and we will be explicit when we anticipate possible confusion.

There are some mappings which are easily seen to be isomtries for the $\mathbb{H}^2$ model; for instance $z \mapsto z + \alpha$ ($\alpha \in \mathbb{R}$), $z \mapsto dz$, ($d \in \mathbb{R}, d > 0$), $z \mapsto -\bar{z}$, and $z \mapsto 1/\bar{z}$. In general the mappings of the form:

$$g(z) : z \mapsto \frac{az + b}{cz + d},$$

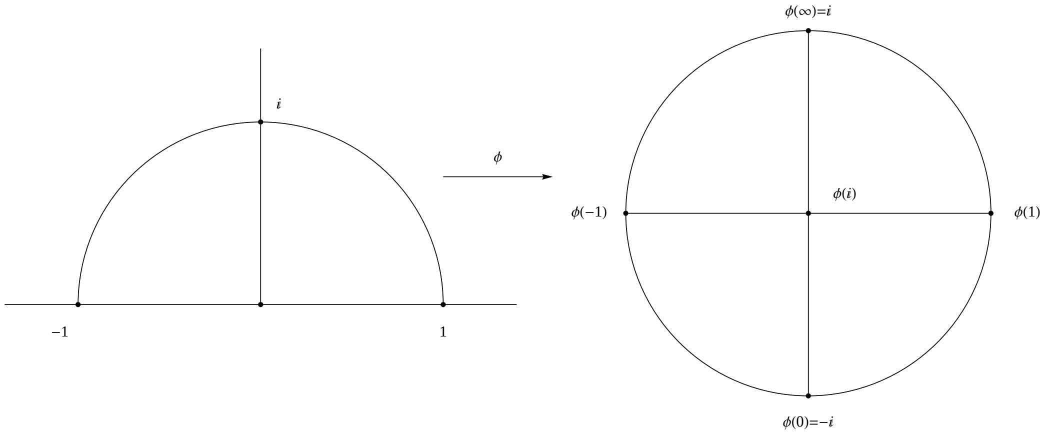

Figure 1: Mapping the hyperbolic plane onto the hyperbolic disc.

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ are hyperbolic isometries (for the $\mathbb{H}^2$ model). This can be seen as follows. We have that

$$g'(z) = \frac{ad - bc}{(cz + d)^2},\ \mathrm{Im}(g(z)) = \frac{ad - bc}{|cz + d|^2}\mathrm{Im}(z),$$

where the latter equality implies the that g leaves $\mathbb{H}^2$ invariant and both equalities together imply that

$$\frac{|g'(z)|}{\mathrm{Im}(g(z))} = \frac{1}{\mathrm{Im}(z)}.$$

This implies that

$$\|g\gamma\| = \int_a^b \frac{|g'(\gamma(t))||\gamma'(t)|}{\mathrm{Im}(g(\gamma(t)))} dt = \|\gamma\|.$$

So we see that ρ is invariant under such mappings, thus that these mappings are $\mathbb{H}^2$ isometries. In fact, all the orientation preserving isometries of $\mathbb{H}^2$ are of such form. Composing such a mapping from the right with the function $z \mapsto -\bar{z}$ yields the orientation reversing isometries. We will refer to mappings such as g mentioned above as *real Möbius transformations*, and we call $ad - bc$ the determinant of the mapping g. Requiring the determinant to be non-zero guarantees that the mapping is non-constant, and requiring it to be positive ensures that it leaves the upper-half plane invariant. Note that different quadruples of a, b, c, d can correspond to the same mapping: where the mapping g is invariant under multiplication of nominator and denominator by the same non-zero constant, the determinant is not. In giving an explicit expression for mappings such as g we assume that they be *normalized*, that is they be written such that they have determinant $ad - bc = 1$. Obviously, the real Möbius transformations with positive determinant act as isometries only for the $\mathbb{H}^2$ model. However, if such g is such an isometry we obtain an expression for an isometry in the $\mathbb{D}^2$ model by conjugating with ϕ, i.e. calculating $\phi g \phi^{-1}$.

2.2 The Hyperbolic lines

We simply define the hyperbolic lines to be the Euclidean lines and semi-circles orthogonal to $\partial\mathbb{H}^2$. In the disk model this translates to the Euclidean lines through the origin and the semi-circles orthogonal to $\partial\mathbb{D}^2$ (since ϕ is conformal $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$). From this definition the following properties of hyperbolic lines are easily established.

1. For every two distinct points, there is a unique geodesic through these points.
2. Two distinct hyperbolic lines intersect in at most one point.
3. The reflection in a hyperbolic line is a hyperbolic isometry.
4. Given any two lines L_1 and L_2, there is a hyperbolic isometry mapping L_1 onto L_2.
5. For every line and every point, there is a unique line through the point and orthogonal to the line.

The third property is easily checked by simply giving an explicit expression of the reflection in a line as a complex function. The explicit expression for reflection in the line which is part of the circle $S(a, r)$ with centre a on the real axis and radius $r > 0$ is

$$f(z) = \frac{-a(-\bar{z}) + r^2 - a^2}{-(-\bar{z}) - a},$$

which has determinant r^2. If the line is a Euclidean line of the form $\mathrm{Re}(z) = c \in \mathbb{R}$, the reflection is of the form

$$f(z) = \frac{-\bar{z} + 2c}{1},$$

which also has positive determinant.

We will now show that there is an isometry mapping any line L onto any other line L'. We do this by showing that there is a mapping of the form

$$g(z) = \frac{az + b}{cz + d}, \text{ where } a, b, c, d \in \mathbb{R}, \text{ and } ad - bc > 0,$$

mapping an arbitrary geodesic L onto the imaginary axis $\mathrm{Re}(z) = 0$. If L is a vertical line of through $\alpha \in \partial\mathbb{H}^2$, then the mapping $z \mapsto z - \alpha$ suffices. If L is not a such a line, let $[\alpha, \beta]$ be a segment of L. Consider the isometries (they obviously leave ds invariant).

$$f : z \mapsto z - \mathrm{Re}(\alpha), \ g : z \mapsto z/|\mathrm{Im}(\alpha)|.$$

The composite gf maps α to i, which is mapped onto the origin of $\mathbb{D}^2$ by ϕ. As rotation around the origin is a $\mathbb{D}^2$-isometry, we may simply rotate until β is on the imaginary axis in the disk model. If we let r denote of the rotation of the disk conjugated by ϕ (and thus acting on $\mathbb{H}^2$). We conclude that the segment $[\alpha, \beta]$ is mapped into the imaginary axis by the isometry $F = rgf$. Similarly, one can construct an isometry G which maps another hyperbolic line L' onto the imaginary axis, thus $G^{-1}F$ maps L onto L'.

The fifth property can be seen to be true by mapping the line onto the imaginary axis and taking the hyperbolic line $\{z \in \mathbb{C} \mid |z| = |w|\}$, which obviously satisfies the claimed property.

There are three possible configurations for a pair of hyperbolic lines in the hyperbolic plane. Two different hyperbolic lines L, N are called

1. *assymptotic* if the intersection of their Euclidean closures is a point at infinity,
2. *disjoint* if the intersection of their Euclidean closures is empty,
3. *intersecting* the intersection is in the hyperbolic plane.

As we will see, these notions are usefull in the classification of hyperbolic isometries.

The following theorem shows that the hyperbolic lines are in fact geodesics (as curves of shortest length).

Theorem 2.1. *The $\mathbb{H}^2$-line segment $[z, w]$ is the curve of minimum length connecting z and w, that is it has hyperbolic length $\rho(z, w)$.*

Proof. By the previous remarks we may assume that $[z, w]$ is a segment of the imaginary axis, thus $z = ip$ and $w = iq$ for some $p, q \in \mathbb{R}$ and we may also assume that $0 < p < q$. Let $\alpha : [a, b] \to \mathbb{H}^2$ be any curve such that $\alpha(a) = z$ and $\alpha(b) = w$. Then the length of α is given by the following integral for which we have thatF

$$\begin{aligned} \|\alpha\| = \int_\alpha ds &= \int_\alpha \frac{\sqrt{dx^2 + dy^2}}{y} \\ &\geq \int_p^q \frac{dy}{y} \\ &= \log(q/p), \end{aligned}$$

The last integral is a lower bound for the length of curves joining z and w, and it is attained by $\gamma(t) = p + t(q - p)$ which is a simple parametrization of the hyperbolic line segement $[z, w]$. Hence the length of $[z, w]$ is $\rho(z, w)$. ☐

Considering the previous proof we see that we have obtained some other noteworthy results, namely that a curve α connecting z and w satisfies $\|\alpha\| = \rho(z, w)$ if and only if α is a parametrization of $[z, w]$ as a simple curve ($\alpha(t_1) = \alpha(t_2) \implies t_1 = t_2$). From the previous prove we can also conclude that for $z, w, y \in \mathbb{H}^2$, we have $\rho(z, w) \leq \rho(z, y) + \rho(y, w)$ with equality if and only if y is in $[z, w]$.

Also we have found an explicit expression for the hyperbolic distance of two points ip and iq: $\rho(ip, iq) = |\log(q/p)|$. From this we can deduce that the distance between a point $w \in \mathbb{D}^2$ and the origin of the disc is given by $\log \frac{1+|w|}{1-|w|}$.

2.3 The Hyperbolic Isometries

We have already seen that the mappings of the form

$$g(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are real numbers such that the 'determinant' $ad - bc$ of g is positive, are hyperbolic isometries.

It is quite well know that any isometry is completely determined by the image of a triangle, and that any isometry can be written as the product of three reflections. If f is an isometry mapping z_1, z_2, z_3 onto w_1, w_2, w_3, then f is the isometry $r_N r_M r_L$, where L, N, M are the lines equidistant from z_1 and w_1, $r_L(z_2)$ and w_2, and $r_M r_L(z_3)$ and w_3, respectively. The reader is referred to [1] for the details of this line of thinking. Obviously, some isometries can be given as a product of one or two reflections. An isometry is orientation preserving if and only if it can be written as the product of two reflections. The orientation preserving isometries form a subgroup in the group of all hyperbolic isometries.

In order to show that all the direct isometries of $\mathbb{H}^2$ are real Möbius transformations with positive determinant, we simply note that all reflections are of such form and that any composition of real Möbius transformations with positive determinant is of such form. Since we may 'normalize' the real Möbius transformations to have determinant equal to 1, the determinant of their product is also equal to 1. We thus have the following theorem:

Theorem 2.2. *The isometries of the $\mathbb{H}^2$-model are of the form*

$$z \mapsto \frac{az+b}{cz+d},\ z \mapsto \frac{a(-\bar{z})+b}{c(-\bar{z}+d},\ \textit{where } a,b,c,d \in \mathbb{R},\ \textit{and } ad-bc=1,$$

which are respectively orientation preserving and reversing. For the $\mathbb{D}^2$-model the explicit expression are

$$z \mapsto \frac{az+\bar{c}}{cz+\bar{a}},\ z \mapsto \frac{a\bar{z}+\bar{c}}{\bar{c}\bar{z}+\bar{a}},\ \textit{where } a,c \in \mathbb{C},\ \textit{and } |a|-|c|=1.$$

The expression for the isometries for the $\mathbb{D}^2$ model can be found by conjugating the expression for the isometries $\mathbb{H}^2$ by ϕ (which maps $\mathbb{H}^2$ onto $\mathbb{D}^2$).

An invertable real 2×2 matrix corresponds to a hyperbolic isometry acting on $\mathbb{H}^2$ under the following convention:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \frac{az+b}{cz+d}.$$

Note that the composition of two isometries corresponds to the product of their matrices. Furthermore, a matrix corresponds to the same isometry if and only if the one matrix is a scalar multiple of the other.

We can categorize the isometries of the hyperbolic plane into the following categories: rotations, limit rotations, translations and glide reflections, of which only the latter is orientation reversing. In terms of orientation preserving isometries, the first three correspond to the elliptic, parabolic and hyperbolic Möbius transformations. It is well known that the squared value of the trace of a matrix corresponding to a Möbius transformation corresponds to the class the transformation is in (see [4]). Explicitly, a Möbius transformation g is

1. parabolic if and only if $\mathrm{tr}^2(g) < 4$,
2. elliptic if and only if $\mathrm{tr}^2(g) = 4$,
3. hyprebolic if and only if $\mathrm{tr}^2(g) > 4$.

If a Möbius tranformation has a strictly complex value of tr^2 then it is called loxodromic. However, these transformations do not act as isometries of the hyperbolic plane.

Since every isometry can be written as a product of at most three reflections, we can classify the hyperbolic isometries by the possible configurations of the lines of reflection. The possible configuration of lines of reflection for a pair of reflections are intersecting, assymptotic and disjoint. Thus there are three classes of orientation preserving isometries in this classification.

Rotations

A hyperoblic isometry g is a *rotation* if it can be written as the product of two reflections in intersecting lines. The point of intersection of these lines is the fixed point of the rotation. It maps the class of lines through the fixed point onto itself and leaves the hyperbolic circles centered at the fixed point invariant. Any rotation is conjugate to a mapping of the form $z \mapsto e^{i\theta}z$, which is a rotation around the origin of the disk. Because the value of $\mathrm{tr}^2(g)$ is given by $4\cos(\theta)$ where θ is the angle of rotation, we see that every rotation is given by a parabolic Möbius transformation.

Limit rotations

A hyperbolic isometry g is a *limit rotation* if it can be written as the product of two reflections in assymptotic lines. The point at which the Euclidan closures of the lines of reflection intresect is the fixed point on the circle at infinity. A limit rotation is always conjugate to a mapping of the form $z \mapsto z + k$, where k is any non-zero real number. It maps the class of lines through ∞ onto itself

Limit rotations have only one fixed point on the circle at infinity. A limit rotation can be written as the product of two assymptotic lines, the point at infinity where the lines meet (in Euclidean sense) is the fixed point at infinity of the limit rotation. Because limit rotations have $\mathrm{tr}^2 = 2$ they are all elliptic Möbius transformations.

Translations

A hyperbolic isometry is *translation* if it can be written as the product of two reflections in disjoint lines. The common orthogonal of the lines of reflection is called the axis of translation of g and is invariant under g. Any translation is conjugate to a mapping of the form $z \mapsto kz$, where $k > 0$ is a positive real number different from 1. A translation has exactly one invariant line, namely the axis of translation, and thus has two fixed points on the circle at infinity (the end points of the axis of translation). Furthermore, it leaves the class of lines orthogonal to the axis of translation invariant. The value of tr^2 is given by $k^2 + 1/k^2 + 2$, hence translations are hyperbolic Möbius transformations.
The above is a classification of all the orientation preserving isometries only, and we are left to deal with the orientation reversing isometries. An orientation preserving isometry is a reflection or the product of three reflections. If one considers the explicit expression for an orientation reversing isometry acting directly on $\mathbb{H}^2$

$$z \mapsto \frac{a(-\bar{z}) + b}{c(-\bar{z}) + d}$$

one may find that the fixed points of g must be equal to their conjugates, and the fixed points are the solutions of a real quadratic equation with positive discriminant and thus are twofold. Hence an orientation reversing isometry has two fixed points on the circle at infinity. Let σ be the reflection in the line L connecting the fixed points, and consider the mapping $f = g\sigma$. Since this is the product of four reflections it is orientation preserving, and since it has two fixed point on the circle at infinity it is a translation with axis L (by the above classifcication). Since σ is self inverse we find $g = f\sigma$.

Glide Reflections

A hyperbolic isometry g is a glide reflection if it can be written as the product of either one or three reflections. As we have just seen, any orientation reversing isometry must have an invariant line and thus we can consider every orientatin reversing isometry as a composition of a translation with a reflection in the axis of translation.

2.4 Hyperbolic polygons

A polygon is a region in the plane whose boundary is a closed polygonal curve. A polygonal curve is a curve that consists entirely of geodesic segments. A polygon can thus be described by it's vertices and the order in which the polygonal curve traverses these vertices. A hyperbolic polygon is such a region in the hyperbolic plane. The vertices of a hyperbolic polygon are allowed to be on the circle at infinity, and if this is the case we say call these vertices *improper vetices*

The simplest polygons are triangles, and as we will see, there is an interesting relation between it's area and it's angle sum. We discuss the situation using the $\mathbb{H}^2$ model. An intersting difference between hyperbolic geometry and the other geometries is that so called assymptotic triangles exist, that is triangles of which two sides are lines through the point at infinity (segments of straight lines orthogonal to the real axis). First we calculate the area of such a triangle.

Theorem 2.3. *The area of a hyperbolic triangle $\Delta_{\alpha,\beta,\gamma}$ with angles α, β, γ is given by:*

$$Area(\Delta_{\alpha,\beta,\gamma}) = \pi - (\alpha + \beta + \gamma).$$

The following image is a picture of a triangle with one vertex at the point at infinity in the $\mathbb{H}^2$-model:

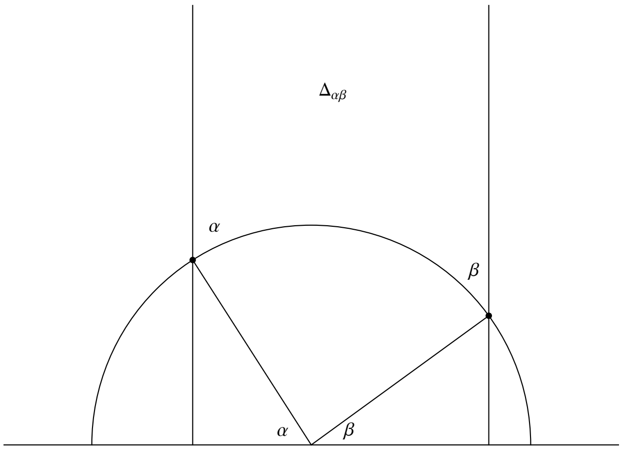

Figure 2: An assymptotic triangle in the hyperbolic plane.

Proof. The area of a region D in the hyperbolic plane $\mathbb{H}^2$ is given by the integral

$$\int_D \frac{dxdy}{y^2}.$$

First we calculate the area of an *assymptotic triangle*, that is a triangle Δ with one point at a point at infinity. We can assume that Δ has vertices A, B, C with interior angles α at A and β at B and where C is at infinity and thus has interior angle 0 and that the two sides through C are Euclidean straight lines intersecting the x-axis at a and b. Furthermore that the edge connecting A and B is part of the unit circle centered at 0, and thus that it is of the form:

$$\{(x, \sqrt{1 - x^2}) \mid a \leq x \leq b\}.$$

Note that the Euclidean lines through A and 0 and B and 0 intersect the x-axis at angles $\pi - \alpha$ and β. The integral becomes

$$\int_\Delta \frac{dxdy}{y^2} = \int_a^b dx \int_{\sqrt{1-x^2}}^\infty \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{1 - x^2}},$$

and substituting $x = \cos\theta$ such that $a = \cos\pi - \alpha$ and $b = \cos\beta$ yields:

$$\int_{\pi-\alpha}^{\beta} -d\theta = \pi - \alpha - \beta.$$

Let Δ_1 be an assymptotic triangle with vertices A and B and let Δ_2 be an assymptotic triangle with vertices A and C, such that B and C are on a Euclidean line orthogonal to $\delta\mathbb{H}^2$ and C is above B. Suppose that the interior angles of Δ_1 are α_1 at A and β at B and the interior angles of Δ_2 are α_2 at A and γ_2 at C. Now $\Delta = \Delta_1 \cap \Delta_2$ is a triangle with vertices A, B and C and interior angles $\alpha = \alpha_1 - \alpha_2$ at A, β at B and $\gamma = \pi - \gamma_2$ at C. We have:

$$\begin{aligned} \text{Area}(\Delta) &= \text{Area}(\Delta_1) - \text{Area}(\Delta_2) \\ &= \pi - \alpha_1 - \beta - (\pi - \alpha_2 - \gamma_2) \\ &= \pi - (\alpha_1 - \alpha_2) - \beta - (\pi - \gamma_2) \\ &= \pi - (\alpha + \beta + \gamma). \end{aligned}$$

□

The previous theorem can be used to calculate the area of a hyperbolic n-gon.

Corollary 2.4. *The area of a hyperbolic n-gon Π with angle sum σ is given by $(n-2)\pi - \sigma$.*

Proof. Choose a point p in the interior of Π such that for every vertex v of Π the geodesic segment connecting p to v is contained in Π. (this is certainly possible for convex polygons). Connecting all the vertices to the point p by such geodesics segments yields a triangulation of the polygon by n triangles $\Delta_1, \ldots, \Delta_n$. The Δ_i have a common vertex at p and denote the interior angles of Δ_i at p by γ_i. Then $\gamma_1 + \ldots + \gamma_n = 2\pi$. If we sum up the other angles, we get the sum σ of the interior angles of Π. Hence we have:

$$\text{Area}(\Pi) = \text{Area}(\Delta_1) + \ldots + \text{Area}(\Delta_n) = n\pi - (2\pi + \sigma).$$

□

We wish to construct a *regular* polygon in the $\mathbb{D}^2$ model. By a *regular polygon centered at the origin* we mean a polygon with vertices $z_0, \ldots, z_n$ such that $\text{Arg}(z_i) - \text{Arg}(z_{i+1})$ and $\rho(0, z_i)$ are constant. For our purposes of constructing surfaces as identification spaces of polygons we require the angle sum of the polygons to be 2π. The constant $\text{Arg}(z_i) - \text{Arg}(z_{i+1})$ is easily seen to be $\frac{2\pi}{n}$, and the interior angles are given by $\frac{\pi}{n}$. The only thing missing in our construction is the Euclidean distance of the vertices to the origin.

In the $\mathbb{D}^2$ model, choose coordinates such that Π_n is centered at 0 and z_0 is on the real axis. Divide Π_n into n isosceles triangles with common vertex 0 (as the centre of $\mathbb{D}^2$). The triangle $\Delta(0, z_0, z_1)$ has at the vertex 0 an interior angle of $2\pi/n$ and the angles at z_0 and z_1 are both equal to π/n. Now we can calculate the hyperbolic length $\rho(0, z_0)$ by using the second cosine rule (see [4]). The second cosine rule is expressed by (see image below):

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

For our triangle we have $c = \rho(0, z_0)$, and $\alpha = 2\pi/n$ and $\beta = \gamma = \pi/n$. This yields:

$$\cosh c = \frac{\cos(2\pi/n)\cos(\pi/n) + \cos(\pi/n)}{\sin(2\pi/n)\sin(\pi/n)}.$$

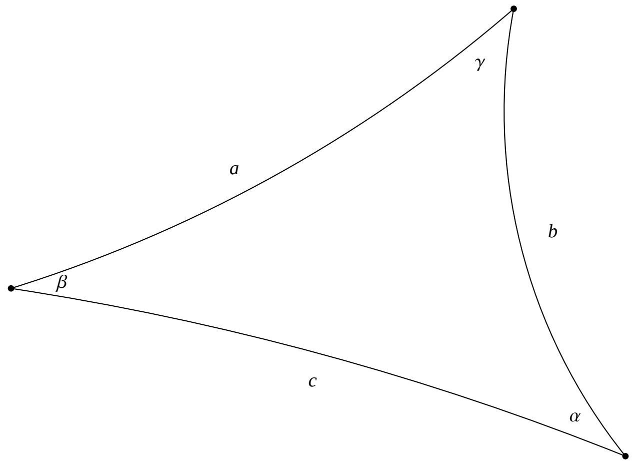

Figure 3: A triangle in the hyperbolic disc and the notation used in stating the second cosine rule.

Using the double angle formulas $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 2\cos^2(x) - 1$, we obtain

$$\cosh c = \frac{\cos^2(\pi/n)}{\sin^2(\pi/n)} = \frac{1}{\tan^2(\pi/n)}.$$

On the other hand, we have that

$$c = \rho(0, z_0) = \log\frac{1+r}{1-r} \implies \cosh(c) = \frac{1+r^2}{1-r^2}$$

where r is the Euclidean distance of z_0 to the origin. We thus have the following:

$$\tan^2(\pi/n) = \frac{1-r^2}{1+r^2} \implies r^2 = \frac{1-\tan^2(\pi/n)}{1+\tan^2(\pi/n)},$$

which can be simplified to:

$$r^2 = \cos^2(\pi/n) - \sin^2(\pi/n) = \cos(2\pi/n).$$

The regular octagon is of special interest to us, since it will be used in the construction of the double torus. For the vertices of the regular octagon centered at the origion we thus have that their Euclidean distance r to the origin in the $\mathbb{D}^2$ model is $\frac{1}{2}^{\frac{1}{4}}$.

2.5 Side-pairing Transformations

A side-pairing transformation for a polygon Π is an isometry which maps one side of Π to another side of Π. We are merely interested in the orientation preserving transformations.

Lemma 2.5. *An orientation preserving isometry is determined by the image of two points.*

Proof. Let f and g be orientation preserving, such that $f(z) = g(z)$ and $f(w) = g(w)$ for two distinct z, w in the hyperbolic plane. Since f and g are isometries, their inverses exist and are orientation preserving also. Thus we find that $g^{-1}f$ is an orientation preserving isometry with two fixed points. By the classification of isometries we know that the only orientation preserving isometry having more than one fixed point is the identity. Hence $f = g$. $\square$

Let p_1 and p_2 be side of the polygon Π, which are segments of the hyperbolics lines L_1 and L_2 respectively. We have seen that for every pair of lines, there is an isometry mapping the one line onto the other. If L_1 and L_2 are disjoint, the isometry is a translation, if the lines are intersecting, the isometry is a rotation and if the lines are assymptotic, the isometry is a limit translation.

3 Geometric Surfaces

In this section we introduce the notions and terminology used in our discussion of surfaces. We give the construction of surfaces by means of identifying the edges of a so-called fundamental polygon. The final goal of this section is to show that every complete and connected geometric surface S of constant curvature can be given as a quotient $\tilde{S}/\Gamma$, where $\tilde{S}$ is the universal covering surface of S and Γ is a discontinuous and fixed-point free group of $\tilde{S}$ isometries.

Definition 3.1. A *geometric surface* S is a surface with a distance function $d_S(p, q)$ defined for all $p, q \in S$, such that for each point $p \in S$ there is an isometry $f : D_\epsilon(\tilde{p}) \to D_\epsilon(p)$, where $\tilde{S}$ is one of $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$ with their usual geometry, and we say that a surface is Euclidean, spherical or hyperbolic (respectively, depending on $\tilde{S}$).

For each point p in S we thus have a neighborhood isometric to a spherical, Euclidean or hyperbolic disc and hence we can measure angles, length and area. A *polygonal path* on S is a path on S that consists of the images of geodesic segments in $\tilde{S}$. A *line segment* on S is a polygonal path such that succesive segments meet at a straight angle. The surface is said to be *complete* if every line segment can be extended indefinetely. The Euclidean plane minus a point is an example of a geometrical surface which is lacking the property of completeness. The property of completeness excludes all surfaces with boundary.

Compact surfaces have been classified, see for instance [1, 2, 3]. The key to this classification is the notion of genus. For orientable surfaces, the genus counts the amount of holes in the surface: e.g. the torus is a genus 1 surface and the sphere is a genus 0 surface. In a non-orientable surface, it counts the amount of so-called crosscaps. The projective plane (the result of identifying anti-podal points on a sphere) is a non-orientable genus 1 surface, and the Klein-bottle is a non-orientable genus 2 surface. The statement of the classification of compact surfaces is that each compact surface is homeomorphic to a sphere with n-handles if it is orientable and homeomorphic to a projective plane with n crosscaps if it is non-orientable. In this text we only deal with orientable surfaces, in particular those with higher genus. When we discuss how to construct surfaces by identifying edges of a polygon, we will see how genus relates to the edge-identification of a polygon.

3.1 The Euler Characteristic

We assume that any compact 2-manifold can be triangulized (actually this is well known, but we wish to refrain from discussing the formalities). If a surface S has a finite triangulization with V vertices, E edges and F faces, then the *Euler characteristic* of the surface S is

$$\chi(S) = V - E + F.$$

The Euler characteristic is invariant under triangulation refinements. Refining a triangulation can be done step by step: in each step adding a new vertex, and connecting the new vertex to a previously existing vertex by a new edge, which increases both V and F by one and E by two.

In the case of compact surfaces, the Euler characteristic is directly related to the notion of genus. In particular if we have an orientable surface of genus g, it has Euler characteristic $2-2g$. By relating the Euler characteristic of a covering surface to it's base surface, we can compute the genus of the covering surface. We will discuss the details when we get to discussing covering spaces.

3.2 Surfaces as Identification Spaces

We construct compact surfaces by gluing the edges of a polygon in pairs. As an example we construct a torus which can be constructed by gluing the edges of a rectangle in pairs. Consider a square and imagine gluing two of it's edges together without 'twisting' the square (otherwise you get what is called a Möbius strip) to obtain a cylinder. Now imagine gluing the two ends of the cylinder together to obtain a torus. This construction can be slightly formalized and generalized to other topological surfaces. If interested reader is referred to [1, 2] for more detailed information.

Consider a polygon Π in the Euclidean plane, i.e. a closed polygonal curve and it's interior. We thus have a set of vertices, geodesic segments between the vertices called edges, and the region enclosed by the polygonal path. If the Π has n vertices, we lable the vertices $v_0, v_1, \ldots, v_{n-1}$ in a counter clockwise manner, and we lable the edges $[v_i, v_{i+1}]$ by e_i where the indices i are taken modulo n. We will use e_i^{-1} to denote the edge $[v_{i+1}, v_i]$, which is the same edge as e_i but with opposite orientation.

To obtain a surface we identify the edges of a polygon in (disjoint) pairs $\{e_i, e_j\}$ called *edge-pairs*. By identifying edges we automatically identify vertices, and this results in (disjoint) sets of identified vertices called *vertex cycles*. The *identification surface* S_π of a polygon Π is a surface which consists of

- Interior points, being the interior points of the polygon,
- pairs $\{e_i, e_j\}$ of identified interior points on the edges and
- vertex cycles, which are sets $\{v_1, \ldots, v_n\}$ of identified vertices.

With a particular surface in mind, we relable the edges of the polygon Π in the following way. If $e_i^{\pm 1}$ is to be identified with $e_j^{\pm 1}$ then we relable both of them with a common lable $a^{\pm 1}$. The surface obtained is called *orientable* if identified edges have opposite orientation, and *non-orientable* otherwise.

As an example we present the torus as an identification space of a square with boundary $aba^{-1}b^{-1}$. This is identification space with the interior points being the interior of the square, with edges a and b being identified with the boundary of the square, and a single vertex v identified with the vertices of the square. Since the identified edges occur with opposite orientations, this is an orientable surface.

We could also have labeled the edge of the square $aba^{-1}c$ to obtain a cylinder with boundaries b and c. A *closed* surface is a surface without boundaries, and hence, when given as an identification space of a polygon, requires a polygon with an even number of sides which all need to be identified.

Consider two pentagons, one with edge labeling $a_1b_1a_1^{-1}b_1^{-1}c$ and one with edge labeling $a_2b_2a_2^{-1}b_2^{-1}c^{-1}$. Following the identifications as per boundary labeling, we obtain two tori with boundary c and c^{-1} respectively. We could then identify c and c^{-1} to obtain what one might call a 'double torus'. It should be intuitively clear that we could also have pasted the pentagons along their c edges to obtain an octagon. Hence we see that a 'double torus' can be obtained by identifying the edges of a octagon with boundary $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$. The Euler charac-

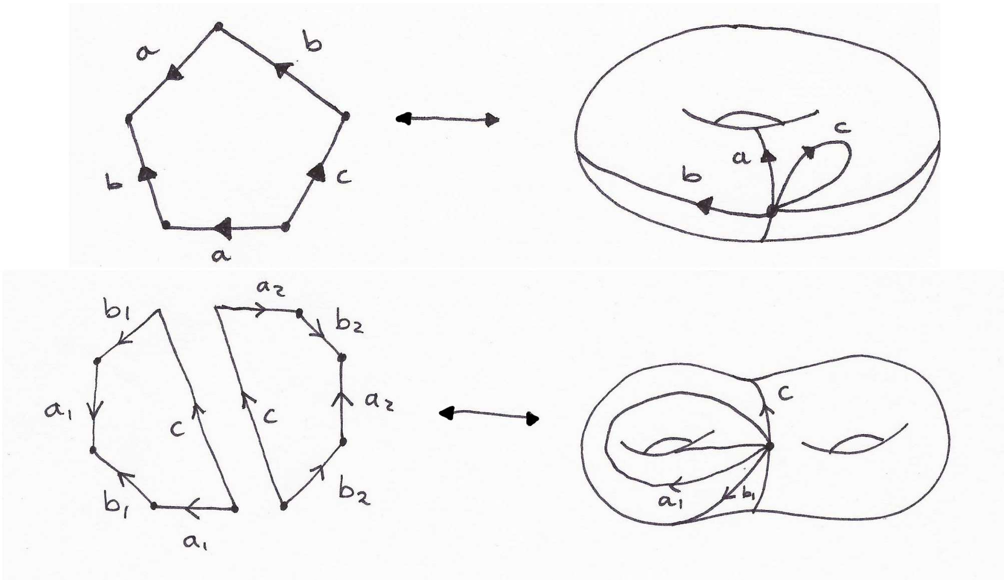

Figure 4: Visualizing the edge identification of an octagon to obtain a double torus.

teristic of surface given as the identification space S_Π of a polygon is easily computed; one just has to count the amount E of pairs of identified edges (edges on the surface) and the amount of V vertex cycles (vertices on the surface), and the Euler characteristic is $\chi(S_\Pi) = V - E + 1$. Various polygons may yield the same topological surface. However, any compact topological surface can be given in *normal form.* This amounts to the following. Each compact orientable surface is homeomorphic to an identification space S_Π, where Π is a polygon with a boundary of the form aa^{-1} or $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1}$. Here each $a_i b_i a_i^{-1} b_i^{-1}$ amounts to a handle of the surface. The genus g of a surface is the amount of handles in a surface, and by the normal form we can see that the Euler characterstic of such a surface is $2 - 2g$. From this we can easily tell the genus of a surface given as the identification space, since we only need to compute the Euler characteristic.

To prove that an identification surface is indeed a surface, one could go on to show that any point in the identification surface has a neighborhood homeomporhic to a disc in $\mathbb{R}^2$. In [1] the reader is assured that any compact (topological) surface can be realized as an identification space of a polygon having a single vertex cycle. Such a surface is not necesarily a geometric surface, but if the polygon satisfies certain side and angle conditions we have the following: If S_Π is a identification surface of a polygon Π such that paired sides have equal length and the interior angles of the vertices in each vertex cycle sum up to 2π, then S_Π is a geometric surface. A formal proof can be found in [1]. The idea of the proof is to construct a neighborhood that is isometric to an open disc in the hyperbolic plane. For edges, these neighborhoods consist of two half-discs and for a vertex cycle containing n vertices of Π they consist of n disc slices.

Theorem 3.2. *Any compact surface can be realized geometrically.*

Proof. If S is constructed by a 2-gon, it is a sphere or a projective plane. If S is constructed by usign a 4-gon, it is either the sphere, the projective plane, the torus or the Klein bottle. Thus we assume that S is realized by a $2n$-gon, where $n > 2$. As mentioned, any surface can be constructed by as an identification space having a single vertex cycle. So S is a geometric surface when the interior angles of the $2n$-gon sum up to 2π. Such a polygon can always be constructed in the hyperbolic plane, as we have shown by giving an explicit construction in the section on hyperbolic polygons. $\square$

From this we see that a geometric genus $g > 1$ surface is necesarily a hyperbolic surface.

3.3 Quotient Surfaces

The construction of a surface via the concept of identification space of a polygon in $\tilde{S} = \mathbb{R}^2, \mathbb{H}^2$, or $\mathbb{S}^2$ suffices to construct all the compact and connected surfaces (as can be seen from the topological classification of surfaces, see [1, 2]). The choice of the particular polygon to construct a surface seems rather arbitrary. We now show how to construct a surface as a set of equivalence classes on $\tilde{S}$.

Let Γ be a group of isometries acting on $\tilde{S}$. The Γ-*orbit* Γx of a point $x \in \tilde{S}$ is defined by

$$\Gamma x = \{gx \mid g \in \Gamma\},$$

and is a subset of $\tilde{S}$. It is straightforward to check that 'being in the same Γ-orbit' is an equivalence relation on $\tilde{S}$, thus we can define the quotient $\tilde{S}/\Gamma$ being the set of Γ-orbits.

The quotient construction works for any such group Γ. However, we would like to define a distance function on $\tilde{S}/\Gamma$ such that the quotient becomes a geometric surface locally isometric to $\tilde{S}$. If we let $S = \tilde{S}/\Gamma$, and we have the standard distance function $d_{\tilde{S}}$ for $\tilde{S}$, we can define d_S as a candidate distance function for S as follows:

$$d_S(\Gamma x, \Gamma y) = \min\{d_{\tilde{S}}(x', y') \mid x' \in \Gamma x,\, y' \in \Gamma y\}.$$

As an example of a case where this definition fails to yield an actual distance function, suppose that Γ is the group acting on $\mathbb{R}^2$ generated by a rotation through an angle $\pi/2$ around the origin O. The quotient can be seen to be a 2-manifold, but the length of a circle centered at the point ΓO, has only a quarter of the length of a Euclidean circle. Hence, the disc neighborhood of ΓO is certainly not isometric to any euclidean disc, nor to a spherical or hyperbolic disc for that matter. Furthermore, to guarantee that in the definition of the quotient distance function an actual minimum exists, the gamma orbits should be without limit points.

Definition 3.3. A group Γ of isometries acting on $\tilde{S}$ is called *discontinuous* if no Γ-orbit has a limit point, and is called *fixed point free* if for any $g \in \Gamma$ not equal to the identity, $gx \neq x$.

We now show that d_S is indeed a distance function for S, when Γ is discontinuous and fixed point free. We have to show that there are indeed $x', y' \in \tilde{S}$ such that the minimum is attained. The set $\{d_{\tilde{S}}(x', y') \mid x' \in \Gamma x,\, y' \in \Gamma y\}$ certainly has an infimum $\epsilon \geq 0$ (since it is bounded from below). If there is no pair x', y' in their respective Γ-orbits such that $d_{\tilde{S}}(x, y) = \epsilon$, then x' would be a limit point for the Γ orbit of y.

We now simply write $x, y, \ldots$ for point of S. First of all, we have that $d_S(x, y) \geq 0$ and that $d_S(x, y) = 0 \iff x = y$. Furthermore, it is obvious that $d_S(x, y) = d_S(y, x)$. The triangle inequality also holds, since assuming points $x, y, z \in S$ satisfying $d_S(x, y) + d_S(y, z) < d_S(x, z)$, gives points $x', y', z' \in \tilde{S}$ violating the triangle inequality for $d_{\tilde{S}}$.

Theorem 3.4. *If Γ is a group of isometries acting on $\tilde{S}$, then Γ is discontinuous and fixed point free if and only if each $x \in \tilde{S}$ has a neighborhood U_x in which each point belongs to a different Γ-orbit.*

Proof. Let $x \in \tilde{S}$. Since Γ is discontinuous, Γx is without limit points so that

$$\inf\{d_{\tilde{S}}(x,y) \mid x, y \in \Gamma x,\ x \neq y\}$$

is strictly positive, and say equal to δ. Since Γ is fixed point free, the image of the disc $D_{\delta/3}(x)$ under any $g \in \Gamma - \{1\}$ is disjoint from the disc, and hence the disc cannot contain more than one point of each Γ-orbit. Since x was arbitrary, any x has a neighborhood in which each point belongs to a different Γ-orbit.

Conversely, suppose that each $x \in \tilde{S}$ has a neighborhood $D(x)$ in which each point belongs to a different Γ-orbit. This immediately contradics non-discontinuity, since any neighborhood of a limit point of a Γ-orbit contains infinetely many points in the same orbit. Now suppose that there is a $g \in \Gamma$ not equal to the identity that has a fixed point p. Any neigborhood of p contains a disc neighboorhood of p, and any point q in that disc neighboorhood is mapped into the disc, since $d_{\tilde{S}}(p,q) = d_{\tilde{S}}(gp, gq) = d_{\tilde{S}}(p, gq)$. This contradicts the hypothesis, thus Γ must be discontinuous and fixed point free. □

An immediate consequence of the previous theorem is the following corollary. Since each $x \in \tilde{S}$ has a neighborhood U_x in which each point belongs to a different Γ-orbit, the Γ orbit map acts as an bijection $U_x \to \Gamma(U_x)$. If we introduce the quotient distance function, Γ becomes a local isometry.

Corollary 3.5. *If Γ is a discontinuous fixed point free group of isometries acting on $\tilde{S}$, then $S = \tilde{S}/\Gamma$ is a geometric surface.*

A proof of a generalization of the previuos statement can also be found in chapter 6 of [4]. Note that in our statement the group is required to be fixed point free, thus excluding rotations (which are not excluded in the proof by Beardon).

3.4 Covering Surfaces

A *covering* of a surface S is a surjective map $f : C \to S$ such that C is a surface and for each point $p \in S$ there is a neighborhood U_p of p such that $f^{-1}(U_p)$ is a union of connected open sets V_i, and the restriction of f to each V_i is a homeomorphism $V_i \to U_p = f(V_i)$. The surface C is called the *covering surface* and the surface S is called the *base surface*. A *geometric covering* is covering that is a local isometry. The cardinality of the set $f^{-1}(p)$ is called the *multiplicity* of the covering f, and is the same for all $p \in S$.

If S is a geometric surface of constant curvature κ, then any covering surface in a geometric covering of S must also be a geometric surface of constant curvature κ. Furthermore, if the surface S has Euler characterstic $\chi(S)$ then a covering surface C in a covering of multiplicity k has euler characteristic $\chi(C) = k\chi(S)$. This can be seen by lifting a triangulation of S to C, where each vertex, edge and face is lifted to k copies. For suppose that we have a triangulation of the surface S, and a covering $f : C \to S$ of multiplicity k. Because f is a local isometry, we can refine the triangulation of S such that each triangle is contained in a neighborhood which is the isometric image of a disc in C. Then each triangle in the triangulation is lifted to k triangles in C. Since f is a local isometry, the configuration of triangles around a vertex is preserved. Hence we get a triangulation of C, with k times as many vertices, edges and faces. Hence we have that $\chi(C) = k\chi(S)$.

Now let S be a compact orientable surface of genus g, and let $f : C \to S$ be a covering of multiplicity k. The Euler characteristic of S is $2 - 2g$. For the genus g' of C we must have that:

$$2 - 2g' = 2k - 2kg \implies g' = 1 + k(g - 1).$$

Thus we see that any covering surface for a covering of multiplicity k of the torus is again a torus, and that for the double torus it is a orientable genus $k+1$ surface.

Are there surfaces which only can only be covered by itself? The answer is yes, and these surfaces are precisely those that are simply connected. As geometric surfaces of constant curvature -1, 0 or 1, only the hyperbolic plane, the Euclidean plane and the sphere remain. This implies that every covering surface C of a surface S can again be covered by one of the planes.

We now show that any compact and complete surface of constant curvature can be given as a quotient surface of a discontinuous group acting isometrically on the the universal coverings surface.Let S be a complete and connected geometric surface of constant curvature 1, 0 or -1, then S is locally isometric to the sphere, the Euclidean plane or the hyperbolic plane, respectively. Let $\tilde{S}$ denote $\mathbb{S}$, $\mathbb{R}^2$ or $\mathbb{H}^2$, according to the curvature of S. We thus assume that for each point $p \in S$, there is a $\tilde{p} \in \tilde{S}$, an $\epsilon > 0$ and an isometry $f : D_\epsilon(\tilde{p}) \to D_\epsilon(p)$. The choice of the point $\tilde{p} \in \tilde{S}$ is rather arbitrary, since we can map any $\tilde{p}$ to any $\tilde{q}$ by an isometry of $\tilde{S}$.

We will show that any complete and connected geometric surface of constant curvature can be given as a quotient of the form $\tilde{S}/\Gamma$ where Γ is a discontinuous and fixed-point free group of isometries of $\tilde{S}$. To this end, we use the *pencil map* to first construct a covering of S by $\tilde{S}$. The pencil map exploits the fact that $\tilde{S}$ is filled by geodesic lines through a fixed point.

Definition 3.6 (The Pencil Map)**.** Let S be a complete and connected geometric surface of constant curvature. Let $\tilde{S}$ be $\mathbb{S}^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$, according to the curvature of S. Choose a point $\tilde{O} \in \tilde{S}$, a point $O \in S$, and an isometry $f : D_\epsilon(\tilde{O}) \to D_\epsilon(O)$. The *pencil map* $P : \tilde{S} \to S$ is defined as follows: For each $\tilde{p}$, define $P(\tilde{p})$ to be the extension of the line segment Op out of O which is the image of $\tilde{O}p \cap D_\epsilon(\tilde{O})$ under f to the distance $d(\tilde{O}, \tilde{p})$ (which is possible by completeness of S).

Each point $\tilde{p} \in \tilde{S}$ is on a unique line through $\tilde{O}$ and the point $\tilde{p}$ is uniquely determined by the line through $\tilde{O}$ and $\tilde{p}$ and the distance $d(\tilde{O}, \tilde{p})$.

Theorem 3.7. *The pencil map $P : \tilde{S} \to S$ has the following properties:*

1. *each $p \in \tilde{S}$ has a neighborhood on which P is a local isometry.*
2. *P is onto S*

We suppose that the reader may find these claims intuitively clear. The surjectiveness can by embedding the polygon realizing the surface in $\tilde{S}$ and since $\tilde{S}$ is filled with geodesics through the origion, the polygon is also filled with geodesics through the origion. The local isometry property holds since we can move the polygon through the plane and see that the pencil map preserves distances inside the polygon. Note that completeness is a crucial property for the surface S for the pencil map construction to work, since we need to extend the lines on S indefinetely. A proof that the pencil map is indeed a surjection and a local isometry can be found in [1].

Definition 3.8 (The covering isometry group)**.** Let $P : \tilde{S} \to S$ be a geometric covering of a geometric surface S by a simply connected surface $\tilde{S}$. The covering isometry group is the group Γ defined by:

$$\Gamma = \{g \in \mathrm{Iso}(\tilde{S}) \mid Px = Pgx, \text{ for all } x \in \tilde{S}\}.$$

To see that this is indeed a group, note that $g \in \Gamma$ implies that $Px = Pgg^{-1}x = Pg^{-1}x$. Furthermore, the product of covering isometries is again a covering isometry. Since the identity transformation is always included in Γ we see that the covring isometry is indeed a group.

Theorem 3.9. *If $Px = Py$ for some $x, y \in \tilde{S}$, then $x = gx$ for some covering isometry $g \in \Gamma$.*

Proof. Since P is a local isometry, there are isometric disc neighborhoods D_x of x and D_y of y and an isometry g such that $gD_x = D_y$. We will show that this g satisfies $Pgx = Px$ for all $x \in \tilde{S}$. Suppose that there is an element $r \in \tilde{S}$ such that $Pgr \neq Pr$. Since P is a local isometry, the set $\{x \in \tilde{S} \mid Pgx = Px\}$ is open, and hence the set $R = \{x \in \tilde{S} \mid pgx \neq px\}$ is closed. Since R is closed, there is a point q in R which has the smallest possible distance to x (of all points of R). Now we construct a sequence of points r_i in $\tilde{S} - R$ converging to q. Since P and g are continous we must have that

$$P(\lim_{i\to\infty} r_i) = Pg(\lim_{i\to\infty} r_i),$$

or: $Pq = Pgq$, which is a contradiction. Therefore R must be empty, or in other words $Pgx = Px$ for all $x \in \tilde{S}$ □

From this it can be seen that any geometric surface can then be realized as a quotient surface. By construction the universal covering via the pencil map P, we have obtained a group Γ such that $P(x) = \pi(x)$ where π is the orbit map of Γ.

Theorem 3.10. *If Γ is a fixed point free and discintinuous group acting on $\tilde{S}$, and Γ' is a subgroup of index k, then $\pi : \tilde{S}/\Gamma' \to \tilde{S}/\Gamma, \Gamma' x \mapsto \Gamma x$ is geometric covering of multiplicity k.*

Proof. Let $\pi_1 : x \to \Gamma x$ and $p_2 : x \to \Gamma'$ be the orbit maps. Let $\Gamma' \subset \Gamma$ be a subgroup. First of all we have to show that $\Gamma' x \mapsto \Gamma x$ is properly defined. To this end, assume that $\Gamma' x = \Gamma' y$. By definition we must have that $x = gy$ for some $g \in \Gamma'$. Since $\Gamma' \subset \Gamma$ is a subgroup, we have that:

$$\pi(\Gamma' x) = \pi(\Gamma' gy) = \Gamma gy = \Gamma y = \pi(\Gamma' y).$$

We continue our proof by showing that this mapping is indeed a geometric covering. Let $x \in \mathbb{D}^2$, then there is an open disc $U_x \subset \mathbb{D}^2$ of x such that the restriction of π_1 to U_x is a local isometry $U_x \to \mathbb{D}^2/\Gamma$ and $\pi_1^{-1}\pi_1(U_x)$ is a disjoint union of open discs $V = \Gamma U_x = \{gU_x \mid g \in \Gamma\}$. Since Γ' is a subgroup of finite index, we have a coset decomposition

$$\Gamma = \Gamma' h_1 \cup \Gamma' h_1 \cup \ldots \cup \Gamma' h_k.$$

Clearly, this gives a partition of V by cosets of Γ_2:

$$V = \Gamma' h_1 U_x \cup \Gamma' h_1 U_x \cup \ldots \cup \Gamma' h_k U_x.$$

Furthermore, for each $i \neq j$ the intersection $\Gamma' h_i U_x \cap \Gamma' h_j U_x = \emptyset$, and each $\Gamma' h_i U_x$ is mapped onto ΓU_x by π. So $\pi^{-1}(U_x)$ is a disjoint union of k open discs in $\mathbb{D}^2/\Gamma'$. Now π clearly is an isometry on each connected componontent of $\pi^{-1}(\pi_1(U_x))$, thus π is indeed a covering map of multiplicity k.

□

3.5 Tesselation of the Plane by the Fundamental Polygon

In this section we will show how the construction of a geometric surface as an identification space and as a quotient surface can be translated into each other. A *fundamental set* for a group Γ acting on $\tilde{S}$ is a subset F of S, such that the interior of S contains precisely one representative of each Γ orbit.

A fundamental domain for a discontinuous fixed point free group Γ of isometries on $\tilde{S}$ is a connected open set $\Pi \subset \tilde{S}$ such that $\tilde{S}$ is tesselated by copies of Π, meaning that:

1. For all $g, h \in \Gamma$ with $g \neq h$, $g\Pi \cap h\Pi = \emptyset$.

2. The union of $g\bar{\Pi}$ over $g \in \Gamma$ is equal to $\tilde{S}$, where $\bar{\Pi}$ is the closere of Π in $\tilde{S}$.

Any compact surface that is contstructed as a quotient of $\tilde{S}$ by a fixed point free and discontinous group, can also be realized as the identification space of a polygon. For any such group we can construct the so-called *Dirichlet polygon.* Let w be a point of $\tilde{S}$. The Dirichlet region with respect to Γ and w is defined by:

$$D(w) = \{x \in \tilde{S} \mid d(x, w) \leq d(x, gw), \text{ for all } g \in \Gamma - \{Id\}\}.$$

This region contains at least one representative of each Γx, and the interior contains at most one. Note that $D(w)$ can be written as the intersection over all $g \in \Gamma$ of the closed half-planes

$$H_g(w) = \{x \in \tilde{S} \mid d(x, w) \leq d(x, gw)\}.$$

Hence $D(w)$ is a closed and convex region with boundary consisting of segments of the lines equidistant to w and gw:

$$L_g(w) = \{x \in \tilde{S} \mid d(x, w) = d(x, gw)\}.$$

We still have to prove that there are only finitely many $L_g(w)$ contributing to the boundary of $D(w)$. Since we have assumed that $\tilde{S}/\Gamma$ is compact, we must have that $D(w)$ is compact which implies that it is contained in a disc of radius ρ. Suppose that infinitely many $L_g(w)$ contribute to the boundary of $D(w)$. Then by defintion of $D(w)$, there are infinitely many points of the form $g(w)$ within a distance of 2ρ of w. This contradicts the assumption that Γ is discontinuous. Thus $D(w)$ is a convex polygon. The group Γ then tells us which sides of $D(w)$ are identified, thus $\tilde{S}/\Gamma$ is now realized as the identification space of the polygon $D(w)$.

On the other hand, suppose that S is realized as the identification space of polygon Π. By using the pencil map, we can lift the polygon to the universal covering surface $\tilde{S}$, and conclude that it is tesselated by copies of Π. Furthermore, each copy is of the form $g\Pi$, where g is a covering isometry. There is a theorem due to Poincaré which says the following:

Theorem 3.11 (Poincaré (1882))**.** *A compact polygon* Π *satisfying the side and angle conditions is fundamental polygon for the group* Γ *generated by the side-pairing transformations of* Π.

This can be proven under conditions which are stronger than the ones we have discussed. It suffices to require that the angle sums of the vertex cycles sum up to an aliquot part of 2π, i.e. to $2\pi/n$ for some positive integer n. This goes into the realm of so-called orbifolds, but we wish to stay in the subrealm of geometric surfaces. A proof can be found in chapter 7.4 of [1] and in chapter 9.8 of [4].

3.6 Group Presentations and Homomorphisms to Finite Cyclic Groups.

A presentation for a group G is an expression of the form

$$< g_1,\ g_2,\ \dots \mid R,\ P,\ Q,\ \dots >,$$

where $g_1,\ g_2,\ \dots$ are such that every $g \in G$ can be written as a product of the $g_1,\ g_2,\ \dots$, and $P,\ Q,\ R,\ \dots$ are products of the $g_1,\ g_2,\ \dots$ such they are equal to the identity element of G. A group G has a finite presentation if the $g_1,\ g_2,\ \dots$ are finite.

Conversely any expression

$$< g_1,\ g_2, \dots g_n \mid R_1, \dots, R_m >$$

is a group. An element of the group is an expression of the form $r_1 r_2 \ldots r_n$, where the $r_i \in \{g_1, g_1^{-1}, \ldots, g_n, g_n^{-1}\}$ and are called *words*, the group operation being concatenation and the identity element being the empty word. The trivial relations are $g_i g_i^{-1} = g_i^{-1} g_i = \mathrm{id}$. For example, the group $\mathbb{Z}_n$ has presentation

$$\mathbb{Z}_n = < a \mid a^n > .$$

Words w_1 and w_2 in $g_1, \ldots, g_n$ are equivalent in G if w_2 can be obtained from w_1 by using the following operations:

1. Insertion or deletion of R_i in the word w_1, and
2. Insertion or deletion of $g_i g_i^{-1}$ or $g_i^{-1} g_i$.

Such groups are called combinatorial groups and have formidable applications in topology, where for instance they are used in describing certain topological invariants (namely groups isomorphic to the groups realizing our geometric surfaces as quotient surfaces). For the details on group presentation and the theory and application of combinatorial groups, we refer the reader to [2].

We are discussing group presentations because will use them in the construction for coverings for quotient surfaces. The group T we use to express the torus as quotient surface $\mathbb{R}^2/T$ has presentation

$$< a, b \mid aba^{-1}b^{-1} > .$$

The group Γ that we will use to construct the double torus as a quotient $\mathbb{D}^2/\Gamma$ has presentation

$$< a, b, c, d \mid ab^{-1}cd^{-1}a^{-1}bc^{-1}d > .$$

What we have in mind is constructing subgroups of these groups as kernels of homomorphisms onto other groups. To this end, we will use the following construction:
Let $G = < g_1, \ldots, g_n \mid R_1, \ldots, R_m >$. We can construct a mapping $f : G \to H$ by setting

$$f(g_i) = a \in H \text{ and } f(g_i^{-1}) = a^{-1} \in H,$$

and by letting $f(w)$ for a word $w = r_1 r_2 \ldots r_n$ be

$$f(w) = f(r_1) f(r_2) \ldots f(r_n).$$

If by using these definitions, $f(R_i) = \mathrm{id} \in H$ for all relations R_i in the presentation of G, then f is a homomorphism. To see that the mapping is indeed properly defined, note that if $u = g_{i_1} \ldots g_{i_n}$ and $v = g_{j_1} \ldots v_{j_m}$ are equal in G, then u can be converted into v by inserting or deleting the trivial relations or the relations R_i's in u. Obviously the image of u under f does not change under these operations, since $f(R_i) = \mathrm{id}$, and thus $f(u) = f(v)$. Furthermore it easy to see that $f(uv) = f(u)f(v)$, $f(u^{-1}) = f(u)^{-1}$, and $f(\mathrm{id}) = \mathrm{id} \in H$.

4 Covering Constructions

As we have seen, subgroups of the group giving a surface as a quotient corresponds to a covering surfaces. In order to find a fundamental polygon for such a subgroup, one could construct the Dirichlet polygon. Another method is by finding a particular coset decompostition. The coset representatives can then be used to find a fundamental polygon for the subgroup. Obviously, the union of images of the original fundamental polygon under the coset representatives is a fundamental set for the subgroup. We will show that one can always find a coset decomposition such that this union is a polygon.

Theorem 4.1. *Let Γ be a discontinous group of isometries of $\tilde{S}$ and Π be a compact fundamental polygon for Γ. If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then there is a coset decomposition*

$$\Gamma = \Gamma' g_1 \cup \Gamma' g_2 \cup \ldots \cup \Gamma' g_n$$

such that

$$\Pi' = g_1\Pi \cup g_2\Pi \cup \ldots \cup g_n\Pi$$

is a fundamental polygon for Γ'.

Proof. Since $\tilde{S}$ is tesselated by copies of Π under the action of Γ, by finiteness of the cosets we can lable each tile $g\Pi$ according to which coset $\Gamma' g_i$ the element $g \in \Gamma$ is in. We conclude that for any coset decomposition of Γ by Γ', the set $\Pi' = g_1\Pi \cup g_2\Pi \cup \ldots \cup g_n\Pi$ is a fundamental region for Γ'. Since the coset labeling consists of finitely many different lables, we can find a sequence R of tiles sharing at least one edge, such that a tile of each lable is included in R. If $f_1, \ldots, f_k$ are all the side-pairings of Π (of which there are only a finitely many, since Π is compact and hence has only finitely many sides), the configuration of the coset labeling around a tile $g\Pi$ is determined by to which cosets the gf_j's belong.

Suppose that the sequence R contains at least two tiles $h_1\Pi$ and $h_2\Pi$ corresponding to the same coset $\Gamma' g_i$. We then wish to drop the tile h_2 from R, but in doing so we may lose the connectedness of R. However, since the same configuration of coset labeling occurs around h_1 and h_2 ($h_1 f_j$ and $h_2 f_j$ belong to the same coset if and only if h_1 and h_2 belong to the same coset, thus the same coset labeling occurs around h_1 and h_2), we can find tiles with the same lable around h_1 as those around h_2. We thus have a region R still containing at least one tile of each coset lable and one tile less of the same lable as h_1. Continuing this process, we obtain a connected region which contains exactly one tile of each coset lable.

We still have to prove that the region R is a polygon, that is a region bounded by a simple curve. Suppose that this is not the case: the boundary of R does not divide the plane into two regions. Then part of the boundary of R must be the boundary of a smaller region R' inside R. But the whole plane is tesselated by copies of R (as we have seen it to be a fundamental region for Γ'), so the region R' must contain a copy of R, and thus a smaller copy of R' and so on. This contradicts the discontinuity of Γ, and thus we conclude that R must be a polygon. $\square$

Next we consider a construction of covering surface for the double torus as an identification space by pasting copies of the normal form fundamental polygons together. This yields a new polygon and we investigate the edges and the vertex cycles.

4.1 Construction of Covering Surfaces for the Double Torus

Let Π be a fundamental polygon with boundary path $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ so that the identification space becomes a orientable genus 2 surface or a double torus. Let g be a side-pairing of Π realizing the edge-identification $a_1 \to a_1^{-1}$. Take Π' to be the polygon $\Pi \cup g\Pi \cup \ldots \cup g^{k-1}\Pi$. We label the vertices of Π as follows: z_0 is the initial vertex of a_0, z_1 the initial vertex of b_0, …, z_7 the inital vertex of d_0^{-1}. The boundary labeling of Π implies the boundary labeling of Π'. We denote de boundary of $g^i\Pi$ by $a_i b_i a_i^{-1} b_i^{-1} c_i d_i c_i^{-1} d_i^{-1}$. Since we are identifying a_0^{-1} with a_1, a_1^{-1} with a_2, …, a_{k-1}^{-1} with a_0, the only sides remaining in the boundary of Π' are a_0 and a_{k-1}^{-1}, which are to be identified and so we relable them respectively to A and A^{-1}. The boundary of Π' becomes:

$$Ab_0a_1b_1 \ldots b_{k-1}A^{-1}b_{k-1}^{-1}c_{k-1}d_{k-1}c_{k-1}^{-1}d_{k-1}^{-1}b_{k-2}^{-1}c^{k-2} \ldots b_0^{-1}c_0d_0c_0^{-1}d_0^{-1},$$

and this can be demonstrated by induction over k, the precise statement being that the following process yields a polygon Π' with aformentioned boundary.

Claim. *Consider the following process:*
Step 1: For each $g^i\Pi$, $0 \leq i \leq k-1$, relable the boundary to $a_i b_i a_i^{-1} b_i^{-1} c_i d_i c_i^{-1} d_i^{-1}$.
Step 2: Identify the side a_i^{-1} with a_{i+1}, for $0 \leq i \leq k-2$ (pasting the polygons).
Step 3: Relable a_0 and a_{k-1}^{-1} to A and A^{-1} respectively.
We claim that this process yields a surface S_k with $V = k$ vertices, $E = 3k+1$ edges and $F = 1$ face, thus a surface with Euler characteristic $\chi(S_k) = -2k = 2-2g \implies g = k+1$: an orientable genus $k+1$ surface.

Proof. The proof is by induction over k. The case $k = 1$ is trivial and we start with $k = 2$. Step 1 yields two polygons with boundary

$$a_0 b_0 a_0^{-1} b_0^{-1} c_0 d_0 c_0^{-1} d_0^{-1}, \text{ and } a_1 b_1 a_1^{-1} b_1^{-1} c_1 d_1 c_1^{-1} d_1^{-1}.$$

Step 2 identifies a_0^{-1} with a_1. The boundary labeling becomes:

$$a_0 b_0 b_1 a_1^{-1} b_1^{-1} c_1 d_1 c_1^{-1} d_1^{-1} b_0^{-1} c_0 d_0 c_0^{-1} d_0^{-1}.$$

Note that this comes down to replacing a_0^{-1} by $b_1 a_1^{-1} b_1^{-1} c_1 d_1 c_1^{-1} d_1^{-1}$. Step 3 then relables a_0 to A and a_1^{-1} to A^{-1}, and this yields the claimed boundary labeling. We do not know right away that this yields a genus 3 surface: we still have to prove that it is in fact a surface, that the interior angles of the vertices in the vertex cycles add up to 2π.

The vertices of the side A are identified with the vertices of the side A^{-1}. The vertices of the side A are identified with all the other vertices of $g^0\Pi$ except those of a_0^{-1}. This yields the vertex cycle:

$$v_0 = \{z_0, z_1, gz_2, gz_3, z_4, z_5, z_6, z_7\}.$$

All the vertices in v_0 have interior angles $\pi/4$ and thus the angle sum is 2π. The remaining vertices are in the vertex cycle

$$v_1 = \{gz_4, gz_5, gz_6, gz_7, z_2 = gz_1, z_3 = gz_0\}.$$

By the pasting the interior angles of $z_2 = gz_1$ and $z_3 = gz_0$ are became $\pi/2$, so the angle sum of v_1 is 2π. So we see that the claim holds for $k = 2$.

Suppose now that the claim holds for $k = n$, thus that this yields a geometric surface S_n with n vertices, $3n+1$ edges. Now consider $k = n+1$ by identifying the edge a_{n-1}^{-1} of $g^{n-1}\Pi$ with a_n of $g^n\Pi$ (pasting $g^n\Pi$ to S_n, along A^{-1}). Step 1 is the same as in the case where $k = n$, except that we have an extra polygon with boundary labeling $a_n b_n a_n^{-1} b_n^{-1} c_n d_n c_n^{-1} d_n^{-1}$. In step 2 we have that by the induction assumption we have that the boundary labeling in the $k = n$ case is:

$$a_0 b_0 b_1 \dots b_{n-1} a_{n-1}^{-1} b_{n-1}^{-1} c_{n-1} d_{n-1} c_{n-1}^{-1} d_{n-1}^{-1} b_{n-2}^{-1} c_{n-2} \dots b_0^{-1} c_0 d_0 c_0^{-1} d_0^{-1}.$$

In the $k = n+1$ case we simply paste the polygon labeled $a_n b_n a_n^{-1} b_n^{-1} c_n d_n c_n^{-1} d_n^{-1}$ to the polgyon obtained in the $k = n$ case, which means replacing a_{n-1}^{-1} by $b_n a_n^{-1} b_n^{-1} c_n d_n c_n^{-1} d_n^{-1}$. Performing step 3 yields the boundary labeling

$$A b_0 b_1 \dots b_{n-1} b_n A^{-1} b_n^{-1} c_n d_n c_n^{-1} d_n^{-1} b_{n-1}^{-1} c_{n-1} d_{n-1} c_{n-1}^{-1} d_{n-1}^{-1} b_{n-2}^{-1} c_{n-2} \dots b_0^{-1} c_0 d_0 c_0^{-1} d_0^{-1}.$$

The induction hypotheses assumes that in S_n, there are n vertex cycles each of which has angle sum 2π. S_{n+1} has the same vertex cycles as S_n, except for the vertex cycles containing the vertices of a_n (in S_{n+1}) and the introduction of the new vertices of the added polygon $g^n\Pi$. The vertex cycles in S_n are:

$$\begin{aligned}
&v_0 = \{z_0,\ z_1,\ g^{n-1}z_2,\ g^{n-1}z_3,\ z_4,\ z_5,\ z_6,\ z_7\}\\
&v_1 = \{z_3 = gz_0,\ z_2 = gz_1,\ gz_4,\ gz_5,\ gz_6,\ gz_7\}\\
&\vdots\\
&v_{n-1} = \{g^{n-1}z_0 = g^{n-2}z_3,\ g^{n-1}z_1 = g^{n-2}z_2,\ g^{n-1}z_4,\ g^{n-1}z_5,\ g^{n-1}z_6,\ g^{n-1}z_7\}
\end{aligned}$$

Pasting $g^n\Pi$ to S_n means that v_0 becomes $\{z_0,\ z_1,\ g^nz_2,\ g^nz_3,\ z_4,\ z_5,\ z_6,\ z_7\}$ and that $v_n\{g^nz_0 = g^{n-1}z_3,\ g^nz_1 = g^{n-1}z_2,\ g^nz_4,\ g^nz_5,\ g^nz_6,\ g^nz_7\}$, and the other v_j $(0 < j < n)$ are unchanged. Both these vertex cycles have angle sum 2π, so S_{n+1} is again a geometric surface. The number of vertices in S_{n+1} is $n+1$, the amount of edges is the amount of edges of S_n plus the 3 new edges b_n, c_n, d_n thus $3(n+1)+1$, and still one face. Hence $\chi(S_{n+1}) = -2(n+1)$. □

4.2 Construction of Fundamental Polygons for Subgroups of a Torus Group

We can construct the torus as a geometric surface of constant curvature by identifying the opposite and oppositely oriented edges of a rectangle. This results in rectangles with boundary labeling $aba^{-1}b^{-1}$. An other option would be to identify the edges of a hexagon by either one of the following labelings $abca^{-1}b^{-1}c^{-1}$, or $aba^{-1}cb^{-1}c^{-1}$ (like libeled edges cannot contain the same vertex, since we require the vertex in the to have a disc neighborhood in the identification space).

To construct the torus as a quotient surface, we start out by choosing a square in the Euclidean plane. We take the square Σ with vertices

$$z_0 = \frac{1}{2}(-1-i),\ z_1 = \frac{1}{2}(1-i),\ z_2 = \frac{1}{2}(1+i),\ z_3 = \frac{1}{2}(-1+i).$$

The mappings

$$g_1 : z \mapsto z+1, g_2 : z \mapsto z+i$$

satisfy $g_1(z_0z_3) = z_1z_2$ and $g_2(z_0z_1) = z_3z_2$, so they are side-pairing transformations for Σ. These side pairings yield the boundary labeling $ab^{-1}a^{-1}b$ for Σ. Note that the inverses in this labeling are arbitrary, they only need to be opposite. We will stick to the convention that the labeling induced by side-pairing transformations g_i is such that $g_i : a \mapsto a^{-1}$. This is only convienient for keeping track of things, and has no content at all.

The group $T =< g_1, g_2 >$ is a discontinuous fixed point free group with fundamental domain Σ. Furthermore, we have that the surface S_Σ construted as an identification space is isometric to $\mathbb{R}^2/T$. The situation is depicted in figure 5.

As we have seen in section 4, a subgroup $H \subset T$ of index k yields a covering $\mathbb{R}^2/T \to \mathbb{R}^2/T$ of multiplicty k. We have discussed two constructions for fundamental polygon for such a group: one by finding a particular coset decomposition, the other by constructing the Dirichlet polygon for a subgroup. First we have to construct a subgroup of finite index, and we can do this by taking the kernel of a homomorphism from T to a finite group. First we construct a subgroup of index 2, by defining the following homomorphism $T \to \mathbb{Z}_2$. Let $f : T \to \mathbb{Z}_2$ be the mapping defined by $f(g_1) = f(g_2) = 1$, since $f(g_1g_2^{-1}g_1^{-1}g_2) = 1$ this mapping is properly defined and furthermore it is a homomorphism. We let $H = \ker(f)$. Since f is surjective, we get $T/H \cong \mathbb{Z}_2$. Since $g_1 \notin H$, we have the following coset decomposition:

$$T = H \cup Hg_1.$$

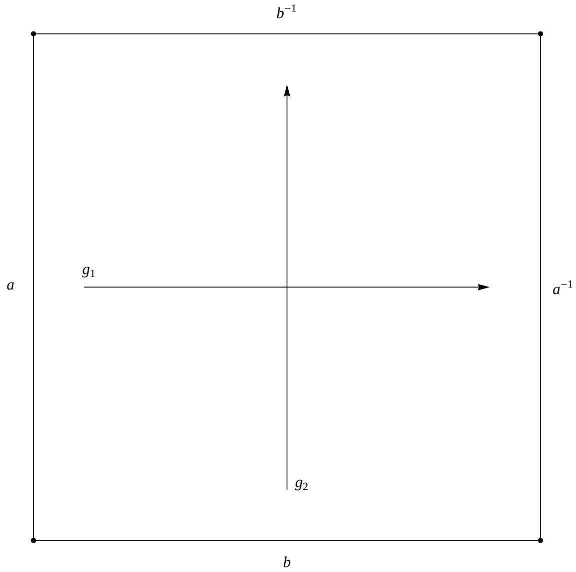

Figure 5: A square in $\mathbb{R}^2$, together with boundary labeling as induced by the side-pairing transformations g_1, g_2.

This yields the fundamental domain $\Sigma \cup g_1\Sigma$ for H. One can also construct the Dirichlet polygon for this subgroup. Note that we have $g_1g_2, g_1^{-1}g_2, g_1g_2^{-1}, g_1^{-1}, g_2^{-1} \in H$, and these are precisely the elements of H that contribute to the sides of the Dirichlet polygon, and these translations are the side-pairings for this polygon (see figure 6). We find that the fundamental polygon as constructed per coset decomposition is not the same as the Dirichlet polygon, but this was not to be expected. Note that the area of the Dirichlet region is twice the area of the fundamental polygon for T. The identification space of this Dirichlet polygon has 2 edges and 1 vertex cycle, and thus the resulting surface has Euler characteristic $1-2+1=0$, and hence has genus 1. This must be the case for any covering of the torus.

Now consider the kernel of the homomorphism $T \to \mathbb{Z}_3$ generated by $g_1, g_2 \mapsto 1$. Here something interesting happens: the Dirichlet polygon is not a rectangle but a hexagon (see figure 7). The elements $g_1g_2^2,\ g_1^2g_2,\ g_1g_2^2,\ g_1g_2^{-1},\ g_1^{-1}g_2^{-2},\ g_1^{-2}g_2^{-1}$ and $g_1^{-1}g_2$ are in the kernel of this homomoprhism, and they are the side-pairings for this Dirichlet polygon. One can check that this Dirichlet polygon has thrice the area of the fundamental polygon for T. The identification space has 3 edges and 2 vertex cycles, and hence has Euler characteristic $2-3+1=0$ and thus the resulting surface is of genus 0.

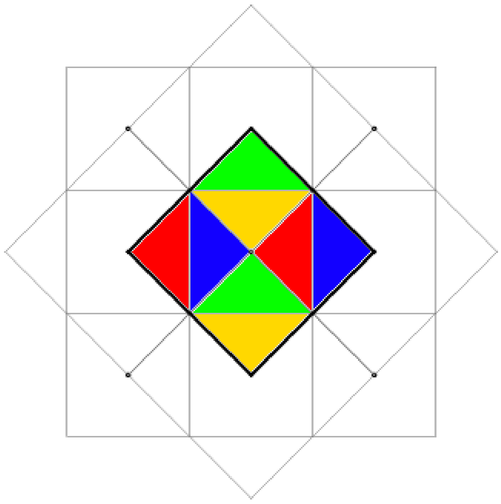

Figure 6: Dirichlet region for the kernel of $g_1, g_2 \to 1 \in \mathbb{Z}_2$ (colored square). The side-pairing transformations are g_1g_2, $g_1^{-1}g_2$, $g_1g_2^{-1}$ and $g_1^{-1}g_2^{-1}$. The tesselation of the fundamental polygon of the covered torus extends to a tesselation of the fundamental polygon for the subgroup such that each color occurs twice. This shows that it corresponds to a covering of multiplicity 2.

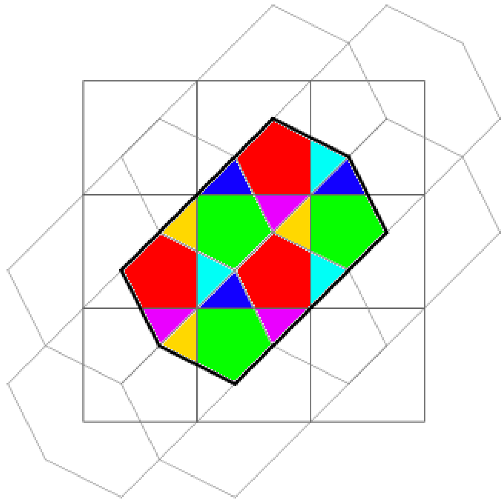

Figure 7: Dirichlet region for the kernel of $g_1, g_2 \to 1 \in \mathbb{Z}_3$ (colored hexagon). The side-pairing transformations are $g_1g_2^2$, $g_1^2g_2$, $g_1g_2^2$, $g_1g_2^{-1}$, $g_1^{-1}g_2^{-2}$, $g_1^{-2}g_2^{-1}$ and $g_1^{-1}g_2$. The tesselation of the fundamental polygon for the subgroup shows that it corresponds to a covering of multiplicity 3.

4.3 Construction of Fundamental Polygons for Subgroups of a Double Torus Group

To construct a compact orientable surface of genus 2, we follow [5] where an explicitit expression for the side-pairing isometries for an octagon with angle sum 2π is given. According to Poincaré's theorem we can start with an octagon Π satisfying the side and angle conditions, determine the side-pairing isometries $g_0, \ldots, g_3$ and their inverses. The group Γ generated by these side-pairings has Π as a fundamental domain, and we have that that $S_\Pi = \mathbb{D}^2/\Gamma$.

In the $\mathbb{D}^2$ model, we construct a regular octagon with angle sum 2π. The vertices are given by (see section 2.4):

$$z_k = 1/\sqrt[4]{2}e^{ik\frac{\pi}{4}}, \text{ where } k = 0, \ldots, 7.$$

In our notation for the vertices of Π, we consider the indices to be modulo 7. Since each side is of the same length and the angle sum is 2π, we get a geometric surface of genus two if the edges are identified in the right manner.

In [5] an explicit expression for side-pairing transformations for an octagon satisfying the side and angle conditions and vertices given by $z_0, \ldots, z_7$ is given:

$$G_k = \frac{-1}{\sqrt{1-|\omega_k|^2}} \begin{pmatrix} 1 & \omega_k \\ \bar{\omega_k} & 1 \end{pmatrix},$$

where ω_k is defined as:

$$\omega_k = \frac{z_{k-1}(1-|z_k|^2) + z_k(1-|z_{k-1}|^2)}{1-|z_k z_{k-1}|^2} = \frac{z_{k-1}+z_k}{1+R^2},$$

where the latter equality follows from our choice of vertices with $|z_k| = R := \sqrt[4]{1/2}$ for all k. In [5] it is asserted that for the ω_k we have $|\omega_k| < 1$. The matrices G_k correspond to the isometries g_k as discussed in section 2.3. These mappings are translations since they obviously have $\mathrm{tr}(g_k)^2 > 4$. Furthermore, the isometries g_k are such that $g_k(z_{k+3}z_{k+4}) = z_k z_{k-1}$. These side-pairings induce the following edge-labeling for Π:

$$a = z_4z_5,\ a^{-1} = z_1z_0,\ b = z_5z_6,\ b^{-1} = z_2z_1,\ c = z_6z_7,\ c^{-1} = z_3z_2,\ d = z_7z_0,\ d^{-1} = z_4z_3$$

The situation is depicted in figure 8.

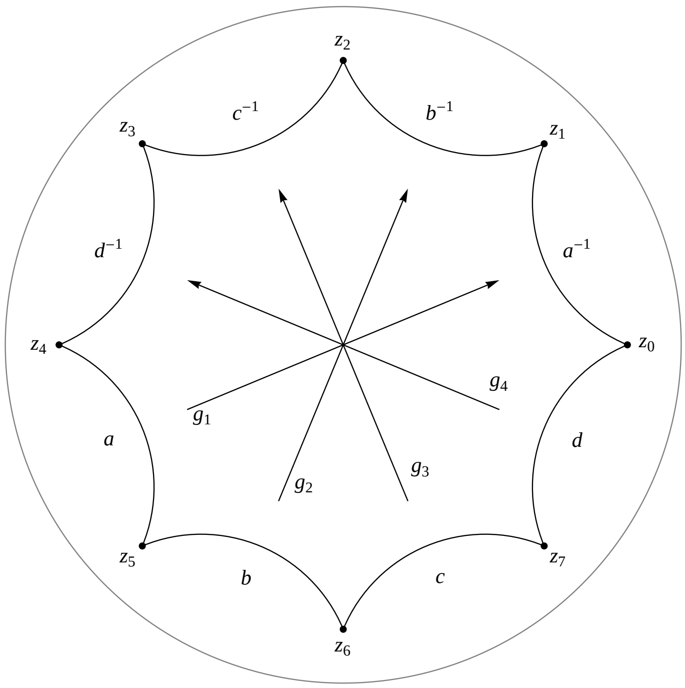

Figure 8: A regular octagon in the $\mathbb{D}^2$ model, together with boundary labeling as induced by the side-pairing transformations g_1, g_2, g_3, g_4.

The $\mathbb{D}^2$ disc is tesselated by copies of Π under the action of Γ. Part of this tesselation is depicted in figure 10. As you can see, there are eight copies of Π around each vertex, and the relation $g_1g_2^{-1}g_3g_4^{-1}g_1^{-1}g_2g_3^{-1}g_4 = \mathrm{id}$ can be seen from this. The labeling of the edges around each vertex is as depicted in 9.

We can construct subgroups of finite index as kernel of homomorphisms from Γ to a finite group (section 4). Using the structure of the kernel, we can generate a list of elements in it. We then consider the images of the disc origin under the action of this list, and construct the equidistant lines from 0 to these images. If we find a closed path around the origin consisting of segments of these equidistant lines, we have found the boundary of the Dirichlet polygon for the

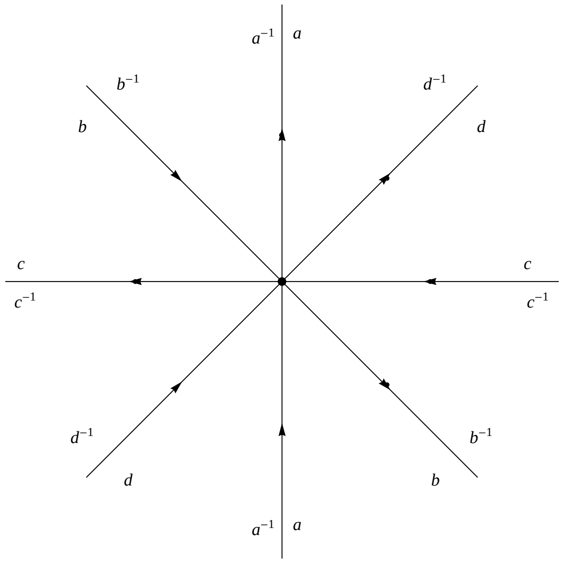

Figure 9: Boundary labeling around a vertex in the tesselation of $\mathbb{D}^2$ by copies of Π. Here you can see that $g_1g_2^{-1}g_3g_4^{-1}g_1^{-1}g_2g_3^{-1}g_4 = \mathrm{id}$ by noticing that it maps the copy Π around the vertex onto itself.

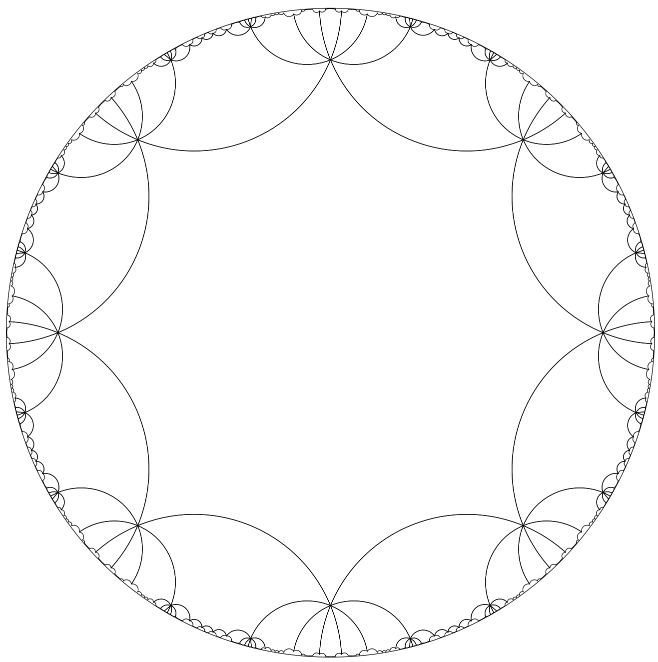

Figure 10: Tesselation of $\mathbb{D}^2$ by copies of the regular octagon under the action of the group generated by the side-pairings g_1, g_2, g_3, g_4.

subgroup. Care should be taken, for it might be possible that there is an element missing from the list that has a smaller length of translation. This would mean that the closed path we have found thus far is not the boundary of the Dirichlet polygon.

4.4 A double covering for the 2-torus

In order to construct a double covering for the 2-torus we define a homomorphism from the group Γ to $\mathbb{Z}_2$. The kernel H of this homomorphism will be a normal subgroup for Γ of index 2, and by section 4 we know that the quotient surface of this group yields a covering of multiplicity 2. To this end, let f be the homomorphism $\Gamma \to \mathbb{Z}_2$ constructed by:

$$f : \Gamma \to \mathbb{Z}_2 : g_1 \to 1, g_2 \to 1, g_3 \to 1, g_4 \to 1,$$

and let $H = \ker(f)$. Now H obviously is a subgroup of index 2. In constructing a fundamental polygon for H we can use the method outlined in the previous sections, or we could construct the Dirichlet polygon. In the case, we have constructed the Dirichlet polygon $D = D(0)$ for the subgroup H. The Dirichlet polygon for H is a 16-gon. It has eight vertices with interior angle $\pi/4$ and 8 vertices with interior angle $\pi/2$. The new vertices turn out to be the images of 0 under the action of the side-pairing transformations of Π. Furthermore, if we triangulate Π as depicted in figure 11 we see that D is Π together with the images of the triangles in the triangulation under the action of the side-pairing transformation of Π (see figure 11).

By using this triangulation, we can figure out the side-pairing transformations for D. Again, these side-pairings yield and edge-labeling which enables us to discuss some properties of the surface constructed as the identification space of D. It obviously is a double covering for $\mathbb{D}^2/\Gamma$, since each triangle in Π occurs twice in D. For instance, we can take a side of D that is part of the $g_1(\Delta)$ as in figure 11. We can see that the transformations $g_4^{-1}g_1^{-1}$ and $g_2g_1^{-1}$ map these sides to some other sides of D. Precoding in such a way we find the following side-pairing transformations and edge-labeling as depicted in figure 11.

$$\begin{array}{llll} g_a = g_1g_4 & g_b = g_2g_1^{-1} & g_c = g_3g_2^{-1} & g_d = g_4g_3^{-1} \\ g_e = g_1^{-1}g_4^{-1} & g_f = g_2^{-1}g_1 & g_g = g_3^{-1}g_2 & g_h = g_4^{-1}g_3 \end{array}$$

Performing the edge-identifications as per the the boundary labeling we find that the surface has three vertex cycles v_0, v_1, v_2:

$$\begin{aligned} v_0 &= \{z_1, z_3, z_5, z_7, z_9, z_{11}, z_{13}, z_{15}\}, \\ v_1 &= \{z_0, z_4, z_8, z_{12}\}, \\ v_2 &= \{z_2, z_6, z_{10}, z_{14}\}, \end{aligned}$$

each with angle sum 2π. Hence we indeed have a geometric surface. The Euler characteristic of the surface is $V - E + F = 3 - 8 + 1 = -4$, and thus the genus is 3. This is precisely what we would expect for a double covering surface for the double torus. We can also discover relations for H by considering these vertex cycles as follows:

$$z_{15} \xrightarrow{g_a} z_1 \xrightarrow{g_b} z_3 \xrightarrow{g_c} z_5 \xrightarrow{g_d} z_7 \xrightarrow{g_e} z_9 \xrightarrow{g_f} z_{11} \xrightarrow{g_g} z_{13} \xrightarrow{g_h} z_{15},$$

$$z_{14} \xrightarrow{g_a} z_2 \xrightarrow{g_c} z_6 \xrightarrow{g_e} z_{10} \xrightarrow{g_g} z_{14} \text{ and } z_0 \xrightarrow{g_b} z_4 \xrightarrow{g_d} z_8 \xrightarrow{g_f} z_{12} \xrightarrow{g_h} z_0,$$

which implies that $g_hg_gg_fg_eg_dg_cg_bg_a = g_gg_eg_cg_a = g_hg_fg_dg_b = \text{id}$ (because they have fixed points). Actually, these relations also follow from the fact that $g_1g_2^{-1}g_3g_4^{-1}g_1^{-1}g_2g_3^{-1}g_4 = \text{id}$ in Γ. By the theory we know that the resulting surface must be a double covering for the 2-torus, but since this polygon satisfies the side and angle conditions we can also conclude this from the triangulation of D.

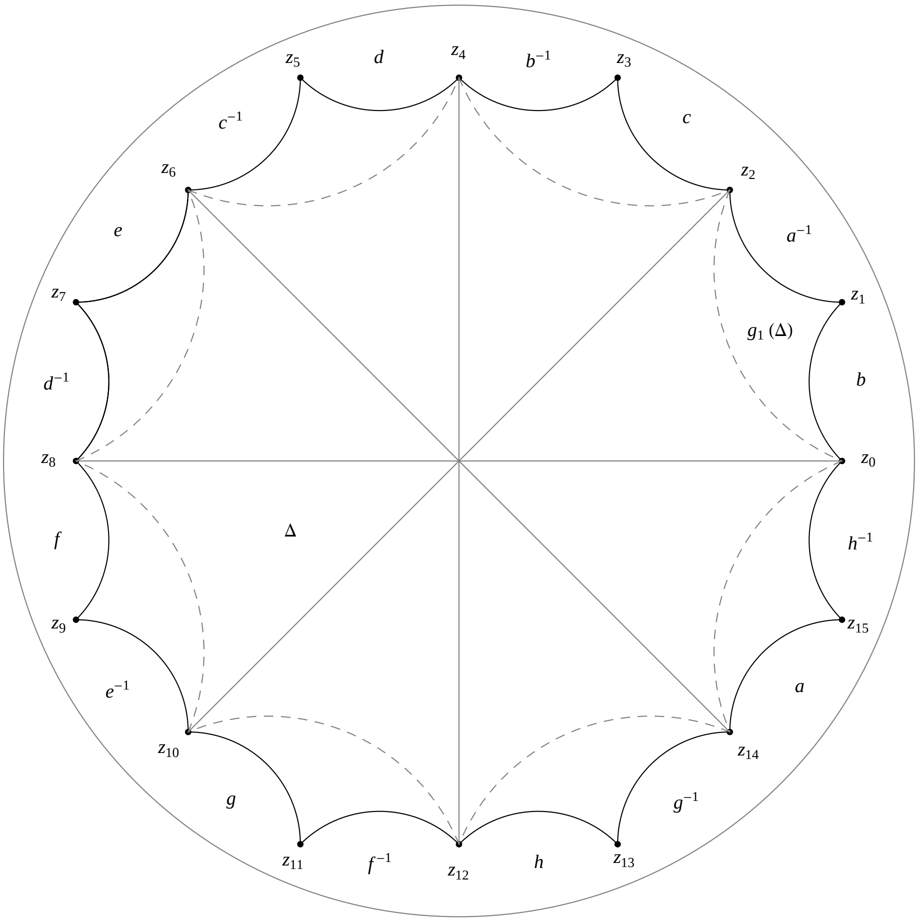

Figure 11: The Dirichlet polygon for $H = \ker(f)$ is the fundamental polygon for Γ together with the images of the triangles in said trianglulation of Π under the side-pairing transformation of Π. This figure shows the triangulation of Π (dotted) and the labeling for D as induced by the side-pairings listed above.

4.5 Dirichlet Polygons for Subgroups of Higher Index

Unfortunately, the construction of Dirichlet polygons for subgroups of Γ constructed along the lines as outlined in section 3.6 gets rather tricky if one is to persue the construction of the Dirichlet polygon in the $\mathbb{D}^2$ model. We have computed the Dirichlet polygon for several other subgroups, but our methods of computation are not sophisticated enough to find the side-pairing transformations for these polygons. We conclude by presenting the following images, for which we have constructed a subgroup of Γ, and plotted some of the equidastant lines from 0 to $h0$ for elements h in this subgroup. As the reader can see, the situation becomes a lot messier in these cases.

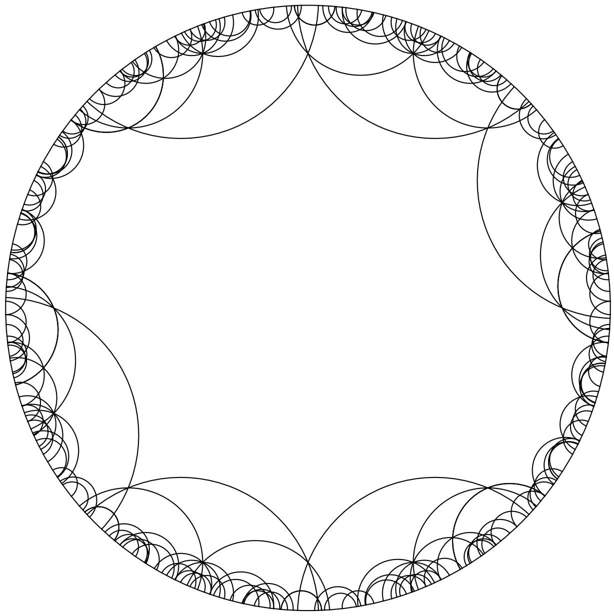

Figure 12: The Dirichlet region $D(0)$ for the subgroup constructed as the kernel of $f : \Gamma \to \mathbb{Z}_2$ as determined by setting $g_1, g_2, g_3 \mapsto 0, g_4 \mapsto 1$. This polygon has 20 sides, so as an identification space we expect it to have 5 vertex cycles.

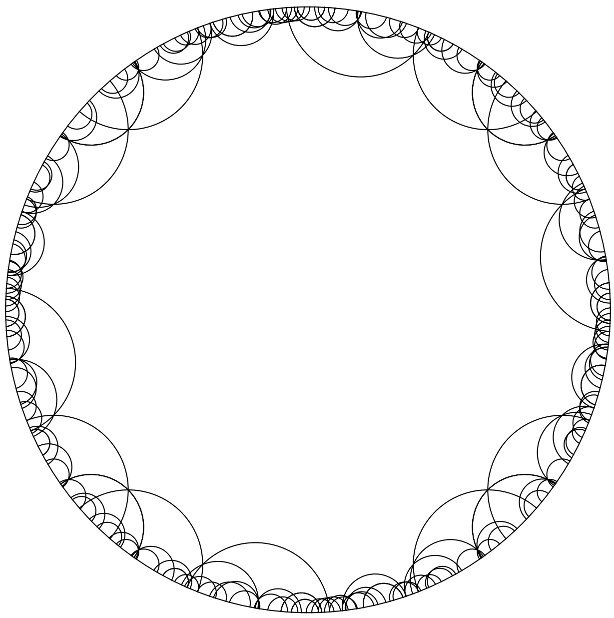

Figure 13: This figure shows the Dirichlet region $D(0)$ for the subgroup constructed as the kernel of $f : \Gamma \to \mathbb{Z}_3$ given by $g_1, g_2, g_3 \mapsto 1, g_4 \mapsto 2$. This is a polygon with 32 sides, and would be a fundamental domain for a covering of multiplicity 3. We would expect the identification space to have 9 vertex cycles.

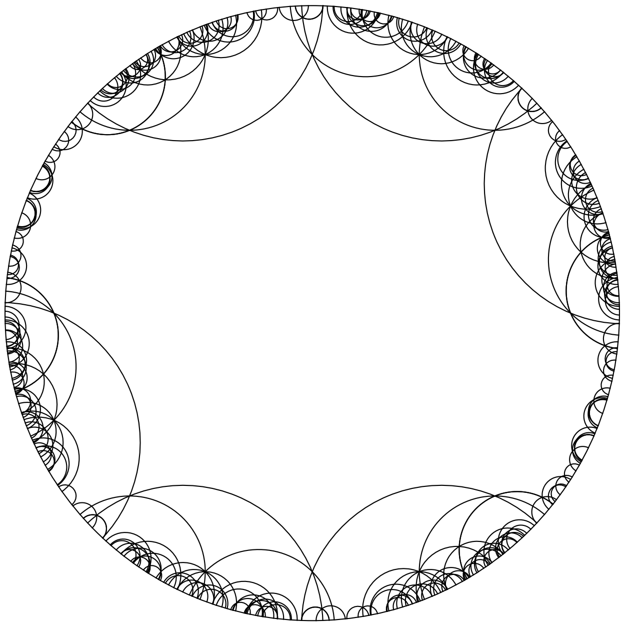

Figure 14: The Dirichlet region $D(0)$ for the subgroup constructed as the kernel of $f : \Gamma \to \mathbb{Z}_3$ as constructed by setting $g_1, g_2, g_3 \mapsto 0, g_4 \mapsto 2$. This is a polygon with 26 sides, so we would expect the identification space of this polygon to have 6 vertex cycles.

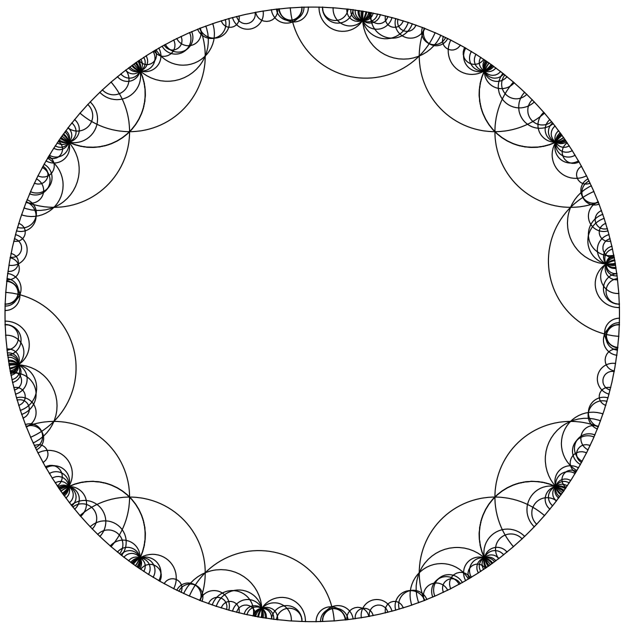

Figure 15: The Dirichlet region $D(0)$ for the subgroup constructed as the kernel of $f : \Gamma \to \mathbb{Z}_4$ as constructed by setting $g_1, g_2, g_3 \mapsto 1, g_4 \mapsto 3$. This is a polygon with 36 sides.

5 Conclusion

The goal of this paper was to find fundamental domains for groups correpsonding to covering surfaces for the double torus. We have discussed two methods for doing this, one by finding a particular coset decomposition of the group of the base surface by the subgroup corresponding to the covering surface. The other by simply constructing the Dirichlet polygon for the subgroup. We outlined the construction of the fundamental polygon for cyclic coverings of the double torus by using the coset decomposition construction, and this enables us to construct a covering of any finite multiplicity. Furthermore, we have constructed the Dirichlet region for several subgroups of a double torus group. First we encountered a nice polygon, and we were able to identify the side-pairing transformations for this polygon. The Dirichlet regions we have constructed for other subgroups seemed somewhat less tractable. It would require further investigation to identify the side-pairing transformations for these polygons. We have not been able to discover any patterns in the construction of Dirichlet regions for subgroups of our double torus group.

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