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Dupin Cyclides

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Abstract

A Dupin cyclide is the envelope of a one-parameter family of spheres tangent to three fixed spheres. By studying the relation between a Dupin cyclide and a torus of revolution, partly from an inversive geometry approach, we prove that the definition of the first is equivalent to being the conformal image of a torus of revolution. Here we define a torus of revolution such that all standard tori fall within the definition. Furthermore, we look into the curvature lines of a Dupin cyclide, where we use amongst others the theorems of Joachimsthal and Meusnier, and prove that the two previous characterizations of a Dupin cyclide are equivalent with being a surface all whose lines of curvature are circles.

Keywords: Dupin Cyclide, inversion, Joachimsthal, Meusnier, circular line of curvature.

1 Introduction and history

1.1 Maxwells construction of a Dupin cyclide

During the 19th century, at the age of sixteen, the French mathematician Charles Pierre Dupin discovered a surface which is the envelope of a family of spheres tangent to three fixed spheres [Bertrand 1888]. In his book *Application de Géométrie* he called these surfaces *cyclides* [Dupin 1822]. Dupin's cyclides have been studied and generalized by many mathematicians including Maxwell (1864), Casey (1871), Cayley (1873) and Darboux (1887). However, the generalized cyclides turned out to have properties quite different from those discovered by Dupin. Therefore the term cyclide has been used for quartic surfaces with the circle at infinity as double curve [Forsyth 1912] and Dupin's cyclides have been called *cyclides of Dupin* or just *Dupin cyclides*. In this article we will use cyclide and Dupin cyclide interchangeably, always referring to Dupin's cyclide.
To give a basic idea of a cyclide we will start with an intuitive way of describing one. A very clear and simple explanation of how to construct a cyclide was given by J. C. Maxwell [Maxwell 1864]. We cite from Boehm [Boehm 1990]:

> "Let a sufficiently long string be fastened at one end to one focus **f** of an ellipse, let the string be kept always tight while sliding smoothly over the ellipse, then the other end `z` will sweep out the whole surface of a cyclide `Z`."

Next Maxwell extends the string by the line `L` which meets the hyperbola confocal to the ellipse, as illustrated in fig. 1. This figure shows that if line `L` meets the hyperbola, both parts of the string are balanced at the point `p`, moreover each fixed point `c` on `L` will sweep out the surface of a cyclide `C` confocal to `Z`. Notice that the vertices of the ellipse are the foci of the hyperbola and vice versa [Boehm 1990]. The chosen ellipse is defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad a \geq b > 0$$

whereas the cyclide is defined uniquely by a, b and a parameter μ which is the length of the string minus a. Two alternatives for a normal form for the implicit equation of a cyclide were developed by Forsyth [Forsyth 1912]:

$$(x^2 + y^2 + z^2 - \mu^2 + b^2)^2 = 4(ax - c\mu)^2 + 4b^2y^2 \tag{1.1}$$

$$(x^2 + y^2 + z^2 - \mu^2 - b^2)^2 = 4(cx - a\mu)^2 - ab^2y^2 \tag{1.2}$$

where $a, b > 0, c, \mu \geq 0$ and $a^2 \geq b^2 \geq c^2$.

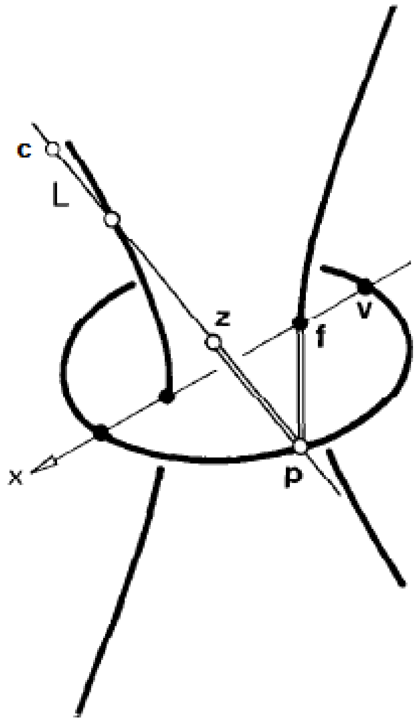

Figure 1: Maxwell's construction of a cyclide [Boehm 1990].

Let us observe the cross-sections of the cyclide Z by the planes $\mathtt{y} = 0$ and $\mathtt{z} = 0$ illustrated in fig. 2. Each of these planes intersect the cyclide in two circles from which many properties of a cyclide can be derived. One can easily obtain their equations by setting $y = 0$ in (1.1) and $z = 0$ in (1.2) [Johnstone 1993]. The smaller circle in the $y = 0$ plane never lies completely inside the larger one, therefore they are called *exterior circles*. On the other hand the smaller circle in the $z = 0$ plane never lies fully outside the larger one and are therefore called *interior circles*. Note that $y = 0$ and $z = 0$ are the two planes of symmetry of the Dupin cyclide in (1.1), also called respectively the *exterior* and *interior planes of symmetry* [Johnstone 1993]. The actual position of the interior and exterior circles determine the shape of a cyclide. From Johnson [Johnstone 1993] we know that the exterior circles intersect if and only if $\mu \geq a$, whereas the interior circles intersect if and only if $\mu \leq c$. We distinguish three types of cyclides:

1. *Ring cyclide* fig. 3(a): In the exterior and interior plane, the circles lie respectively completely out- and inside one another. Moreover $c < \mu < a$.

2. *Spindle cyclide* fig. 3(b): The circles in the exterior plane intersect and in the interior plane the smaller circle lies completely inside the larger one. This implies that $\mu \geq a$.

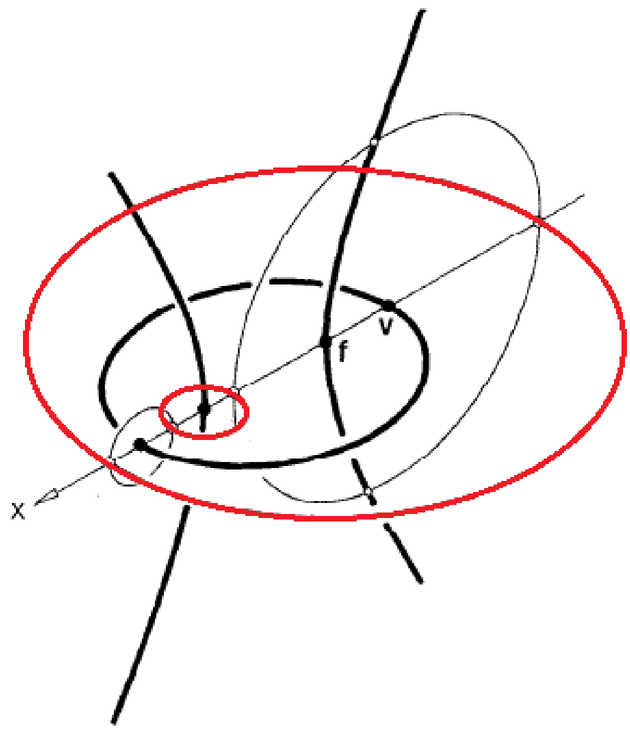

Figure 2: Exterior (black, non-bold) and interior (red) circles of cyclide Z [Boehm 1990].

3. *Horned cyclide* fig. 3(c): In the exterior plane, the smaller circle lies completely outside the larger one, while the two circles situated in the interior plane intersect. Thus, $\mu \leq c$.

1.2 Contents

In this article we will look at three different ways to characterize a Dupin cyclide. Based on one definition of a cyclide, we will derive two other characterizations. Moreover, we will verify their equivalence. To do so, we need some knowledge about inversion theory, which will be explained in the section two. In the subsequent section we will define a Dupin cyclide as the envelope of a one-parameter family of spheres tangent to three fixed spheres, moreover we will prove both the theorem that a Dupin cyclide is a conformal image of the torus of revolution as well as the equivalent relation of this theorem and our definition of a cyclide. In the last section we will start with some theory on curvature lines and envelopes after which we then prove our third characterization of a cyclide, which states that a Dupin cyclide is a surface whose lines of curvature are all circles. We will accomplish this by first inquire into surfaces with lines of curvature in one system, whereafter we switch to surfaces with circular lines of curvature in both systems. Furthermore we will complete the proof of the equivalence of the three characterizations of a Dupin cyclide.

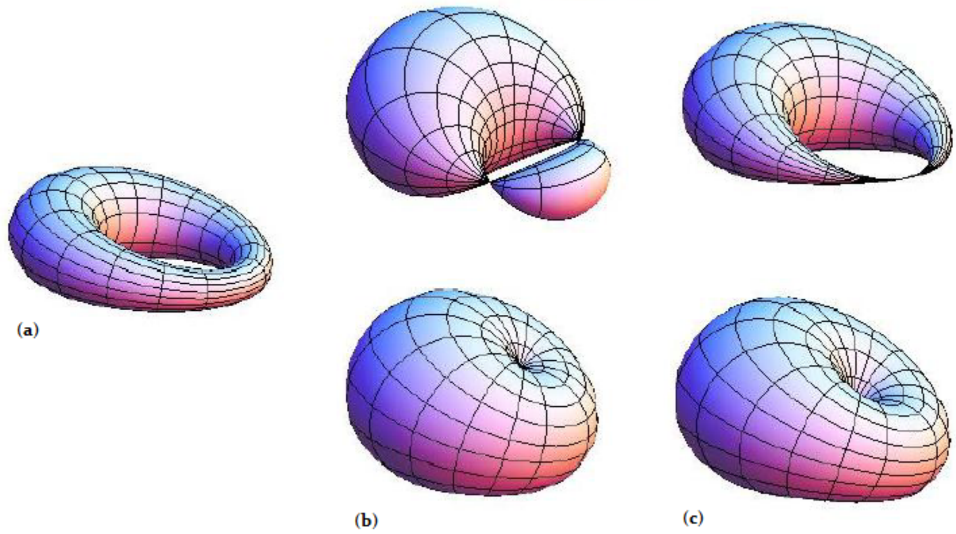

Figure 3: A ring cyclide (a), two spindle cyclides (b) and two horn cyclides (c).

2 Inversion Theory

In the next section we are interested in the conformal image of a Dupin cyclide. Previously we described the Dupin cyclide as the envelope of the one-parameter family of spheres tangent to three fixed spheres, which will become our definition of the cyclide. Therefore it will be convenient to have some knowledge of transformations that preserve angles and map spheres into spheres. In this section we will present some basics of such transformations, called *inversions*, which are reflections in a sphere or circle. The inversion theory presented here is derived from [Blair 2000] and [BrEsGra 1999], where extensive material on this subject can be found.

As reflection in a line maps a point from one side of the line to the other side, by inversion in a circle C with center O and radius r a point p lying inside the circle is mapped to a point outside the circle and vice versa. The inverse of the point p is notated as p' and is situated on the line segment Op such that $Op \cdot Op' = r^2$, see fig. 4. The circle C is called the *circle of inversion* and by the definition we just mentioned, inversion maps points on C onto themselves. Moreover, the center O is called the *center of inversion* and has no image under inversion and no point in the plane is mapped onto O. However, if we adjoin one *point at infinity* to the Euclidean plane, this defines the image under inversion of O and vice versa. Clearly inversion in a circle is its own inverse; the inverse of p' with respect to C is again p. And since the inversion map is invertible, it is bijective considering $f(O) = \infty$. If we let C be the unit circle, we get the following definition.

Definition 2.1. Inversion *of a point* $\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \backslash \{O\}$ *in the unit circle* $C = \{\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \| x^2 + y^2 = 1\}$ *is the function* $f : \mathbb{R}^2 \backslash \{O\} \longrightarrow \mathbb{R}^2$,

$$f(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^2}.$$

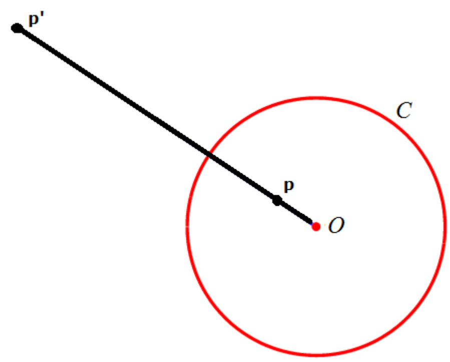

Figure 4: Inversion of a point p in a circle C with center O and radius r s.t. $Op \cdot Op' = r^2$.

We have seen that under inversion points inside the circle of inversion are mapped outside this circle and vice versa. We can extend this to saying that a point p_1, closer to the center of inversion than a point p_2, is mapped onto a point p_1' further away from the center than p_2' and vice versa. Since points on a line or circle through the center of inversion can be chosen arbitrarily close to this center, their image under inversion can be arbitrary far from the center and must therefore be situated on a line. Moreover, points on a line can be chosen arbitrary far from the center of inversion and therefore their image under inversion can be arbitrary close to it. Consequently they must lie on a line or circle through the center. These remarks lead to the following theorem that states that inversion maps lines and circles onto lines and circles. Proves of this theorem can be found in the two references brought forward previously in this section.

Theorem 2.2. *Inversion in a circle C with center O maps*

1. *a line through O onto itself.*
2. *a line not passing through O onto a circle passing through O.*
3. *a circle passing through O onto a line not passing through O.*
4. *a circle not passing through O onto a circle not passing through O.*

Besides that inversion in a circle maps circles and lines to circles and lines, it has another interesting property, namely that it preserves angles between curves, but it reverses their orientation. A transformation is said to be *conformal* if it preserves the magnitude of angles between any two curves at each point of its domain, which gives us the following theorem.

Theorem 2.3. *Inversion in a circle is a conformal map.*

As mentioned previously, we are interested in the conformal transformation that maps spheres to spheres. Therefore we will now generalize inversion in a circle to inversion in a sphere.

Definition 2.4. *Let $S_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$ denote the sphere centered at x_0 with radius r. Then inversion in $S_r(x_0)$ is the mapping $g : \mathbb{R}^n \backslash \{x_0\} \longrightarrow \mathbb{R}^n$ defined by*

$$g(x) = x_0 + r^2 \frac{x - x_0}{\|x - x_0\|^2}.$$

$S_r(x_0)$ is called the sphere of inversion, x_0 the center of inversion and r the radius of inversion.

Note that since g maps $\mathbb{R}^n \backslash \{x_0\}$ onto itself, $g \circ g$ is defined on $\mathbb{R}^n \backslash \{x_0\}$ and is the identity.

Like circle inversion, inversion in a sphere is a conformal map. So in particular it preserves tangency and orthogonality. Moreover sphere inversion satisfies the following properties.

Theorem 2.5. *Inversion in a sphere $S_r(x_0)$ with radius r and center x_0 maps*

1. *a line through x_0 onto itself.*
2. *a line not passing through x_0 onto a sphere passing through x_0.*
3. *a sphere passing through x_0 onto a line not passing through x_0.*
4. *a sphere not passing through x_0 onto a sphere not passing through x_0.*

Proof. We will prove the last property. Let without loss of generality the unit sphere be the sphere of inversion. So the radius of inversion $r = 1$ and the center of inversion is the origin $x_0 = O$. Then the inversion map reduces to $g : \mathbb{R}^n \backslash \{O\} \longrightarrow \mathbb{R}^n$

$$g(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Let $S_R(y_0)$ with radius R and center $y_0 = (a, b, c) \in \mathbb{R}^3$ be an arbitrary sphere, not through O. And define the inverse of a point $(x, y, z) \in S_R(y_0)$ as (X, Y, Z), thus

$$g(x, y, z) = \frac{(x, y, z)}{x^2 + y^2 + z^2} := (X, Y, Z).$$

And since g is its own inverse, we also have $g(X, Y, Z) = (x, y, z)$. Therefore we can write

$$X = \frac{x}{x^2 + y^2 + z^2}, \quad Y = \dots \quad and \quad x = \frac{X}{X^2 + Y^2 + Z^2}, \quad y = \dots \tag{2.1}$$

Define $d = a^2 + b^2 + c^2 - R^2$, then the general equation for $S_R(y_0)$ becomes

$$0 = x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d. \tag{2.2}$$

Since $S_R(y_0)$ is not passing through O, $d \neq 0$.
For the image after inversion of $S_R(y_0)$ to be a sphere, we must be able to write its equation in a similar way as (2.2).
With (2.1) we express

$$\begin{aligned} X^2 + Y^2 + Z^2 &= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{2ax + 2by + 2cz - d} \\ &= \frac{X^2 + Y^2 + Z^2}{2aX + 2bY + 2cZ - d(X^2 + Y^2 + Z^2)}. \end{aligned}$$

Rewriting and dividing by $d \neq 0$ gives

$$X^2 + Y^2 + Z^2 = 2\frac{a}{d}X + 2\frac{b}{d}Y + 2\frac{c}{d}Z - \frac{1}{d}. \tag{2.3}$$

Hence we know that the image of $S_R(y_0)$ after inversion in the unit sphere is a sphere with center

$$y_0' = (\frac{a}{d} + \frac{b}{d} + \frac{c}{d}) = \frac{y_0}{d} = \frac{y_0}{\|y_0\|^2 - R^2}$$

and radius

$$R' = \sqrt{\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2 - \frac{1}{d}} = \frac{R}{\|y_0\|^2 - R^2}. \tag{2.4}$$

If this sphere would pass through the origin, we would get from (2.3) that $0 = -\frac{1}{d}$, implying that $0 = 1$, which is most certainly not true. Therefore we can conclude that the sphere, being the image of $S_R(y_0)$ after inversion in the unit sphere, is not passing through O. This completes the proof. □

3 Dupin Cyclides

We have discussed the inversion map in the previous section, giving us the ability to study the image under inversion of a Dupin cyclide in the current section. In order to give a proper definition of a cyclide, we begin with a view definitions.

Consider a family of curves, all determined by a value of one parameter. The curve that is tangent to every curve in the family is called the *envelope* of the family. However, as mentioned in the previously section, we are interested in the envelope of a family of spheres. Therefore we first need the following definition of a *one-parameter family of surfaces.*

Definition 3.1. *A* one-parameter family of surfaces *is the family defined by an equation of the form*

$$F(x, y, z, a) = 0,$$

where each surface is determined by a value of the parameter a.

A *characteristic* of a one-parameter family of surfaces is the limiting curve of intersection of two coincident members of the family, whose parameters approach a common value. So the characteristic of a one-parameter family of spheres is the intersection curve of two coincident spheres, which is a circle. As a varies, the locus of families of characteristic curves belonging to different parameters defines a surface that is called the *envelope* of the family of surfaces.

Definition 3.2. *The equation of the curve of intersection of two surfaces of a one-parameter family of surfaces corresponding to the values a and a' of the parameter is defined by:*

$$F(x, y, z, a) = 0, \qquad \frac{F(x, y, z, a') - F(x, y, z, a)}{a' - a} = 0, \tag{3.1}$$

The limiting form of this curve that arises as a' approaches a is the characteristic *of the surface of parameter a, whose equations are*

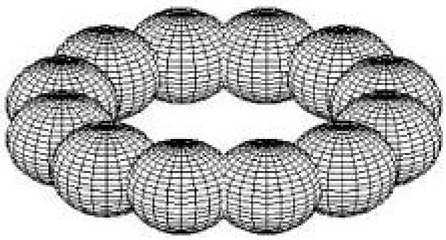

Figure 5: A one-parameter family of spheres of constant radius and center on a given circle.

$$F(x, y, z, a) = 0, \quad \frac{\partial F(x, y, z, a)}{\partial a} = 0. \tag{3.2}$$

The envelope *of the family of surfaces can be obtained by eliminating the parameter a from the equations in (3.2).*

Consider a one-parameter family of spheres of constant radius and with their center on a given circle, whose radius is greater than the radius of the spheres. Then, as can be seen in fig. 5, the envelope of this family is a standard torus of revolution. To illustrate the computation of an envelope of a one-parameter family we give the following example.

Example 1. Envelope of a one-parameter family of spheres. *Let the equation of a one-parameter family of spheres of the same radius be defined as*

$$F(x, y, z, a) = (2x - a)^2 + y^2 + z^2 - r^2 = 0.$$

Then it can easily be seen that derivation with respect to the parameter a gives the expression

$$\frac{\partial F(x, y, z, a)}{\partial a} = -2(2x - a) = 0,$$

so that elimination of the parameter a from this equation yields $a = 2x$. By substituting this expression for a in the first equation we obtain the equation of the evolute

$$F(x, y, z, a = 2x) = y^2 + z^2 - r^2 = 0,$$

which is the cylinder of radius r and whose axis is the x-axis.

With these definitions we are now able to understand the definition of a Dupin cyclide, which reads as follows.

Definition 3.3. *A Dupin cyclide is the envelope surface of a one-parameter family of spheres tangent to three fixed spheres.*

In what is to come in this section, we will look at the image under inversion of a Dupin cyclide, where we always link back to definition 3.3.

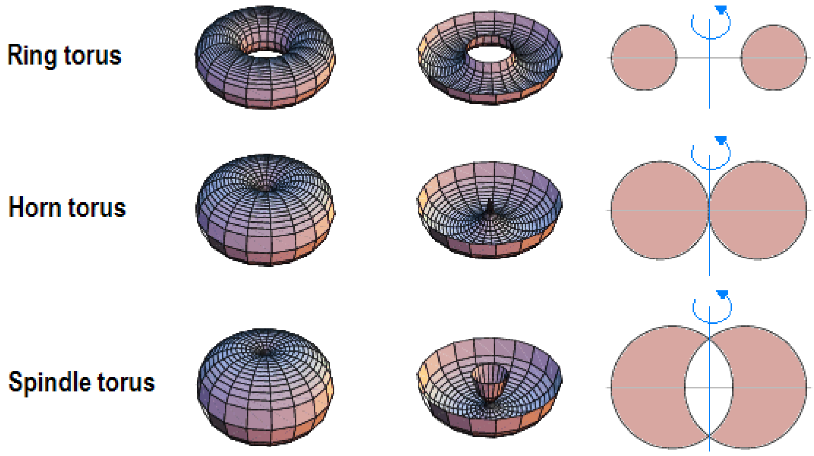

Figure 6: Various tori of revolution with their middle cutaway and cross-section showing the axis of revolution.

3.1 The conformal image of a Dupin cyclide.

With a torus of revolution we mean a torus obtained by rotating a circle around an axis in the plane of the circle, where this axis can intersect the circle. As is shown in fig. 6, we obtain a *ring torus* if the circle and axis are distinct, while the tangent circle and axis produce a *horn torus* and finally the intersecting axis and sphere lead to a so called *spindle torus*, all of which we consider as different tori of revolution. With this notion we will study the image under inversion of a torus of revolution, moreover we will prove in this section that the image under inversion of the torus of revolution in $\mathbb{R}^3$ is a Dupin cyclide in $\mathbb{R}^3$ and vice versa.

Theorem 3.4. *The image of a Dupin cyclide under inversion is a torus of revolution.*

Proof. By definition 3.3, we can define a cyclide as the envelope of a one-parameter family of spheres tangent simultaneously to three fixed spheres, say $S^{'}, S^{''}$ and $S^{'''}$.
Consider the plane determined by the centers of $S^{'}, S^{''}, S^{'''}$. This plane cuts the three spheres in three circles, respectively $C^{'}, C^{''}, C^{'''}$.

Let C be the circle orthogonal to these three circles and x_0 be a point situated inside one of the circles $C^{'}, C^{''}, C^{'''}$ and on C. If we let g be an inversion in the sphere $S_r(x_0)$ and C to be the equator of a sphere S, then according to theorem 2.5 S is transformed by g into a straight line L not passing through x_0.

We placed the center of inversion inside one of the circles $C^{'}, C^{''}, C^{'''}$, which means that no of the spheres $S^{'}, S^{''}, S^{'''}$ are passing through it. Then according to theorem 2.5 the three spheres are transformed into spheres not passing through x_0. Moreover, orthogonality is preserved by this transformation, which tells us that the images of $S^{'}, S^{''}$ and $S^{'''}$ under this inversion must be orthogonal to L. Hence the three fixed spheres are mapped onto three spheres $g(S^{'}), g(S^{''}), g(S^{'''})$ whose centers are on L.

Observe the family F of spheres tangent simultaneously to $S^{'}, S^{''}$ and $S^{'''}$. Since x_0 is located inside one of the circles $C^{'}, C^{''}, C^{'''}$ and therefore in one of the fixed spheres, the spheres of F can not pas through x_0. As tangency is preserved by this transformation, the images of the spheres of F under inversion are spheres not through x_0 and tangent to the three spheres $g(S^{'}), g(S^{''}), g(S^{'''})$, whose centers are on L.

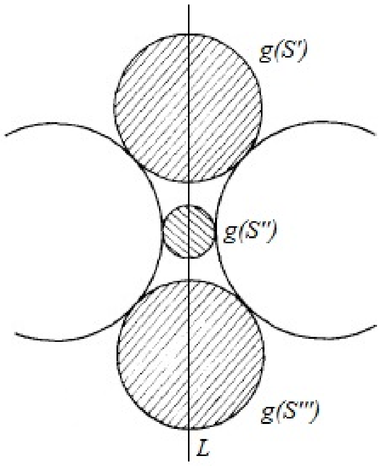

Figure 7: Intersection of the three fixed spheres after inversion and the plane of centers

Fig. 7 illustrates the intersection of the images of the three fixed spheres after inversion with the plane determined by their centers. The intersection of this plane with the images after inversion of the spheres of F are indicated by the empty circles. The radius of these spheres is constant and their centers lie on a circle. Consequently the envelope of this family of spheres is a torus with L as axis. Moreover, since this torus is a one-parameter family of spheres tangent to three fixed spheres, it is a Dupin cyclide.
Hence the given envelope of a one-parameter family of spheres, all tangent to three fixed spheres, is mapped under inversion onto a torus of revolution.

□

Note that if we allow the tangency of the spheres of F to the three fixed spheres, beside exterior to be interior, that we can obtain a whole different image, as is shown in fig. 8. If we let the two empty circles go all the way around, still remaining tangent to every fixed sphere, we obtain a spindle torus, which is in particular a spindle cyclide.
Having seen this, we can argue the other way, which is almost the same reasoning. However, we will limit ourselves to $\mathbb{R}^3$.

Theorem 3.5. *The conformal image of the torus of revolution in $\mathbb{R}^3$ is a Dupin cyclide in $\mathbb{R}^3$.*

Proof. Let $T \subset \mathbb{R}^3$ be our torus of revolution. As we saw in the proof of theorem 3.4, T is the envelope of a one-parameter family F' of spheres tangent to three fixed spheres, say S^1, S^2, S^3. Let g be an inversion with respect to $S_r(x_0)$ with x_0 contained inside S^1, S^2 or S^3, so non of them nor the spheres of F' can pas through it. Then according to theorem 2.5, g maps the three fixed spheres onto the spheres $g(S^1), g(S^2), g(S^3)$ not

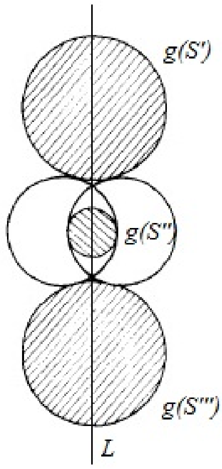

Figure 8: Tangency of spheres of F can be interior as well as exterior inducing a spindle torus.

through x_0. Moreover, since g preserves tangency, it maps any sphere of F' onto a sphere tangent simultaneously to $g(S^1), g(S^2), g(S^3)$. Hence the image under inversion of the torus of revolution is an envelope of a one-parameter family of spheres tangent to three fixed spheres and therefore a Dupin cyclide. □

Not only we can obtain a Dupin cyclide via the inversion of a torus of revolution, but also the three different types of tori give rise to three different types of Dupin cyclides. This can easily be seen if we make use of our result in the proof of theorem 2.5, where we found the expression (2) for the radius of a sphere not passing through the origin after inversion in the unit sphere centered at the origin.

Corollary 3.6. *The image under inversion of*

1. *a ring torus in* $\mathbb{R}^3$ *is a ring cyclide in* $\mathbb{R}^3$.
2. *a horn torus in* $\mathbb{R}^3$ *is a horn cyclide in* $\mathbb{R}^3$.
3. *a spindle torus in* $\mathbb{R}^3$ *is a spindle cyclide in* $\mathbb{R}^3$.

Proof. Let $T \subset \mathbb{R}^3$ be a torus of revolution, by the proof of theorem 3.4 being the envelope of the one-parameter family F' of spheres tangent to three fixed spheres. Let g be an inversion with respect to $S_r(x_0)$ with x_0 contained inside S^1, S^2 or S^3, so non of them nor the spheres of F' can pas through it. Then we know from theorem 3.5 that the image under inversion of T is a Dupin cyclide in $\mathbb{R}^3$, where we denote the one-parameter family of spheres tangent to three fixed spheres and whose envelope is the cyclide by F.
Moreover, without loss of generality we can choose our coordinate system such that x_0 is the origin and let the radius of inversion be equal to one. Then the sphere of inversion becomes $S_r(x_0) = S_1(O)$ so that we know from the proof of theorem 2.5 that a sphere $S_R(y_0)$ with radius R and center y_0 of the family F' is mapped under inversion to a sphere not through O and with radius R' defined as

$$R' = \frac{R}{\|y_0\|^2 - R^2}.$$

So if we consider T to be a ring torus we know that all the spheres of the one-parameter family F' must be of radius different from zero and therefore all spheres of F' are mapped under inversion in $S_1(O)$ onto spheres not through O and with radius different from zero. Hence T is mapped under this inversion onto a ring cyclide.
When we consider T to be a horn torus, one of the spheres of the family F' must be equal to a point, i.e. having a radius equal to zero. Therefore the Dupin cyclide being the image of T under inversion in $S_1(O)$ must be the envelope of a one-parameter family of spheres of which one sphere is of radius zero, making it a horn cyclide.
Finally if we let T be a spindle cyclide, two of the spheres of the family F' must be equal to two different points, which is why two of the spheres of the one-parameter family F must be equal to two different points. Therefore the image after inversion in $S_1(O)$ of T must be a spindle cyclide.

□

3.2 Lines of curvature.

In this section we are interested in the lines of curvature of a Dupin cyclide. Our aim is to proof that a necessary and sufficient condition that all lines of curvature of a surface be circles is that this surface is a Dupin cyclide. Since we have already proved that a cyclide is the inversion of a torus, it will be easy to show that the curvature lines of a cyclide are circular, given that inversion preserves lines of curvature. Before we come to that, we first discuss the concepts of curvature, curvature lines and focal surfaces. On the basis of these definitions we can develop a proper notice of the structure of a cyclide, so that we can describe the shape of the focal surface of a cyclide and compare it to that of an arbitrary surface of which all lines of curvature are circular.

3.2.1 Curvature, principal curves and focal surface.

Let $\alpha(s)$ be a *plane curve* in the connected and orientable surface $M \subset \mathbb{R}^3$, parameterized by arc length; $\|\alpha'(s)\| = 1$. Then, as is well known, the tangent $T = \alpha'$ is the *unit tangent vector field* on α and its derivative $T' = \alpha''$ is called the *curvature vector field* of α, since it measures the way α is turning in $\mathbb{R}^3$. A measure of the rate of change of the tangent T when moving along the curve is the *curvature* $\kappa(s) = \|T'(s)\|$. So the sharper α is turning, the larger κ is. The direction in which α is turning is given by the unit vector field $N = T'/\kappa$, which is called the *principal normal vector field* of α. If $\kappa > 0$, then we call the three mutually orthogonal unit vector fields T, N and B the *Frenet frame field* on α, where $B = T \times N$ on α is called the *binormal vector field* of α. Since α is a plane curve, the torsion $\tau(s)$, which measures how sharp the curve is twisting, is zero. Therefore we have the following corresponding *Frenet formulas*:

$$T' = \kappa N$$
$$N' = -\kappa T$$
$$B' = 0$$

Where the shape of a curve can be measured by its curvature and torsion, the shape of a surface $M \subset \mathbb{R}^3$ can be described by a certain linear operator defined on each tangent plane of M. This linear operator is called the *shape operator* S and is the negative derivative of the unit normal vector field U.

Definition 3.7. *Let* $\boldsymbol{p}$ *be a point on a connected and orientable surface* $M \subset \mathbb{R}^3$*,* $\boldsymbol{v}$ *any tangent vector to* M *at* $\boldsymbol{p}$ *and* U *a unit normal vector field on a neighborhood of* $\boldsymbol{p}$ *in* M*. The* shape operator S_p *of* M *at* $\boldsymbol{p}$ *derived from* U *is the linear operator on the tangent plane of* M *at* $\boldsymbol{p}$ $S_p : T_p(M) \longrightarrow T_p(M)$ *defined as*

$$S_p(\boldsymbol{v}) = -D_{\boldsymbol{v}}U.$$

Consider a curve α in the surface $M \subset \mathbb{R}^3$, oriented by the choice of a unit normal vector field U. Clearly the bending of M influences the shape of curves on it. At each point on M the component of acceleration α'' normal to the surface is given by the inner product $\alpha'' \cdot U$, which is the *normal curvature* or in other words, the curvature of the corresponding normal section. Since the velocity α' is always tangent to M we have

$$\alpha' \cdot U = 0,$$

U restricted to the curve α. Differentiating both sides of the previous equation and substituting the well known fact that $S(\alpha') = -U'$ gives us

$$\alpha'' \cdot U = -U' \cdot \alpha' = S(\alpha') \cdot \alpha'. \tag{3.3}$$

So the component of acceleration normal to the surface depends only on the shape operator and the velocity α' of M. Hence this component in a point $\mathbf{p}$ is the same for all curves on M which have the same velocity $\mathbf{v}$ at $\mathbf{p}$. Generalizing this result by reducing $\mathbf{v}$ to a unit tangent vector $\mathbf{u}$, we get a expression for the normal curvature.

Definition 3.8. *The* normal curvature $k(\boldsymbol{u})$ *of a surface* $M \subset \mathbb{R}^3$ *in the* $\boldsymbol{u}$ *direction, where* $\boldsymbol{u}$ *is a unit vector tangent to* M *at a point* $\boldsymbol{p}$ *is the number* $k(\boldsymbol{u}) = S(\boldsymbol{u}) \cdot \boldsymbol{u}$*.*

With the foregoing results we are now able to prove the theorem of Meusnier [Meusnier 1785, p. 477-510], which will prove to be particularly useful in section 3.2.3.

Theorem 3.9. *Let* α *be a unit-speed curve on a regular surface* $M \subset \mathbb{R}^3$ *with initial velocity* $\alpha'(0) = \boldsymbol{u}$*, where* $\boldsymbol{u}$ *is a unit vector tangent to* M *at a point* $\boldsymbol{p}$*. If* ϑ *is the angle between the plane of the normal section, determined by* $\boldsymbol{u}$ *and unit normal vector field* $U(\boldsymbol{p})$*, and the osculating plane, determined by* $\boldsymbol{u}$ *and the principal normal* $N(0)$*, then*

$$k(\boldsymbol{u}) = \kappa(0) cos(\vartheta), \tag{3.4}$$

where $\kappa(0)$ *is the curvature of* α *at* $\alpha(0) = \boldsymbol{p}$*.*

Proof. From equation (3.3) we know that $\alpha'' \cdot U = -U' \cdot \alpha' = S(\alpha') \cdot \alpha'$. Using this and the fact that $T' = \alpha'' = \kappa N$ we get

$$\begin{aligned} k(\mathbf{u}) = S(\mathbf{u}) \cdot \mathbf{u} &= S(\alpha'(0)) \cdot \alpha'(0) \\ &= \alpha''(0) \cdot U(\mathbf{p}) \\ &= \kappa(0) N(0) \cdot U(\mathbf{p}) \\ &= \kappa(0) \left\| N(0) \right\| \left\| U(\mathbf{p}) \right\| cos(\vartheta) \\ &= \kappa(0) cos(\vartheta). \end{aligned}$$

□

We can write $k = \frac{1}{r}$, where r is the radius of normal curvature and since $\kappa = \frac{1}{R}$, where R is the radius of curvature of α, we can rephrase equation (3.4) into the expression

$$R = rcos(\vartheta). \tag{3.5}$$

An geometric interpretation can be obtained as follows: Consider the sphere with radius equal to the radius of curvature of the normal section r and with center the corresponding center of curvature, then equation (3.5) utters that the circle being cut off the sphere by the osculating plane of the curve α, is the circle of curvature of the curve. In this manner we can cast the theorem of Meusnier into the form: *All curves on a surface passing through a given point* $\boldsymbol{p}$ *and having the same velocity at* $\boldsymbol{p}$ *also have the same normal curvature at* $\boldsymbol{p}$ *and their osculating circles form a sphere.*

As mentioned, the theorem of Meusnier will come in handy in section 3.2.3 where we deal with surfaces with one system of some special curves. Currently our interest is in this special curve γ on a surface $M \subset \mathbb{R}^3$, which travels in directions for which the bending of M in $\mathbb{R}^3$ takes its extreme values. This curve is called a *principal curve* or *curvature line* and is defined as follows.

Definition 3.10. *A* line of curvature *is a regular curve* γ *in a surface* $M \subset \mathbb{R}^3$ *with the condition that the velocity* γ' *of* γ *points in a principal direction. The* principal directions *of* M *at point* $\boldsymbol{p}$ *are the directions of the maximum and minimum values of the normal curvature of* M *at* $\boldsymbol{p}$*, which are called the* principal curvatures*, denoted by* k_1 *and* k_2*.*

If the normal curvature $k(\mathbf{u})$ is constant in the directions of all the unit tangent vectors $\mathbf{u}$ at a point $\mathbf{p}$ of $M \subset \mathbb{R}^3$, then $\mathbf{p}$ is called a *umbilic* point. Since $k(\mathbf{u})$ is constant, its maximum and minimum values must be the same, $k_1 = k_2$, and therefore there is only one principal direction. This leads us to the following theorem of which a proof can be found in [O'Neil 2006, p. 213].

Theorem 3.11. *Let* $\boldsymbol{p}$ *be a point of* $M \subset \mathbb{R}^3$*. If* $\boldsymbol{p}$ *is not umbilic, then there are exactly two principal directions and these are orthogonal.*

Therefore we can state that through every nonumbilic point $\mathbf{p}$ of M there are exactly two mutually orthogonal intersecting lines of curvature. We say that the orthogonal lines of curvature of all the point of M form two *systems of principal curves*.
In a moment we will see that the evolute is the locus of the so called centers of curvature, which we therefore discuss first. We define the *osculating circle* of a curve β on a surface $M \subset \mathbb{R}^3$ as the limiting position of the circle lying in the plane obtained by N and T and passing through three nearby points of β while one point is approached by the other two, fig. 9.
So for each point $\mathbf{p}$ on the curve β there is a corresponding osculating circle with its center on the principal normal N at distance $R = 1/\kappa$ from $\mathbf{p}$, where κ is the curvature at $\mathbf{p}$. The center of the osculating circle is the *center of curvature* and R is called the *curvature radius*. The centers of curvature corresponding to all the points of a curvature line γ on M form the envelope of the normals to M along γ [Valiron, p. 61]. Let ζ denote the envelope of the normals to M along γ, then the locus of the curves ζ corresponding to all the curvature lines of one system engender a surface Σ (fig. 10); i.e. the locus of centers of one system of curvature of M. In a similar way we obtain another surface Σ' from the

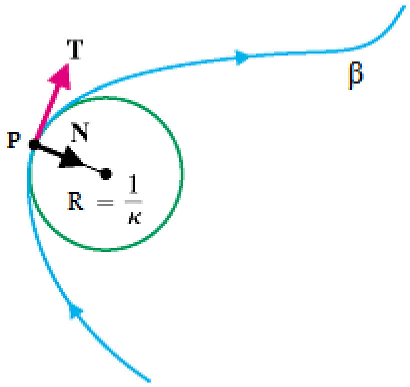

Figure 9: Osculating circle.

envelope of the normals along the curvature lines of the second system. Together the two surfaces form the so called *surface of centers* or *focal surface.*
A third denomination of the surface of centers is rooted in a different explanation of the concept. By definition the *evolute of a curve* is the curve that is tangent to all the normals to the first. Therefore consider two curvature lines γ_1 and γ_2 on the surface M through the point $\mathbf{p} \in M$. Since the normals to a surface along a line of curvature form a developable surface, the evolutes of γ_1 and γ_2 are curves on respectively Σ and Σ'. Consequently Σ and Σ' are the set of evolutes of the curvature lines on M, which is why together they are called the *evolute* of the surface M as well.

Definition 3.12. *The* evolute *of a surface is the locus of centers of both systems of curvature. Equivalently, it is the locus of envelopes of the normals along the curvature lines of both systems.*

3.2.2 Lines of curvature of a Dupin cyclide.

In what is to come in this section we want to proof that a surface being a Dupin cyclide is a necessary and sufficient condition that this surface has circular lines of curvature. Here we begin with one part of the proof, namely that all lines of curvature of a cyclide are circles. For the proof we make use of theorem 3.5 where we showed that a Dupin cyclide is the image under inversion of a torus. Clearly all lines of curvature of a torus are circular, so if we could prove that the lines of curvature are preserved under inversion, then we are done. With the following theorem of Joachimsthal we are indeed able to do so.

Theorem 3.13. Theorem of Joachimsthal. *If two surfaces intersect at a constant angle along a curve γ, then this curve is a line of curvature of both surfaces or neither. Conversely, if the curve of intersection of two surfaces is a line of curvature of both, then these surfaces meet under constant angle.*

Proof. Let M_1 and M_2 be two surfaces in $\mathbb{R}^3$ intersecting under positive angle and with unit normal vector fields U_1 respectively U_2. The intersection curve γ is regular and for all points $\mathbf{p} \in \gamma$ $\vartheta(\mathbf{p})$ denotes the angle between the normals $U_1(\mathbf{p})$ and $U_2(\mathbf{p})$.

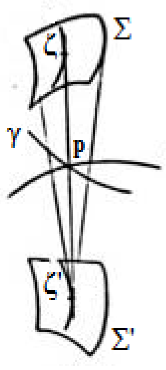

Figure 10: Sheets of the focal surface.

Define the unit speed parametrization of γ as $\alpha : I \longrightarrow \mathbb{R}^3$ with I an interval and tangent vector $T(t) = \alpha'(t)$. We will use simplified notations like

$$U_i' = \frac{d}{dt} U_i(\alpha(t))$$

to improve the readability.
Let the shape operators of M_1 and M_2 in $\alpha(t)$ respectively be defined as

$$S_1 : T_{\alpha(t)} M_1 \longrightarrow T_{\alpha(t)} M_1$$

$$S_2 : T_{\alpha(t)} M_2 \longrightarrow T_{\alpha(t)} M_2.$$

Now let γ be a curvature line of M_1 and M_2. Then by definition $\alpha'(t) = T(t)$ always points in a principal direction and therefore T is a principal vector of both surfaces at all $\mathbf{p} \in \gamma$. Therefore we have $S_i(T) = k_i T$, where k_i is the principal curvature of M_i at $\mathbf{p}$. Since T is also a tangent vector to both M_1 and M_2 we have

$$U_i \cdot T = 0$$

which implies that

$$U_1 \cdot S_2(T) = U_1 \cdot k_2 T = k_2 U_1 \cdot T = 0$$

and in a same manner we get

$$U_2 \cdot S_1(T) = k_1 U_2 \cdot T.$$

By definition $U_i = -S_i(T)$, then with the product rule we find that for all $\mathbf{p} \in \gamma$

$$(U_1 \cdot U_2)' = U_1' \cdot U_2 + U_1 \cdot U_2' = -S_1(T) \cdot U_2 - U_1 \cdot S_2(T) = 0.$$

So $U_1 \cdot U_2$ is constant and therefore

$$cos(\vartheta) = \frac{U_1 \cdot U_2}{\|U_1\| \, \|U_2\|} = U_1 \cdot U_2$$

is constant, implying that $\vartheta(\mathbf{p})$ is constant for all $\mathbf{p} \in \gamma$.

Conversely, given that the angle $\vartheta(\mathbf{p})$ between the normals $U_1(\mathbf{p})$ and $U_2(\mathbf{p})$ is constant for all points $\mathbf{p} \in \gamma$, $cos(\vartheta) = U_1 \cdot U_2$ is constant and therefore $(U_1 \cdot U_2)' = 0$.

1. Assume that γ is a curvature line on M_1, then we know from above that

$$U_2 \cdot S_1(T) = 0,$$

or equivalent that

$$0 = (U_1 \cdot U_2)' = -U_1 \cdot S_2(T)$$

implying that

$$U_1 \cdot S_2(T) = 0.$$

Since

$$U_2 \cdot S_2(T) = 0,$$

$S_2(T)$ is perpendicular to U_1 as well as to U_2 along γ, which means that $S_2(T)$ and T are collinear for all points $\mathbf{p} \in \gamma$. Hence T is a unit vector in a principal direction, i.e. α' always points in a principal direction and thus by definition γ is a curvature line on M_2.

2. Let now, given that $\vartheta(\mathbf{p})$ is constant for all points $\mathbf{p} \in \gamma$, γ not be a curvature line on M_1. Assume that γ is indeed a curvature line on M_2, then

$$0 = (U_1 \cdot U_2)' = -U_2 \cdot S_1(T).$$

Hence $S_1(T)$ and T are collinear and consequently γ is a curvature line on M_1, which causes a contradiction. Therefore given that γ is no curvature line on M_1 and $\vartheta(\mathbf{p})$ is constant for all points $\mathbf{p} \in \gamma$, γ cannot be a curvature line on M_2.

Therefore the intersection curve of two surfaces intersecting under constant angle, is a line of curvature on both surfaces or neither. □

Consider a line of curvature γ of the surface $M \subset \mathbb{R}^3$. Then the normals at M along the curve γ form a congruence and the developable surface of this congruence is called a *normality*. Since the normals to M are orthogonal to it, the surface is orthogonal to its normalities.

Theorem 3.14. *Inversion maps lines of curvature of one surface onto lines of curvature of the image surface.*

Proof. Let M be a surface in $\mathbb{R}^3$ and Γ the normality corresponding to a line of curvature γ. Since inversion preserves angles, it would preserve the orthogonality between M and Γ. So if M', Γ' and γ' respectively denote the images under inversion of M, Γ and γ, then M' is orthogonal to Γ' along γ'. So M' and the inverse surface of the normality meet at constant right angle.
As γ' is a line of curvature of the surface of the normality Γ', we now know from Joachimsthal's theorem that it is also a line of curvature on M'. □

As we noticed above, with the completion of this proof we also have fulfilled the proof of the following lemma.

Lemma 3.15. *All lines of curvature of a Dupin cyclide are circular.*

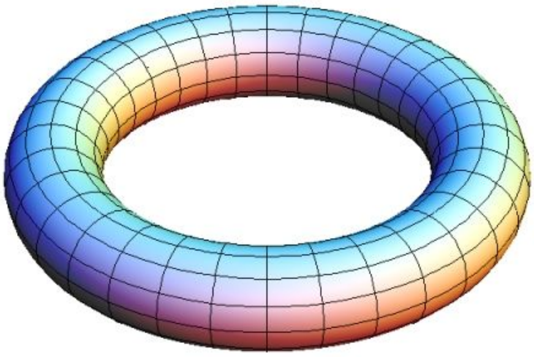

Figure 11: Circular lines of curvature in both systems on a torus.

Proof. From theorem 3.4 we know that a Dupin cyclide is the image under inversion of a torus. The lines of curvature in both systems of a torus are circular, as is displayed in fig. 3.2.2. As we have seen before, we can define the torus as the envelope of a one-parameter family of spheres tangent to three fixed spheres. One of the systems of curvature of the torus is given by the characteristics of these spheres, which are clearly circles.
That the characteristics form one system of lines of curvature, is because the normals to the torus are equal to the normals to the spheres along the characteristics. Accordingly, they form a congruence meeting the torus at constant right angle. Since the characteristics of the spheres are lines of curvature on the surfaces of the normalities, Joachimsthal's theorem tells us that they must be lines of curvature on the torus.
The torus can therefore be considered as engendered by a mobile circumference, i.e., by infinitely near intersecting spheres. The lines of curvature of the other system intersect orthogonal with the lines of curvature of the first system. Since inversion preserves angles and maps intersecting spheres onto intersecting spheres and lines of curvature onto lines of curvature it follows that the lines of curvature in both systems of a Dupin cyclide are circular.

□

3.2.3 Circular lines of curvature in one system.

Before we study surfaces with circular lines of curvature in both systems, let us first consider surfaces with one system of circular lines of curvature. Throughout the section we make use of the works of [O'Neil 2006, p. 203-210], [Picard, p. 424-428] and [Eisenhart 1909, p. 310-314].

Lemma 3.16. *The normals to a surface along a circular line of curvature form a cone of revolution.*

Proof. Let $M' \subset \mathbb{R}^3$ be a surface with circular lines of curvature in one system and let γ be one of these circles. From theorem 3.13 of Joachimsthal it follows that the normals to M along γ and the plane of the circle meet under constant angle. Considering that γ is a circle, this implies that the normals meet in a point situated on the axis of the circle γ and therefore form a right circular cone. □

Remark 1. *In addition to lemma 3.16 we have to mention that the cone formed by the normals to the surface along a circular line of curvature γ becomes a cylinder or plane if the vertex is situated at respectively infinity or in the plane of the circle γ.*

Lemma 3.16 helps to form an accurate picture of a surface with circular lines of curvature.

Lemma 3.17. *A surface has circular lines of curvature in one system iff the surface is the envelope of a one-parameter family of spheres, the locus of whose centers lie on a curve.*

Proof. Let Γ_1 be the system of circular lines of curvature of the surface $M \subset \mathbb{R}^3$ and $\gamma \in \Gamma$ one of the circular lines of curvature which passes through the point $\mathbf{p} \in M$. According to lemma 3.16 the normals to γ form a right circular cone.
Consider the normal section α of M through $\mathbf{p}$ defined as the intersection of M and the plane P determined by unit tangent vector in $\mathbf{p}$ $\mathbf{u_p}$ and the unit normal vector field $U(\mathbf{p})$. Let ϑ denote the constant angle between the plane of the circle γ and the plane P, then according to the theorem of Meusnier 3.5

$$R(\mathbf{p}) = r(\mathbf{p})cos(\vartheta) \tag{3.6}$$

where $R(\mathbf{p})$ is the radius of curvature of $\gamma(\mathbf{p})$ and $r(\mathbf{p})$ is the radius of normal curvature at $\mathbf{p}$. From section 3.2 we know that since $\alpha'(\mathbf{p}) = \gamma'(\mathbf{p})$, it follows from the theorem of Meusnier that the circle γ and the osculating circle of $\alpha(\mathbf{p})$, whose radius is the radius of normal curvature $r(\mathbf{p})$ and center $I_N(\mathbf{p})$ the center of normal curvature, form a sphere with radius $r(\mathbf{p})$ and center $I_N(\mathbf{p})$. Moreover, equation (3.6) implies that the center of normal curvature $I_N(\mathbf{p})$ has to be on the axis of the circle γ. And since the normals to γ form a right circular cone, the center of the sphere is the vertex of the cone.
So the locus of envelopes of normals to M along all curvature lines of Γ, which defines the corresponding sheet of the evolute, is the locus of all the centers of these curvature lines engendering a curve. Moreover, the sphere through γ with as radius the radius of normal curvature and as center the vertex of the cone engendered by the normals to γ is tangent to the surface M along γ. Therefore M is the envelope of a family of spheres dependent of the same parameter as the osculating circle α, since it is the spheres characteristic, and whose centers lie on a curve being one of the sheets of the evolute.

Conversely, the normals to the envelope along the circular characteristics form also a cone of revolution and this cone cuts the envelope surface at right angles. Moreover the circular characteristics are lines of curvature of the corresponding cones. So by Joachimsthal's theorem these characteristics are the lines of curvature in one system of the envelope surface. □

Since the locus of envelopes of normals to a surface along all curvature lines of one system define the corresponding sheet of the evolute, it follows from the proof of lemma 3.17 that the locus of the centers of the enveloped spheres is one sheet of the evolute of the surface.

3.2.4 Circular lines of curvature in both systems.

From lemma 3.17 it follows that a surface with circular lines of curvature in both systems is in two ways the envelope of a one-parameter family of spheres, where the locus of the

centers of these spheres is a curve. Hence both sheets of the evolute are curves. However, to obtain a better understanding of the nature of these curves, we will work from a surface with circular lines of curvature in one system.
Considering a surface $M' \subset \mathbb{R}^3$ with circular lines of curvature in one system. We know that one sheet of the evolute is formed by the locus of the centers of the spheres enveloped by M'. For the other sheet, let us focus on the envelope of the normals along the circular lines of curvature of the second system. Lemma 3.16 tells us that the normals to M' along a circular line of curvature of the second system also form a cone of revolution. Accordingly, the second sheet of the evolute is the envelope of a family of cones of revolution formed by the normals to M' along its circular lines of curvature. The envelope of a one-parameter family of surfaces is the locus of the characteristics of the family. According to [Peternell 1997, p. 33] the characteristics of a one-parameter family of cones are conics, so in particular this holds for right circular cones. Therefore we have that the second sheet of the evolute of a surface with circular lines of curvature in one system, is the locus of a family of conics. This leads us to the next theorem.

Theorem 3.18. *Both sheets of the evolute of a surface with circular lines of curvature are conics.*

Proof. Consider a surface $M \subset \mathbb{R}^3$ with circular lines of curvature in both systems. The above discussion tells us that both sheets of the evolute have to be the locus of a family of conics. Furthermore, in consequence of lemma 3.17 both sheets of the evolute must be curves and in combination with the previous statement this implies that both sheets of the evolute are conics. □

Continuing to consider the surface $M \subset \mathbb{R}^3$ with circular lines of curvature in both systems, we see that in each point $\mathbf{p}$ on M' two curvature lines from different systems intersect. Consequently the two corresponding spheres are tangent in $\mathbf{p}$. And since any line of curvature of one system is intersected by all curvature lines of the other system, each sphere of the first family is tangent to all the spheres of the second family [Picard, p. 427].

Corollary 3.19. *A surface with circular lines of curvature in both systems is a Dupin cyclide.*

Proof. From lemma 3.17 it follows that a surface with circular lines of curvature in both systems is in two ways the envelope of a one-parameter family of spheres. Moreover an arbitrary sphere of the first family is tangent to an arbitrary sphere of the second family according to the above. Therefore we can choose three arbitrary spheres of one family that are simultaneously tangent to all spheres of the other family. This defines a Dupin cyclide. □

Together lemma 3.15 and corollary 3.19 form the desired proof of the following theorem.

Theorem 3.20. *All lines of curvature on a surface are circular iff this surface is a Dupin cyclide.*

So since a Dupin cyclide is a surface all of whose lines of curvature are circles it follows from theorem 3.18 that both sheets of its evolute are conics. Moreover it can be shown that these conics lie in perpendicular planes, each of them passing through the foci of the other [Picard, p. 427-248], which is characteristic for a pair of focal conics. Therefore from theorem 3.20 it results that a surface of which the sheets of the evolute be a pair of focal conics necessarily has to have circular lines of curvature in both systems, i.e. be a Dupin cyclide, and vice versa.

4 Conclusion

The basis of this article was the definition that Dupin originally gave to his cyclide; the envelope of a one-parameter family of spheres tangent to three fixed spheres. Viewed from here we have studied the relation between a Dupin cyclide and the torus of revolution, where we considered a torus of revolution to be all tori obtained by rotating a circle around an axis in the plane of the circle, where the axis can intersect the circle. Therefore all standard tori, i.e. ring torus, horn torus and spindle torus, fall under this denomination. By using inversion in a sphere we showed that the torus of revolution is the conformal image of a cyclide of Dupin and furthermore that the conformal image of the torus of revolution in $\mathbb{R}^3$ is a cyclide of Dupin in $\mathbb{R}^3$. In addition we have specified the images under inversion of the various tori were we found that the image under inversion of

- a ring torus in $\mathbb{R}^3$ is a ring cyclide in $\mathbb{R}^3$.
- a horn torus in $\mathbb{R}^3$ is a horn cyclide in $\mathbb{R}^3$.
- a spindle torus in $\mathbb{R}^3$ is a spindle cyclide in $\mathbb{R}^3$.

With use of Joachimsthal's theorem we showed that lines of curvature of one surface are mapped under inversion onto lines of curvature of the image surface. Therefore we were able to make a proper study of the lines of curvature of a Dupin cyclide, resulting in the conclusion that all lines of curvature of a Dupin cyclide are circular. Finally we were interested in the answer to the question whether each surface with circular line of curvature is a Dupin cyclide. We tackled this problem by first considering surfaces with circular lines of curvature in one system where we could conclude, with help of Meusnier's theorem, that a surface has circular lines of curvature in one system if and only if the surface is the envelope of a one-parameter family of spheres, where the locus of the centers of the spheres lies on a curve. When this result is applied to a surface with circular lines of curvature in both systems it follows that a surface with circular lines of curvature in both systems is in two ways the envelope of a one-parameter family of spheres. These results together with the outcome that the normals to a surface along a circular line of curvature form a cone of revolution led to the inference that both sheets of the evolute of a surface with circular lines of curvature must be conics. Observing the obtained surface being the envelope of a one-parameter family of spheres in two different ways so that became clear that an arbitrary sphere of the first family is tangent to an arbitrary sphere of the second family led us to conclude that a surface with circular lines of curvature in both systems is a Dupin cyclide.
Summarizing we showed that the definition of a Dupin cyclide being the envelope of a one-parameter family of spheres tangent to three fixed spheres is equivalent to the two characterizations; *the conformal image of a torus of revolution* and *a surface all whose lines of curvature are circles.*

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